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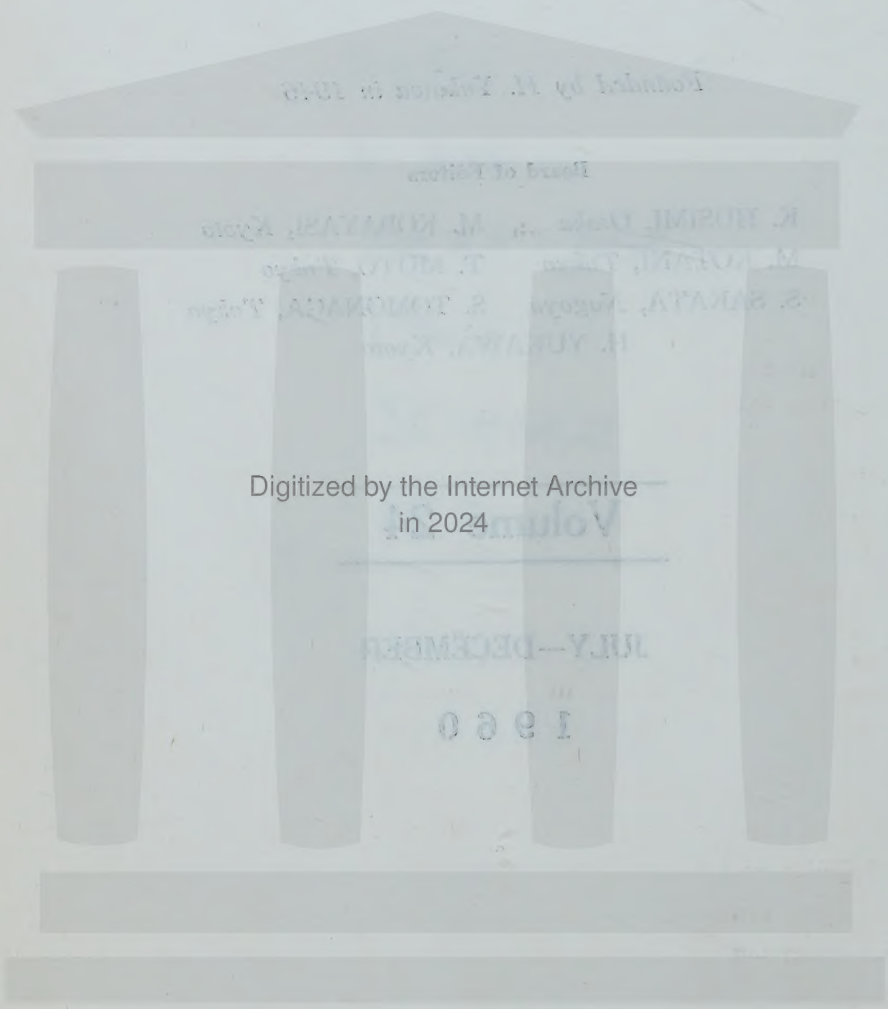

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Abandonment of the Assumption of the Displacement Invariance and Replacement of the Assumption of Field by the Assumption of Particle

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We describe a field theory by two parameter spaces in order to separate the two functions fulfilled by the space-time, i.e. the representation of the degree of freedom of the dynamical system and the specification of the order of measurements. After the two functions of space-time are separated, we abandon the assumption of the displacement invariance and assume the momentum-conservation law of finite "range".* Further, we replace the assumption of the field by the assumption of the particle. Then the domain of the applicability of theory is extended so as to be able to describe the elementary particles with finite degree of freedom. Some possible observable effects peculiar to the theory of finite degree of freedom are discussed. The difference between the concept of space-time distance in the present field theory and that in our theory is mentioned.

§ 1. Introduction

Field theory describes elementary particles and their interactions as a dynamical system of field variables which are functions of space-time coordinate, on the basis of the principle of the special theory of relativity and the quantum mechanics. In the theory, space-time fulfils two distinct functions. The first one is the representation of a degree of freedom of the dynamical system. The second one is the specification of the order of measurements in the dynamical principle. (The order is specified by space-like surfaces in space-time.) We may characterize the concept of space-time by these two functions. In order that a parameter space can fulfil the two functions, it is necessary that the dynamical principle is invariant under the translation group in the parameter space, i.e. the displacement invariance. The displacement invariance is not a mathematical theorem but a physical assumption in the concept of the movement in physics.

The principle of the special theory of relativity requires the invariance of theory under the inhomogeneous Lorentz group, i.e. the rotation group plus the translation group. While the requirement of invariance under the rotation group has been shown to be very useful for unified understanding of the experimental data, the requirement of the displacement invariance merely derives the momentum conservation law. The derived conservation law has the accuracy of zero " range."*

* We define "range" by g in Eq. (3·4) and Eq. (4·10).

We wish to mention in this paper that when we abandon the assumption of displacement invariance and assume a momentum conservation law of finite " range,"* the domain of the applicability of theory is effectively extended without losing the merit of field theory.

In order to abandon the assumption of displacement invariance, we have to describe a theory by two separated parameter spaces, each of which fulfils each one of the two functions of space-time respectively. Each separated space no longer deserves to be called space-time because it fulfils only a part of functions of the space-time. In the theory described by the two parameter spaces, it seems natural to replace the assumption of the field in field theory by that of the existence of the elementary particle.

Previously, we have attempted to construct a theory which is based on the assumption of the existence of elementary particles and does not rely on the concept of the space-time characterized by the above-mentioned two functions and accordingly also on the concept of the field.[1] In this paper, we start from the assumptions in field theory and replace some of them by weaker assumptions, and arrive at the theory attempted previously. The theory is shown to contain the present field theory as a special limit. We discuss about the examples of dynamical systems (i.e. the system of elementary particles with finite degree of freedom) which can not be described by field theory but can be described by our theory. The possibility of experimental test for the existence of such a dynamical system is also mentioned.

Physical theories degenerate if the assumptions restricting them are replaced by weaker ones, when the assumptions are consistent, independent and supported by experimental data. But when the assumptions are not consistent, they have to be replaced, in some cases, by weaker ones. Our consideration based on such idea would not be meaningless in a sense that it would give a possibility of being free from the inconsistency** of field theory caused by the infinite degree of freedom and it would stimulate the reexamination of the necessity of the fundamental assumptions in field theory.

§ 2. Assumptions in field theory and introduction of the third picture

Field theory is based on the following assumptions: (I) the special theory of relativity, (II) quantum mechanics and (III) the existence of fields.

(I) a. The theory is described by a parameter space called space-time which is a Minkowskian space, i.e. a four-dimensional real vector space with the inner product

$$x \cdot y \equiv x^\mu y_\mu = g_{\mu\nu} x^\mu y^\nu = -x^0 y^0 + x^1 y^1 + x^2 y^2 + x^3 y^3. \qquad (2 \cdot 1)$$

* See the foot-note of the preceding page.

** Quantum mechanics is described by a separable vector space (Hilbert space) and, therefore, is applied only to the dynamical system of finite degree of freedom. Field theory applies quantum mechanics to the dynamical system of infinite degree of freedom.

(I) b. The theory is invariant under the inhomogeneous Lorentz group (L),

$$x'^{\mu}=a_{\nu}{}^{\mu}x^{\nu}+a^{\mu},$$
$$a_{\lambda}{}^{\mu}a^{\lambda\nu}=g^{\mu\nu};\ a_{\nu}{}^{\mu},\ a^{\mu}\ \text{real},$$
$$\det(a_{\nu}{}^{\mu})>0,\quad a_{0}{}^{0}>0. \tag{2·2}$$

(II) a. When a dynamical system is given being described by a certain number of variables, v_i, a Hilbert space corresponds to the system and a dynamical variable is represented by a linear operator. Measurable quantities are functions of dynamical variables and represented by Hermitian operators called observables. The result of measurement of an observable is one of eigenvalues of the observable. A complete set of commuting observables exists.

(II) b. 1. Measurements have an order. The order of measurements is specified by space-like surfaces in space-time.

(II) b. 2. In the successive measurements specified by two parallel space-like planes,

$$\sigma_1:\ n\cdot x+\tau_1=0;\ \sigma_2:\ n\cdot x+\tau_2=0, \tag{2·3}$$

where n is a time-like unit vector and τ is a parameter, when a set of results $a=\{a_r\}$ of measurement belonging to σ_1 of a complete set of commuting observables $\{\zeta_r\}$ is obtained, the expectation value of the result of measurement σ_2 of an observable ζ is

$$(\Psi_a,\ e^{-iP\cdot\xi}\zeta e^{iP\cdot\xi}\Psi_a), \tag{2·4}$$

where P_μ is the observable representing the total energy-momentum of the system, $\xi^\mu=n^\mu(\tau_2-\tau_1)$ and Ψ_a is the common eigenvector of $\{\zeta_r\}$ associated with a.

(III) a. The dynamical system is described by field variables which are functions of space-time coordinate having several components $\phi_\beta(x)$ which transform linearly under L-group (2·2) as

$$\phi_\beta(x')=S_\beta{}^{\alpha}\phi_\alpha(x). \tag{2·5}$$

(III) b. 1. The equations exist which determine $\phi_\beta(x)$ at an arbitrary x by the field variables and their first derivatives on a space-like surface in space-time. (The equations are called the coupled field equations.)

(III) b. 2. An observable P_μ exists which expresses the coupled field equations in the form

$$i\partial_\mu\phi_\beta(x)=P_\mu\phi_\beta(x)-\phi_\beta(x)P_\mu\equiv[P_\mu,\ \phi_\beta(x)]. \tag{2·6}$$

(III) b. 3. The observable representing the total energy-momentum is defined by P_μ.

(III) c. 1. A complete set of commuting observables $\{\zeta(x/\sigma)\}$* exists, of

---

* x/σ means x on σ.

which $\zeta(x/\sigma)$'s are functions of field variables $\phi_\beta(x)$ on a space-like surface σ in space-time.

(III) c. 2. $$[\zeta(x/\sigma),\ \zeta(x'/\sigma)]=0 \tag{2·7}$$

(the micro-causality).

The consistency of (I) with (II) and (III) requires the existence of a unitary representation $U(L)$ of L-group which gives

$$\phi_\beta'(x)=U(L)\phi_\beta(x)U^{-1}(L),\ \phi_\beta'(Lx)=S_\beta{}^\alpha\phi_\alpha(x). \tag{2·8}$$

For this, it is necessary and sufficient that the Hermitian operators $\mathcal{P}^\mu$, $\mathcal{M}^{\mu\nu}=-\mathcal{M}^{\nu\mu}$ exist which generate the infinitesimal $U(L)$ and satisfy

$$\begin{aligned}&[\mathcal{P}^\mu,\ \mathcal{P}^\nu]=0,\\ &i[\mathcal{M}^{\mu\nu},\ \mathcal{P}^\lambda]=\mathcal{P}^\mu g^{\nu\lambda}-\mathcal{P}^\nu g^{\mu\lambda},\\ &i[\mathcal{M}^{\mu\nu},\ \mathcal{M}^{\rho\sigma}]=g^{\mu\rho}\mathcal{M}^{\nu\sigma}-g^{\mu\sigma}\mathcal{M}^{\nu\rho}+g^{\nu\sigma}\mathcal{M}^{\mu\rho}-g^{\nu\rho}\mathcal{M}^{\mu\sigma}.\end{aligned} \tag{2·9}$$

From (2·6)

$$\mathcal{P}^\mu=P^\mu,\ \mathcal{M}^{\mu\nu}=M^{\mu\nu}, \tag{2·10}$$

and from (2·9)

$$[P^\mu,\ P^\nu]=0 \tag{2·11}$$

are required.

When $\phi_\beta^R(x)$ which is defined by a canonical transformation

$$\phi_\beta{}^R(x)=R\phi_\beta(x)R^{-1},\ \mathcal{P}^R=R\mathcal{P}, \tag{2·12}$$

is transformed under L-group by a unitary representation as

$$\phi_\beta{}^{R'}(x)=U^R(L)\phi_\beta{}^R(x)U^{R-1}(L),\ \phi_\beta{}^{R'}(Lx)=S_\beta{}^\alpha\phi_\alpha{}^R(x), \tag{2·13}$$

we can use $\phi_\beta^R(x)$ instead of $\phi_\beta(x)$ as dynamical variables. We call the freedom expressed by (2·12) and (2·13) the "picture". The picture characterized by (2·10) is called the Heisenberg picture. Another known picture is called the interaction picture (Tomonaga picture). $U^R(L)$ in this picture is generated by $P^{(0)\mu}$, $M^{(0)\mu\nu}$ which are determined by the free field equations obtained by omitting the coupling terms from the coupled field equations. It is characterized by

$$\mathcal{P}^\mu=P^{(0)\mu},\ \mathcal{M}^{\mu\nu}=M^{(0)\mu\nu}. \tag{2·14}$$

When R is taken to be

$$R=e^{iP\cdot x}, \tag{2·15}$$

the variation of $\phi_\beta^R(x)$ under the infinitesimal L-transformation is

$$\begin{aligned}\delta\phi_\beta{}^R(x)&=1/2[\Sigma_\beta^{\alpha\mu\nu}\phi_\alpha{}^R(x)+(x^\mu\partial^\nu-x^\nu\partial^\mu)\phi_\beta{}^R(x)]\alpha_{\mu\nu}-\partial^\mu\phi_\beta{}^R(x)\alpha_\mu\\ &=1/2\,\Sigma_\beta^{\alpha\mu\nu}\phi_\alpha{}^R(x)\alpha_{\mu\nu}\end{aligned} \tag{2·16}$$

(the infinitesimal part of S_β^α is expressed by $\delta_\beta^\alpha+\Sigma_\beta^\alpha$). Then the infinitesimal generator of $U^R(L)$ is formed by

$$\mathcal{P}^\mu=0,\ \mathcal{M}^{\mu\nu}=N^{\mu\nu}, \tag{2·17}$$

where $N^{\mu\nu}$ is defined by

$$i[N^{\mu\nu},\ \phi_\beta{}^R(x)]=\Sigma_\beta^{\alpha\mu\nu}\phi_\alpha{}^R(x). \tag{2·18}$$

Since $\mathcal{P}^\mu$, $\mathcal{M}^{\mu\nu}$ in (2·17) satisfy (2·9), R in (2·15) defines the third picture characterized by (2·17).

From (2·6), an observable $\zeta(x)$ satisfies

$$\zeta(x+\xi)=e^{-iP\cdot\xi}\zeta(x)\,e^{iP\cdot\xi} \tag{2·19}$$

for an arbitrary vector ξ. Then (II) b. 2 is restated for the system of fields as follows. When a set of results $a=\{a(x/\sigma)\}$ is obtained in the measurement of a complete set of observables $\{\zeta(x/\sigma)\}$ on a space-like plane σ, the expectation value of the result of the next measurement of an observable $\zeta(x/\sigma+\xi)$ (ξ is arbitrary but $\xi^0>0$) is

$$(\Psi_a[\sigma],\ \zeta(x/\sigma+\xi)\Psi_a[\sigma]), \tag{2·20}$$

where $\Psi_a[\sigma]$ is the common eigenvector of $\{\zeta(x/\sigma)\}$ associated with a.

In order to separate the two functions of space-time, we describe* the field theory by using two Minkowskian parameter spaces, in which an L-transformation of one space always accompanies the same L-transformation of the other space. We denote vectors in the two spaces by x and ξ respectively. We define $\phi_\beta(x)$ in x-space in accordance with (III) a and (III) b, hence σ in (2·20) is a plane in x-space. We represent ξ in (2·20) by a vector in ξ-space. Then, (2·20) is written as

$$(\Psi_a[\sigma],\ \zeta(x/\sigma,\xi)\Psi_a[\sigma]). \tag{2·21}$$

Since $\zeta(x,\xi)$ is given by (2·19), $\zeta(x,\xi)$ is considered to be a function of $\phi_\beta(x,\xi)$ satisfying

$$\begin{aligned} i\frac{\partial}{\partial x^\mu}\phi_\beta(x,\xi)&=[P_\mu,\ \phi_\beta(x,\xi)],\\ i\frac{\partial}{\partial\xi^\mu}\phi_\beta(x,\xi)&=[P_\mu,\ \phi_\beta(x,\xi)]. \end{aligned} \tag{2·22}**$$

$\phi_\beta(x,\xi)$ is transformed by L-group in x-space (L_x) and L-group in ξ-space (L_ξ) (although L_ξ is always accompanied with L_x) as

$$\phi_\beta'(x,\xi)=U(L_x)\phi_\beta(x,\xi)U^{-1}(L_x),\ \phi_\beta'(L_x x,\xi)=S_\beta{}^\alpha\phi_\alpha(x,\xi), \tag{2·23}$$

---

* In the field theory which assumes the displacement invariance, ξ-space is naturally redundant. Rewriting field theory by using the separated x and ξ-spaces is serviceable for the understanding of the theory in § 3 and § 4.

** $\phi_\beta(x,\xi)\equiv\phi_\beta(x+\xi)$.

and

$$\phi_\beta''(x,\hat{\xi})=U(L_\xi)\phi_\beta(x,\hat{\xi})U^{-1}(L_\xi),\quad \phi_\beta''(x,L_\xi\hat{\xi})=S_\beta{}^\alpha\phi_\alpha(x,\hat{\xi}) \tag{2·24}$$

respectively. $\phi_\beta(x,\hat{\xi})$ is considered to be a variable in the Heisenberg picture for both L_x and L_ξ. We can use variables in the other picture defined by a canonical transformation

$$\phi_\beta{}^M(x,\hat{\xi})=M(x,\hat{\xi})\phi_\beta(x,\hat{\xi})M^{-1}(x,\hat{\xi}),\quad \Psi^M=M(x,\hat{\xi})\Psi. \tag{2·25}$$

The picture which is the interaction picture for L_x and the third picture (2·17) for L_ξ exists. In this picture,

$$\begin{aligned} i\frac{\partial}{\partial x^\mu}\phi_\beta{}^M(x,\hat{\xi})&=[P_\mu^{(0)},\ \phi_\beta{}^M(x,\hat{\xi})],\\ i\frac{\partial}{\partial\hat{\xi}^\mu}\phi_\beta{}^M(x,\hat{\xi})&=0, \end{aligned} \tag{2·26}$$

and the observable $\zeta^M(x,\hat{\xi})$ is independent of $\hat{\xi}$. Then, (2·21) is expressed by

$$(\Psi_a{}^M[\sigma,\hat{\xi}_2],\ \zeta^M(x/\sigma)\Psi_a{}^M[\sigma,\hat{\xi}_2]),\ \Psi_a{}^M[\sigma,\hat{\xi}_2]=\exp[iP^M[\sigma]\cdot(\hat{\xi}_2-\hat{\xi}_1)]\Psi_a{}^M \tag{2·27}$$

where $P_\mu^M[\sigma]$ is the observable P_μ in (2·22) expressed by the variables satisfying (2·26) and $\Psi_a^M[\sigma]$ is the common eigenvector of $\{\zeta^M(x,\sigma)\}$ associated with $a=\{a(x/\sigma)\}$. This is a rewriting of the usual field theory in the third picture.

§ 3. Abandonment of the assumption of the displacement invariance

We replace, in this section, the assumption of the displacement invariance in field theory by the assumption of a momentum-conservation law of finite range g.* As the assumption of the displacement invariance is abandoned, we describe the theory by the two parameter spaces, x-space and $\hat{\xi}$-space, each of which fulfils each one of the two functions of space-time. Our theory is obtained by replacing the assumptions (I), (II) and (III) of field theory by the following (I′), (II′) and (III′).

(I′) a. The theory is described by two parameters x and $\hat{\xi}$ which are vectors in two separate Minkowskian spaces. The L-transformation (2·2) in x-space accompanies the L_0-transformation

$$L_0:\hat{\xi}'^\mu=a_\nu{}^\mu\hat{\xi}^\nu \tag{3·1}$$

in $\hat{\xi}$-space, and conversely the L-transformation in $\hat{\xi}$-space accompanies the L_0-transformation in x-space.

(I′) b. The theory is invariant under L-group in x-space and L-group in $\hat{\xi}$-space.

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* Eq. (3·4) and Eq. (4·10).

(II′) a. The same as (II) a.

(II′) b. 1. Measurements have an order. The order of measurements is specified by vectors in $\hat{\xi}$-space.

(II′) b. 2. When a set of results $a=\{a_r\}$ of measurement belonging to $\hat{\xi}_1$ of a complete set of commuting observables $\{\zeta_r\}$ is obtained, the expectation value of the result of the next measurement belonging to $\hat{\xi}_2(\hat{\xi}_2{}^0-\hat{\xi}_1{}^0\geq 0)$ of an observable ζ is

$$(\Psi_a(\hat{\xi}_2), \zeta\Psi_a(\hat{\xi}_2)),\ \Psi_a(\hat{\xi}_2)=\exp[iP\cdot(\hat{\xi}_2-\hat{\xi}_1)]\Psi_a, \tag{3·2}$$

where Ψ_a is the common eigenvector of $\{\zeta_r\}$ associated with a and P_μ is an observable representing the total energy-momentum. We define $\exp[iP\cdot\hat{\xi}]$ by Feynman's operator calculus.[2]

(III′) a. The same as (III) a, except that "space-time" is replaced by "x-space".

(III′) b. 1. The equation which determines $\phi_\beta(x)$ at an arbitrary x by $\phi_\beta(x)$ and its first derivative on a space-like surface in x-space exists. (Such equation is called the free field equation.)

(III′) b. 2. An observable $P_\mu{}^{(0)}$ exists which expresses the free field equation in the form

$$i\partial_\mu\phi_\beta(x)=[P_\mu^{(0)},\ \phi_\beta(x)]. \tag{3·3}$$

(III′) b. 3. The observable representing the total energy-momentum has the form

$$P_\mu[\sigma]=P_\mu^{(0)}+P_\mu^{(1)}[\sigma],\quad P_\mu^{(1)}[\sigma]=\int_\sigma I_{\mu\nu}(x)\exp\left[-\frac{1}{2}g^2x^2\right]d\sigma^\nu, \tag{3·4}$$

where $I_{\mu\nu}(x)$ is a function of field variables and g is a real number.

(III′) c. The same as (III) c, except that "space-time" is replaced by "x-space."

The consistency of (I′) with (II′) and (III′) requires the relation (2·9) for $\mathcal{P}^\mu=P^{(0)\mu}$, $\mathcal{M}^{\mu\nu}=M^{(0)\mu\nu}$ which are determined by the free field equation. But $[P^\mu, P^\nu]=0$ is not required, because x-space and $\hat{\xi}$-space in (I′) are independent with respect to their translations contrary to the case in § 2. Therefore, $P_\mu{}^{(1)}[\sigma]$ with $g\neq 0$ is not excluded.

In (III′), free field equations are assumed but coupled field equations are not assumed. Hence $P_\mu{}^{(0)}$ is defined but P_μ is not defined. P_μ is assumed* in (3·4). When we assume $I_{\mu\nu}(x)$ equals the interaction part of the energy-momentum tensor in the interaction picture in the usual theory, P_μ in the limit $g=0$ coincides with the total energy-momentum vector in the usual theory. When we assume

$$I_{\mu\nu}(x)=g_{\mu\nu}H(x) \tag{3·5}$$

---

* We can assume $P_\mu{}^{(1)}$. It is due to the replacement of the assumptions of field theory by weaker ones. In the theory in § 4, we can assume even $P_\mu{}^{(0)}$.

with an invariant function $H(x)$ which does not contain the derivative of the field variable, $P_\mu[\sigma]$ in $g\to 0$ coincides with that of the usual theory which assumes the interaction Lagrangian $-H(x)$.

From (III′) b. 1 and (III′) b. 2, every observable is expressed as a function of variables on a fixed space-like plane σ_1 in x-space, i.e. $\zeta[\phi(x/\sigma_1)]=\zeta(x/\sigma_1)$, hence in (II′) b. 2, we can take $\{\zeta_r\}=\{\zeta(x/\sigma_1)\}$, $\zeta=\zeta(x_1/\sigma_1)$ and $\{a_r\}=\{a(x/\sigma_1)\}$. Then (3·2) is expressed by

$$(\Psi_a[\sigma_1,\hat{\xi}_2],\ \zeta(x_1/\sigma_1)\Psi_a[\sigma_1,\hat{\xi}_2]),$$
$$\Psi_a[\sigma_1,\hat{\xi}_2]=\exp[iP[\sigma_1]\cdot(\hat{\xi}_2-\hat{\xi}_1)]\Psi_a[\sigma_1], \tag{3·6}$$

where $\Psi_a[\sigma_1]$ is the common eigenvector of $\{\zeta(x/\sigma_1)\}$ associated with $a=\{a(x/\sigma_1)\}$. (3·6) coincides with the formula (2·27) in field theory when $\sigma_1=\sigma$ and

$$P_\mu[\sigma]=P_\mu^{\,M}[\sigma]. \tag{3·7}$$

It should be noted, here, that σ in field theory is a plane in the space-time and specifies the order of measurements but σ_1 in (3·6) is a plane in x-space independent of the order of measurements from the assumption (II′) b. 1. However, as we show in the following, the expectation value in (2·27) depends only on a set of numbers $\{a(x/\sigma)\}$, i. e.

$$(\Psi_a^{\,M}[\sigma_1,\hat{\xi}_2],\ \zeta^M(x_1/\sigma_1)\Psi_a^{\,M}[\sigma_1,\hat{\xi}_2])=(\Psi_a^{\,M}[\sigma_2,\hat{\xi}_2]\zeta^M(x_2/\sigma_2)\Psi_a^{\,M}[\sigma_2,\hat{\xi}_2]), \tag{3·8}$$

when $a(x_1/\sigma_1)=a(x_2/\sigma_2)$ for all x_1 and x_2 which are connected by an L-transformation $x_2^{\,\mu}=a_\nu^{\,\mu}x_1^{\,\nu}+a^\mu$. Accordingly, when (3·7) is assumed in (III′) b.3, (3·6) and (2·27) are physically equivalent. Thus, the theory of (I′), (II′) and (III′) contains field theory as a special case.*

For the proof of (3·8), we have to show it only in the case that x_1 and x_2 (and hence σ_1 and σ_2) are connected by an infinitesimal L-transformation

$$x_2^{\,\mu}=x_1^{\,\mu}+\alpha_\nu^{\,\mu}x_1^{\,\nu}+\alpha^\mu. \tag{3·9}$$

Since

$$\zeta^M(x_2)=\zeta^M(x_1)+(x^\mu\partial^\nu-x^\nu\partial^\mu)\zeta^M(x_1)\alpha_{\mu\nu}-\partial^\mu\zeta^M(x_1)\alpha_\mu, \tag{3·10}$$

(3·8) is proved from the following relations,

$$\zeta^M(x_2)=V\zeta^M(x_1)V^{-1},\ \Psi_a^{\,M}[\sigma_2]=V\Psi_a^{\,M}[\sigma_1], \tag{3·11}$$
$$\Psi_a^{\,M}[\sigma_2,\hat{\xi}_2]=V\exp[iP^M[\sigma_1]\cdot(\hat{\xi}_2-\hat{\xi}_1)]V^{-1}V\Psi_a^{\,M}[\sigma_1]=V\Psi_a^{\,M}[\sigma_1,\hat{\xi}_2], \tag{3·12}$$

* This means that the assumptions (I′), (II′) and (III′) are weaker than (I), (II) and (III) of field theory. Therefore, new inconsistencies which have not been contained in field theory may not appear in the theory of (I′), (II′) and (III′). The same argument holds for the theory of (I″), (II″) and (III″) in § 4.

$$V=1+i\left(\frac{1}{2}L_{\mu\nu}^{(0)}\alpha^{\mu\nu}+P_{\mu}^{(0)}\alpha^{\mu}\right),$$

where $P_{\mu}^{(0)}$ and $L_{\mu\nu}^{(0)}$ are known operators called the momentum and the orbital-angular momentum operators for the free field.

In (3·12), we have used $[P_{\mu}, P_{\nu}]=0$ (2·11) and the condition for the localized interaction

$$[I_{\mu\nu}(x), I_{\mu\nu}(x')]=0 \quad \text{for } (x-x')^2>0. \tag{3·13}$$

When (2·11) is not satisfied (the interaction (3·4) with $g\neq 0$) or (3·13) is not satisfied (the non-localized interaction), we have the case in which a special plane σ_1 have to be specified in (3·6). A special plane $\sigma_N(0)$ is introduced in (4·15) in § 4.

In (III′) c, it is assumed that

$$[\zeta_r(x), \zeta_s(x')]=0 \quad \text{for } (x-x')^2>0, \tag{3·14}$$

but $\zeta(x)$ in (3·14) is a function of free field variables, hence (3·14) is not the microcausality assumed in field theory. The assumption of (3·14) and the positive definiteness of $P_0^{(0)}$ lead to the relation between spin and statistics only for the field variables satisfying free field equation.

§ 4. Replacement of the assumption of field by the assumption of particle

The field variable in (I′), (II′) and (III′) in § 3 is the same operator with that of the usual theory in the interaction picture and expressed as

$$\phi_{\beta}(x)=\sum_{p,i}[a(p,i)u_{\beta}^{i}(p)e^{ip\cdot x}+b^*(p,i)v_{\beta}^{i}(p)e^{-ip\cdot x}],\ p^2=-m^2. \tag{4·1}$$

We can use $a(p,i)$ and $b^*(p,i)$ as dynamical variables, instead of $\phi_{\beta}(x)$. In this case, the degree of freedom expressed by a plane σ_1 in x-space is replaced by a degree of freedom expressed by a surface $p^2=-m^2$ specified by an intrinsic quantity m of the particle. The fact that every degree of freedom is specified by the intrinsic quantity of the particle seems to show that the description based on the existence of the particle is the more direct one than that based on the existence of the field.

We replace the assumption of the field by the assumption of the particle. We assume further that every degree of freedom of the system consists of a finite number of points in order that the Hilbert space is clearly defined and any divergence never appears in the theory. The theory contains field theory in a limit of infinite degree of freedom. The theory is obtained by replacing (I′), (II′) and (III′) in § 3 by the following (I″), (II″) and (III″).

(I″) a. The theory is described by parameters of points in coordinate spaces Σ^1, Σ^2, ⋯, expressing degrees of freedom of the system and a parameter ξ in a Minkowskian space.

The degree of freedom expressed in Σ^i is the subspaces $\Sigma^i(\sigma_1^i)$, $\Sigma^i(\sigma_2^i)$, $\cdots$, specified by real numbers σ_k^i called intrinsic quantities.

Σ^m(one of Σ^i) is a Minkowskian space. $\Sigma^m(m_k)$ (degree of freedom of momentum) is a set of discrete points p which lie on a surface

$$p^2=-m_k^2 \tag{4·2}$$

in Σ^m with a density given by

$$dn=\mu_k Q\,\frac{d^3p}{p_0}\exp[-\lambda_k^2(N\cdot p)^2], \tag{4·3}*$$

where λ_k, μ_k, Q are positive real constants and N_μ is a unit time-like vector in Σ^m.

$\Sigma^{(u)}(J_k, J_k')$ (degree of freedom of spin) is defined by a set of q points which distinguish the functions of $p\in\Sigma^m(m_k)$

$$u_\beta^i(p),\ i=1, 2, \cdots, q;\ \beta=1, 2, \cdots, (2J_k+1)(2J_k'+1), \tag{4·4}$$

which are transformed by an irreducible representation $D(J_k, J_k')$ of L_0-group (3·1) in Σ^m and defined by an independent solution of the invariant homogeneous equations,

$$\sum_\beta L_{\alpha\beta}(p)u_\beta(p)=0,\quad \alpha=1, 2, \cdots, (2J_k+1)(2J_k'+1). \tag{4·5}$$

The L_0-transformation (3·1) in Σ^m accompanies the L_0-transformation in ξ-space. The L-transformation (2·2) in ξ-space accompanies the L_0-transformation in Σ^m.

(I″) b. The theory is invariant under L_0-group in Σ^m.

(II″) a. The same as (II′) a. (=(II) a).

(II″) b. The same as (II′) b.

(III″) a. 1. A set of intrinsic quantities $k=(\sigma^1, \sigma^2, \cdots)$ is called "kind". A kind specifies a set of degrees of freedom $(\Sigma(\sigma_k^1), \Sigma(\sigma_k^2), \cdots)$.

(III″) a. 2. We define operators $a_k(s)$ and $a_k^*(s)$ which are specified by the coordinates of kind k

$$s=(s^1, s^2, \cdots),\ s^i\in\Sigma^i(\sigma_k^i) \tag{4·6}$$

and satisfy

$$[a_k(s), a_{k'}^*(s')]_\pm=\delta_{kk'}\delta_{ss'},$$
$$[a_k(s), a_{k'}(s')]_\pm=[a_k^*(s), a_{k'}^*(s')]_\pm=0. \tag{4·7}$$

$[A, B]_\pm=AB\pm BA$. When $+(-)$ is assumed in (4·7), k is called Fermi-Dirac (Bose-Einstein).

(III″) a. 3. The dynamical system is described by variables $a_k(s)$ and $a_k^*(s)$.

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* The physical meaning of the constant vector N_μ is discussed in § 5.

(III″) b. 1. The observable representing the total energy-momentum has the form

$$P_\mu = \sum_{k,s} p_\mu a_k^*(s) a_k(s) + P_\mu^{(1)} \quad (\equiv P_\mu^{(0)} + P_\mu^{(1)}). \tag{4·8}$$

(III″) b. 2. $a_k(s)$ and $a_k^*(s)$ appear in $P_\mu^{(1)}$ always in the combinations

$$a_k(s)u^s(p) + a_k^*(\bar{s})\bar{u}^{\bar{s}}(\bar{p}) \text{ and } a_k^*(s)u^{s*}(p) + a_k(\bar{s})\bar{u}^{\bar{s}*}(\bar{p}), \tag{4·9}$$

where s and $\bar{s}$ are two sets of coordinates of k (4·6) which correspond one to one and are distinguished in an invariant way under L_0-group in Σ^m. (s and $\bar{s}$ are called coordinates of particle and antiparticle respectively.)

(III″) b. 3. $P_\mu^{(1)}$ consists of products of $a_k(s)$ and $a_k^*(s)$ and each product has the factor

$$\sqrt{\pi}\, g^{-3} \exp[-\{(\sum_i \pm p^i)^2 + (N \cdot \sum_i \pm p^i)^2\}/g^2], \tag{4·10}$$

where N_μ is the vector in (4·3) and p^i with $+(-)$ sign is the momentum coordinate in $a(s)$ $(a^*(s))$.

In (I″) a (4·3), a special direction N_μ is introduced in Σ^m in order to make finite the degree of freedom of momentum. As we shall mention in § 5, the existence of N_μ has some possible observable effects in the phenomena of energy larger than a certain critical value determined by λ_k. In the limit $\lambda_k = 0$, (4·3) coincides with the state density in momentum space in the usual theory in an invariant normalization. The vector N_μ also serves for the invariant expression of the momentum conservation law of finite range in (4·10) and for the specification of a special plane $\sigma_N(0)$ (mentioned under Eq. (3·13)) in (4·15). The finite density (4·3) requires g of the order of

$$g \gtrsim (\bar{\lambda}/\bar{\mu}) Q^{-1/3}, \quad (\bar{\lambda} = \text{Max. } \lambda_k, \ \bar{\mu} = \text{Min. } \mu_k). \tag{4·11}$$

We can not assume $g = 0$, although we can assume g to be arbitrarily small by assuming Q to be sufficiently large.

The canonical transformation associated with L_0-group,

$$a_k'(s) = U(L_0) a_k(s) U^{-1}(L_0), \ a_k^{*\prime}(s) = U(L_0) a_k^*(s) U(L_0)$$

is generated by the Hermitian operator $\mathcal{M}^{\mu\nu}$ satisfying (2·9),

$$\mathcal{M}^{\mu\nu} = -i \sum_{k,s} a_k^*(s)[p^\mu(\partial/\partial p_\nu) - p^\nu(\partial/\partial p_\mu)] a_k(s). \tag{4·12}$$

In (III″) b, the form of P_μ is assumed, since we have neither a coupled field equation nor a free field equation. $P_\mu^{(0)}$ in (4·8) coincides in the limit $\lambda_k = 0$ with $P_\mu^{(0)}$ (2·14) in field theory except for the case excluded in field theory, i.e. the case of half-integer-spin boson.

For the half-integer-spin boson, $P_\mu^{(0)}$ is defined in field theory from the field equation as $(\lambda_k \to 0)$

$$P_{k\mu}^{(0)}=\sum_{s} p_{\mu}(a_k^*(s)a_k(s)-a_k(\bar{s})a_k^*(\bar{s})), \qquad (4\cdot13)$$

while (III″) assumes

$$P_{k\mu}^{(0)}=\sum_{s} p_{\mu}(a_k^*(s)a_k(s)+a_k^*(\bar{s})a_k(\bar{s})). \qquad (4\cdot13')$$

The half-integer-spin boson is not excluded by the positive definiteness of $P_0^{(0)}$ in the theory of (I″), (II″) and (III″).

We assume the existence of antiparticles ((III″) b. 2) and the conservation of momentum ((III″) b. 3). We have not, however, assumptions related to the concept of the field, i.e. the assumptions of the displacement invariance and the microcausality. We can introduce a parameter x in a Minkowskian space and define operators of function of x as

$$\phi_{k\beta}(x)=(2\pi)^{-2/3}(\mu_k Q)^{-1/2}\sum_{s}(a_k(s)u_{\beta}^{\ i}(p)e^{ip\cdot x}+a_k^*(\bar{s})u_{\beta}^{\ \bar{r}}(p)e^{-ip\cdot x}) \qquad (4\cdot14)$$

and its Hermitian conjugate. We assume that L_0-transformation in Σ^m accompanies L_0-transformation in x-space, and L-transformation in x-space accompanies L_0-transformation in Σ^m. It is not required, however, that observables are functions (or functionals) of $\phi_{k\beta}(x)$ and satisfy the relation (3·14), hence (I″), (II″) and (III″) can not exclude the integer-spin fermion. Thus, the theory of (I″), (II″) and (III″) can not imply the relation of spin and statistics.

The x-space and the operators introduced in (4·14) are useful for a concise expression of $P_{\mu}^{(1)}$. The following $P_{\mu}^{(1)}$ satisfies the assumptions in (III″) b.

$$P_{\mu}^{(1)}[\sigma_N(0)]=\int_{\sigma_N(0)} I_{\mu\nu}(x)\exp\left[-\frac{1}{2}g^2x^2\right]d\sigma^{\nu}, \qquad (4\cdot15)$$

where $I_{\mu\nu}(x)$ is a function of (4·14) and $\sigma_N(0)$ is the plane being perpendicular to N and containing $x=0$. (N_{μ} is defined also in x-space, because L_0 in Σ^m accompanies L_0 in x-space.) $P_{\mu}^{(1)}[\sigma_N(0)]$ is the same operator with $P_{\mu}^{(1)}[\sigma]$ in (3·4) with $\sigma=\sigma_N(0)$ and in the limit $\lambda_k=0$, except for the system in which half-integer-spin bosons and integer-spin fermions exist. (When $P_{\mu}[\sigma]=P_{\mu}^{M}[\sigma]$ (3·7), the theory does not depend on the special choice of the plane $\sigma_N(0)$ because the relation of the form of (3·8) holds in this case.) Accordingly, the theory of (I″), (II″) and (III″) coincides in the limit $\lambda_k=0$ with the theory of (I′), (II′) and (III′) except for the system excluded in the theory of (I′), (II′) and (III′). Thus, the theory of (I″), (II″) and (III″) contains the usual field theory as a special limit.

§ 5. Possible observable effects peculiar to the theory of (I″), (II″) and (III″)

1. *High energy electron-electron scattering*

Referring to the differential cross section of high energy electron-electron elastic scattering, we consider the physical meaning of the existence of a special direction N_{μ} in Σ^m ((4·3), (5·3)).

We assume $P_\mu^{(1)}$ in (III″) b for the system of electrons and photons in the form of (4·15) as

$$P_\mu^{(1)}=\int_{\sigma_N(0)} g_{\mu\nu}H(x)d\sigma^\nu,\quad H(x)=ie\bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x). \tag{5·1}$$

We consider the problem in the limit $g=0$ ($Q\to\infty$ in (4·11)). $\bar{\psi}(x)$, $\psi(x)$ and $A_\mu(x)$ are operators of (4·14) and the same operators with the electron and photon field variables in field theory in the interaction picture except for the density of degree of freedom of momentum, e.g

$$\psi_\beta(x)=(2\pi)^{-3/2}(\mu_e Q)^{-1/2}\sum_p\sum_i\left(a(p,i)u_\beta^{\,i}(p)e^{ip\cdot x}+b^*(p,i)v_\beta^{\,i}(p)e^{-ip\cdot x}\right). \tag{5·2}$$

The summation over the momentum coordinate is replaced by

$$\begin{aligned}\sum_p&=\mu_e Q\int\frac{d^3p}{p_0}\exp[-\lambda_e^{\,2}(N\cdot p)^2]\quad\text{and}\\ \sum_k&=\mu_\gamma Q\int\frac{d^3k}{k_0}\exp[-\lambda_\gamma^{\,2}(N\cdot k)^2]\end{aligned} \tag{5·3}$$

for electrons and photons respectively. We take

$$\mu_e=m,\quad \mu_\gamma=1,\quad p^2=-m^2,\quad k^2=0. \tag{5·4}$$

The transition probability between two two-electron states is given by (II″) b (=(II′) b) as

$$\begin{aligned}&W_{ab}=|\langle b|\exp[iP\cdot\hat{\xi}]|a\rangle|^2,\\ &|a\rangle=a^*(p_1,i_1)a^*(p_2,i_2)|0\rangle,\quad |b\rangle=a^*(p_1',i_1')a^*(p_2',i_2')|0\rangle.\end{aligned} \tag{5·5}$$

$|0\rangle$ is the common eigenstate of a complete set of observables $\{a_k^*(s)a_k(s)\}$ associated with a set of eigenvalues $\{0\}$. Since $\hat{\xi}$-space satisfies all the conditions to be satisfied by x-space in (4·14), $P_\mu^{(1)}$ is expressed also in $\hat{\xi}$-space. Then,

$$P_\mu=P_\mu^{(0)}+\int_{\sigma_N(0)} g_{\mu\nu}H(\hat{\xi})d\sigma^\nu(\hat{\xi}),$$

$$H(\hat{\xi})=ie\bar{\psi}(\hat{\xi})\gamma^\mu\psi(\hat{\xi})A_\mu(\hat{\xi}),$$

$$e^{iP\cdot\hat{\xi}}\equiv\exp[iP^{(0)}\cdot\hat{\xi}/2]\sum_n\frac{i^n}{n!}\int_{\sigma_N(-\xi/2)}^{\sigma_N(\xi/2)}\cdots\int d^4\hat{\xi}_1\cdots d^4\hat{\xi}_n P(H(\hat{\xi}_1)\cdots H(\hat{\xi}_n))$$
$$\times\exp[-iP^{(0)}\cdot\hat{\xi}/2]. \tag{5·6}$$[2]

Since operators ψ, $\bar{\psi}$ are expressed in (5·6) in $\hat{\xi}$-space, we can define in $\hat{\xi}$-space the cross section in the usual way. The transition probability per unit volume in $\hat{\xi}$-space is defined in the scattering approximation as (We consider a calculation

using the following w_{ab} to be an approximation.)

$$w_{ab}=\lim_{\xi^0\to\infty}|\langle b|e^{iP\cdot\xi}|a\rangle|^2/\lim_{p\to 0}\int e^{ip\cdot\xi}\,d^4\xi \tag{5·7}$$

and calculated from (5·6) in the lowest order as,

$$w_{ab}=(2\pi)^4\delta_4(p_1+p_2-p_1'-p_2')\,|K|^2,$$

$$K=e^2(2\pi)^{-6}(mQ)^{-2}i\left[\frac{(\bar{u}_2'\gamma_\lambda u_2)(\bar{u}_1'\gamma^\lambda u_1)}{(p_1-p_1')^2}\exp[-\lambda_\gamma^2(N\cdot p_1-p_1')^2]\right.$$

$$\left.-\frac{(\bar{u}_2'\gamma_\lambda u_1)(\bar{u}_1'\gamma^\lambda u_2)}{(p_1-p_2')^2}\exp[-\lambda_\gamma^2(N\cdot p_1-p_2')^2]\right]. \tag{5·8}$$

The factor containing λ_γ comes from

$$\langle 0|P(A_\mu(x)A_\nu(x')|0\rangle=-g_{\mu\nu}\frac{i}{(2\pi)^4}\int_F\exp[ik(x-x')-\lambda_\gamma^2(N\cdot k)^2]\,\frac{d^4k}{k^2}. \tag{5·9}$$

The cross section is defined by

$$d\sigma=\frac{p_{10}p_{20}}{((p_1\cdot p_2)^2-m^4)^{1/2}}\frac{((2\pi)^3mQ)^2}{p_{10}p_{20}}w_{ab}(mQ)^2\exp[-\lambda_e^2(N\cdot p_1')-\lambda_e^2(N\cdot p_2')]$$

$$\times\frac{d^3p_1'}{p_{10}'}\,\frac{d^3p_2'}{p_{20}'}. \tag{5·10}$$

In the coordinate system $N=(0,0,0,1)$ and in the case $p_2=0$ in the system, (5·10) becomes, for $\gamma\gg 1$,

$$\frac{d\sigma}{d\Omega^*}=\frac{r_0^2}{2\gamma}\left[A^2\csc^4\frac{\theta^*}{2}+B^2\sec^4\frac{\theta^*}{2}+\frac{1}{2}(A^2+B^2)+AB\csc^2\frac{\theta^*}{2}\sec^2\frac{\theta^*}{2}\right.$$

$$\left.-\left(A^2\csc^2\frac{\theta^*}{2}+B^2\sec^2\frac{\theta^*}{2}\right)\right]CD, \tag{5·11}$$

$$A=\exp\left[-\lambda_\gamma^2\gamma^2m^2\sin^4\frac{\theta^*}{2}\right],\quad B=\exp\left[-\lambda_\gamma^2\gamma^2m^2\cos^4\frac{\theta^*}{2}\right],$$

$$C=\exp\left[-2\lambda_e^2\gamma^2m^2\sin^4\frac{\theta^*}{2}\right],\quad D=\exp\left[-2\lambda_e^2\gamma^2m^3\cos^4\frac{\theta^*}{2}\right],$$

$$\cos\theta^*\equiv\frac{2-(\gamma+3)\sin^2\theta}{2+(\gamma-1)\sin^2\theta},\quad d\Omega^*\equiv\frac{8\cos\theta(\gamma+1)}{[2+(\gamma-1)\sin^2\theta]^2}d\Omega,$$

where r_0 is the classical electron radius, θ is the angle of scattering and γ is the energy of the incident electron in units of $m(p_{10}=\gamma m)$.

The L_0 (homogeneous Lorentz)-transformation is the relation between the descriptions by two observers also in our theory. Accordingly, the cross section for the observer in the coordinate system other than that of $N=(0,0,0,1)$ is obtained from (5·11) by an L_0-transformation. Although (5·11) is a covariant equation

for L_0-group, $d\sigma$ depends on the components of the vector N. Therefore, if λ_γ or λ_e is not zero, two observers having a relative velocity close to the light velocity observe different cross sections. We can observe this difference by comparing the very high energy scattering of two electrons with p_1 and $p_2=0$ and the scattering of two electrons with p_1^* and $p_2^*=-p_1^*$, where p_1^* and p_2^* are connected with p_1 and p_2 by an L_0-transformation ($p_1^*=L_0p_1$, $p_2^*=L_0p_2$).

If, for instance, we assume that we are in the coordinate system of $N=(0, 0, 0, 1)$ (or we are moving against the system of $N=(0, 0, 0, 1)$ with a velocity very much smaller than the light velocity), then the angular distribution for the case $p_1=-p_2$ does not depend on λ_γ and λ_e (λ_e reduces the total cross section) and is identical with that of the Møller scattering as we can see directly from (5·8) and (5·10), but for the case $p_2=0$, the angular distribution depends on λ_γ, λ_e and γ. Fig. 1 shows the deviation from the Møller scattering in two cases, A. $\lambda_\gamma \gg \lambda_e$, $\lambda_\gamma^2 m^2\gamma^2=5$; B. $\lambda_\gamma=\lambda_e=\lambda$, $\lambda^2 m^2\gamma^2=0.2$. At present,[3] non-zero λ_γ and λ_e less than 10^{-14} cm are not excluded.

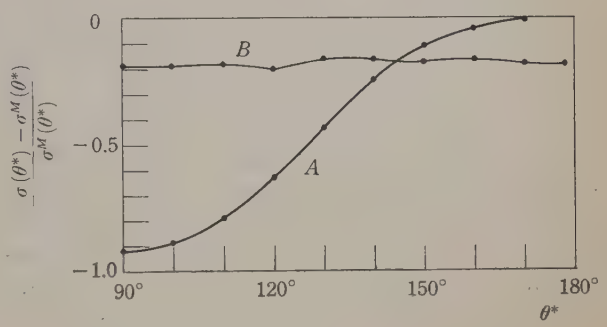

Fig. 1.

Generally, we can detect, in principle, the special direction N by comparing the transitions with two initial states which are connected by an L_0-transformation, if λ_k is not zero. The upper limit of values λ_k can be given but the non-zero λ_k are not excluded at present. We are very anxious to know whether the direction N or non-zero λ_k really exists.

2. *The self-energy of the electron**

The self-energy of the electron is given from (5·6) in the usual way by $\Sigma(p)$ in

$$|a\rangle=a^*(p, i)|0\rangle, \quad |b\rangle=a^*(p', i')|0\rangle,$$

$$\langle a|\ \frac{e^2}{2}\int_{-\infty}^{\infty}\!\!\int dx_1 dx_2 P(H(x_1)H(x_2))|b\rangle=ie^2\delta_4(p-p')\,\Sigma'(p)(u^{i'}(p')u^{i}(p)). \tag{5·12}$$

By using (5·9) and

$$\langle 0|T(\psi(x_1)\bar{\psi}(x_2))|0\rangle=\frac{-1}{(2\pi)^4}\int_F \exp[ip\cdot(x_1-x_2)-\lambda_e^2(N\cdot p)^2]\,\frac{\gamma\cdot p+im}{p^2+m^2}\,d^4p, \tag{5·13}$$

* The self-energy of the photon will be investigated elsewhere by using quantum electrodynamics with nonvanishing photon mass. (F. Coester, Phys. Rev. **83** (1951), 798.)

$$\Sigma(p)=2\int_0^1 dz\int_F dk\,\frac{im(1+z)-\gamma\cdot k}{[k^2+z^2m^2]^2}\exp[-\lambda_\gamma^2(N\cdot k)^2-\lambda_e^2(N\cdot p+k)^2]. \quad (5\cdot14)$$

$\Sigma(p)$ depends on p and $\Sigma(p)\to 0$ when $|\boldsymbol{p}|\to\infty$. For small $|\boldsymbol{p}|$,

$$\Sigma(p)=\Sigma(0)+2\lambda_e^4(N\cdot p)\int_0^1 dz\int_F d^4k\,\frac{(\gamma\cdot k)(N\cdot k)^3}{[k^2+z^2m^2]^2}\exp[-(\lambda_\gamma^2+\lambda_e^2)(N\cdot k)^2]+\cdots,$$

$$\Sigma(0)=2\int_0^1 dz\int_F dk\,\frac{im(1+z)}{[k^2+z^2m^2]^2}\exp[-(\lambda_\gamma^2+\lambda_e^2)(N\cdot k)^2]. \quad (5\cdot15)$$

$\Sigma(0)$ is observed as the increment of mass. The mass of the low energy physical electron is then given by $m+\delta m$, $\delta m=\Sigma(0)$.

§ 6. The concept of the space-time distance

In the theory of (I′), (II′) and (III′) (§ 3) or (I″), (II″) and (III″) (§ 4), it is assumed that the theory is described by the parameter $\hat{\xi}$ which is a vector in a Minkowskian space in which the L_0-transformation is accompanied by the L-transformation in x-space (I′) or L_0-transformation in Σ^m (I″). We can remove the assumption of $\hat{\xi}$-space from the theories of (I′), (II′) and (III′) and (I″), (II″) and (III″), if we express the dynamical principle (II′) b= (II″) b in the following way: Measurements have an order

$$O_1,\ O_2,\ \cdots. \quad (6\cdot1)$$

When a set of results $a=\{a_r\}$ of measurement belonging to O_i of a complete set of commuting observables $\{\zeta_r\}$ is obtained, the expectaion value of the result of measurement belonging to O_{i+1} of an observable ζ is

$$\langle a|\exp[-i\sum_\mu P_\mu C_\mu(O_i,\ O_{i+1})]\zeta\exp[i\sum_\mu P_\mu C_\mu(O_i,\ O_{i+1})]|a\rangle, \quad (6\cdot2)$$

where $|a\rangle$ is the common eigenvector of $\{\zeta_r\}$ associated with $a=\{a_r\}$, P_μ is the observable representing the total energy-momentum of the system and $C_\mu(O_i, O_{i+1})$ is undetermined parameters specified by (O_i, O_{i+1}).

The dynamical principle containing undetermined parameters $C_\mu(O_i, O_{i+1})$ ((6·1), (6·2)) and the assumption that the theory is invariant under L_0-group in x-space ((I′) b) or in Σ^m ((I″) b) imply the assumption of the $\hat{\xi}$-space in (I′) or in (I″). It may become clearer in the expression (6·2) (which is used in the theory[1] attempted previously) that the theory in § 3 or in § 4 does not rely on the space-time characterized by two functions mentioned in § 1.

The space-time distance is defined in the theory in § 3 or in § 4 by a mean value $\overline{C_\mu(O_i, O_{i+1})}$ of $C_\mu(O_i, O_{i+1})$ in (6·2) which fits the results of a number of measurements belonging to O_i and O_{i+1}. In our theory, the concept of space-time distance resides in the results of measurements. It depends on the measured ob-

servables. For instance, in the results of measurements of the observables $\{\zeta_r\}$ which commute with $\boldsymbol{P}$ ($[P_i, \zeta_r]=0$ for $i=1, 2, 3$, $[P_0, \zeta_r] \neq 0$), $C_i(O_i, O_{i+1})$ for $i=1, 2, 3$ are completely undetermined (when $[P^\mu, P^\nu]=0$), hence in the results of these measurements only time-distance exists and space-distance does not exist. $C_\mu(O_i, O_{i+1})$, $\mu=1, 2, 3, 4$ are only undetermined parameters which are introduced to express the relations between the results of measurements belonging to the order (O_i, O_{i+1}). Also in field theory, the measured space-time distance is a mean value $\overline{x_{i+1}-x_i}$ of $x_{i+1}-x_i$ in (2·4) which fits the results of a number of measurements belonging to the order x_i and x_{i+1}. However, the space-time distance $x_{i+1}-x_i$ exists before it is measured, because the existence of x_{i+1} and x_i is assumed in field theory in the definition of the dynamical system to express the degree of freedom of the system. In the theory in § 3 and § 4, space-time distance does not exist before it is measured, because $C_\mu(O_i, O_{i+1})$ or ξ-space is not used to define the dynamical system. Therefore, the fluctuation of $\overline{C_\mu(O_i, O_{i+1})}$ is the fluctuation of the space-time distance itself.

References

1) T. Tati, Nuovo Cimento **4** (1956), 75.
T. Tati, Prog. Theor. Phys. **18** (1957), 235.
2) R. P. Feynman, Phys. Rev. **84** (1951), 108.
3) S. D. Drell, Ann. Phys. **4** (1958), 75.

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On the Test of Global Symmetry

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An attempt to investigate the concept of Global Symmetry on the strong interactions in pion-baryon system is presented. To this end, the absorption processes of K^--meson by proton are investigated at low energies. The general relations of complex phases in K^--P absorption are also presented. We obtain rather conclusive evidence for inconsistency of Global Symmetry with the present experiments. The effects of the intermediate $\overline{K}$-N states and Λ states are also discussed. It should be noted that this conclusion is derived only by kinematical considerations and not by details of dynamics in the model.

§ 1. Introduction

Recently, experiments to investigate and test some model of the strong interactions of elementary particles have become available. However, since we have not yet anything useful to get into details of dynamics in the model of strong interactions, it is necessary to draw results from experiments only by kinematical considerations and not by details of dynamics in the model.

The concept of Global Symmetry on the strong interactions in pion-baryon system was proposed by many authors[1] independently. And many attempts[2] have been made to test this symmetry, but no definite conclusions seem to have been derived up to the present. Among many attempts, Kawarabayashi investigated the Global Symmetry using the K^--proton absorption processes. But as he neglected the intermediate Λ-state, his conclusion may not be taken as conclusive.

We will take into account the effect of this Λ-state in Section 3. To test Global Symmetry, it may be desirable to investigate directly the scattering of pion on hyperon and the pion-nucleon scattering. However, it may be quite hopeless at present to make an experiment on direct pion-hyperon scattering. So one has to ask for some processes having the virtual effects of pion-hyperon interactions. For this reason, we consider the absorption processes of K^--meson by proton at low energies. In Section 2, we will sketch a method to test Global Symmetry under the following assumptions.

I) The effects of the intermediate K^--N states are neglected. This assumption may be plausible, for the K-baryon coupling would be weaker than the π-baryon coupling by about order of unity. These effects will be considered in Sections 3 and 4.

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II) Processes $\pi+\Sigma \rightleftarrows \pi+\Lambda$ are neglected. If the branching ratio of Λ production is very small, this assumption will be accepted (see Section 3).

III) Only S-wave is taken into account. Present experimental results show that the S-wave is dominant in the low energy region. In Section 3, by using the unitarity of S-matrix, the general and exact relation of complex phases in K^--P reaction are presented. In Section 4, we will present a method to test Global Symmetry, using the relation given in Section 3, and the experimental data are analysed by the method given in this section.

§ 2. A Sketch of testing Global Symmetry

We consider the following processes,

$$\begin{aligned} K^-+P &\to K^-+P \\ &\to \Sigma^{\pm,0}+\pi^{\mp,0} \\ &\to \Lambda+\pi^0. \end{aligned} \tag{1}$$

Transition matrix for the processes (1) may be written as

$$\langle \pi Y|T|K^-P\rangle=\langle \pi Y|S|\pi Y\rangle\langle \pi Y|T^\dagger|K^-P\rangle, \tag{2}$$

where unitarity of S-matrix and assumption I) are used. Using assumptions II), III) and the invariance under the Wigner time reversal, we obtain the following relation for the phase in eigen-channel of Eq.(2),

$$e^{i\Delta_I}=-e^{2i\delta_I}e^{-i\Delta_I}, \quad \text{or} \quad \Delta_I=\delta_I+(1/2)\,\pi+n\pi, \tag{3}$$

where I is total isospin, Δ_I and δ_I represent the absorption phase and the π-Y scattering phase shift in the I-state, respectively. If we consider the π-Y scattering in the final state of the processes (1), by Global Symmetry, π interacts with isodoublet $Y=(\Sigma^+, Y^0)$ and $Z=(Z^0, \Sigma^-)$ in the same form as π interacts with nucleon, where $Y^0=\frac{1}{\sqrt{2}}(\Lambda^0-\Sigma^0)$ and $Z^0=\frac{1}{\sqrt{2}}(\Lambda^0+\Sigma^0)$. On the other hand, π-Σ scattering occurs in the ordinary isospin states, $I=0$, 1, 2 and π-Λ scattering does in $I=1$ state, so we can obtain the relation between these two cases, i.e. the ordinary isospin and isodoublet cases. We will denote the π-Σ and π-Λ scattering phase shifts as δ_0, δ_1, δ_2, and δ_1^Λ respectively. π-Y and π-Z scattering phase shifts are denoted by $\alpha_{1/2}$ and $\alpha_{3/2}$ like the phase shifts in the π-N S-wave scattering. The scattering S-matrix can be expanded by the projection operator of the $I_d=1/2$ and 3/2 states, where I_d is total isospin in the case of isodoublets,

$$S=P_{1/2}e^{2i\alpha_{1/2}}+P_{3/2}e^{2i\alpha_{3/2}}, \tag{4a}$$

$$P_{1/2}=1/3(1-\boldsymbol{\tau}\boldsymbol{I}), \tag{4b}$$

$$P_{3/2}=1/3(2+\boldsymbol{\tau}\boldsymbol{I}). \tag{4c}$$

where $\boldsymbol{\tau}$ is the isospin operator of Y or Z and $\boldsymbol{I}$ is that of pions.

Isospin states of $(\pi\text{-}\Sigma)$ and $(\pi\text{-}\Lambda)$ states are denoted by

$$\begin{aligned} &|\pi\Sigma;\ I=0\rangle=|\Sigma 0\rangle,\\ &|\pi\Sigma;\ I=1\rangle=|\Sigma 1\rangle,\\ &|\pi\Sigma;\ I=2\rangle=|\Sigma 2\rangle,\\ &|\pi\Lambda;\ I=1\rangle=|\Lambda 1\rangle. \end{aligned} \tag{5}$$

The results of the operations of $\boldsymbol{\tau I}$ to the states (5) are given as follows;

$$\begin{aligned} &\boldsymbol{\tau I}|\Sigma 0\rangle=-2|\Sigma 0\rangle,\\ &\boldsymbol{\tau I}|\Sigma 1\rangle=-|\Sigma 1\rangle+\sqrt{2}\,|\Lambda 1\rangle,\\ &\boldsymbol{\tau I}|\Sigma 2\rangle=|\Sigma 2\rangle,\\ &\boldsymbol{\tau I}|\Lambda 1\rangle=\sqrt{2}\,|\Sigma 1\rangle. \end{aligned} \tag{6}$$

Using (5) and (6), we can evaluate the S-matrix element for the processes (1);

$$e^{2i\delta_0}=\langle\Sigma 0|S|\Sigma 0\rangle=e^{2i\alpha_{1/2}}, \tag{7a}$$

$$e^{2i\delta_1}=\langle\Sigma 1|S|\Sigma 1\rangle=1/3(2e^{2i\alpha_{1/2}}+e^{2i\alpha_{3/2}}), \tag{7b}$$

$$e^{2i\delta_2}=\langle\Sigma 2|S|\Sigma 2\rangle=e^{2i\alpha_{3/2}}, \tag{7c}$$

$$e^{2i\delta_1{}^{\Lambda}}=\langle\Lambda 1|S|\Lambda 1\rangle=1/3(e^{2i\alpha_{1/2}}+2e^{2i\alpha_{3/2}}). \tag{7d}$$

From (7b) or (7d), we obtain

$$\cos 2(\alpha_{1/2}-\alpha_{3/2})=1. \tag{8}$$

From (7a) and (7b), we get the following relation,

$$\cos 2(\delta_1-\delta_0)=2/3+1/3\cos 2(\alpha_{1/2}-\alpha_{3/2}). \tag{9}$$

Combining (8) and (9), we obtain

$$\cos 2(\delta_1-\delta_0)=1,$$

that is, $|\delta_1-\delta_0|=n\pi$, $n=0, 1, 2, \cdots\cdots$. Then, if we take $n=0$,

$$|\delta_1-\delta_0|=0. \tag{10}$$

Even if we took $n\neq 0$, the following discussions would not be altered. (3) and (10) lead to

$$|\delta_1-\delta_0|=|\Delta_1-\Delta_0|=0. \tag{11}$$

After all, we have shown that the difference of absorption phases between $I=0$ and $I=1$ is equal to that of scattering phase shifts and that the difference vanishes exactly.

We will get some information about the phase difference $\Delta_1-\Delta_0$ by the analysis of experimental results and investigate how the relation (11) is satisfied. $N(\Sigma^-)$, $N(\Sigma^+)$, $N(\Sigma^0)$ and $N(\Lambda)$ denote the probabilities producing Σ^-, Σ^+, Σ^0 and Λ in

the processes (1). And we write the ratio of absorption matrix elements as $M_1/M_0 \equiv re^{i(\Delta_1-\Delta_0)}$. Then we get the following relations by charge independence,

$$\frac{N(\Sigma^+)}{N(\Sigma^-)} = \frac{2/3-2\sqrt{2/3}\, r\cos(\Delta_1-\Delta_0)+r^2}{2/3+2\sqrt{2/3}\, r\cos(\Delta_1-\Delta_0)+r^2}, \tag{12}$$

and

$$\frac{N(\Sigma^+)+N(\Sigma^-)}{2N(\Sigma^0)} = 1+\frac{3}{2}r^2. \tag{13}$$

Using the experimental results by Alvarez[3] *et al.*,

$$N(\Sigma^-) : N(\Sigma^+) : N(\Sigma^0) : N(\Lambda) = 8 : 4 : 4 : \lesssim 1,$$

we get

$$|\Delta_1-\Delta_0|_{exp} \simeq 70^\circ. \tag{14}$$

The most recent experimental data[4] show that

$$|\Delta_1-\Delta_0|_{exp} = 62^\circ \pm 4^\circ. \tag{14'}$$

These results are quite inconsistent with the relation (11). In Fig. 1, we have shown the relation between $N(\Sigma^+)/N(\Sigma^-)$ and $\frac{N(\Sigma^+)+N(\Sigma^-)}{2N(\Sigma^0)}$ taking $\cos(\Delta_1-\Delta_0)$ as parameter. Circles are the experimental result by Alvarez *et al.* and that reported at Kiev Conference[4] (1957). If Global Symmetry is valid, the experimental data must be on the curve of $\cos(\Delta_1-\Delta_0)=1$. However, the experimental data are quite far from this curve. It will be quite difficult to understand the very large phase difference (14) and (14') with the errors due to assumptions I) and II). Indeed, the effects of $\bar{K}$-N states and reaction, $\pi+\Sigma \rightleftarrows \pi+\Lambda$ do not sufficiently alter the above conclusion (see Section 4). Therefore, we obtain rather conclusive evidence for inconsistency of Global Symmetry with the present experiments.

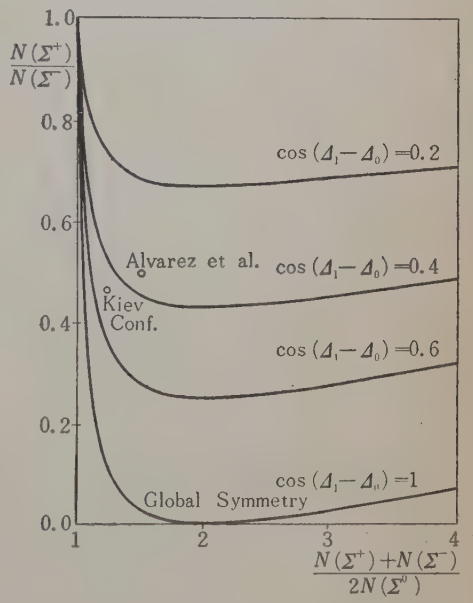

Fig. 1. $N(\Sigma^+)/N(\Sigma^-)$ and $\{N(\Sigma^+)+N(\Sigma^-)\}/2N(\Sigma^0)$ taking $\cos(\Delta_1-\Delta_0)$ as a parameter

§ 3. General and exact relations of complex phases and amplitudes in K^--P reactions

We will take into account the effects of $\bar{K}$-N states and $\pi+\Sigma \rightleftarrows \pi+\Lambda$ processes. To derive the general relations of complex phases and amplitudes, however,

we will use only the unitarity of S-matrix with the invariance under the Wigner time reversal.

In the processes (1), there are two isospin states, $I=0$ and $I=1$. The S-matrix of the $I=0$ state has two channels and can be expressed by the invariance under the Wigner time reversal as

$$S_0 = \begin{array}{c} \begin{matrix} \bar{K}\text{-}N & \pi\text{-}\Sigma \end{matrix} \\ \begin{pmatrix} A_0 e^{2i\delta_0{}^K} & C_0 e^{i\Delta_0} \\ C_0 e^{i\Delta_0} & B_0 e^{2i\delta_0} \end{pmatrix} \end{array}, \tag{15}$$

where all parameters are real, and A_0 and B_0 are positive. $\delta_0{}^K$ and δ_0 are real scattering phase shifts in $\bar{K}$-P and π-Σ scattering respectively, and Δ_0 is the absorption phase in π-Σ production. Using the unitarity of the S-matrix, we obtain following relations:

$$A_0{}^2+C_0{}^2=B_0{}^2+C_0{}^2=1, \tag{16a}$$

$$A_0 e^{-2i\delta_0{}^K+i\Delta_0}+B_0 e^{2i\delta_0-i\Delta_0}=0. \tag{16b}$$

B_0 and δ_0 have relation respectively to A_0 and Δ_0 by (16),

$$A_0=B_0, \tag{17a}$$

$$\Delta_0=\delta_0+\delta_0{}^K+\pi/2. \tag{17b}$$

By the same way, the S-matrix of the $I=1$ state can be expressed as

$$S_1 = \begin{pmatrix} A_1 e^{2i\delta_1{}^K} & C_1 e^{i\Delta_1} & C_\Lambda e^{i\Delta_\Lambda} \\ C_1 e^{i\Delta_1} & B_1 e^{2i\delta_1} & B' e^{i\delta'} \\ C_\Lambda e^{i\Delta_\Lambda} & B' e^{i\delta'} & B_\Lambda e^{2i\delta_\Lambda} \end{pmatrix}, \tag{18}$$

where all parameters are real, and A_1, B_1 and B_Λ are positive. $\delta_1{}^K$, δ_1 and δ_Λ are real scattering phase shifts in $\bar{K}$-N, π-Σ and π-Λ scattering, respectively. Δ_1 and Δ_Λ are absorption phases of π-Σ and π-Λ productions in K^--P reactions. Using the unitarity of the S-matrix, we obtain the following relations:

$$A_1{}^2+C_1{}^2+C_\Lambda{}^2=1, \tag{19a}$$

$$C_1{}^2+B_1{}^2+B'^2=1, \tag{19b}$$

$$C_\Lambda{}^2+B'^2+B_\Lambda{}^2=1, \tag{19c}$$

$$A_1 C_1 e^{i\Delta_1-2i\delta_1{}^K}+C_1 B_1 e^{2i\delta_1-i\Delta_1}+C_\Lambda B' e^{i\delta'-i\Delta_\Lambda}=0, \tag{19d}$$

$$A_1 C_\Lambda e^{i\Delta_\Lambda-2i\delta_1{}^K}+C_1 B' e^{i\delta'-i\Delta_1}+B_\Lambda C_\Lambda e^{2i\delta_\Lambda-i\Delta_\Lambda}=0, \tag{19e}$$

$$C_1 C_\Lambda e^{i\Delta_\Lambda-i\Delta_1}+B_1 B' e^{i\delta'-2i\delta_1}+B_\Lambda B' e^{2i\delta_\Lambda-i\delta'}=0. \tag{19f}$$

Now, we have to solve Eq. (19), i.e. unknown parameters such as B_1, B', B_Λ, δ_1, δ' and δ_Λ must be expressed in terms of known parameters A_1, C_1, C_Λ, δ_1^K, Δ_1 and Δ_Λ.

To solve Eq. (19), we will take into account the experimental evidence that C_Λ is vanishingly small. If we choose $C_\Lambda=0$ in (18), we obtain

$$B'=0, \tag{20a}$$

$$\delta_1=\Delta_1-\delta_1^K-\pi/2. \tag{20b}$$

(20a) tells us that if the branching ratio of Λ-production is very small, the processes $\pi+\Sigma\rightleftarrows\pi+\Lambda$ can be neglected. We can obtain the exact solution of Eq. (19), using the phase relation (20b). Eq. (19) has only one solution,

$$\delta_1=\Delta_1-\delta_1^K-\pi/2, \tag{21a}$$

$$\delta_\Lambda=\Delta_\Lambda-\delta_1^K, \tag{21b}$$

$$\delta'=\Delta_1+\Delta_\Lambda-2\delta_1^K+\pi, \tag{21c}$$

and

$$B_1=\frac{A_1-C_\Lambda^2/C_1^2}{1+C_\Lambda^2/C_1^2}, \tag{22a}$$

$$B_\Lambda=(1-A_1)+\frac{A_1-C_\Lambda^2/C_1^2}{1+C_\Lambda^2/C_1^2}=(1-A_1)+B_1, \tag{22b}$$

$$B'=\frac{C_\Lambda}{C_1}(A_1+B_\Lambda). \tag{22c}$$

§ 4. Test of Global Symmetry

We can test Global Symmetry using (17), (21) and (22). To this end, we transform the S-matrix (18) by the following unitary matrix,

$$U=\begin{bmatrix}1 & 0 & 0\\ 0 & \sqrt{2/3} & -\sqrt{1/3}\\ 0 & \sqrt{1/3} & \sqrt{2/3}\end{bmatrix}. \tag{23}$$

The bases in (18) are transformed by U as follows;

$$|K^- - P\rangle \rightarrow |K^- - P\rangle, \tag{24a}$$

$$|\pi-\Sigma\rangle \rightarrow \sqrt{1/3}\{\sqrt{2}\,|\pi-\Sigma\rangle-|\pi-\Lambda\rangle\}, \tag{24b}$$

$$|\pi-\Lambda\rangle \rightarrow \sqrt{1/3}\{|\pi-\Sigma\rangle+\sqrt{2}\,|\pi-\Lambda\rangle\}. \tag{24c}$$

Then, the S-matrix (18) are transformed by U into the following form;

$$S_1' = \begin{vmatrix} A_1 e^{2i\delta_1^K}, & \sqrt{2/3}C_1 e^{i\Delta_1} & \sqrt{1/3}C_1 e^{i\Delta_1} \\ & -\sqrt{1/3}C_A e^{i\Delta_A}, & +\sqrt{2/3}C_A e^{i\Delta_A} \\ \sqrt{2/3}C_1 e^{i\Delta_1} & 1/3[2B_1 e^{2i\delta_1} & 1/3[\sqrt{2}B_1 e^{2i\delta_1} \\ -\sqrt{1/3}C_A e^{i\Delta_A}, & +B_A e^{2i\delta_A} - 2\sqrt{2}B' e^{i\delta'}], & -\sqrt{2}B_A e^{2i\delta_A} + B' e^{i\delta'}] \\ \sqrt{1/3}C_1 e^{i\Delta_1} & 1/3[\sqrt{2}B_1 e^{2i\delta_1} & 1/3[B_1 e^{2i\delta_1} \\ +\sqrt{2/3}C_A e^{i\Delta_A}, & -\sqrt{2}B_A e^{2i\delta_A} + B' e^{i\delta'}], & +2B_A e^{2i\delta_A} + 2\sqrt{2}B' e^{i\delta'}] \end{vmatrix} \tag{25}$$

Before proceeding to test Global Symmetry, we suppose the processes (1) as follows. The incident probability of K^--P states is distributed into final states in accordance with the unitarity of the S-matrix (the conservation of probability). After the initial probability is distributed into final states, we will assume that final pion-hyperon states proceed to scatter under Global Symmetry, having the complex scattering phase shifts due to other opening channels. This assumption may be accepted if the imaginary parts of phase shifts in pion-hyperon scattering are very small (i.e. ρ_0 and ρ_1 are almost equal to unity). Indeed, these values, ρ_0 and ρ_1, are almost equal to unity (see Table II). Then, the real parts of phase shifts in pion-hyperon scattering may be regarded as the phase shifts in pion-nucleon scattering.

Now, we will impose Global Symmetry on the transformed S-matrix (25) with (26). We then obtain

$$S_1' = \begin{bmatrix} A_1 e^{2i\delta_1^K}, & \sqrt{2/3}C_1 e^{i\Delta_1} & \sqrt{1/3}C_1 e^{i\Delta_1} \\ & -\sqrt{1/3}C_1 e^{i\Delta_A}, & +\sqrt{2/3}C_A e^{i\Delta_A} \\ \sqrt{2/3}C_1 e^{i\Delta_1} & & \\ -\sqrt{1/3}C_A e^{i\Delta_A}, & \rho_0 e^{2i\alpha_{1/2}}, & 0 \\ \sqrt{1/3}C_1 e^{i\Delta_1} & & \\ +\sqrt{2/3}C_A e^{i\Delta_A}, & 0, & \rho_1 e^{2i\alpha_{3/2}} \end{bmatrix}. \tag{26}$$

Equating (25) and (26), we obtain

$$1/3\{2B_1 e^{2i\delta_1} + B_A e^{2i\delta_A} - 2\sqrt{2}B' e^{i\delta'}\} = \rho_0 e^{2i\alpha_{1/2}}, \tag{27a}$$

$$1/3\{B_1 e^{2i\delta_1} + 2B_A e^{2i\delta_A} + 2\sqrt{2}B' e^{i\delta'}\} = \rho_1 e^{2i\alpha_{3/2}}, \tag{27b}$$

$$1/3\{\sqrt{2}B_1 e^{2i\delta_1} - \sqrt{2}B_A e^{2i\delta_A} + B' e^{i\delta'}\} = 0. \tag{27c}$$

By the same way, the following relation in the case of $I=0$ state can be obtained,

$$B_0 e^{2i\delta_0} = \rho_0 e^{2i\alpha_{1/2}}. \tag{28}$$

Combining Eqs. (27) and (28), we obtain

$$B_0 B_1 \cos 2(\delta_1 - \delta_0) = 1/3\{2\rho_0^2 + \rho_0\rho_1 \cos(\alpha_{1/2} - \alpha_{3/2})\}, \tag{29}$$

and

$$B_0^2 = \rho_0^2, \tag{30a}$$

$$\rho_1^2 = \rho_0^2 + 3(B_A^2 - B_1^2). \tag{30b}$$

ρ_0 and ρ_1 can be expressed in terms of experimentally known quantities by (17), (22) and (30). The values of ρ_0 and ρ_1 are listed in Table II.

In the very low energy region, K^--P scatterings are well described by the scattering length approximation,

$$k \cot \eta_I^K = \frac{1}{a_I + ib_I}, \quad \eta_I^K = \delta_I^K + i\gamma_I^K, \tag{31}$$

where k is the incident K^--meson momentum in the c.m. system, and the unit $\hbar = c = y = 1$ is used ($y = 10^{-13}$ c.m.).

The values[5] of a_I and b_I are listed in Table I.

Table I. K^--P zero energy scattering amplitudes reported at CERN Conference in 1958

	a_0+ib_0	a_1+ib_1
sol. 1	$0.28+i0.54$	$1.19+i0.22$
sol. 2	$1.28+i0.71$	$0.43+i0.18$

With this approximation, A_I^2 can be expressed in terms of a_I and b_I,

$$A_I^2 = \frac{a_I^2 + b_I^2\left(1 - \frac{1}{kb_I}\right)^2}{a_I^2 + b_I^2\left(1 + \frac{1}{kb_I}\right)^2}. \tag{32}$$

ρ_1 can also be expressed in terms of a_I and b_I, and $\cos 2(\delta_1^K - \delta_0^K)$ are almost equal to unity in the very low energy region ($T_K \lesssim 50$ Mev.). The values of A_I and ρ_1 are listed in Table II.

Table II. A_0, A_1 and ρ_1 as function of center-of-mass momentum k

	sol. 1			sol. 2		
k	A_0	A_1	ρ_1	A_0	A_1	ρ_1
0.0	1.00	1.00	1.00	1.00	1.00	1.00
0.1	0.948	0.987	0.974	0.944	0.984	0.957
0.2	0.900	0.981	0.963	0.907	0.971	0.932
0.3	0.855	0.978	0.959	0.881	0.960	0.916
0.4	0.813	0.976	0.958	0.862	0.950	0.906
0.5	0.773	0.947	0.959	0.847	0.942	0.898

In estimation of ρ_0 and ρ_1, the branching ratios in the Report[5] at Kiev Conference (1959) are used. These values in Table II show that ρ_0 and ρ_1 are almost equal to unity, so $\alpha_{3/2}$ and $\alpha_{1/2}$ may be regarded as S-wave scattering phase shifts in the case of pion-nucleon. If we use the large phase difference $\Delta_1 - \Delta_0$ of (14) or (14′), the left-hand side of (29) is always negative, but the right-hand side is always

positive,* in the very low energy region. Then we can conclude that the effects of intermediate $\bar{K}$-N and Λ-states cannot alter the conclusion in Section 2. After we had completed the present work, we recieved the Physical Review Vol. **115**, No. 6, containing a paper by M. Ross and G. L. Shaw. In this paper, entitled "*Test of Global Symmetry in $\bar{K}$-P Reactions*", they essentially neglected Λ-effects. But if this neglect is accepted, the fact that Global Symmetry does not hold is self-evident (see Eqs. (8) and (20a)),

$$\langle \pi\Sigma\,;\,1|S|\pi\Lambda\,;\,1\rangle = \sqrt{2}/3(e^{2i\alpha_{1/2}} - e^{2i\alpha_{3/2}}) = 0.^{**}$$

Even if Global Symmetry is valid, the pion-hyperon scattering will be regarded as pion-nucleon scattering, if and only if ρ_0 and ρ_1 are almost equal to unity.

This is the reason why we have discussed the effects of intermediate $\bar{K}$-N and Λ-states.

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References

1) M. Gell-Mann, Phys. Rev. **106** (1957), 1296.
J. Schwinger, Phys. Rev. **104** (1956), 1164.
J. Tiomno, Nuovo Cimento **6** (1957), 69.
2) D. Amati and B. Vitale, Nuovo Cimento **9** (1958), 895.
D. Ito, S. Minami and H. Tanaka, Nuovo Cimento **10** (1958), 554.
G. Feldman and P. T. Matthews, Phys. Rev. **109** (1957), 546.
K. Kawarabayashi, Prog. Theor. Phys. **22** (1959), 451 (L).
3) L. W. Alvarez *et al.*, Nuovo Cimento **5** (1957), 1026.
4) L. W. Alvarez, *Report at Kiev Conference* (1959).
5) R. H. Dalitz, *Report at CERN Conference* (1958).

* We use $\alpha_{3/2}=(-0.101\pm0.015)$ and $\alpha_{1/2}=(0.167\pm0.023)$.

** W are indebted to Dr. A. Kanazawa who kindly pointed out this fact to us.

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On Ratios of the Renormalized and the Bare Coupling Constants in the Beta Decay Interaction

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The ratio of the renormalized and the bare coupling constants in the β-decay interaction is calculated in two ways. The first is evaluation of the contribution from the process in which a neutron and an antiproton, which virtually exchange an infinite number of pions, are annihilated into an electron-neutrino (or antineutrino) pair. For this purpose a dispersion relation for the vertex function of the β-decay is assumed. Using this relation the ratio is calculated in the "ladder" approximation of Federbush, Goldberger and Treiman. The result is that the ratio is smaller than unity. The second way is calculating the ratio so as to include the main contribution from virtual nucleon pairs. For this purpose we derive an equation for the vertex operator of the β-decay. The vertex operator is calculated using Goto and Machida's approximation. The result is that the ratio for the axial vector coupling constants is larger than unity, while that for the vector coupling constants is smaller than unity. The experimental value for the ratio ($\sim 1\cdot 2$) is reproduced at a rather small cutoff energy ($\sim$ one-pion rest mass).

§ 1. Introduction

The question of whether the ratio g_A/f_A, where g_A is the renormalized coupling constant in the β-decay interaction and f_A is the bare axial vector coupling constant, could be determined to be equal to or larger than unity has recently been raised in connection with the conserved vector current proposed by Feynman and Gell-Mann.[1] Recent experimental values[2] of the ratio g_A/g_V, where g_V is the renormalized vector coupling constant, is somewhat larger than unity. On the other hand, the μ-decay suggests that the effect of renormalization is very small for g_V, since only leptons are associated in the μ-decay, and experimental values for g_V and the vector coupling constant in the μ-decay are very close to each other. If g_V is equal to f_V, the bare vector coupling constant, as Feynman and Gell-Mann suggested, and $f_A=f_V$ from the viewpoint of the universal Fermi interaction, g_A should be larger than f_A. Old calculations[3] showed that $g_A/g_V>1$ when $f_A=f_V$, but $g_A/f_A<1$ and $g_V/f_V<1$.

Recently Fujii and Furuichi[4] have shown that the intermediate three-pion state has favorable contribution to the renormalization of the axial vector coupling constant. Their evaluation includes a nucleon loop which virtually interacts with three pions and an electron-neutrino pair. The virtual nucleon pair may therefore make favorable contribution to the renormalization of the axial vector coupling constant.

In this note it will be suggested that the virtual nucleon pairs which cannot be reduced to ordinary nucleons in the sense of Feynman diagram have an effect of increasing the ratio g_A/f_A. In § 2, calculations are made to evaluate the contribution from the process in which a neutron and an antiproton, which virtually exchange an infinite number of pions, are annihilated into an electron and neutrino (or antineutrino) pair. To do this, a dispersion formula for the vertex function of the β-decay interaction is assumed. Physical implication of this formula is discussed. Regarding this formula as an integral equation for the vertex function, we shall evaluate the form factor in the "ladder" approximation.[5] In § 3, the effect of virtual nucleon pairs are investigated, and it will be shown that the effect is to increase the ratio g_A/f_A. Discussions and conclusions are made in § 4.

§ 2. The calculation in the ladder approximation

In this section we shall calculate with the "ladder" approximation which has been used by Federbush, Goldberger and Treiman to invetigate the electromagnetic structure of the nucleon. To do this, we shall first derive the dispersion formula for the form factor of the β-interaction. The notation in this section is exactly the same as those in Goldberger and Treiman's paper.[6]

The importance of the dispersion relations presumably lies in the fact that in several cases they can be directly tested by experiments. In other cases it is at least possible to extract approximate relations between observable quantities. This may be the reason why the dispersion relations have been used to express the relations between observable quantities. In our case, however, it is absolutely necessary that the dispersion relations include the unobservable quantity f_A or f_V. Thus it is the first problem that the bare coupling constants are to be defined.

We shall define the bare coupling constant, f_A, at the high energy limit of the form factor $a(\xi)$, that is, $\lim_{\xi\to\infty} a(\xi)=f_A$. This has not been proved, but based on the following speculation: The higher the energy transfer is, the nearer to the nucleon core the electron-neutrino pair is emitted, and at the high energy transfer limit the lepton pair is emitted by the nucleon core itself. The statements similar to the above were made concerning the high energy limit behavior in the quantum field theory sometime ago by several authors, but have not been proved rigorously. Rather, at the high energy limit approximate calculations[7] showed that this limit value was not finite but zero. The only exception is the evaluation[8] in the Lee model which can be solved exactly. In the following, therefore, we shall assume the above speculation.

We assume the dispersion formula as follows:

$$a(\xi)=f_A+\frac{1}{\pi}\int\frac{\mathrm{Im}\,a(-\xi')}{\xi'+\xi-i\varepsilon}d\xi'. \tag{1}$$

Usually, the dispersion relation for $a(\xi)$ has been written with one subtraction because of the uncertainty at infinity. In our case, however, we have assumed

that $a(\xi)$ tends to f_A at infinity. Im $a(-\varepsilon')$ should tend to zero, since otherwise Eq. (1) is meaningless. Then actually Eq. (1) shows $a(\xi)$ is f_A at infinity. If we assumed $a(\xi)$ is zero at infinity, f_A might be dropped. This corresponds, however, to the assumption that there is no nucleon core that emits a lepton pair, and it may be meaningless to disscuss the ratio of the renormalized to the bare coupling constants.

Note that substitution of

$$f_A = g_A - \frac{1}{\pi}\int \frac{\mathrm{Im}\, a(-\xi')}{\xi' - i\varepsilon} d\xi', \tag{2}$$

which is obtained from Eq. (1) at $\xi=0$ into Eq. (1) itself leads to Goldberger and Treiman's[6] dispersion formula for $a(\xi)$.

If we write Im $a(-\varepsilon')$ as

$$\mathrm{Im}\, a(-\xi') = \tan\varphi_1(\xi')\, \mathrm{Re}\, a(-\xi'), \tag{3}$$

the solution of Eq. (1) may be of the form

$$a(\xi) = f_A(\xi) \exp\int d\xi' \frac{\varphi_1(\xi')}{\xi' + \xi - i\varepsilon}. \tag{4}$$

The evaluation of $\varphi_1(\xi')$ is readily performed in the "ladder" approximation employed by Federbush, Goldberger and Treiman,[5]

$$\varphi_1(\xi) = \tan^{-1}\left[\frac{g^2}{32\pi}\left(\frac{\xi - 4m^2}{\xi}\right)^{1/2}\right]\theta(\xi - 4m^2)\,\theta(\Lambda^2 - \xi), \tag{5}$$

where $\theta(\Lambda^2-\xi)$ is introduced to assure Im $a(\xi)$ at infinity be zero. $f_A(\xi)$ is assumed to be $f_A Z_2$ at $\xi=0$. Z_2 can be evaluated in a similar way if we notice that it is just the Z_1-factor of the isoscalar part of the electromagnetic form factor of nucleon because of Ward's identity. Using the dispersion formula of the form Eq. (1), the first term, f_A, being replaced by unity, we obtain

$$Z_2 = \exp\left\{-\int_{(2m)^2}^{\Lambda^2} d\xi' \frac{\tan^{-1}[3g^2/32\pi[(\xi'-4m^2)/\xi']^{1/2}]}{\xi'}\right\}. \tag{6}$$

Finally, we obtain

$$g_A/f_A = Z_2 \exp\left\{+\int_{(2m)^2}^{\Lambda^2} d\xi' \frac{\tan^{-1}[g^2/32\pi[(\xi'-4m^2)/\xi']^{1/2}]}{\xi'}\right\}. \tag{7}$$

Similarly, we obtain

$$g_V/f_V = Z_2 \exp\left\{-\int_{(2m)^2}^{\Lambda^2} d\xi' \frac{\tan^{-1}[g^2/32\pi[(\xi'-4m^2)/\xi']^{1/2}]}{\xi'}\right\}. \tag{8}$$

It may be clear that $g_V/f_V < g_A/f_A < 1$. The numerical values are given in Fig. 1.

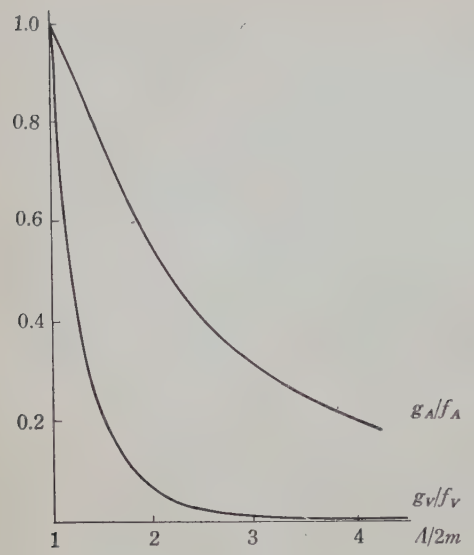

Fig. 1. The cutoff dependence of the ratios in the "ladder" approximation.

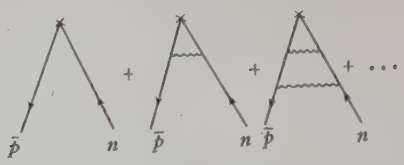

Fig. 2. The Feynman diagrams corresponding to the "ladder" approximation.

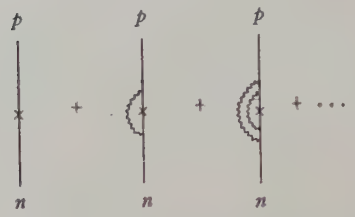

Fig. 3. The Feynman diagrams for the ordinary β-decay, corresponding to Fig. 2.

It is to be remarked that in the "ladder" approximation a neutron and antiproton pair is annihilated into a lepon pair, exchanging virtual pions (Fig. 2). This includes a virtual nucleon pair, but this picture can be reduced to an ordinary neutron β-decay (Fig. 3) in the sense of Feynman diagram.

Another point to be remarked is that if we calculate only to the second order in g (to two terms in Fig. 2), we have

$$\mathrm{Im}\, a(-\xi)=\frac{g^2}{32\pi}\left(\frac{\xi-4m^2}{\xi}\right)^{1/2}\theta(\xi-4m^2)\theta(\Lambda^2-\xi), \tag{9}$$

and substitution of Eq. (9) into Eq. (1) yields

$$g_A/f_A=1+\frac{g^2}{32\pi}\left\{\log\left(\frac{\Lambda}{m}\right)^2-2\right\} \tag{10}$$

at large Λ, i.e. logarithmic divergent. On the other hand, from Eq. (7) we obtain (for comparison with Eq. (7), we put $Z_2=1$)

$$g_A/f_A\sim\left(\frac{\Lambda}{m}\right)^{\tan^{-1}g^2/32\pi}\sim\left(\frac{\Lambda}{m}\right) \tag{11}$$

at large Λ, i.e. linearly divergent. Thus, we can say that the sum of logarithmic divergent terms is linearly divergent.

§ 3. The effect of virtual nucleon pairs

We will investigate in this section the effect of virtual nucleon pairs following

the method that has been used by Goto and Machida[9] to investgate the electromagnetic structure of the nucleon. We first define the vertex operator $\Gamma'(p, n; q)$ of the β-decay interaction by the following equation:

$$\bar{u}(p)\Gamma'_{\mu}(p, n; q)u(n) = -\bar{u}(p)\int d^4x d^4y \times \\ \times e^{ipx}\overrightarrow{D}_x\langle 0|T\psi_p(x)A_{\mu}\bar{\psi}_n(y)|0\rangle\overleftarrow{D}_y e^{-iny}u(n), \tag{12}$$

where

$$\overrightarrow{D}_x = \gamma\cdot\partial_x + m, \tag{13a}$$

$$\overleftarrow{D}_y = -\gamma\cdot\overleftarrow{\partial}_y + m, \tag{13b}$$

$$A_{\mu} = f_A Z_2 \bar{\psi}_p(0) i\gamma_{\mu}\gamma_5\psi_n(0), \tag{14}$$

$$q_{\mu} = n_{\mu} - p_{\mu} \tag{15}$$

The notation other than the above is the same as that used by Goldberger and Treiman.[6] The $\Gamma'_{\mu}(p, n; q)$ is related to matrix element $\langle p|A_{\mu}|n\rangle$ by the equation

$$\langle p|A_{\mu}|n\rangle = (2\pi)^{-3}(m^2/p_0 n_0)\bar{u}(p)\Gamma'_{\mu}(p, n; q)u(n). \tag{16}$$

Performing the indicated differentiations, we obtain

$$-\overrightarrow{D}_x\langle 0|T\psi_p(x)A_{\mu}\bar{\psi}_n(y)|0\rangle\overleftarrow{D}_y = \delta(x)f_A i\gamma_{\mu}\gamma_5\delta(y) \\ + \int d^4x' \Sigma^*(x-x')S_F'(x')f_A i\gamma_{\mu}\gamma_5\delta(y) \\ + \int d^4y' \delta(x)f_A i\gamma_{\mu}\gamma_5 S_F'(y')\Sigma^*(y'-y) \\ -\langle 0|T\eta_p(x)A_{\mu}\bar{\eta}_n(y)|0\rangle, \tag{17}$$

where

$$S_F'(x-y) = i\langle 0|T\psi(x)\bar{\psi}(y)|0\rangle, \tag{18}$$

$$\int d^4x' \Sigma^*(x-x')S_F'(x'-y) = i\langle 0|T\eta(x)\bar{\psi}(y)|0\rangle, \tag{19a}$$

$$\int d^4y' S_F'(x-y')\Sigma^*(y'-y) = i\langle 0|T\psi(x)\bar{\eta}(y)|0\rangle, \tag{19b}$$

$$\eta(x) = \overrightarrow{D}_x\psi(x), \tag{20a}$$

and

$$\bar{\eta}(y) = \bar{\psi}(y)\overleftarrow{D}_x. \tag{20b}$$

In Eqs. (19a) and (19b), Σ^* is a proper self energy operator with δm, the mass renormalization constant, subracted, and defined by

$$\overrightarrow{D}_x S_F'(x-y) = \delta(x-y) + \int d^4x' \Sigma^*(x-x')S_F'(x'-y'), \tag{21a}$$

$$S_F'(x-y)\overleftarrow{D}_y = \delta(x-y) + \int d^4y' S_F'(x-y')\Sigma^*(y'-y). \tag{21b}$$

Using Eq. (18), we may readily find Eqs. (19a) and (19b).

In this way we obtain

$$-\int d^4x d^4y \overrightarrow{D}_x \langle 0|T\psi_p(x) A_\mu \bar{\psi}_n(y)|0\rangle \overleftarrow{D}_y$$

$$= f_A i\gamma_\mu\gamma_5 + \Sigma^*(p) S_F'(p) f_A i\gamma_\mu\gamma_5 + f_A i\gamma_\mu\gamma_5 S_F'(n)\Sigma^*(n)$$

$$-\int d^4x d^4y e^{ipx}\langle 0|T\psi_p(x) A_\mu \bar{\psi}_n(y)|0\rangle e^{-iny}, \tag{22}$$

where

$$\Sigma^*(p) = \int d^4x e^{ipx}\Sigma^*(x), \tag{23}$$

and

$$S_F'(p) = \int d^4x e^{ipx} S_F'(x). \tag{24}$$

Since

$$\bar{u}(p)\Sigma^*(p) S_F'(p) = 0, \tag{25a}$$

and

$$S_F'(n)\Sigma^*(n) u(n) = 0, \tag{25b}$$

in view of Eqs. (21a) and (21b), we finally obtain

$$\bar{u}(p)\Gamma_\mu(p, n; q) u(n) = \bar{u}(p) f_A i\gamma_\mu\gamma_5 u(n)$$

$$-\bar{u}(p)\int d^4x d^4y e^{ipx}\langle 0|T\psi_p(x) A_\mu \bar{\psi}_n(y)|0\rangle e^{-iny}. \tag{26}$$

The vertex operator for the vector coupling may be obtained just in the same manner, except for $f_A i\gamma_\mu\gamma_5$ being replaced by $f_V\gamma_\mu$.

Eq. (26), which corresponds to Eq. (2·5) of reference 9), is rigorous and no approximations have been made. We further remark that Eq. (26) is applicable to the μ-capture without any change.

Thus our task may be to evaluate the second term of Eq. (26) so as to estimate the effect of virtual nucleon pairs. This may be accomplished by using Goto and Machida's approximation[9] in which the main contributions from the virtual nucleon pairs are included. The calculations are much complicated, but the final result is very simple and given by

$$\bar{u}(p)\Gamma_\mu(p, n; q) u(n) = f_A \bar{u}(p) i\gamma_\mu\gamma_5 u(n)$$

$$-f_A \frac{g'^2}{(2\pi)^4}\int d^4k\, \bar{u}(p)\gamma_5 S_F(p-k) i\gamma_\mu\gamma_5 S_F(n-k)\gamma_5 \Delta_F(k) u(n), \tag{27}$$

where

$$g'^2 = \left\{\frac{9}{4} + \frac{1}{2}\left(\frac{8}{\pi}\right)^2\right\} g^2. \tag{28}$$

It may be seen that Eq. (27) has the same form as the second order perturbation theory, except for g^2 being replaced by $g'^2(>g^2)$, hence the effect of virtual nucleon pairs is to increase the pion-nucleon coupling constant. The ratio g_A/f_A is readily obtainable, and

$$g_A/f_A=1+\frac{g'^2}{4\pi}\{I_1(\Lambda/m)-I_2(\Lambda/m)\} \tag{29}$$

where

$$I_1(\Lambda/m)=\frac{1}{4\pi}\frac{(\Lambda/m)^2}{(\Lambda/m)^2-(\mu/m)^2}\int_0^1 dt\, t\log\frac{t^2+(\Lambda/m)^2(1-t)}{t^2+(\mu/m)^2(1-t)}, \tag{30}$$

$$I_2(\Lambda/m)=\frac{1}{4\pi}\frac{(\Lambda/m)^2}{(\Lambda/m)^2-(\mu/m)^2}\int_0^1 dt\, t^3\left\{\frac{1}{t^2+(\mu/m)^2(1-t)}-\frac{1}{t^2+(\Lambda/m)^2(1-t)}\right\}, \tag{31}$$

$\Lambda=$ Feynmann's cutoff momentum, and $\mu=$pion mass. Similarly we have, for the vector coupling constant,

$$g_V/f_V=1-\frac{g'^2}{4\pi}\{I_1(\Lambda/m)+I_2(\Lambda/m)\}. \tag{32}$$

Since $I_1(\Lambda/m)\geq I_2(\Lambda/m)\geq 0$, we conclude that

$$g_A/f_A\geq 1\geq g_V/f_V. \tag{33}$$

The cutoff dependence of the ratios is given in Fig. 4.

In conclusion we may say that the effect of virtual nucleon pairs is to increase the ratio g_A/f_A although we should not take the above result too seriously for the interactions between the P-wave pion and the nucleon are neglected in the above approximation. Experimental value $g_A/f_A\sim 1.2$ is obtained at low cutoff, $\Lambda\sim\mu$.*

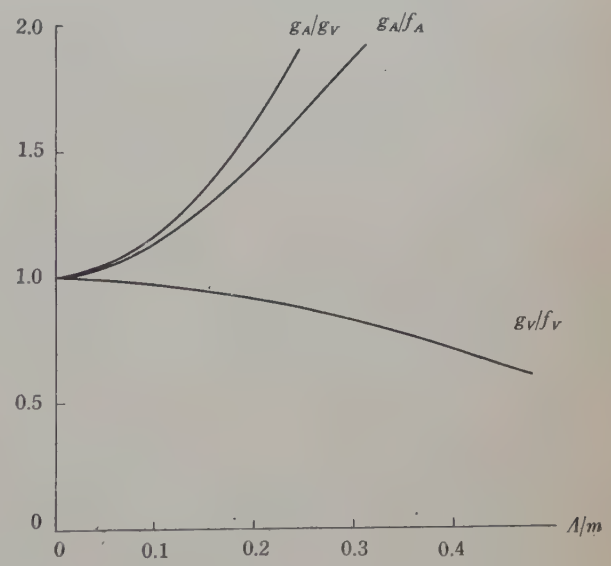

Fig. 4. The cutoff dependence of the ratios in Goto and Machida's approximation.

§ 4. Discussion and conclusion

It has been shown in § 3 that the vertex operator, Eq. (27), which includes the main contribution from the virtual nucleon pairs, has the form of that in the

* In Fig. 4 we have assumed $g^2/4\pi=15$. On the other hand, which of the two coupling constants, $g^2/4\pi$ or $g'^2/4\pi$, should be identified with the observed constant is not clear.[9] In case of $g'^2/4\pi=15$, the larger value of cutoff energy, $\Lambda\sim 2\mu$, is necessary to reproduce the experimental value of g_A/f_A. The qualitative features however, may not be changed.

second order perturbation theory expressible by the Feynmann diagrams in Fig. 5. On the other hand, the ratio g_A/f_A in the second order perturbation theory is smaller than unity, irrespective of the magnitude of cutoff momentum. One might, therefore, think that our result may be misleading.

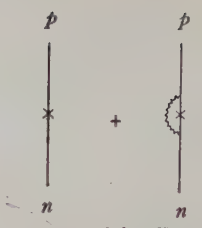

Fig. 5. The Feynman diagrams in the second order perturbation theory.

In general, the ratio of the renormalized and the bare coupling constants is expressible as $Z_1^{-1}Z_2$. In the second order perturbation theory the first factor, Z_1^{-1}, corresponds to the Feynman diagrams in Fig. 5. In this approximation the value of Z_1^{-1} is such that $Z_1^{-1} \geq 1$. The second factor, Z_2, is the bare state probability and the second order perturbation theoretical evaluation yields the result $Z_2 \leq Z_1$. Then the ratio is such that $Z_2 Z_1^{-1} \leq 1$. On the other hand, our calculation yields the ratio at once, hence $Z_1^{-1}Z_2$ itself *corresponds* to the Feynman diagram in Fig. 5. In the strict sense, therefore, correspondence of our calculation to the Feynman diagrams is only superficial.

We have calculated the ratio g_A/f_A with the "ladder" approximation in § 2 and with Goto and Machida's approximation in § 3. Both approximations include nucleon pairs and the results are quite different. As noticed in § 2, the "ladder" approximation which may be expressible in Fig. 2 is reducible to Fig. 3 in the sense of Feynman diagram, which does not include nucleon pairs. We may speculate, therefore, that the difference of the results in § 2 and in § 3 might come from whether nucleon pairs not reducible to ordinary nucleons exist or not.

Acknowledgements

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References

1) R. P. Feynman and M. Gell-Mann, Phys. Rev. **109** (1958), 193.
2) A. I. Alikhanov, *Proceedings of the Ninth Conference on High Energy Nuclear Physics*, Part III (1959).
3) R. J. Finkelstein and S. A. Moszkovski, Phys. Rev. **95** (1954), 1695.
S. S. Gershtein and Ia. B. Zel'dovich, Ž.E.T.F. (U.S.S.R) **29** (1955) 698; Soviet Phys. JETP **2** (1957), 576.
M. Ross, Phys. Rev. **104** (1956), 1736.
4) Y. Fujii and S. Furuichi, Prog. Theor. Phys. **23** (1960), 251.
5) P. Federbush, M. L. Goldberger and S. B. Treiman, Phys. Rev. **112** (1958), 642.
6) M. L. Goldberger and S. B. Treiman, Phys. Rev. **111** (1958), 354.
Dispersion theoretical approach to the ratio g_A/g_V has been made also by K. Symanzik, Nuovo Cimento **11** (1959), 269.
7) G. Källen, *CERN Symposium 1956*, Vol. 2, p. 187.
M. Gell-Mann and F. E. Low, Phys. Rev. **95** (1954), 1300.
H. Umezawa et al., Nuovo Cimento **3** (1956), 772; **4** (1956), 1461.
I. Ya. Pomeranchuk, V. V. Suderov and K. A. Ter-Martirosyan, Phys. Rev. **103** (1956), 784.
8) G. Källen, *Consistency problems in quantum electrodynamics*, CERN 57-43.
9) S. Goto and S. Machida, Prog. Theor. Phys. **20** (1958), 216.

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Electromagnetic Mass Difference of Elementary Particles*

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We propose to examine whether the observed mass difference between the members of each charge multiplet, for both baryons and mesons, can be explained in terms of an electromagnetic self-energy. The electromagnetic form factor is introduced to play the role of cutoff factor, and the isospace transformation property of each individual elementary particle is taken into account for the phenomenologically assumed form factor.

The following results are obtained from the e^2-order self-energy calculation: (i) The observed neutron-proton mass difference ($2.5m_e$) may be explained using the suitable form factor which is assumed to agree with the empirical one for the momentum transfer below 1 Bev/c. The desirable behavior of the form factor for the higher momentum transfer is shown with a few examples, which may suggest some information about the inner structure of the nucleon. The calculated mass difference has no direct close connection with the root-mean-square radius of the charge or magnetic moment distribution, but strongly depends upon the higher momentum behavior of the form factor. (ii) The similar argument seems favorable for understanding that the observed $\Sigma^- - \Sigma^+$ mass difference is considerably larger ($13m_e$), while $(M_{\Sigma^+}+M_{\Sigma^-})/2-M_{\Sigma^0}$ is comparatively small ($\sim 1.5m_e$). (iii) Both the observed $\pi^\pm-\pi^0$ mass difference ($9m_e$) and the observed $K^0(\bar{K}^0)-K^\pm$ mass difference ($8m_e$) may also be understood, if the suitable form factors are assumed according to the isospace transformation property of each meson. (iv) Although no experimental data have been made available about the $\Xi^0-\Xi^-$ mass difference with sufficient accuracy, the argument similar to that for the nucleon leads to some conjectures.

Finally, the strong interaction corrections to the mass shift are estimated for the baryons and mesons in the lowest approximation of g^2. This effect cannot be thoroughly neglected, but in most cases it seems to be regarded as a small correction to the main electromagnetic effect.

§ 1. Introduction

Up to the present, there have been measured the following mass differences,**

$$M_n - M_p = (2.53 \pm 0.02)\, m_e,$$

$$M_{\Sigma^-} - M_{\Sigma^+} = (13.4 \pm 0.7)\, m_e,$$

---

* The main features of this work were partly reported at the 11th annual meeting of the Physical Society of Japan (Sendai, July 17, 1956), at the 13th annual meeting (Kyoto, October 18, 1958), and at the 14th annual meeting (Hiroshima, October 11, 1959).

** L. W. Alvarez, Proc. 9th Conference on High Energy Nuclear Physics (Kiev, 1959) Part III, 1. This article came to us after our work had been finished by using the older experimental data. Most of our arguments, however, need not be changed by the new data except for the improved accuracy of the mass of Σ^0 of the latter.

$$
\begin{aligned}
&M_{\Sigma^-}-M_{\Sigma^0}=(8.7\pm0.8)\,m_e,\\
&M_{\Xi^-}-M_{\Xi^0}\simeq(15\pm17)\,m_e,\\
&m_{\pi^\pm}-m_{\pi^0}=(9.0\pm0.3)\,m_e,\\
&m_{K^0(\bar{K}^0)}-m_{K^\pm}=(7.6\pm1.2)\,m_e.
\end{aligned}
\tag{1·1}
$$

The order of magnitude of these mass differences is not far and away beyond 1/137 times of the rest mass of the corresponding particle. This fact suggests that the origin of these mass differences appears to be purely electromagnetic. Thus it might be of interest to take the viewpoint that the mass levels of individual baryons and mesons may be determined by the charge independent strong interactions, and that the smaller mass shifts between the members of the same charge multiplet are given by the electromagnetic interaction which violates the charge independence. The purpose of the present work is to examine whether all the observed mass differences (1·1) can be systematically understood as the electromagnetic effect, by means of the phenomenological treatment.

In 1953 the present author pointed out[1] the possibility that the negative self-energy due to the magnetic moment interaction could explain the rest mass of the proton being lighter than that of the neutron, introducing the phenomenological Pauli moment. Our calculation[2),3)] in the e^2-order perturbation approximation indicated that the observed $n-p$ mass difference could be understood if the straight cutoff was made at the momentum $|\boldsymbol{k}|\simeq 1$ Bev/c. Similar attempts were also made independently by Oishi,[3),4)] Feynman and Speisman,[5)] and Petermann.[6)] These authors used the Feynman cutoff technique or something like that, but their qualitative results were quite akin to ours. At that time there was no experimental evidence about the electromagnetic structure of the nucleon, so the cutoff factor was introduced only as a mathematical convenience in those early works, except that Petermann took into account the momentum dependence of the vertex correction in the eg^2-approximation. In 1955 the high energy electron scattering experiment at Stanford[7)] showed the evidence for the finite size of the proton. From that time on, experimental information about the electromagnetic structure of the nucleon has been piled up.[8)] We then recalculated[9)] the nucleon mass difference, attempting to use the empirical form factor thus established as a cutoff factor, and found that the observed value of nucleon mass difference and even its sign could not be reproduced so long as the experimental form factor was assumed to be correct at all regions of the momentum transfer. Afterwards similar analyses were made by several persons,[10),11)] and again this difficulty has recently been stressed by Kato and Takeda.[12)] As has been reported by us,[13)] however, one need not be surprised at such a discrepancy. That is because the empirical form factor has been established only for the momentum transfer $|\boldsymbol{k}|\lesssim 1$ Bev/c and its validity for $|\boldsymbol{k}|>1$ Bev/c has never been justified, while the quadratically divergent part of the self-energy expression is fairly sensitive to the behavior of the form factor

at larger $|\boldsymbol{k}|$. As a matter of fact, we have shown[13] that a suitable form factor can be conceived so that it may explain not only the scattering experiment up to 650 Mev but also the observed nucleon mass difference. A detailed discussion of this problem will be found in § 3.

As soon as the experimental data on the Σ-hyperon mass difference were published, an attempt to explain this mass difference was made likewise, the straight cutoff being used by the author[14] and the Feynman cutoff by Marshak and others.[15] The result indicates that the cutoff momentum $|\boldsymbol{k}| \simeq M_\Sigma c$ may produce the observed Σ mass difference, if somewhat large anomalous magnetic moments of the Σ are assumed. In § 4, assuming the trial Σ-hyperon form factor to be very close to that used for the nucleon and using the sufficiently small magnetic moment of the Σ^0, we shall show a possibility of explaining the Σ mass difference, contrary to Kato and Takeda's criticism.[12] Our argument seems favorable for explaining the fact that the $\Sigma^- - \Sigma^+$ mass difference is considerably larger while $(M_{\Sigma^+} + M_{\Sigma^-})/2 - M_{\Sigma^0}$ is comparatively small. In § 5, a speculation on the Ξ-hyperon mass difference will be made, although it has not yet been experimentally established with sufficient accuracy.

As for the π-meson mass difference, Feynman,[16] Oishi,[4] and Petermann[6] independently made the e^2-order perturbation calculation using the Feynman cutoff factor or something like that, and showed that the reasonable cutoff parameter could explain the observed mass difference. It will be shown in § 6 that the rather simply assumed π-meson form factor explains the observed π mass difference. On the other hand, the K-meson mass difference, which has recently been observed at Berkeley,[17] has the opposite sign contrary to Gasiorowicz and Petermann's expectation.[18] We have pointed out[19] that there is a possibility of understanding the opposite sign, if the isospace transformation property of the K-meson is taken into account. A few examples of the trial K-meson form factor which explains the observed K mass difference, will be discussed in § 7.

In § 8, the contribution to the electromagnetic mass shift through strong interactions will be estimated, in the g^2-approximation, for all the baryons and mesons.

§ 2. Expression of electromagnetic self-energy with form factor

The experimental electromagnetic form factor of the nucleon refers only to the space-like part of the invariant momentum transfer k. Generally speaking, the form factor may depend on the time-like part of k^2 as well as the space-like part of it, just as the Feynman cutoff factor. We are going to assume, however, that the form factor works essentially as a cutoff factor for the space-like k^2, considering that such a form factor is tractable in examining the high momentum behavior which turns out to be responsible for the trend of the calculated mass difference. Use of the form factor including the time-like k^2 does not change the qualitative

features of the following discussions.

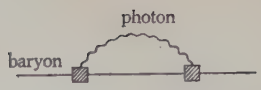

Fig. 1. The e^2-order self-energy of a baryon.

All the baryons are assumed to be Dirac particles with the Pauli magnetic moment. The form factor $F_1(k)$ is associated with the Dirac charge e and intrinsic magnetic moment $e/2M$, and the form factor $F_2(k)$ with the anomalous Pauli magnetic moment $\mu e/2M$.* Here M is the rest mass of the baryon. The e^2-order electromagnetic self-energy of a baryon represented by the diagram in Fig. 1 is calculated using these form factors. Now the integration over k_4 can be carried out independently of the form factor, and the contribution becomes**

$$\delta M = \frac{\alpha}{\pi} M \int_0^\infty dl \left\{ F_1^2(l)\, I(l) - \left[3F_1(l)\, F_2(l) + \frac{5}{4} F_2^2(l) \right] J(l) \right\}, \qquad (2\cdot1)$$

where $\alpha = e^2/4\pi = 1/137$, $l = |\boldsymbol{k}|/M$, and

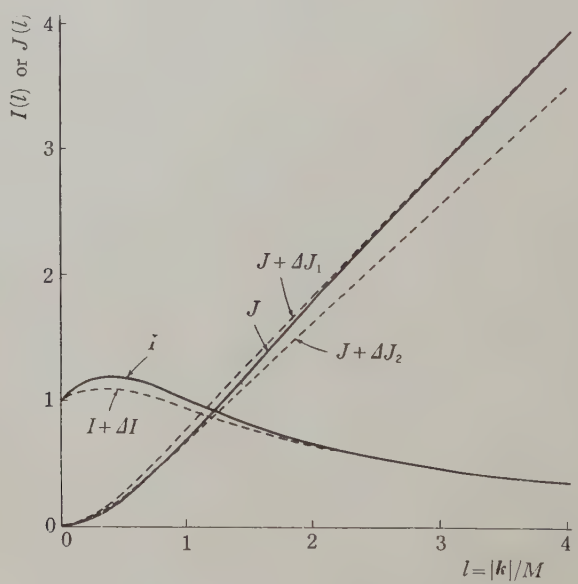

Fig. 2. The solid lines show the $I(l)$ and $J(l)$ functions for a baryon with respect to $l = |\boldsymbol{k}|/M$. The dashed lines represent these functions including the surface term contributions ΔI, ΔJ_1 and ΔJ_2. ΔJ_1 is the surface term contribution of the Dirac-Pauli self-energy and ΔJ_2 that of the Pauli-Pauli self-energy.

---

* The natural unit $\hbar = c = 1$ is used henceforth throughout this paper.

** The expression (2·1) is directly given by the corresponding equation in our early report (ref. 2) or 3)), if the surface term is omitted before the integration over $|\boldsymbol{k}|$ is done. Afterwards, the same expression, apart from the form factor, was also obtained by K. Huang (Phys. Rev. **101** (1956), 1173).

$$I(l)=2/\sqrt{1+l^2}-(\sqrt{1+l^2}-l),$$
$$J(l)=\sqrt{1+l^2}-1/\sqrt{1+l^2}. \qquad (2\cdot2)$$

$I(l)$ comes from the positive self-energy due to the Dirac-Dirac interaction, and $-J(l)$ the negative self-energy due to the Dirac-Pauli and Pauli-Pauli interactions. Unless the form factor is introduced, the integrals of $I(l)$ and $J(l)$ over l are respectively logarithmically and quadratically divergent. Fig. 2 shows the behaviors of these functions with respect to l. In (2·1) the surface term contribution[20] arising from the translation of momentum variable has been omitted, but its effect on the numerical result is not very large in most cases.

Fig. 3. The e^2-order self-energy of a meson.

The K-meson is assumed to be a spin 0 particle as well as the π-meson. The e^2-order electromagnetic self-energy of a meson represented by the diagrams in Fig. 3 is calculated by introducing the meson form factor. Omitting the surface term again, we have the contribution

$$\delta m=\frac{\alpha}{2\pi}m\int_0^\infty dl\,F_1^2(l)\,K(l), \qquad (2\cdot3)$$

where m is the rest mass of the meson, $l=|\boldsymbol{k}|/m$, and

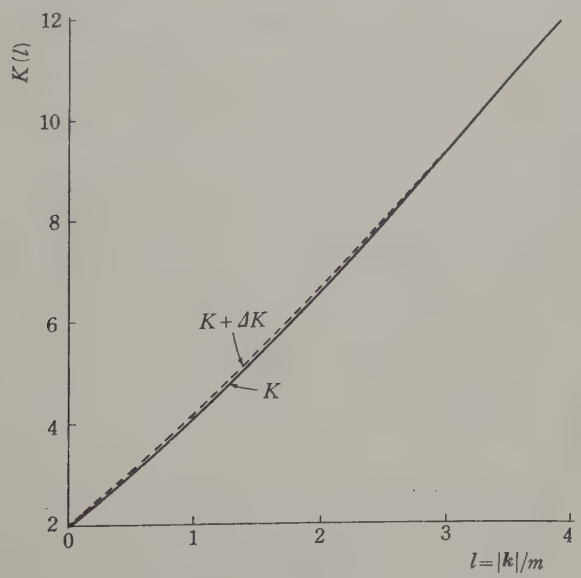

Fig. 4. The solid line shows the $K(l)$ function for a meson with respect to $l=|\boldsymbol{k}|/m$. The dashed line represents this function including the surface term contribution ΔK.

$$K(l)=2l+\sqrt{1+l^2}+1/\sqrt{1+l^2}. \tag{2·4}$$

Unless the form factor is introduced, the integral of $K(l)$ is quadratically divergent, as shown in Fig. 4. Either Klein and Gordon's formalism or Duffin and Kemmer's formalism naturally leads to the same final result of $K(l)$.

For the convenience of our later discussions, let us represent the form factor as the sum of isoscalar and isovector parts.

Nucleon :

$$\left.\begin{aligned} F_1^{(N)}(l)&=F_{1S}^{(N)}(l)+\tau_3 F_{1V}^{(N)}(l)\\ F_2^{(N)}(l)&=F_{2S}^{(N)}(l)+\tau_3 F_{2V}^{(N)}(l)\end{aligned}\right\} \quad l=|\boldsymbol{k}|/M_N, \tag{2·5}$$

$$\left.\begin{aligned} F_{1S}^{(N)}(0)&=F_{1V}^{(N)}(0)=1/2\\ F_{2S}^{(N)}(0)&=(\mu_p+\mu_n)/2=-0.060\\ F_{2V}^{(N)}(0)&=(\mu_p-\mu_n)/2=1.853\end{aligned}\right\}. \tag{2·6}$$

Σ-hyperon :

$$\left.\begin{aligned} F_1^{(\Sigma)}(l)&=F_{1S}^{(\Sigma)}(l)+I_3 F_{1V}^{(\Sigma)}(l)\\ F_2^{(\Sigma)}(l)&=F_{2S}^{(\Sigma)}(l)+I_3 F_{2V}^{(\Sigma)}(l)\end{aligned}\right\} \quad l=|\boldsymbol{k}|/M_\Sigma, \tag{2·7}$$

$$\left.\begin{aligned} &F_{1S}^{(\Sigma)}(0)=0,\ F_{1V}^{(\Sigma)}(0)=1\\ &F_{2S}^{(\Sigma)}(0)=(\mu_{\Sigma^+}+\mu_{\Sigma^-})/2=\mu_{\Sigma^0}\\ &F_{2V}^{(\Sigma)}(0)=(\mu_{\Sigma^+}-\mu_{\Sigma^-})/2\end{aligned}\right\}. \tag{2·8}*$$

Ξ-hyperon :

$$\left.\begin{aligned} F_1^{(\Xi)}(l)&=F_{1S}^{(\Xi)}(l)-\tau_3 F_{1V}^{(\Xi)}(l)\\ F_2^{(\Xi)}(l)&=F_{2S}^{(\Xi)}(l)-\tau_3 F_{2V}^{(\Xi)}(l)\end{aligned}\right\} \quad l=|\boldsymbol{k}|/M_\Xi, \tag{2·9}$$

$$\left.\begin{aligned} F_{1S}^{(\Xi)}(0)&=F_{1V}^{(\Xi)}(0)=-1/2\\ F_{2S}^{(\Xi)}(0)&=(\mu_{\Xi^-}+\mu_{\Xi^0})/2\\ F_{2V}^{(\Xi)}(0)&=(\mu_{\Xi^-}-\mu_{\Xi^0})/2\end{aligned}\right\}. \tag{2·10}$$

π-meson :

$$F_1^{(\pi)}(l)=I_3 F_V^{(\pi)}(l), \quad l=|\boldsymbol{k}|/m_\pi, \tag{2·11}$$

$$F_V^{(\pi)}(0)=1. \tag{2·12}$$

K-meson :

$$\left.\begin{aligned} F_1^{(K)}(l)&=F_S^{(K)}(l)+\tau_3 F_V^{(K)}(l)\\ F_1^{(\bar{K})}(l)&=-F_S^{(K)}(l)+\tau_3 F_V^{(K)}(l)\end{aligned}\right\} \quad l=|\boldsymbol{k}|/m_K, \tag{2·13}$$

---

* The fact that $(\mu_{\Sigma^+}+\mu_{\Sigma^-})/2=\mu_{\Sigma^0}$ holds independently of the perturbation approximation, was pointed out by the author[14] and also by Marshak and others.[15] (2·7) obviously gives the relation extended for $l\neq 0$, $[F_2^{(\Sigma^+)}(l)+F_2^{(\Sigma^-)}(l)]/2=F_2^{(\Sigma^0)}(l)$.

$$F_S^{(K)}(0)=F_V^{(K)}(0)=1/2. \tag{2·14}$$

Here $\tau_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, $I_3=\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}$, and μ denotes the anomalous magnetic moment of the baryon in units of the corresponding baryon magnetons. Since $F_1(0)$ represents the total charge of the particle divided by an elementary charge, Q/e, the above relations of $F_1(l)$ for $l=0$ are reduced to the well-known Nishijima and Gell-Mann relation, $Q/e=\frac{1}{2}(S+\mathcal{N})-I_z$, where S is the strangeness quantum number, $\mathcal{N}$ is the number of baryons minus the number of antibaryons, and $I_z=\frac{1}{2}\tau_3$ or I_3. The vanishing of $F_S^{(\pi)}(l)$ in (2·11) even for $l\neq 0$ is due to the fact that the strangeness of the π-meson is assigned to be zeno and that the charge conjugate π^+-and π^--mesons belong to the same charge triplet, as commonly assumed. The above particular situation on the π-meson form factor generally holds true for neither the other isovector particles nor the isospinor particles.

Hofstadter and others [21),22)] examined several models of the nucleon form factor to analyze the high energy electron scattering data. In Table I, let us list several types of these models together with the asymptotic behavior at the high momentum transfer. In order that the form factor may serve as a convergent factor for the divergent integral, the product of $F_i(l)F_j(l)$ in the self-energy expression has to satisfy the condition:

$$F_i(l)F_j(l)\to l^\nu \text{ when } l\to\infty,$$
$$\begin{cases}\nu<0 & \text{for logarithmically divergent integral,}\\ \nu<-2 & \text{for quadratically divergent integral.}\end{cases} \tag{2·15}$$

Table I

Model	$\rho(r)$	$F(l)$	Asymptotic behavior
Gaussian (G)	e^{-r^2}	e^{-l^2}	e^{-l^2}
Hollow exponential (HE)	re^{-r}	$(4/3)(1+l^2)^{-3}-(1/3)(1+l^2)^{-2}$	l^{-4}
Exponential (EXP)	e^{-r}	$(1+l^2)^{-2}$	l^{-4}
Yukawa II (YII)	$r^{-1}e^{-r}$	$(1+l^2)^{-1}$	l^{-2}
Yukawa I (YI)	$r^{-2}e^{-r}$	$l^{-1}\tan^{-1}l$	l^{-1}
Clementel and Villi (CV)	$r^{-1}e^{-r}-\delta(r)$	$(1-l^2)/(1+l^2)$	const.

§ 3. Nucleon mass difference

Introducing the nucleon form factors (2·5–6) into the self-energy expression (2·1), we obtain the nucleon mass difference,

$$M_n-M_p=(\delta M_N)_{\tau_3=-1}-(\delta M_N)_{\tau_3=+1}\equiv \Delta M_1+\Delta M_2, \tag{3·1}$$

where

$$\left.\begin{aligned} \Delta M_1 &= -\frac{\alpha}{\pi} M_N \int_0^\infty dl\, 4F_{1S}^{(N)} F_{1V}^{(N)} I(l) \\ \Delta M_2 &= \frac{\alpha}{\pi} M_N \int_0^\infty dl\,(6F_{1S}^{(N)} F_{2V}^{(N)} + 6F_{2S}^{(N)} F_{1V}^{(N)} + 5F_{2S}^{(N)} F_{2V}^{(N)}) J(l) \end{aligned}\right\} \quad (3\cdot 2)$$

Hofstadter and others[8] have shown that both form factors, EXP and CV, with the root mean square radius of 0.8 fermi,

$$\left.\begin{aligned} F_{EXP}(l) &= 1/(1+1.2085\, l^2)^2 \\ F_{CV}(l) &= (1-0.4028\, l^2)/(1+2.0142\, l^2) \end{aligned}\right\} \quad l=|\boldsymbol{k}|/M_N \qquad (3\ 3)$$

fit the high energy electron scattering data up to 650 Mev, under the assumption $F_{1S}=F_{1V}=F_{2V}$. Use of F_{CV}^2, however, is excluded for the present purpose on account of the condition (2·15). F_{EXP}^2 and $F_{CV}F_{EXP}$ can be used in (3·2), but they do not produce the observed mass difference.[9),10),11),13)] Now it should be remembered that these empirical form factors have been established only for the momentum transfer $l \lesssim 1$ ($|\boldsymbol{k}| \lesssim 1$ Bev/c), and that the validity of their high momentum behaviors for $l>1$ has never been justified.

As to ΔM_1, $I(l)$ decreases as $1/l$ for large l, therefore the behavior of the form factor below $l \lesssim 1$ determines the major part of its numerical value with the negative sign. Thus we may safely adopt the empirical form factor, e.g. F_{EXP}, in ΔM_1 and obtain $\Delta M_1 \simeq -2.1 m_e$. Since $J(l)$ linearly increases as l for large l, ΔM_2 is considerably sensitive to the high momentum behavior of the form factor, and the empirical form factor ought not to be inserted into ΔM_2 without any justification. At present we cannot get any information about the high momentum behavior of the form factor or the inner part of the distribution function, either from reliable theoretical study or from available experimental data. We now attempt to make a trial function which agrees with the empirical form factor below $l \lesssim 1$ but generally differs from it above $l \gtrsim 1$ so as to explain the observed mass difference.

We are going to discuss the problem from two different but rather simple standpoints, although there may be many other possible cases.

(i) *The first standpoint:* It is assumed that every component of the form factor $F_i(l)$ can be written as $F_i(0)$ multiplied by a trial function $F_A(l)$. It follows that

$$F_{1S}^{(N)}(l) = F_{1V}^{(N)}(l) = \frac{1}{2} F_A(l), \qquad (3\cdot 4)$$

$$F_{2S}^{(N)}(l) = -0.060\, F_A(l), \qquad (3\cdot 5)$$

$$F_{2V}^{(N)}(l) = 1.853\, F_A(l). \qquad (3\cdot 6)$$

Two examples of the trial form factor F_A are shown in Fig. 5, in which F_{AI} and F_{AII} are assumed to have the positive and negative signs for large l respectively.

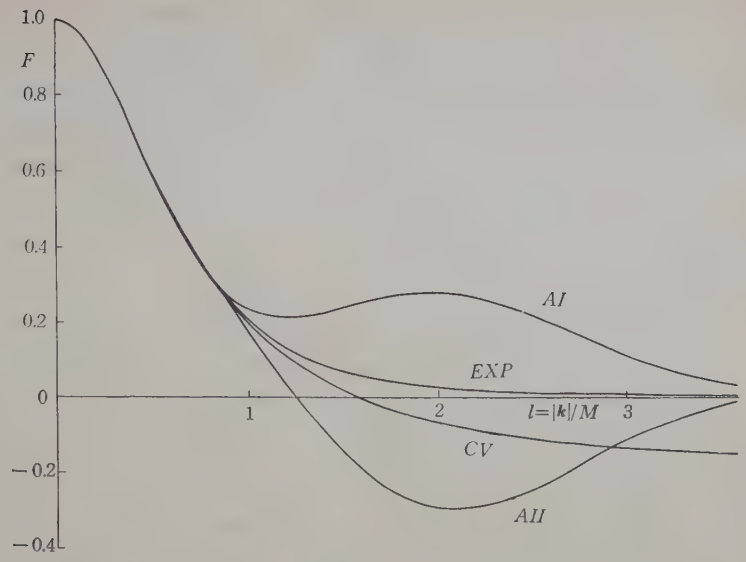

Fig. 5. The trial nucleon form factors F_{AI} and F_{AII} are shown together with F_{EXP} and F_{CV}. For example, they are given by
$F_{AI}=1/(1+2.5395\,l^2)+0.1284\,l^4 e^{-0.5l^2}$,
$F_{AII}=[(4/3)/(1+0.6893\,l^2)^3-(1/3)/(1+0.6893\,l^2)^2]-0.1795\,l^4 e^{-0.5l^2}$.

F_{EXP} and F_{CV} are also shown in the figure for the sake of comparison. If either F_{AI} or F_{AII} is used in (3·4–6), we have $\Delta M_2 \simeq 4.8 m_e$. As the use of the trial form factor for ΔM_1 does not appreciably affect its numerical value ($\Delta M_1 \simeq -2.3 m_e$), we thus obtain the observed value of the mass difference, $\Delta M_1+\Delta M_2 \simeq 2.5 m_e$. As is seen from (3·5), we have here assumed the quite small value of $|F_{2S}(l)|$ even for large l. Then the term of $6F_{1S}F_{2V}$ in (3·2) primarily contributes to ΔM_2, while the remaining terms of $6F_{2S}F_{1V}$ and $5F_{2S}F_{2V}$ have an effect on the total contribution only within about 10% so far as the common form factor F_A is taken for every component. Therefore the condition

$$F_{1S}^{(N)}(l)\,F_{2V}^{(N)}(l)>0 \quad \text{for large } l \tag{3·7}$$

is necessary in order to get the correct mass difference from the first standpoint. The condition (3·7) excludes the use of $F_{AI}F_{AII}$ for $F_{1S}F_{2V}$ in (3·2). We have also assumed above that $F_{2S}F_{1V}<0$ and $F_{2S}F_{2V}<0$ for large l, but these inequalities are not always needed and one of them or both may be positive, in which case ΔM_2 undergoes only small change. Furthermore, the assumption $F_{1S}=F_{1V}$ in (3·4) is not always necessary for explaining the nucleon mass difference. We may indeed assume that $F_{1S}\neq F_{1V}$ for large l, which leads to a nonvanishing inner charge distribution of the neutron with a sufficiently small r.m.s. radius so as to be compatible with the available experimental data.[11),23)]

(ii) *The second standpoint:* It is assumed that both the isoscalar and isovector parts of the form factor for large l behave similarly with the same order of magnitude in the absolute value, without regard to their behaviors at small l. The smallness of the observed value of $|F_{2S}(0)| = |\frac{1}{2}(\mu_p + \mu_n)| = 0.06$ may be understood qualitatively as a result of minor π-meson contribution to the isoscalar part of $F_2(l)$ for small l. $|F_{2S}(l)|$ for large l, however, might become as large as the isovector part of $F_2(l)$, on account of the considerable contributions from the other heavier particles as well as a large number of π-mesons. Now let us conceive a model of F_{2S} which is close to $-0.06F_{EXP}$ for small l but behaves as $\frac{1}{2}F_A$ for large l. Two examples of such a trial form factor are shown in Fig. 6, where F_{BI} and F_{BII} become close to F_{AI} and F_{AII} for large l respectively. Furthermore, both $F_{1S}(l)$ and $F_{2V}(l)$ are also assumed to behave as $\frac{1}{2}F_A$ for large l. ΔM_2 is then given by the sum of three contributions from $6F_{1S}F_{2V}$, $6F_{2S}F_{1V}$ and $5F_{2S}F_{2V}$, each of them being of the same order of magnitude, roughly $\sim 1.5 m_e$ in the absolute value. Therefore the observed mass difference can be explained only under the condition

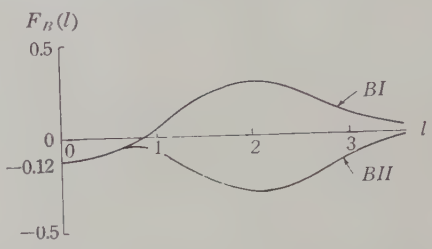

Fig. 6. Two examples of F_B are shown.

$$\text{all of } F_{1S}^{(N)}F_{2V}^{(N)},\ F_{2S}^{(N)}F_{1V}^{(N)} \text{ and } F_{2S}^{(N)}F_{2V}^{(N)} > 0 \quad \text{for large } l. \tag{3·8}$$

This restriction means that all of F_{1S}, F_{2S} and F_{2V} should have the same sign for large l, just as either F_{AI} or F_{AII}. Instead of $\frac{1}{2}F_A$, however, F_A itself may be taken as an asymptotic function of the form factor. In this case, each of the above three contributions roughly amounts to $\sim 5m_e$ in the absolute value, therefore the observed mass difference can be explained only under the condition

$$\text{any two of } F_{1S}^{(N)}F_{2V}^{(N)},\ F_{2S}^{(N)}F_{1V}^{(N)} \text{ and } F_{2S}^{(N)}F_{2V}^{(N)} > 0 \text{ and the remaining one } < 0 \quad \text{for large } l. \tag{3·8′}$$

It should be noted that there is no other possibility than either (3·8) or (3·8′) to explain the nucleon mass difference, insofar as the second standpoint is concerned.

The trial form factors, F_{AI} and F_{AII}, may be arbitrarily represented by a linear combination of the same or different type of functions with various parameters. As is shown in Fig. 5, for example, F_{AI} may be written as a sum of the YII function with the r.m.s. radius of 0.82 fermi and the modified G function without affecting the r.m.s. radius, and F_{AII} as a combination of the HE function with the r.m.s. radius of 0.78 fermi and the modified G function with negative sign. Fig. 7 shows the graphs of the square of proton form factor, $F^{(p)2}$, versus k^2 for these trial AI and AII models together with the EXP and CV models, which are to be compared with the experimental points by Hofstadter and others.[8] Here $F^{(p)2}$ has been defined by $(F_{1S}+F_{1V})^2 = [(F_{2S}+F_{2V})/1.79]^2$.

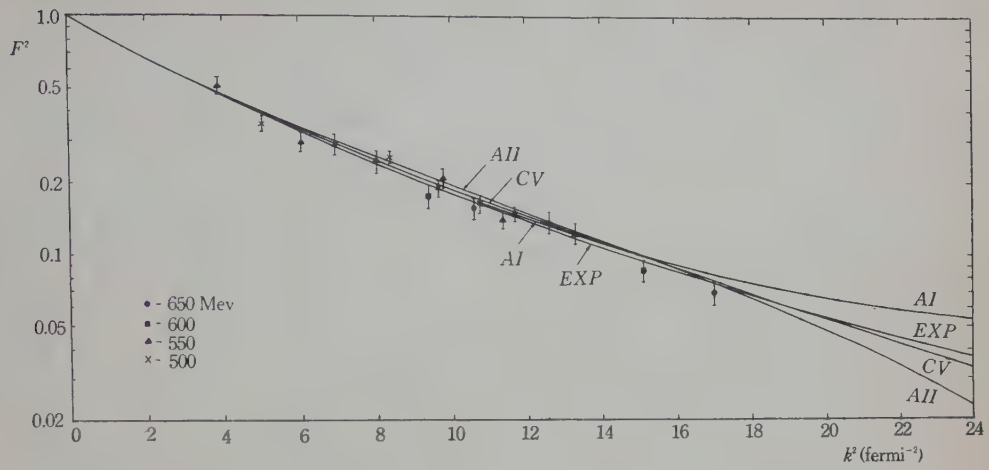

Fig. 7. $F^{(p)2}$ is plotted against k^2 in units of fermi^{-2}. The trial form factors F_{AI} and F_{AII} are compared with those of the *EXP* and *CV* models. The experimental points show the electron-proton scattering data between 500 and 650 Mev measured by Hofstadter et al.

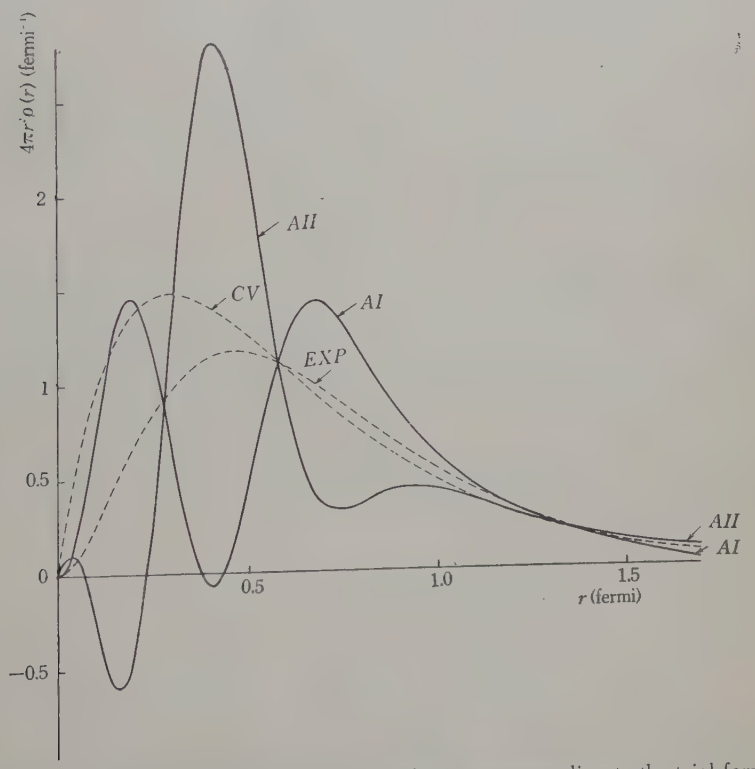

Fig. 8. The nucleon distribution functions ρ_{AI} and ρ_{AII} corresponding to the trial form factors F_{AI} and F_{AII} assumed in Fig 5 are shown as examples, together with ρ_{EXP} and ρ_{CV}. The detailed behaviors at the inner region should not be taken seriously.

The corresponding spatial distribution functions, $\rho_{AI}(r)$ and $\rho_{AII}(r)$, are given by the Fourier transforms of $F_{AI}(k)$ and $F_{AII}(k)$ respectively. Fig. 8 indicates $4\pi r^2\rho(r)$ for these trial functions. Those for the EXP and CV models are also shown in the figure. Needless to say, the detailed behavior of the trial distribution functions, ρ_{AI} and ρ_{AII}, at the inner region should not be taken very seriously, because we have just shown them as arbitrary examples among a large number of possible models. The future experimental and theoretical works, which are intended to investigate the inner structure of the nucleon, will not only discriminate between these models but also test the standpoints presented above.

§ 4. Σ-hyperon mass difference

Let us consider first the $\Sigma^- - \Sigma^+$ mass difference calculated from (2·1),

$$M_{\Sigma^-} - M_{\Sigma^+} = (\delta M_\Sigma)_{I_3=-1} - (\delta M_\Sigma)_{I_3=+1} \equiv \Delta M_1 + \Delta M_2, \tag{4·1}$$

where ΔM_1 and ΔM_2 have a formal resemblance to those for the nucleon (3·2), except that the form factor components satisfy the different conditions because of the isovector nature of the Σ.

If the first standpoint is taken, it follows that

$$F_{1S}^{(\Sigma)}(l) = 0, \tag{4·2}$$

$$F_{1V}^{(\Sigma)}(l) = F_A(l), \tag{4·3}$$

$$F_{2S}^{(\Sigma)}(l) = \frac{1}{2}(\mu_{\Sigma^+} + \mu_{\Sigma^-}) F_A(l) = \mu_{\Sigma^0} F_A(l), \tag{4·4}$$

$$F_{2V}^{(\Sigma)}(l) = \frac{1}{2}(\mu_{\Sigma^+} - \mu_{\Sigma^-}) F_A(l), \tag{4·5}$$

where $l = |\boldsymbol{k}|/M_\Sigma$ and F_A is assumed to be similar to the trial nucleon form factor. On account of (4·2), ΔM_1 vanishes and ΔM_2 has the contributions only from $F_{1V}F_{2S}$ and $F_{2S}F_{2V}$. If we assume a sufficiently small value for $|\mu_{\Sigma^0}|$ following Kato and Takeda,[12] ΔM_2 becomes too small to explain the observed $\Sigma^- - \Sigma^+$ mass difference ($13 m_e$), even though the high momentum behavior of F_A is taken into account.

We should recollect that the charge independence requires $F_{1S}^{(\Sigma)}(0) = 0$, but this is not the case for general l, i. e.

$$F_{1S}^{(\Sigma)}(l) \neq 0 \quad \text{for } l \neq 0, \tag{4·6}$$

which implies a nonvanishing charge distribution of the Σ^0 in contrast with the π^0. This effect was ignored in the previous calculations.[12),14),15] To take (4·6) into account is nothing but a partial application of the second standpoint in a sense. We then assume that $F_{1S}(l)$ behaves as $F_A(l)$ for large l, which

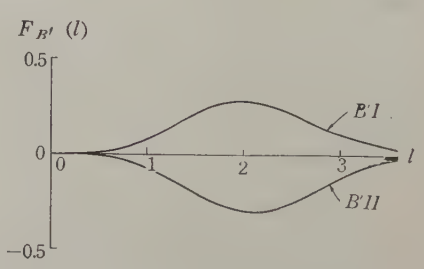

Fig. 9. Two examples of $F_{B'}$ are shown.

is named as $F_{B'}$ and its behavior is shown in Fig. 9. In this case it follows from (4·3) and (4·6) that ΔM_1 does not vanish but is small enough as compared with the nucleon case, since $|F_{1N}(l)|$ is still small for $l \ll 1$. In order to explain the large $\Sigma^- - \Sigma^+$ mass difference, the condition

$$F_{1N}^{(\Sigma)}(l)\, F_{2Y}^{(\Sigma)}(l) > 0 \quad \text{for large } l \tag{4·7}$$

is necessary to hold, because the term $F_{1N}F_{2Y}$ provides the main contribution to ΔM_2. In fact, assuming the reasonable values of $\mu_{\Sigma^+} - \mu_{\Sigma^-} \simeq 3.7$ and $|\mu_{\Sigma^0}| \sim 0.08$ in units of hyperon magnetons, which were used in Kato and Takeda's estimation,[12] we obtain $\Delta M_1 + \Delta M_2 \simeq (13 \pm 1)\, m_e$, where the upper or lower sign corresponds to the unknown sign of μ_{Σ^0}.

Now we consider the problem entirely from the second standpoint. It is assumed that $|F_{1N}|$ and $|F_{2N}|$ as well as $|F_{1Y}|$and$|\ F_{2Y}|$ behave as $|F_A|$ for large l, however small or large they may be at $l \ll 1$. The argument for ΔM_1 in the last paragraph holds true also in this case. As regards ΔM_2, however, each of three parts, $6|F_{1N}F_{2Y}|$, $6|F_{2N}F_{1Y}|$ and $5|F_{2N}F_{2Y}|$, has the contribution with the same order of magnitude ($\sim 5m_e$). Consequently, the large $\Sigma^- - \Sigma^+$ mass difference can be obtained only under the condition:

$$\text{all of } F_{1N}^{(\Sigma)}F_{2Y}^{(\Sigma)},\ F_{2N}^{(\Sigma)}F_{1Y}^{(\Sigma)} \text{ and } F_{2N}^{(\Sigma)}F_{2Y}^{(\Sigma)} > 0 \quad \text{for large } l. \tag{4·8}$$

If the asymptotic function of the form factors is assumed to be roughly twice as large as F_A instead of F_A itself, the condition:

$$\text{any two of } F_{1N}^{(\Sigma)}F_{2Y}^{(\Sigma)},\ F_{2N}^{(\Sigma)}F_{1Y}^{(\Sigma)} \text{ and } F_{2N}^{(\Sigma)}F_{2Y}^{(\Sigma)} > 0 \text{ and the remaining one } < 0 \quad \text{for large } l, \tag{4·8'}$$

is necessary to hold. There is no other possibility than either (4·8) or (4·8′) to explain the $\Sigma^- - \Sigma^+$ mass difference from the second standpoint. It should be interesting to note that the conditions (4·7) and (4·8) or (4·8′) are formally equal to those for the nucleon, (3·7) and (3·8) or (3·8′). This resemblance of the conditions between nucleon and Σ does not seem trivial, but might suggest some implication of the inner structure.

Next we shall discuss the mass of Σ^0. It is convenient to calculate the mass difference,

$$\frac{1}{2}\,(M_{\Sigma^+} + M_{\Sigma^-}) - M_{\Sigma^0} = \frac{\alpha}{\pi}\, M_\Sigma \int_0^\infty dl \left[F_{1Y}^{(\Sigma)2} I(l) - \left(3F_{1Y}^{(\Sigma)} F_{2Y}^{(\Sigma)} + \frac{5}{4} F_{2Y}^{(\Sigma)2} \right) J(l) \right], \tag{4·9}$$

which is to be compared with the observed value $(2 \pm 1)\, m_e$. The expression (4·9) is less ambiguous than that of $M_{\Sigma^-} - M_{\Sigma^0}$ or $M_{\Sigma^+} - M_{\Sigma^0}$, for it does not contain the product of the isoscalar and isovector parts. If the choice (4·3) is taken, the first term of (4·9) including $I(l)$ provides the contribution $\sim 3m_e$, without respect

to the first or second standpoint. In order to explain the observed mass difference $(2\pm1)m_e$, the second term of (4·9) including $J(l)$ must have contribution $(1\pm1)m_e$. Either consideration from the first or second standpoint similarly indicates that there is a possibility of understanding the observed value of $\frac{1}{2}(M_{\Sigma^+}+M_{\Sigma^-})-M_{\Sigma^0}$ only under the condition.

$$F_{1V}^{(\Sigma)}(l)F_{2V}^{(\Sigma)}(l)<0 \quad \text{for large } l. \tag{4·10}$$

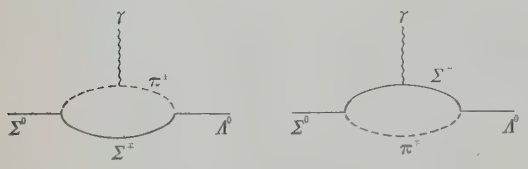

Fig. 10. The eg^2-order vertices which produce the Pauli moment like interaction between Σ^0 and Λ^0.

Owing to the condition (4·10), the two contributions of $F_{2S}F_{1V}$ and $F_{2S}F_{2V}$ to ΔM_2 turn out to have the opposite signs, but the foregoing argument on the $\Sigma^- - \Sigma^+$ mass difference under the condition (4·7) need not be changed. The condition (4·10), however, contradicts (4·8); consequently (4·8′) must be taken so that the mass difference between Σ^-, Σ^0 and Σ^+ may be consistently understood from the second standpoint.

Fig. 11. The e^2-order self-energy of the Σ^0 due to the virtual dissociation into a Λ^0 and a photon.

We have ignored in the above the effect of the virtual dissociation of the Σ^0 into a Λ^0 and a photon. Fig. 10 shows examples of the corresponding Feynman diagrams of the eg^2-order vertices which produce the vertex factors $F_1^{(\Sigma\Lambda)}(l)$ and $F_2^{(\Sigma\Lambda)}(l)$. The e^2-order self-energy due to this effect shifts the mass of Σ^0 by an amount,

$$\delta M_{\Sigma^0}\simeq-\frac{\alpha}{\pi}M_\Sigma\int_0^\infty dl\left(3F_1^{(\Sigma\Lambda)}F_2^{(\Sigma\Lambda)}+\frac{5}{4}\right)F_2^{(\Sigma\Lambda)2}J^{(\Sigma\Lambda)}(l), \tag{4·11}$$

where the small contribution including $I^{(\Sigma\Lambda)}$ is neglected because of $F_1^{(\Sigma\Lambda)}(0)=0$, and $-J^{(\Sigma\Lambda)}$ represents the main contribution from the diagram shown in Fig. 11. If the relative parity between Σ and Λ is even, $J^{(\Sigma\Lambda)}(l)$ may be approximated to $J(l)$. The lowest order perturbation calculation by using the charge independent interaction suggests that $|F_2^{(\Sigma\Lambda)}(0)|\gg|F_{2S}^{(\Sigma)}(0)|=|\mu_{\Sigma^0}|$, and then the effect seems to make the mass of Σ^0 lower, even for the first standpoint. Thus there is a possibility for explaining the observed value of $\frac{1}{2}(M_{\Sigma^+}+M_{\Sigma^-})-M_{\Sigma^0}$ by taking account of the shift of (4·11), even if the condition (4·10) is not satisfied. The somewhat detailed discussion will be made elsewhere.

§ 5. Ξ-hyperon mass difference

From (2·1) the Ξ mass difference may be calculated as

$$M_{\Xi^0}-M_{\Xi^-}=(\delta M_\Xi)_{\tau_3=+1}-(\delta M_\Xi)_{\tau_3=-1}\equiv\Delta M_1'+\Delta M_2, \tag{5·1}$$

where ΔM_1 and ΔM_2 are formally equal to those of the nucleon (3·1). As both

the Ξ-hyperon and the nucleon are isospinor particles, the situation might be quite similar to each other.

Let us now consider the Ξ mass difference from the first standpoint. If the strong π-baryon interaction is common to the nucleon and the Ξ, i.e. $g^2_{\Xi\Xi\pi}\simeq g^2_{NN\pi}$, and if the K-baryon interaction is also symmetric between nucleon and Ξ or does not predominantly contribute to the form factor for small l, we may expect that

$$F_2^{(\Xi)}(l)\simeq -F_2^{(N)}(l) \quad \text{for small } l. \tag{5·2}$$

The relation (5·2) would be checked by measuring the magnetic moment of the Ξ in the future. The choice of the form factor similar to (3·4–6) gives $\Delta M_1\simeq -3m_e$ and $\Delta M_2\simeq 7m_e$, and then we have $M_{\Xi^0}-M_{\Xi^-}\simeq 4m_e$. On the contrary, if the π-baryon interaction is such as $g^2_{\Xi\Xi\pi}\ll g^2_{NN\pi}$, and if other situations are same as the above, it follows that $M_{\Xi^0}-M_{\Xi^-}\lesssim 0$. We might thus expect that the future experimental data on the Ξ mass difference would test the validity of the above alternative assumption on the π-baryon coupling constants, so far as we discuss the problem from the first standpoint.

Next we shall conjecture the possible cases from the second standpoint. If both the π-baryon and K-baryon interactions are symmetric between nuclon and Ξ, and in addition if the relative parity between nucleon and Ξ is even, the Ξ-hyperon form factor as a function of $l=|\boldsymbol{k}|/M_\Xi$ would become quite similar to the nucleon form factor even for large l. Then it may be expected that $M_{\Xi^0}-M_{\Xi^-}\simeq (M_n-M_p)M_\Xi/M_N\simeq 4m_e$. On the other hand, if the opposite sign of the mass difference, $M_{\Xi^0}-M_{\Xi^-}<0$, is measured by the future experiment, the condition

$$6F_{1S}^{(\Xi)}F_{2V}^{(\Xi)}+6F_{2S}^{(\Xi)}F_{1V}^{(\Xi)}+5F_{2S}^{(\Xi)}F_{2V}^{(\Xi)}<0 \quad \text{for large } l \tag{5·3}$$

should be satisfied in order to explain the mass difference. The sign of the inequality (5·3) is opposite to that for the nucleon, (3·8) or (3·8′). Such a different condition might be expected, if at least one of the π-baryon and K-baryon interactions has no symmetry between nucleon and Ξ, or if the relative parity between nucleon and Ξ is odd, even though the coupling constants are common between them.

§ 6. π-meson mass difference

Introducing the π-meson form factor (2·11–12) into the self-energy expression (2·3), we may calculate the π-meson mass difference,

$$m_{\pi^\pm}-m_{\pi^0}=(\delta m_\pi)_{I_3=\pm 1}-(\delta m_\pi)_{I_3=0}=\frac{\alpha}{2\pi}m_\pi\int_0^\infty dl\,F_V^{(\pi)2}(l)K(l), \tag{6·1}$$

where $l=|\boldsymbol{k}|/m_\pi$. Since $K(l)$ is always positive, (6·1) is obviously positive definite without respect to the explicit behavior of the form factor. That is, a charged π meson ought to be heavier than a neutral one in agreement with the observation.

This fact seems plausible from the intuitive viewpoint that a classical charge distribution has positive self-energy. It should be emphasized, however, that the vanishing of the isoscalar part of the π-meson form factor, namely, the vanishing charge distribution of the π^0 as a result of the strangeness assignment by Nishijima and Gell-Mann's rule, is responsible for the positive sign of $m_{\pi^\pm}-m_{\pi^0}$.

We can explain the observed value $m_{\pi^\pm}-m_{\pi^0}=9m_e$, using a rather simple form factor. Fig. 12 shows the calculated π mass difference versus the r.m.s. radius of the charge distribution for several models, *G*, *HE*, *EXP* and *YII*. The r.m.s. radius of 0.6 to 0.8 fermi is required to explain the mass difference, as long as we take these simple models. According to the dispersion theoretical investigations, Goldberger and others[24] have obtained the r.m.s. radius of 0.41 fermi for the π-meson, although there is no direct experimental confirmation about this. If a somewhat smaller r.m.s. radius, say $\sim$0.4 fermi, is confirmed by future experiments, we can conceive the suitable form factor with an arbitrarily smaller r.m.s. radius so that it may explain the observed π mass difference, because (6·1) is fairly sensitive to the high momentum behavior of the form factor.

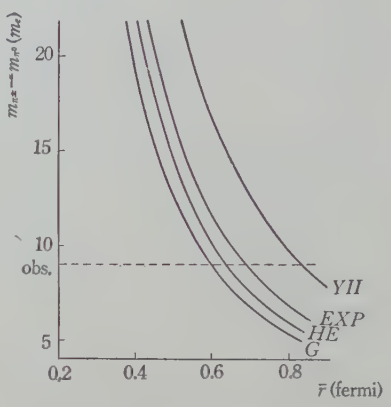

Fig. 12. The calculated π-meson mass difference is shown against the r.m.s. radius of the charge distribution for the *G*, *HE*, *EXP* and *YII* models.

§ 7. *K*-meson mass difference

From (2·3) the *K*-meson mass difference may be calculated as

$$m_{K^\pm}-m_{K^0(\bar{K}^0)}=(\delta m_K)_{\tau_3=+1}-(\delta m_K)_{\tau_3=-1}=(\delta m_{\bar{K}})_{\tau_3=-1}-(\delta m_{\bar{K}})_{\tau_3=+1}$$

$$=\frac{\alpha}{2\pi}m_K\int_0^\infty dl\,4F_S^{(K)}(l)F_V^{(K)}(l)K(l), \qquad (7\cdot1)$$

where $l=|\boldsymbol{k}|/m_K$. (7·1) contains the product F_SF_V, in contrast with F_V^2 in (6·1). Since $K(l)$ increases as l for large l, the integral (7·1) strongly depends upon the sign of the product F_SF_V for large l. Thus the negative sign of the observed mass difference, $m_{K^\pm}-m_{K^0(\bar{K}^0)}\simeq-8m_e$, can be explained only under the condition

$$F_S^{(K)}(l)F_V^{(K)}(l)<0 \quad \text{for large } l. \qquad (7\cdot2)$$

As a matter of fact, the observed value of the *K* mass difference may be produced by using the suitable form factor, which satisfies not only (2·14) but (7·2). Let us now try to conceive a suitable model of the form factor so that it may explain the observed mass difference at the r.m.s. radius of $\sim$0.4 to 0.5 fermi.

Such an example is arbitrarily represented as a linear combination of the same and/or different kind of functions with various parameters. For example, F_{CI} is assumed to be always positive for every range of l, and F_{CII} negative for some large value of l, so that the product $F_{CI}\,F_{CII}$ may satisfy (7·2) and may be used for $4F_S F_V$. Fig. 13 shows the calculated K mass difference against the r.m.s. radius of the charge distribution for two trial models F_C and $F_{C'}$ with different parameters. At the r.m.s. radius of $\sim$0.4 fermi for the model C or at $\sim$0.5 fermi for the model C', we get the observed K mass difference. These values of the r.m.s. radius, however, cannot be taken seriously, for the result strongly depends on the used model.

The Fourier transform of the trial K-meson form factor provides the corresponding charge distribution $\rho(r)$. The inner behavior of the trial distribution function should be regarded as an arbitrary example among many possible models. The somewhat complicated charge distribution of the K^0 is shown in Fig. 14 as

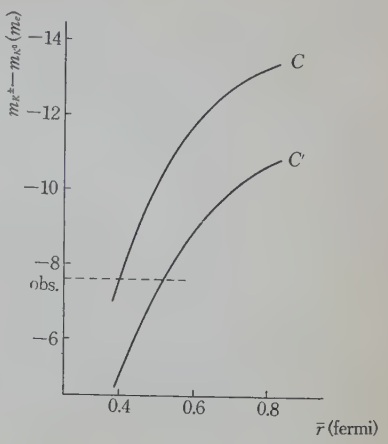

Fig. 13. The calculated K-meson mass difference is shown against the r.m.s. radius of the suitably assumed charge distributions ρ_C and $\rho_{C'}$. For example, the trial form factors F_{CI} and F_{CII} are assumed to be given by $F_{CI,II}=1\Big/\left(1+\frac{a^2}{12}l^2\right)^2 \pm bl^4 e^{-l^2}$, where $a/m_K=\sqrt{\langle r^2\rangle}$. The model C corresponds to the choice of parameters, $b=0.6$ and $c=0.5$, while the model C' to $b=0.1$ and $c=0.25$.

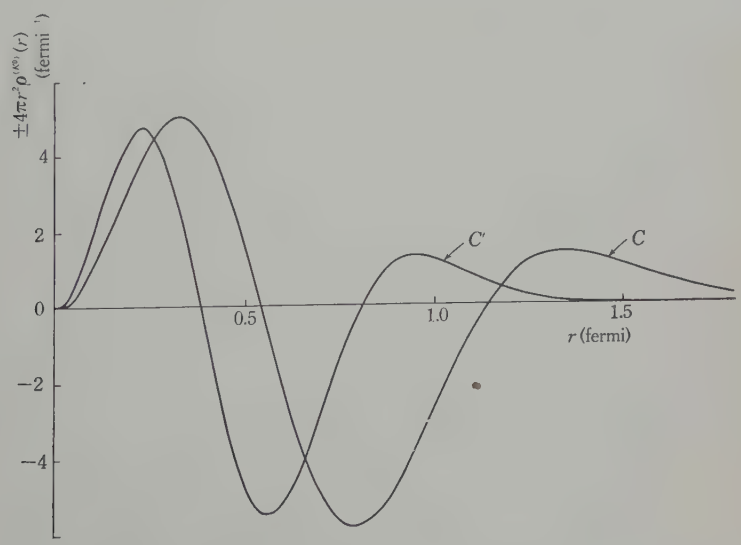

Fig. 14. The charge distribution of the K^0 for the two examples ρ_C and $\rho_{C'}$ is shown. The value of $\pm 4\pi r^2 \rho^{(K^0)}(r)$ is taken as ordinate in the figure.

examples with the parameters chosen for the models C and C'. In the present circumstances, trying to discuss the inner charge distribution is so speculative that we shall not go further but conjecture only on some qualitative features. The inequality (7·2) suggests that the spatial charge distribution might have the property,

$$\rho_S(r)\rho_V(r)<0 \quad \text{for small } r. \tag{7·3}$$

The inner charge density of the K^0 is then proportional to $|\rho_S|+|\rho_V|$ while that of the K^+ proportional to $||\rho_S|-|\rho_V||$. Consequently the stronger charge concentration of the K^0 at the inner region may have the larger positive self-energy in agreement with the observed mass difference. The Nishijima and Gell-Mann rule informs us that the K^0 can virtually dissociate into a pair of one kind of baryon and the other kind of antibaryon, such as (p and $\bar{\Sigma}^+$), (Σ^- and $\bar{\Xi}^-$), etc., whereas on the contrary the π^0 dissociates into the same kinds of baryon-antibaryon pair, such as (p and $\bar{p}$), ($\Sigma^\pm$ and $\bar{\Sigma}^\pm$), etc. Therefore we may expect that the K^0 has a fairly complicated charge distribution unlike the π^0, and such an inner charge distribution seems to be responsible for the opposite sign of the observed K-meson mass difference.

Nakano and Miyachi[25] previously calculated the K-meson self-energy, taking into account an additional electromagnetic interaction of the type, $\partial_\mu\phi^*\partial_\nu\phi(\partial_\mu A_\nu - \partial_\nu A_\mu)$, which was derived from the eg^2-order vertex. The self-energy due to this effect partly covers the contributions of the diagrams in Fig. 3, if the suitable form factor is introduced. Although this effect gives rise to a new additional contribution, their result of the lowest order calculation by using the Feynman cutoff factor seems insufficient to explain the correct sign of the K mass difference only by itself. This is because the electromagnetic self-energy due to the sum of the ordinary electromagnetic interaction and the additional one is still positive apart from the form factor. Needless to say, if we introduce the form factor behaving like (7·2) into both the ordinary and additional interactions, we have the negative contribution to the K mass difference. The additional contribution, however, may be regarded as a small correction as far as the lowest order calculation is concerned.

§ 8. Strong interaction corrections

Considerable parts of the electromagnetic self-energy with the strong interaction corrections may be comprised in the self-energy diagrams of Figs. 1 and 3 in which suitable form factors are introduced, because the rigorous form factor, if it were known, would include the vertex corrections of all orders in g^2. Examples of such diagrams of the lowest order in g^2 are shown in Figs. 15–16 (a) for baryons and mesons. There are some other self-energy contributions from the strong interaction which are not contained in the diagrams of Figs. 1 and 3. These diagrams of the lowest order are shown in Figs. 15–16 (b) and (c). Among these contributions the essential parts of the self-energy diagrams in Figs. 15–16 (b) may be replaced

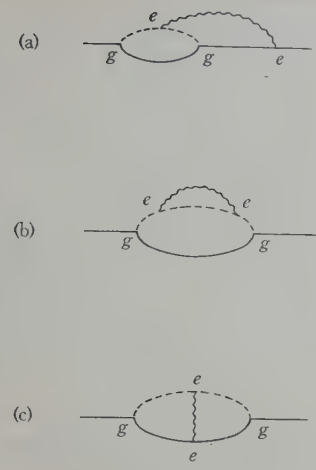

Fig. 15. The baryon self-energy diagrams of the e^2g^2-order.

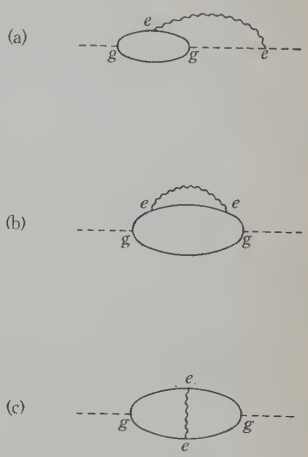

Fig. 16. The meson self-energy diagrams of the e^2g^2-order.

by the contributions from the diagrams, in which the observed electromagnetic mass difference of the particles in the intermediate states is taken into account.[26] In the following we are going to study whether this type of strong interaction correction appreciably contributes to the main electromagnetic mass difference.

In the lowest order approximation the charge independent strong interactions provide the mass shifts for the diagrams of Figs. 15–16 (b),

$$\begin{aligned}\Delta(M_n-M_p) =& -(M_n-M_p)\,(g^2_{NN\pi}/4\pi)\,(\partial I_{NN\pi}/\partial M_N)\\ &+2(M_{\Sigma^-}-M_{\Sigma^+})\,(g^2_{N\Sigma K}/4\pi)\,(\partial I_{N\Sigma K}/\partial M_\Sigma)\\ &-(m_{K^0}-m_{K^\pm})[(g^2_{N\Sigma K}/4\pi)\,(\partial I_{N\Sigma K}/\partial m_K)-(g^2_{N\Lambda K}/4\pi)\,(\partial I_{N\Lambda K}/\partial m_K)],\end{aligned} \tag{8·1}$$

$$\begin{aligned}\Delta(M_{\Xi^0}-M_{\Xi^-}) =& -(M_{\Xi^0}-M_{\Xi^-})\,(g^2_{\Xi\Xi\pi}/4\pi)\,(\partial I_{\Xi\Xi\pi}/\partial M_\Xi)\\ &-2(M_{\Sigma^-}-M_{\Sigma^+})\,(g^2_{\Xi\Sigma K}/4\pi)\,(\partial I_{\Xi\Sigma K}/\partial M_\Sigma)\\ &-(m_{K^0}-m_{K^\pm})[(g^2_{\Xi\Sigma K}/4\pi)\,(\partial I_{\Xi\Sigma K}/\partial m_K)-(g^2_{\Xi\Lambda K}/4\pi)\,(\partial I_{\Xi\Lambda K}/\partial m_K)],\end{aligned} \tag{8·2}$$

$$\begin{aligned}\Delta(M_{\Sigma^-}-M_{\Sigma^+}) =& (M_{\Sigma^-}-M_{\Sigma^+})\,(g^2_{\Sigma\Sigma\pi}/4\pi)\,(\partial I_{\Sigma\Sigma\pi}/\partial M_\Sigma)\\ &+2(M_n-M_p)\,(g^2_{\Sigma NK}/4\pi)\,(\partial I_{\Sigma NK}/\partial M_N)\\ &-2(M_{\Xi^0}-M_{\Xi^-})\,(g^2_{\Sigma\Xi K}/4\pi)\,(\partial I_{\Sigma\Xi K}/\partial M_\Xi)\\ &+2(m_{K^0}-m_{K^\pm})[(g^2_{\Sigma\Xi K}/4\pi)\,(\partial I_{\Sigma\Xi K}/\partial m_K)-(g^2_{\Sigma NK}/4\pi)\,(\partial I_{\Sigma NK}/\partial m_K)],\end{aligned} \tag{8·3}$$

$$\begin{aligned}&\Delta[1/2(M_{\Sigma^+}+M_{\Sigma^-})-M_{\Sigma^0}]\\ &=-[1/2(M_{\Sigma^+}+M_{\Sigma^-})-M_{\Sigma^0}](g^2_{\Sigma\Sigma\pi}/4\pi)\,(\partial I_{\Sigma\Sigma\pi}/\partial M_\Sigma)\\ &+(m_{\pi^\pm}-m_{\pi^0})[(g^2_{\Sigma\Lambda\pi}/4\pi)\,(\partial I_{\Sigma\Lambda\pi}/\partial m_\pi)-(g^2_{\Sigma\Sigma\pi}/4\pi)\,(\partial I_{\Sigma\Sigma\pi}/\partial m_\pi)],\end{aligned} \tag{8·4}$$

$$\Delta(m_{\pi^\pm}-m_{\pi^0})=2[1/2(M_{\Sigma^+}+M_{\Sigma^-})-M_{\Sigma^0}][(g^2_{\Lambda\Sigma\pi}/4\pi)(\partial I_{\Lambda\Sigma\pi}/\partial M_\Sigma) - (g^2_{\Sigma\Sigma\pi}/4\pi)(\partial I_{\Sigma\Sigma\pi}/\partial M_\Sigma)], \tag{8·5}$$

$$\Delta(m_{K^0}-m_{K^\pm})=(M_n-M_p)[(g^2_{\Lambda NK}/4\pi)(\partial I_{\Lambda NK}/\partial M_N)-(g^2_{\Sigma NK}/4\pi)(\partial I_{\Sigma NK}/\partial M_N)] + (M_{\Xi^0}-M_{\Xi^-})[(g^2_{\Lambda\Xi K}/4\pi)(\partial I_{\Lambda\Xi K}/\partial M_\Xi)-(g^2_{\Sigma\Xi K}/4\pi)(\partial I_{\Sigma\Xi K}/\partial M_\Xi)] +2(M_{\Sigma^-}-M_{\Sigma^+})[(g^2_{\Sigma\Xi K}/4\pi)(\partial I_{\Sigma\Xi K}/\partial M_\Sigma)-(g^2_{\Sigma NK}/4\pi)(\partial I_{\Sigma NK}/\partial M_\Sigma)]. \tag{8·6}$$

Some contributions to the mass difference from several diagrams just cancel out each other. The above equations have been expressed by only the nonvanishing remainders. In these equations I_{XYZ} denotes

$$I_{XYZ}=-\frac{1}{4i\pi^2}\int d^4q\,\Gamma\frac{1}{\boldsymbol{p}-\boldsymbol{q}-M_Y}\Gamma\frac{1}{q^2-m_Z^2} \tag{8·7}$$

for the baryon X satisfying $(\boldsymbol{p}-M_X)\psi=0$, or

$$I_{XYZ}=\frac{1}{8i\pi^2}\int d^4p\,\mathrm{Sp}\left[\Gamma\frac{1}{\boldsymbol{p}-M_X}\Gamma\frac{1}{\boldsymbol{p}-\boldsymbol{q}-M_Y}\right] \tag{8·8}$$

for the boson Z satisfying $(q^2-m_Z^2)\phi=0$, where Γ stands for 1 or $i\gamma_5$ in accordance with the scalar or pseudoscalar coupling of the interaction (XYZ). In Tables II–III, the calculated values of $\partial I/\partial M$ and $\partial I/\partial m$ appearing in Eqs. (8·1–6) are listed. On account of the logarithmic divergence of $\partial I/\partial M$, the straight cutoff at M_N has been made for simplicity. The same straight cutoff has been made also for $\partial I/\partial m$, though

Tabl II. $\partial I/\partial M$ and $\partial I/\partial m$ for baryons

Γ	$\frac{\partial I_{NN\pi}}{\partial M_N}$	$\frac{\partial I_{\Sigma\Sigma\pi}}{\partial M_\Sigma}$	$\frac{\partial I_{\Xi\Xi\pi}}{\partial M_\Xi}$	$\frac{\partial I_{N\Sigma K}}{\partial M_\Sigma}$	$\frac{\partial I_{\Sigma NK}}{\partial M_N}$	$\frac{\partial I_{\Sigma\Xi K}}{\partial M_\Xi}$	$\frac{\partial I_{\Xi\Sigma K}}{\partial M_\Sigma}$
1	···	···	···	0.049	0.172	0.070	0.132
$i\gamma_5$	0.025	0.015	0.011	0.004	0.026	0.005	0.013

Γ	$\frac{\partial I_{\Sigma\Lambda\pi}}{\partial m_\pi}$	$\frac{\partial I_{\Sigma\Sigma\pi}}{\partial m_\pi}$	$\frac{\partial I_{N\Lambda K}}{\partial m_K}$	$\frac{\partial I_{N\Sigma K}}{\partial m_K}$	$\frac{\partial I_{\Sigma NK}}{\partial m_K}$	$\frac{\partial I_{\Sigma\Xi K}}{\partial m_K}$	$\frac{\partial I_{\Xi\Lambda K}}{\partial m_K}$	$\frac{\partial I_{\Xi\Sigma K}}{\partial m_K}$
1	0.605	···	0.121	0.110	0.288	0.136	0.256	0.219
$i\gamma_5$	−0.018	−0.021	−0.025	−0.026	−0.013	−0.022	−0.014	−0.018

Table III. $\partial I/\partial M$ for mesons

Γ	$\frac{\partial I_{\Lambda\Sigma\pi}}{\partial M_\Sigma}$	$\frac{\partial I_{\Sigma\Sigma\pi}}{\partial M_\Sigma}$	$\frac{\partial I_{\Lambda NK}}{\partial M_N}$	$\frac{\partial I_{\Sigma NK}}{\partial M_N}$	$\frac{\partial I_{\Lambda\Xi K}}{\partial M_\Xi}$	$\frac{\partial I_{\Sigma\Xi K}}{\partial M_\Xi}$	$\frac{\partial I_{N\Sigma K}}{\partial M_\Sigma}$	$\frac{\partial I_{\Xi\Sigma K}}{\partial M_\Sigma}$
1	0.223	···	0.095	0.089	0.054	0.051	0.075	0.056
$i\gamma_5$	0.308	0.288	0.100	0.091	0.089	0.084	0.110	0.078

it has no divergence, in order to keep a consistency of the result. At any rate the calculated values are not very sensitive to the cutoff momentum.

Strictly speaking, we have to solve a set of simultaneous equations (8·1–6) regarding the six kinds of mass differences as unknown quantities, after we add the electromagnetic mass differences calculated in the preceding sections to the right-hand side of each equation. Tables II–III indicate, however, that the strong interaction corrections are not very large as compared with the observed mass difference. We may then estimate the strong interaction corrections by inserting the observed values of mass difference (1·1) into the right-hand sides of Eqs. (8·1–6), instead of solving the simultaneous equations.

As for the baryon mass shifts, the scalar coupling contribution turns out to be generally comparable to the observed mass difference while the pseudoscalar coupling contribution is sufficiently small. It should be remembered, however, that the lowest order perturbation calculation may in general provide the overestimated result for the baryon current contribution, and that the straight cutoff technique may also lead to oversupply as compared with other cutoff techniques. Assuming that $g_\pi^2/4\pi \simeq 15$, $g_K^2/4\pi \simeq 1$ and $|M_{\Xi^0} - M_{\Xi^-}| \simeq M_n - M_p$, and considering the above situations, we may conjecture from Eqs. (8·1–3) that $|\Delta(M_n - M_p)| \lesssim 0.5 m_e$, $|\Delta(M_{\Xi^0} - M_{\Xi^-})| \lesssim 1 m_e$ and $|\Delta(M_{\Sigma^-} - M_{\Sigma^+})| \lesssim 1 m_e$. Here we have made a reduction by a factor of about one-fifth for the possible overestimation. If the relative parity between Σ and Λ is even, the coupling $(\Sigma\Lambda\pi)$ is pseudoscalar and then (8·4) gives a rather small contribution, $|\Delta[\frac{1}{2}(M_{\Sigma^+} + M_{\Sigma^-}) - M_{\Sigma^0}]| \lesssim 0.5 m_e$. On the contrary, if the relative parity between Σ and Λ is odd, and if the scalar coupling constant $g_{\Sigma\Lambda\pi}^2$ is also assumed to be of the same order of magnitude as the pseudoscalar g_π^2, we have an appreciable contribution, $\Delta[\frac{1}{2}(M_{\Sigma^+} + M_{\Sigma^-}) - M_{\Sigma^0}] \lesssim 10 m_e$, even after we have made the reduction for overestimation.

As for the boson mass shifts, both the scalar and pseudoscalar coupling contributions are of the same order of magnitude and have the same signs, and then they appear to produce considerable mass shifts. The $(\Sigma\Sigma\pi)$ and $(\Lambda\Sigma\pi)$ interactions, however, destructively contribute to $\Delta(m_{\pi^\pm} - m_{\pi^0})$, and the resultant mass shift becomes smaller so long as $g_{\Sigma\Sigma\pi}^2 \simeq g_{\Lambda\Sigma\pi}^2$ is assumed. For $\Delta(m_{K^0} - m_{K^\pm})$ there is no contribution from the π-baryon interaction, and the (ΣNK) interaction and the other K-baryon interactions destructively contribute to the resultant mass shift so long as all g_K^2's are taken to be common. Moreover, we should remark that if the pair suppression effect is taken into account, the result in the lowest order perturbation approximation may be reduced to some extent. Considering these situations and remembering again the use of the straight cutoff technique, we may conjecture from Eqs. (8·5–6) that $|\Delta(m_{\pi^\pm} - m_{\pi^0})| \lesssim 1 m_e$ and $|\Delta(m_{K^0} - m_{K^\pm})| \lesssim 0.5 m_e$.

No numerical estimate on the self-energy of the type shown in Figs. 15–16 (c) has been made in this report, for such an effect seems small enough judging from the rough estimate by Kato and Takeda.[12]

Although reliability of the perturbation approximation for the strong inter-

action should be problematical and although, in addition, the estimate has been made only for the special types of the lowest order diagrams, the above study seems to suggest that the strong interaction effect cannot be thoroughly neglected, yet in most cases it may be regarded as a small correction to the major electromagnetic effect.

§ 9. Concluding remarks

In the preceding sections, it has been shown that most of the observed mass differences (1·1) can be understood as the effect of electromagnetic self-energy. It has been stressed there that the high momentum behavior of the isoscalar and isovector parts of the form factor plays an important role in explaining the mass difference. Some trial examples of the form factor have been conceived in the hope that at least their qualitative features might be tested by experimental and/or theoretical investigations in the future, though there are naturally many other possibilities for choosing the trial functions. As is seen from the spectral representation of the form factor by using the dispersion relation,[24),27)] the isoscalar and isovector parts may in general have individually different behaviors.

With respect to the requirements of the relativistic invariance and of the gauge invariance, the form factors assumed in the present work are clearly unsatisfactory. A generalization of the noncovariant form factor to the covariant one is directly possible, and the use of the covariant form factor does not essentially change the qualitative results of our argument. As to the gauge invariance requirement, we may take the gauge transformation defined by Chrétien and Peierls,[28)] so far as the two-point form factor is concerned.

We are now going to show an intuitive version of the argument made for the nucleon mass difference. According to our original attempt,[1)] the nucleon mass difference is explained as a result of the predominance of the negative self-energy due to the magnetic moment distribution over the positive self-energy due to the charge distribution,

$$M_n-M_p=\delta M_n-\delta M_p\simeq -e^2\left\langle\frac{1}{r}\right\rangle+[(1+\mu_p)^2-\mu_n^2]\left(\frac{e}{2M_N}\right)^2\left\langle\frac{1}{r^3}\right\rangle, \qquad (9\cdot1)$$

where the classical self-energies are assumed to be $e^2\langle 1/r\rangle$ for the charge distribution, and $-\mu^2(e/2M_N)^2\langle 1/r^3\rangle$ for the magnetic moment distribution. If we assume that $\langle 1/r\rangle\simeq 1/\bar{r}$ and $\langle 1/r^3\rangle\simeq 1/\bar{r}^3$ where $\bar{r}=\sqrt{\langle r^2\rangle}$ denotes the r.m.s. radius of the distribution, the observed mass difference is then explained by taking the value of $\bar{r}\lesssim 1/M_N\simeq 0.2$ fermi, which obviously contradicts the experimental data of the electron scattering. However, if the inner behavior of the distribution function is suitably conceived so that it may give $\langle 1/r\rangle\gtrsim 1/\bar{r}$ and $\langle 1/r^3\rangle\gg 1/\bar{r}^3$, the observed mass difference can be explained by (9·1) at the value of $\bar{r}\simeq 0.8$ fermi. The similar intuitive explanation for the other particles is also possible, if the isospace transformation property of the individual particle is taken into account.

It has been suggested that the considerable largeness of the $\Sigma^- - \Sigma^+$ mass

difference seems to be connected with the isovector nature of the Σ. It has also been shown that the opposite signs of the $\pi^{\pm}-\pi^{0}$ and $K^{\pm}-K^{0}$ mass differences are consistent with their strangeness assignment.

The lowest order estimation of the mass shift due to the strong π-baryon and K-baryon interactions suggests that the effect may be regarded as a small correction in most cases. Some care should be taken of $\frac{1}{2}(M_{\Sigma^+}+M_{\Sigma^-})-M_{\Sigma^0}$, to which we may have an appreciable contribution from the $(\Sigma\Lambda\pi)$ interaction, in case of the opposite relative parities of Σ and Λ.

* * *

After completion of this work, several recent articles treating of related subjects have come to our notice. Hofstadter (Nuovo Cimento **12** (1959), 63) has demonstrated that the proton form factor fitting the electron scattering data can be given by some cutoff proton models unlike the exponential model formerly described by him.[8] The distribution functions of his cutoff models have a somewhat strong behavior in the inner region, just like our trial model proposed in the present report. Katayama and others (Prog. Theor. Phys., to be published) have shown that the nucleon mass difference can be explained by using the trial form factor which has a negative strong behavior at the high momentum region. Their result is qualitatively equivalent to ours given by the form factor F_{III} for the first standpoint. Hiida and others (Prog. Theor. Phys. **22** (1959), 247; 351) have tried to calculate the nucleon form factor based on the perturbation theory. Sakurai (Phys. Rev. **114** (1959), 1152) has given some prediction as to the Ξ mass difference. His argument is qualitatively close to ours for the first standpoint, though he has not considered the form factor. Riazuddin (Phys. Rev. **114** (1959), 1184) has attempted to deduce the π-meson charge radius from the observed π mass difference. As has been stated in our report, such a result strongly depends on the used model, because the calculated mass difference is sensitive to the inner charge distribution but not to the r.m.s. radius. Bransden and Moorhouse (Phys. Rev. Letters **2** (1959), 431) have calculated the $\Sigma^{-}-\Sigma^{+}$ mass shift which comes from the K-meson mass difference in the intermediate states through the strong interaction. This effect is included in our estimation, though we have made the straight cutoff. Our result indicates that the mass differences of $n-p$, $\Xi^{0}-\Xi^{-}$ and $\Sigma^{-}-\Sigma^{+}$ in the intermediate states also give similar contributions, but the resultant correction seems insufficient to explain the observed Σ mass difference by this effect alone.

Acknowledgments

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References

1) H. Katsumori, Memoirs of Osaka Gakugei Univ. B, No. 2 (1953), 28.
2) H. Katsumori, Soryushiron-kenkyu (Mimeographed circular in Japanese) **6** (1954), 1242.
3) Y. Oishi and H. Katsumori, Prog. Theor. Phys. **12** (1954), 109.
4) Y. Oishi, Soryushiron-kenkyu **6** (1954), 975.
5) R. P. Feynman and G. Speisman, Phys. Rev. **94** (1954), 500.
6) A. Petermann, Helv. Phys. Acta **27** (1954), 441.
7) R. Hofstadter and R. W. McAllister, Phys. Rev. **98** (1955), 217.
8) R. Hofstadter, F. Bumiller and M. R. Yearian, Rev. Mod. Phys. **30** (1958), 482.
This review article provides references to the related experimental works.
9) H. Katsumori, A report at the 11th annual meeting of the Physical Society of Japan (Tohoku Univ., 1956).
10) K. Hiida and M. Sawamura, Prog. Theor. Phys. **18** (1957), 451.
11) M. Cini, E. Ferrari and R. Gatto, Phys. Rev. Letters **2** (1959), 7.
12) M. Kato and G. Takeda, Supplement of Prog. Theor. Phys. No. 7 (1959), 35.
13) H. Katsumori, A report at the 13th annual meeting of the Physical Society of Japan (Kyoto Univ., 1958).
14) H. Katsumori, Prog. Theor. Phys. **17** (1957), 803; ibid. **18** (1957), 375.
15) E. C. G. Sudershan and R. E. Marshak, Phys. Rev. **104** (1956), 267; R. Marshak, S. Okubo and G. Sudershan, ibid. **106** (1957), 599.
16) R. P. Feynman, *International Conference on Theoretical Physics* (Paris, 1950). See also ref. 5).
17) A. H. Rosenfeld et al., Phys. Rev. Letters **2** (1959), 110; F. S. Crawford et al., ibid. **2** (1959), 112.
18) S. Gasiorowicz and A. Petermann, Phys. Rev. Letters **1** (1958), 457.
19) H. Katsumori, Nuovo Cimento **14** (1959), 1381.
20) H. Enatsu, Prog. Theor. Phys. **6** (1951), 643.
21) R. Hofstadter, Rev. Mod. Phys. **28** (1956), 214. See also ref. 8).
22) E. Clementel and C. Villi, Nuovo Cimento **4** (1956), 1207.
23) L. I. Schiff, Rev. Mod. Phys. **30** (1958), 462.
24) P. Federbush, M. L. Goldberger and S. B. Treiman, Phys. Rev. **112** (1958), 642; J. Bernstein and M. L. Goldberger, Rev. Mod. Phys. **30** (1958), 465.
25) T. Nakano and S. Miyachi, Soryushiron-kenkyu **14** (1957), 641.
26) H. Katsumori and K. Shimoura, Memoirs of Osaka Gakugei Univ. B, No. 8 (1959), 76.
27) C. F. Chew, R. Karplus, S. Gasiorowicz and F. Zachariasen, Phys. Rev. **110** (1958), 265.
28) M. Chrétien and R. E. Peierls, Proc. Roy. Soc. **A223** (1954), 468.

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A Possible Interpretation of High Energy Nuclear Events

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A method of analysing the data of high energy jet showers in nuclear emulsions is proposed. We discuss the possibilities to ruduce the nucleon-nucleon collisions to pion-nucleon interactions. If we use the experimental data on $\pi^- \cdot p$ collision at 5 Gev as the pion-nucleon interaction, we can estimate the energy and angular distribution of emitted pions in rather low energy jet showers ($\lesssim$ 100 Gev) and the obtained results fairly agree with the experiments. As to the high energy jet showers ($\gtrsim$ 100 Gev) we can conclude that almost all of collisions are regarded as peripheral ones and that the properties of the pion-nucleon interaction reduced from the nucleon-nucleon collision are similar to those of the pion-nucleon interaction actually observed. It is also suggested that the nucleon has a core whose radius is about a nucleon Compton wave length.

§ 1. Introduction

In the last few years, experimental data on the multiple production of the particles in the nucleon-nucleon collisions have been accumulated in the high energy region. However, we have only a few pieces of information on the properties of elementary interaction because of the complication of the phenomena. We should, instead of directly comparing the predictions of specific theories with experiments, rearrange empirical data so as to get some clues to understanding the mechanism of high energy collisions. Thus, it is desirable in investigating the high energy interactions between two nucleons to establish a method for the analysis of these data.

Recently, Ezawa[1] has proposed a method of the analysis by introducing the concept of impact parameter into the nuclear collision. He assigned this impact parameter to each event and classified the experimental data by using this parameter. In this sense, his method gives a possibility to separate the head-on collisions from the peripheral ones in experiments.

Several authors[2] have attempted to analyse the peripheral collisions separating

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them from the central ones. For example, Okun' and Pomerančuk pointed out that if nucleons collide with each other with a large angular momentum the Feynman diagrams including only the exchange of one pion contribute predominantly to the matrix element of this process. Therefore we can obtain knowledge on the vertex parts through which the pion interacts with the nucleon and produces other pions or strange particles.

An attempt to analyse events by Williams-Weizsäcker's method has been proposed by Černavskij,[3] who applied Fermi-Landau's and Heisenberg's theories to the interaction part between pions and a nucleon and obtained the results in disagreement with experiment. The conclusions derived from these analyses, however, are strongly affected by what theory one applies to the interaction part between pions and a nucleon.* Thus, it would be impossible to conclude that Williams-Weizsäcker's method is not applicable although his calculated results are in disagreement with experiment. We may say that no theories for the interaction between pion and nucleon are reliable in the high energy region.

In this paper we shall show a possibility to interpret a nucleon-nucleon collision as the sum of two pion-nucleon collisions which are caused by the interaction between a nucleon and the cloud of its partner. We shall show our fundamental idea in § 2. Using the data on $\pi^- - p$ collisions at 5 Gev, we shall investigate the nucleon-nucleon collision with rather low energy (10 Gev ~ 300 Gev in the laboratory system) in § 3. In § 4, we shall generalize the method developed in § 2 and § 3 to an extremely high energy region (more than 100 Gev in the laboratory system). Finally in § 5, we shall give a possible suggestion to the momentum distribution of pions in the proper field of nucleon. The discussions in § 5 are closely related to the structure of the nucleon.

§ 2. The fundamental idea

The nucleon has the proper field around itself due to the interaction between the nucleon and the pion. We consider what rôle the pion in the proper field plays in a nucleon-nucleon collision. The dissociation time of a pion from the nucleon in its proper field is, in the order of magnitude, ω^{-1} or so, where ω is the frequency of the pion.** If the nucleon moves with a Lorentz factor γ_c the dissociation time of the above considered pion is of the order of γ_c/ω in this Lorentz frame. We consider the collision of two nucleons in their center-of-mass system (hereafter we denote this system by *C.M.S.*). In *C.M.S.* let the Lorentz factor of a colliding nucleon be γ_c. The time during which two nucleons pass by each

* For example, if we apply Fermi-Landau's theory[4] to the interaction part of pions and a nucleon, the angular distribution of emitted pions does not agree with experiment, and if Heisenberg's theory[5] is applied, the number of emitted pions and the value of transverse component of the momentum of emitted pions do not agree with experiment.

** In this paper we use the unit $\hbar = c = 1$.

other is $(\mu\gamma_c)^{-1}$ in the order of magnitude, where μ is the pion mass. If the dissociation time of the pion whose frequency is ω is so long that it satisfies the condition

$$\gamma_c/\omega \gg 1/(\mu\gamma_c), \tag{2·1}$$

we may consider that there is no interference among the contributions caused by the pions having different frequencies.

In this case the nucleon-nucleon collisions may be interpreted as the pion-nucleon collisions.

If the frequency of an exchanged pion does not satisfy the relation (2·1) the above method may not be valid.

If there is a core in a nucleon, the frequency of the pion in proper field is limited by an upper bound. For example, let the extent of the nucleon core be $\sim M^{-1}$ (M is the mass of the nucleon), then the frequency of the pion exchanged during a collision is at most of the order of magnitude M and always satisfies the relation (2·1) as the Lorentz factor γ_c increases. In this case, we can divide the nuclear collision into two parts: one is caused by the core-core interaction and the other the peripheral one, which can be reduced to the pion-nucleon interaction.

In the following we attempt to analyse high energy nucleon-nucleon collisions under the assumption that the condition (2·1) holds. In particular, we want to try to reduce the collision between two nucleons to the pion-nucleon collision.

Let the momentum and energy of a pion be $\boldsymbol{q}$ and ω in the nucleon rest system. When this nucleon moves with velocity $-v_c$ ($v_c=\sqrt{\gamma_c^2-1}/\gamma_c$) in *C.M.S.*, the momentum, $\boldsymbol{k}$, and energy, ϵ, of the pion in *C.M.S.* are as follows:

$$\begin{aligned} k_{\parallel} &= \gamma_c(q_{\parallel} - v_c\omega) \\ k_{\perp} &= q_{\perp} \\ \epsilon &= \gamma_c(\omega - v_c q_{\parallel}) \end{aligned} \tag{2·2}$$

where $k_{\parallel}$ and $k_{\perp}$ ($q_{\parallel}$ and $q_{\perp}$) are parallel and perpendicular components of momentum $\boldsymbol{k}(\boldsymbol{q})$ to the direction of the nucleon motion. If q is $\sim\mu$ and $\gamma_c \gg 1$, we can ignore $k_{\perp}$ compared with $k_{\parallel}$. Now we consider the center of mass system composed of this pion and another nucleon which moves with the velocity v_c in *C.M.S.* Denote this system by π-*N.S.* (See Fig. 1.)

According to our assumption there are two π-*N.S.* which move in opposite directions with the Lorentz factor γ' in *C.M.S.* Further, we assume the secondary pions are emitted independently in each π-*N.S.*

In *C.M.S.*, π-*N.S.* moves with the velocity v' (and the Lorentz factor γ'). From the relativistic kinematics, we obtain

$$v' = \frac{\sqrt{\gamma_c^2-1}(1-\omega/M)+\gamma_c q_{\parallel}/M}{\gamma_c(1+\omega/M)-\sqrt{\gamma_c^2-1}\,q_{\parallel}/M}. \tag{2·3}$$

As above mentioned, we ignore $k_{\perp}$ here. When $\gamma_c \gg 1$, γ' is as follows:

$$\gamma' = (1+\omega^*/M)/2\sqrt{\omega^*/M} \quad (\gamma_c \gg 1), \tag{2·4}$$

$$\omega^* = \omega - q_{\parallel}\,.$$

It is interesting to note that the Lorentz factor of π-$N.S.$, γ', is independent of the incident energy γ_c but only depends on the energy and momentum of the pion which is lost from the nucleon. We can get the order of magnitude of γ', putting $q \approx 0$

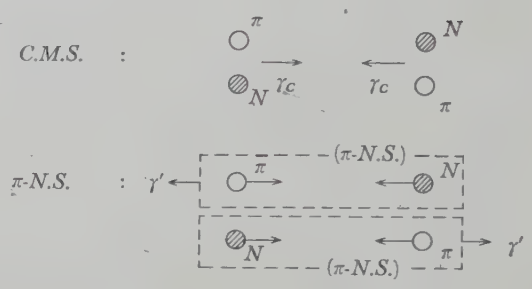

Fig. 1. Schematic drawing of π-$N.S.$
N: Nucleon π: pion

$$\gamma' \approx 1.5. \tag{2·5}$$

If the value of γ_c is not so large γ' can be written from Eq. (2·3) when $q \approx 0$ and $\omega \approx \mu$

$$\gamma' = \frac{1+\mu/M}{2\sqrt{\mu/M}}\{1+(1+\mu/M)^2/(4\gamma_c^2\mu/M)\}^{-1/2} \approx 1.5\cdot(1+1.2\gamma_c^{-2})^{-1/2}. \tag{2·4'}$$

From Eq. (2·4') we can see that γ' depends little on γ_c when $q \lesssim \mu$.

Here, we crudely discuss the angular distribution of the emitted pions in $C.M.S.$ If the number of emitted pions is so large that the nucleon is almost at rest in π-$N.S.$, the secondary pions may be emitted in a nearly isotropic way in this system. (In the present stage, we have no interest in whether such an assumption is valid or not because we are going to discuss the behavior of the secondary pions roughly. See §§ 3 and 4.)

If the emitted pions distribute isotropically in π-$N.S.$ the anisotropy factor σ is[6),7)]*

$$\sigma^2 = 0.39^2 + \{\log(\gamma' + \sqrt{\gamma'^2-1})\}^2. \tag{2·6}$$

From Eqs. (2·5) and (2·6) we obtain the anisotropy factor

$$\sigma \approx 0.57. \tag{2·7}$$

According to the Polish Group the average value of σ in the range of $2 \lesssim \gamma_c \lesssim 140$ is

$$\langle \sigma_{\mathrm{exp}} \rangle \approx 0.6.^{8)} \tag{2·8}$$

The agreement between (2·7) and (2·8) seems to show that our model fairly reproduces the nature of the angular distribution of emitted pions in nucleon-nucleon collisions.

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* The anisotropy factor σ is defined by

$$\sigma^2 = \langle (\log \tan \bar{\vartheta}/2)^2 \rangle$$

where the bracket $\langle\ \rangle$ expresses the average for all emitted particles and $\bar{\vartheta}$ is the emitted angle of the pions in $C.M.S.$ If the angular distribution is isotropic σ is 0.39. See Appendix I and compare with references 6) and 7).

Of course we must not strictly admit the above discussion because σ of each shower distributes widely, from 0.2 to 1.0, and to put $\gamma'=1.5$ has its meaning only in the sense of average. In fact, the momentum of the pion which plays a rôle in the collision is not at all constant and so the value of γ' should be found in a wide range at around $\gamma'\approx 1.5$. Moreover, we have assumed in deriving Eq. (2·7) that the angular distribution of emitted pions in π-*N.S.* is isotropic, but actually the secondary pions are not always produced isotropically in their *C.M.S.*

Taking the above circumstances into account we shall estimate the value of γ' and then the momenta of the exchanged pions from the value of σ in §4 and § 5.

§ 3. The analysis of the experiments. I ($\gamma_c \lesssim 10$)

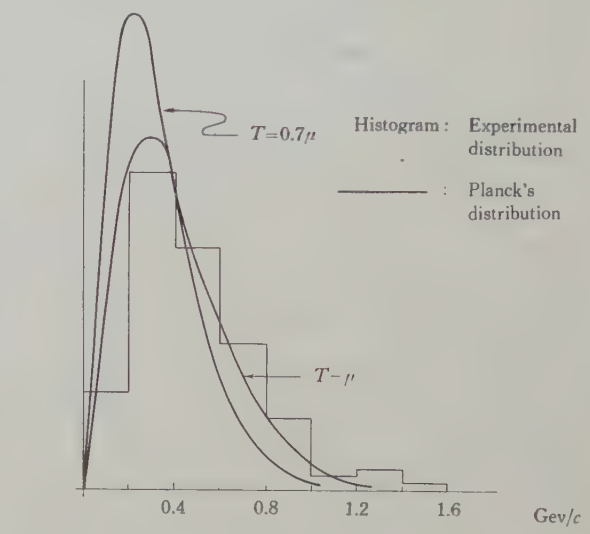
Employing the experimental data of π^-–p collision at 5 Gev as the pion-nucleon interaction in π-*N.S.*,[9] we analyse here the experimental data of the nucleon-nucleon collisions surveyed by Kaneko et al.[10] (the energies of the incident nucleons in these data are less than 300 Gev in the laboratory system and their mean value is about 30 Gev). In the experiments surveyed by Kaneko et al., γ_c is nearly equal to 3.8 on the average. Substituting $\gamma_c\approx 3.8$ into Eq. (2·4′) the value of γ' becomes about 1.4, and then the total energy in π-*N.S.* is about 3 Gev. On the other hand, in the experiment on the π^-–p interaction quoted above the total energy in their *C.M.S.* is also about 3 Gev. Consequently, it is justified to use the experiment on the π^-–p interaction at 5 Gev in analysing the nucleon-nucleon collisions surveyed by Kaneko et al.

The number of emitted pions in the π^-–p interactions at 5 Gev is 3.3 on the average and we may expect that about 6.6 pions are produced from the two π-*N.S.*'s on the basis of our model. This expected value, however, is rather smaller than the number of emitted pions in a nucleon-nucleon collision at about 30 Gev which is nearly 10 averaged over all events. This disagreement is probably caused by the conscious selection of the events, which have rather large multiplicities, from the whole events of nucleon-nucleon collisions.

Therefore, the events in the π^-–p collisions at 5 Gev whose multiplicities are larger than the average one and are nearly 5 should be used for our purpose. In order to select the events with multiplicity 5 from the whole π^-–p collisions we assume that the distribution of charge states of emitted pions is determined only by their statistical weights. With this assumption we can calculate the momentum and angular distribution of emitted pions from π^-–p collisions with multiplicity 5 using the experimental data at 5 Gev. The calculated results are shown in Fig. 2. According to Fig. 2, the momentum distribution of emitted pions can be represented fairly well by the Planck distribution with the temperature $T=\mu\sim 0.7\mu$. As to the angular distribution it is nearly isotropic with a slight peak in the forward direction of the incident negative pion.

The experimental data for the nucleon-nucleon collision surveyed by Kaneko et al. are shown in Fig. 3. The distribution of the transverse momentum P_T of the emitted pions fits fairly well to the Planck one with $T \approx 0.7\mu$.

However, the energy distribution of emitted pions in *C.M.S.* does not quite fit to the Planck distribution with $T \approx 0.7\mu$. The disagreement is particularly remarkable in the high energy region. An energy distribution of the Heisenberg type ($\propto E^{-2}\,dE$) has a similar tendency to the experiments at the high energy region, but P_T-distribution of this type does not agree with experiment as to the position of its peak. In these senses, the distribution of momenta $\boldsymbol{p}, f(\boldsymbol{p})d\boldsymbol{p}$, slightly differs from the isotropic one, $f(p)\,p^2\,dp\,d\Omega$. Practically, however, it may be said that the angular distribution of emitted pions is isotropic.



a) Momentum distribution

b) Experimental angular distribution

Fig. 2. The momentum and angular distribution of the emitted pions in π^--p collisions at 5 Gev

We shall show how we can explain the nature of the nucleon-nucleon collisions basing on the model given in § 2. The interaction between the pion and nucleon in π-*N.S.*, which moves with a Lorentz factor γ' in *C.M.S.*, is expressed by the π^-–p interaction at 5 Gev. For simplicity we assume the distribution of emitted pions is an isotropic Planck distribution with $T=0.7\mu$ in π-*N.S.** Therefore the distribution of emitted pions in a nucleon-nucleon collision in *C.M.S.* is obtained by letting the isotropic Planck distribution ($T=0.7\mu$) move with the Lorentz factor γ'.

Let the energy and momentum of emitted pions in *C.M.S.* be E and P, respectively. By means of the Lorentz transformation the energy distribution of emitted pions in *C.M.S.*, $f(E)$, is written as follows:

* Here we do not assume any specific model of the interaction mechanism in a pion-nucleon collision. The Planck distribution is chosen only because it is simple and easy to treat mathematically.

$$f(E)\propto\frac{\gamma' E}{T}\log\frac{1-e^{-\gamma'(E+v'P)/T}}{1-e^{-\gamma'(E-v'P)/T}}$$
$$+\frac{\gamma' v}{T}\log(1-e^{-\gamma'(E+v'P)/T})$$
$$\times(1-e^{-\gamma'(E-v'P)/T})$$
$$+2\sum_{n=1}^{\infty}\frac{e^{-n\gamma' E/T}}{n^2}\sinh\frac{\gamma' v' P}{T} \qquad (3\cdot1)$$

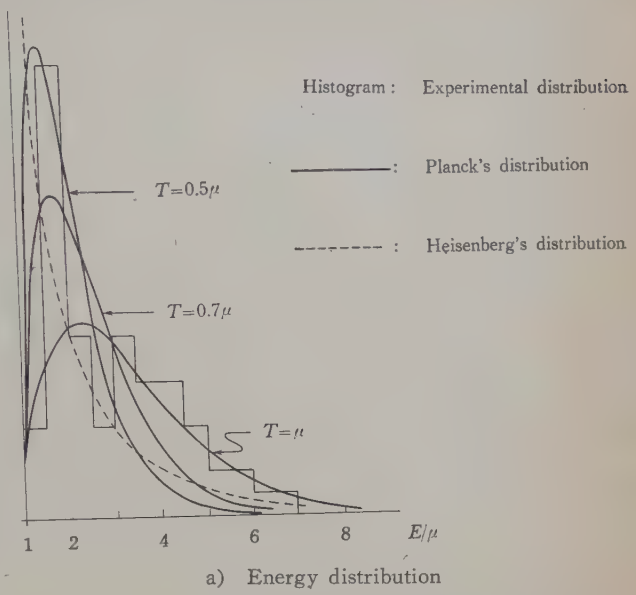
where $v'=\sqrt{\gamma'^2-1}/\gamma'$. (See Appendix II.) In the limit of $\gamma'\to1(v'\to0)$ $f(E)$ tends to the ordinary Planck distribution with the temperature T. The energy distribution $f(E)$ are shown in Fig. 4 for various values of γ'. The energy distributions (3·1) with $\gamma'=1.2\sim1.5$ and $T=0.7\mu$ does not seem to disagree with the experimental one except for the sharpness of the peak (at $E=\mu\sim2\mu$).

The angular distribution, which is obtained from the two isotropic Planck distributions moving in the opposite direction with the Lorentz factor γ', is written as follows:

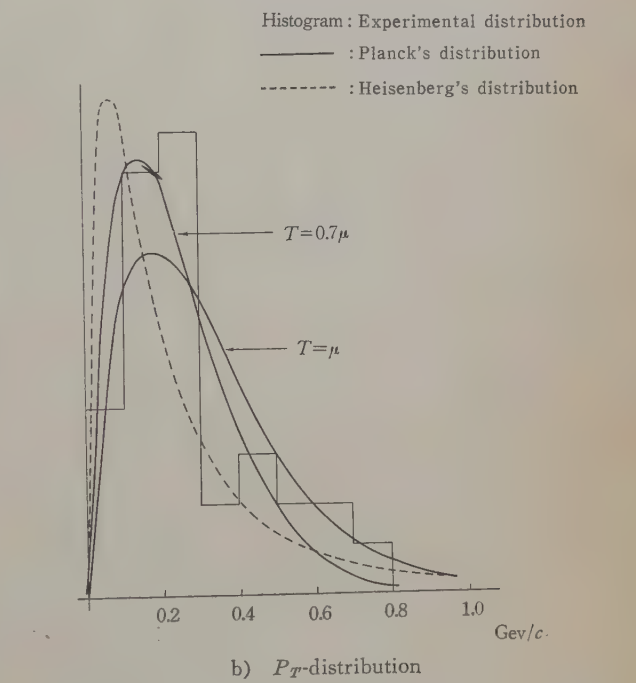
$$g(\bar{\vartheta})=\frac{F(\gamma' z)}{2\gamma'^2F(z)}+\frac{\gamma' z^3}{2F(z)}$$
$$\times\sum_{\nu=1}^{\infty}\frac{2\nu+1}{2^\nu\nu!}(v'\cos\bar{\vartheta})^{2\nu}$$
$$\times\Phi_\nu(\gamma' z), \qquad (3\cdot2)$$

where

$$\Phi_\nu(\gamma' z)=\sum_{n=1}^{\infty}(n\gamma' z)^{\nu-1}\left\{K_\nu(n\gamma' z)+\frac{2K_{\nu+1}(n\gamma' z)}{n\gamma' z}\right\},$$



a) Energy distribution



b) P_T-distribution

Fig. 3. The energy and P_T-distribution of the emitted pions in nucleon-nucleon collision at ~30 Gev

$$F(z)=z^2\sum_{m=0}^{\infty}\frac{K_2(z(1+m))}{1+m},$$

$$z=\mu/T$$

and K_ν is the Kelvin function of ν-th order. (See Appendix II.) From (3·2), we can estimate the anisotropy factor σ defined by (2·6), namely

$$\sigma^2=0.39^2+\frac{\gamma' z^3}{F(z)}\times\sum_{\nu=1}^{\infty}\frac{v'^{2\nu}}{\nu!\,2^\nu}[\log(\sqrt{\nu}+\sqrt{\nu+1})]^2\Phi_\nu. \qquad (3\cdot3)$$

(See Appendices I and II.) The values of σ are listed in Table I for $T=0.7\mu$. Since the series in the expression (3·3) hardly converge and the terms up to $\nu\sim15$ contribute to (3·3), the σ for $\gamma'=2.0$ and 2.4 have been estimated by means of the extrapolation method for ν.

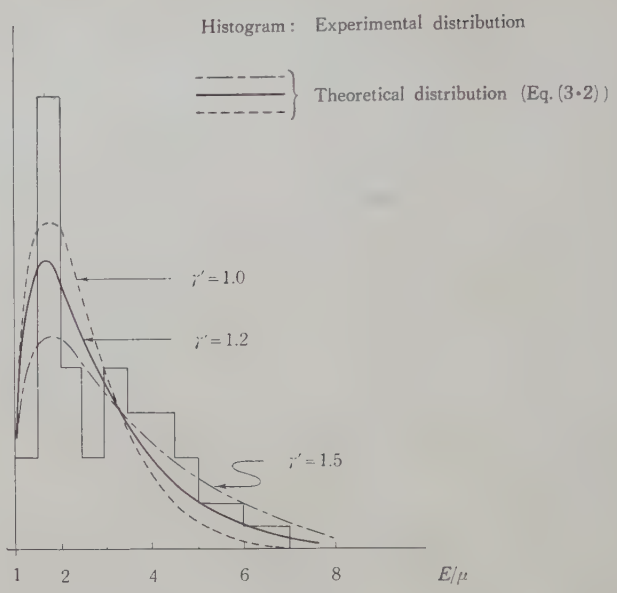


Fig. 4. The calculated energy distribution from Eq. (3·2)

Table I. The anisotropy factor with $T=0.7\mu$

σ : The anisotropy factor calculated from Eq. (3·3)
σ_c : The anisotropy factor calculated from Eq. (2·6)

γ'	1.2	1.6	2.0	2.4
σ	0.48	0.60	~0.7	~0.8
σ_c	0.45	0.60	0.70	0.76

Since σ is still not so large for the $\gamma'=1.2\sim1.6$ as seen from Table I, the angular distribution (3·2) may be said to look a little like an isotropic one. Here, in calculating the values of σ in Table I, the angular distribution of emitted pions in π-*N.S.* has been assumed to be isotropic instead of the actual one in π^-–p interactions at 5 Gev which has a slight peak in the forward direction of the incident pions. The direction of motion of the π-*N.S.* in *C.M.S.* may be the same as that of the nucleon and opposite to that of the pion, because in most cases the momentum of the nucleon is larger than that of the pion in *C.M.S.* We may expect, therefore, that if the experimental distribution of emitted pions in π^-–p collisions at 5 Gev were used for the behaviors of emitted pions in π-*N.S.* it leads to a more satisfactory result; namely, the calculated angular distribution of emitted pions in

the nucleon-nucleon collision is almost isotropic as shown in the experiments. (See Fig. 5.)

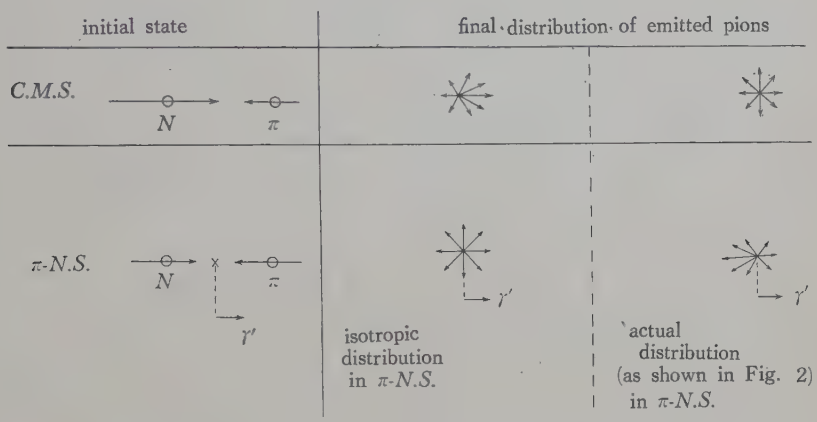


Fig. 5. The schematic explanation of the angular distribution of emitted pions

In conclusion, we may say that the angular and momentum distributions calculated on the basis of our model reproduce the feature of the experiments surveyed by Kaneko et al. However, the discussions in this section are still inconclusive because of the poor statistics of the experiments. In order to test our model more precisely experimental data with the energy near 30 Gev must be accumulated.

The values of anisotropy factor σ calculated from Eq. (2·6) are written in Table I besides the values from Eq. (3·3). These values coincide with each other except for small γ'. Thus, it may be crudely said that Eq. (2·6) is correct for any energy distribution of pions in their center of mass system and so we shall use Eq. (2·6) hereafter.

Differences between σ and σ_c for small γ' are due to the following circumstance: In deriving Eq. (2·6) Čiok et al. assumed the velocities of emitted pions are near the light velocity, while in Eq. (3·3) we have assumed the Planck distribution for emitted pions.

§ 4. The analysis of experiments II. ($\gamma_c \lesssim 10$)

In this section we generalize the method developed in previous sections to very high energy nuclear events.*

First, we estimate the Lorentz factor of π-*N.S.*, γ'. We denote the Lorentz factor of the system in which emitted pions distribute isotropically (we call it *M*-system, hereafter) by $\bar{\gamma}$, in distinct from γ'. Naturally it is not necessary γ' to be equal to $\bar{\gamma}$.

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* An idea similar to ours has been proposed by Hayakawa, (*Cosmic Radiation* (in Japanese), (Iwanami Shoten, Tokyo, (1959), 149).

We can get the value of $\bar{\gamma}$ for each event from its angular distribution; for example, if we assume the angular distribution of emitted pions to be $((m+1)/2)\times\cos^m\bar{\vartheta}(m=\text{even})$, we obtain $\bar{\gamma}$ from the relation

$$\bar{\gamma}=\sqrt{m/2+1}. \tag{4·1}$$

(See Appendix I.)

It is easily shown that we can determine γ' from $\bar{\gamma}$ and the inelasticity K, if we assume that the nucleon retains its direction during the collision and changes its momentum only along the shower axis. Then the momentum of the nucleon in *C.M.S.* after the collision, P_f, is

$$P_f=(1-K)\gamma_c M. \tag{4·2}$$

Eq. (4·2) is correct only if $\gamma_c\gg 1$ and K is not so near to 1. We denote the magnitude of the momentum, the angle and the energy of emitted pion in *C.M.S.*, *M*-system and π-*N.S.* by $(p_i, \vartheta_i, \epsilon_i)$, $(\bar{p}_i, \bar{\vartheta}_i, \bar{\epsilon}_i)$ and $(p_i', \vartheta_i', \epsilon_i')$, respectively. From the definition of *M*-system, the equation

$$\sum_i \bar{p}_i\cos\bar{\vartheta}_i=0 \tag{4·3}$$

should be satisfied, where the summation should be carried out over all pions that compose the *M*-system. From the definition of the π-*N.S.* we find

$$P_f'+\sum_i p_i'\cos\vartheta_i'=0 \tag{4·4}$$

where P_f' is the momentum of the nucleon in π-*N.S.* after the collision and is expressed as

$$P_f'=\gamma'(1-K)(1-v')\gamma_c M. \tag{4·5}$$

The total energy of emitted pions in *C.M.S.* is given by

$$\sum_i \epsilon_i=K\gamma_c M. \tag{4·6}$$

According to the Lorentz transformations we have

$$\begin{aligned} p_i\cos\vartheta_i&=\bar{\gamma}(\bar{p}_i\cos\bar{\vartheta}_i+\bar{v}\bar{\epsilon}_i)\\ \epsilon_i&=\bar{\gamma}(\bar{\epsilon}_i+\bar{v}\bar{p}_i\cos\bar{\vartheta}_i), \end{aligned} \tag{4·7}$$

$$\begin{aligned} p_i'\cos\vartheta_i'&=\gamma'(p_i\cos\vartheta_i-v'\epsilon_i)\\ \epsilon_i'&=\gamma'(\epsilon_i-v'p_i\cos\vartheta_i). \end{aligned} \tag{4·8}$$

From Eqs. (4·7), (4·8) and (4·3) one finds that

$$\sum_i p_i'\cos\vartheta_i'=\gamma'(v-v')\sum_i \epsilon_i \tag{4·9}$$

where we use the relation $\gamma\sum_i\bar{\epsilon}_i=\sum_i\epsilon_i$. From Eqs. (4·4), (4·5) and (4·6) we obtain

$$v'=1-K(1-\bar{v}). \tag{4·10}$$

Using (4·10) we can determine the value of γ' from K and $\bar{\gamma}$ for each nuclear jet shower. Then ω^* in Eq. (2·4) can be written in terms of γ' thus obtained:

$$\omega^*/M=(1-v')/(1+v'). \tag{4·11}$$

The total energies of emitted pions in M-system and π-*N.S.* are given by Eq. (4·6)

$$\langle\bar{\epsilon}\rangle/M\equiv\sum_i\bar{\epsilon}_i/M=K\gamma_c/\bar{\gamma} \tag{4·12}$$

and from Eqs. (4·8), (4·7), (4·3) and (4·6)

$$\langle\epsilon'\rangle/M\equiv\sum_i\epsilon_i'/M=K\gamma_c\gamma'(1-v'\bar{v}), \tag{4·13}$$

respectively. Taking (4·4) into account, we get the total energy in π-*N.S.* as

$$E'=M\gamma_c/\gamma'. \tag{4·14}$$

The inelasticity of the nucleon in π-*N.S.*, K', is given by

$$K'=\gamma'\gamma_c(1-v')K/\{\gamma'\gamma_c(1-v')-1\}. \tag{4·15}$$

When v' is not so close to 1 we may put $K'\approx K$.

Therefore we can obtain the various quantities in π-*N.S.* for each event from its angular distribution and inelasticity. In this case, however, we can choose either sign of $\bar{v}$, plus or minus. As shown in Fig. 6, there are two nucleons N_1 and N_2 and two lumps of the pions M_1 and M_2 with the velocity $\pm\bar{v}$ in both of which pions are isotropically emitted, in final state. If the one π-*N.S.* consists of N_1 and M_1 and then another consists of N_2 and M_2 we use positive $\bar{v}$ in all formula above given. We denote the quantities thus determined by the ones with suffix 1, for example, γ_1', ω_1^*, ···. If the two π-*N.S.* are composed of the combination (N_1, M_2) and (N_2, M_1) the sign of $\bar{v}$ must be taken to be minus, and in this case the calculated quantities are expressed by adding suffix 2, say γ_2', ω_2^*, ···. It will be decided later on to choose which combination is suitable for actual cases.

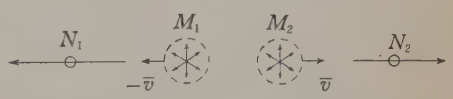


Fig. 6. The state after collision
N_1, N_2 : nucleons after collision with the momentum $\pm P_f$.
M_1, M_2 : lumps of pions in which they distribute isotropically.

Now, we try to analyse the experimental data of jet showers in nuclear emulsion, mainly the showers observed by the Bristol group.[11] From the experimental values of n_s,* γ_c, K and m for each event we can get the quantities characterizing the nature of the π-*N.S.*, γ', ω^*, $\langle\bar{\epsilon}\rangle$, ···. Their numerical values are listed in Table II.

As seen from Table II, the values of γ_1' are quite large and ω_1^*/M corresponding to them are very small. This means the energies of pions lost from

* n_s is the number of shower particles.

Table II

Event	$N=3\,n_s/2$	γ_c	K††	m	$\bar{\gamma}$	$2\langle\bar{\varepsilon}\rangle/NM$	γ_1'	ω_1^*/M	γ_2'	ω_2^*/M
Schein[19]	21	100	0.12	18	3.2	0.36	9.0	0.0031	1.56	0.12
P7†	17	28	0.36	4	1.7	0.69	2.8	0.0340	1.07	0.49
P19	4	30	0.19	78	6.3	0.45	14.5	0.0012	1.28	0.26
P15	6	30	0.21	16	3.0	0.70	6.5	0.0060	1.24	0.26
P13	9	33	0.21	38	4.5	0.31	9.7	0.0027	1.25	0.26
P10	18	33	0.41	8	2.2	0.67	3.4	0.0220	1.03	0.63
P16	14	45	0.24	10	2.5	0.63	4.9	0.0110	1.20	0.30
P5	45	67	0.24	2	1.4	0.51	2.7	0.0370	1.24	0.25
P2	30	87	0.36	18	3.2	0.66	5.2	0.0093	1.05	0.54
P20N1	48	110	0.14	2	1.4	0.47	3.5	0.0210	1.54	0.14
P24	14	112	0.14	194	9.9	0.24	26.5	0.0004	1.62	0.16
P20	5	142	0.19	1000	22.4	0.48	51.4	0.0001	1.27	0.23
P25	35	180	0.06	2	1.4	0.44	5.4	0.0088	2.26	0.05

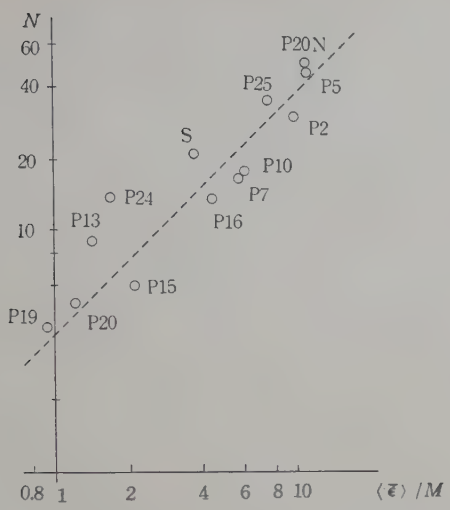
†: The events (P7~P25) were observed by the Bristol group.
††: K in Bristol experiments are the rate of the energies of emitted pions to available energy.

nucleons are very large. For example, we see $\omega_1^*/M\sim 0.034$ for event P7 (the initial energy $\gamma_c=28$) and the corresponding energy of exchanged pion, ω_1, is larger than 2.5μ.* For event P19, which is initiated by a nucleon with the initial energy $\gamma_c=30$, the value of ω_1^*/M is nearly 0.0012 and leads to $\omega_1 \gtrsim 50\mu$. As to the initial energies both events differ little from each other. Nevertheless, 17 particles are produced in the P7 event, while only 4 particles are found in the P19 event. It is quite strange that the number of emitted pions in event P19 is less than the one in event P7 in spite of the larger energy loss of nucleon in the former ($\omega \gtrsim 50\mu$) than the one in the latter ($\omega \gtrsim 2.5\mu$). Such a strange feature is generally observed in any choice of two events with the same initial energies in Table II.

In addition to that, it is evidently unreasonable from our standpoint to admit such large values of ω, because the losses of these high energy pions leads to give to the nucleon a strong recoil, hence Williams-Weizsäcker's method will become illegitimate. In this case we must treat the problem quantum-mechanically and should not accept the classical picture which is our standpoint. According to the above-mentioned reasons, the combinations (N_1, M_1) and (N_2, M_2) are not suitable.

Next we consider another type of combination, (N_1, M_2) and (N_2, M_1). From Table II, the values of γ_2' are almost constant and range from 1.2 to 1.5. The corresponding energies of pions are small as expected in § 2. This fact enables our point of view stated in § 2 to be reliable. It is thus verified that one can reduce the nucleon-nucleon interaction to a pion-nucleon one and investigate the latter interaction from high energy jet showers.

* From the relation $\omega^*=\omega-q_{\parallel}$, we can easily show $\omega \geq (\omega^{*2}+\mu^2)/2\omega^*$.



N: Total number of emitted pions in the jet shower.

Fig. 7. $N-\langle\bar{\epsilon}\rangle/M$ plot

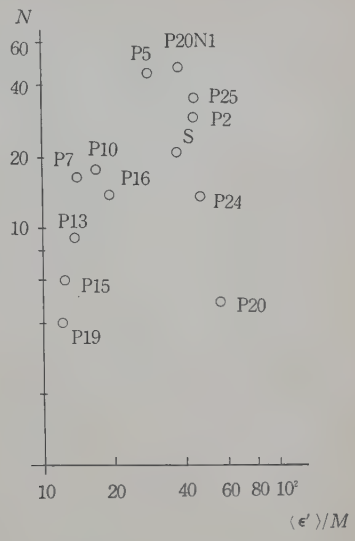


Fig. 8. $N-\langle\epsilon'\rangle/M$ plot

The mean energies of emitted pions in M-system, $\langle\bar{\epsilon}\rangle/N$, are listed in the seventh column in Table II. The fact that these values are nearly constant for various showers shows that the pions are produced as a lump in which the pions move isotropically and have nearly the same energy for any lump irrespective of the process of formation of lump. Fig. 7 shows this circumstance; the rest energies of pion lumps ($=\langle\bar{\epsilon}\rangle$) linearly depend on the number of emitted pions. This fact agrees with the well-known results obtained by the two-center model or fire-ball model.[7),13),14)]

The numbers of emitted pions, N, are plotted in Fig. 8 taking the total energies of emitted pions in π-$N.S.$, $\langle\epsilon'\rangle$, as the variable. In this figure we hardly recognize any correlation between them.

In Fig. 9 we plot the number of the emitted pions, N, against E_2', the total energy (pions'+nucleons') in π-$N.S.$ obtained from (4·14), for the events listed in Table II. We also plot the corresponding number of emitted pions for jet showers initiated by the secondary particles in primary showers which were

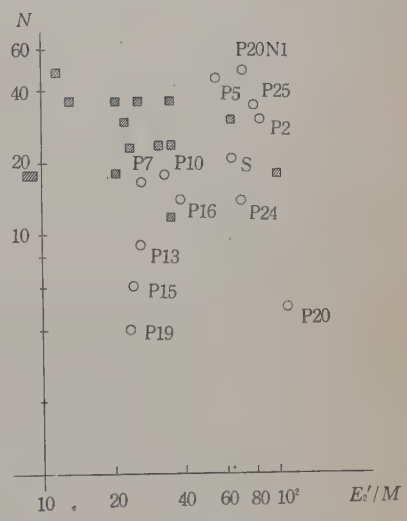


Fig. 9. $N-E_2'/M$ plot

▨: Secondary jet shower observed by the Bristol group. In this case, $N=3n_s$, n_s=the number of shower particles.

observed by the Bristol group.* The plotted points are localized nearly in the same region for both kinds of data.

We may conclude that our model is correct at extremely high energies in the same way as in the rather lower energy region discussed in § 3.

§ 5. The momentum distribution of pion in nucleon proper field

In the previous section it is noted that the energy of the exchanged pion (speaking more precisely, $\omega^*=\omega-q_1$) can be derived for each event from $\bar{\gamma}$ and K. The distribution of ω^* derived from experimental data might give some clue to investigate the structure of the nucleon.

From Eqs. (4·10) and (4·11) we obtain

$$\alpha \equiv \frac{\omega^*}{M}=K\frac{\bar{\gamma}+\sqrt{\bar{\gamma}^2-1}}{2\bar{\gamma}-K(\bar{\gamma}+\sqrt{\bar{\gamma}^2-1})}. \tag{5·1}$$

If we assume that K and $\bar{\gamma}$ are independent of each other, we can obtain the distribution function of α, $f(\alpha)$, from the distribution function of K and γ determined experimentally. Further, if we assume that the distribution of the momentum of the exchanged pion is equal to that of the pions in the proper field of a nucleon, we can infer the latter, $\rho(q)$, from $f(\alpha)$.

The distribution function of $\bar{\gamma}$, $P(\bar{\gamma})$, can be derived from the distribution of σ experimentally determined by the Polish group,[8] who have obtained σ of each jet shower in an energy region $140\geq\gamma_c\geq 2$. This distribution of σ is given in Fig. 10.** Eq. (2·6), together with the distribution of σ, enables us to get $P(\bar{\gamma})$ which is shown in Fig. 11.

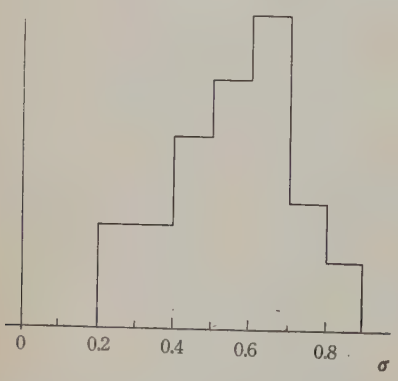


Fig. 10. Distribution of anisotropy factor σ

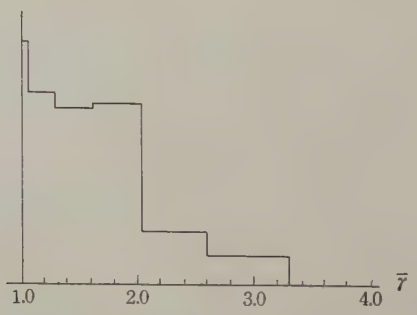


Fig. 11. Distribution of $\bar{\gamma}$, $P(\bar{\gamma})$
In obtaining $P(\bar{\gamma})$, we have used only the distribution of σ which corresponds to $\sigma\gtrsim 0.39$, because Eq. (2·6) shows $\sigma\gtrsim 0.39$.

* The secondary particles in primary jets are generally considered to be pions. In order to definitely pick out the secondary jets brought by pions we selected the events that have the inelasticity $\gtrsim 0.7$.

** Though the mean value of σ slowly varies with γ_c, we neglected this dependence on γ_c and obtained the distribution function of σ from all events.

On the other hand, we know little of the distribution of K, $Q(K)$.[11] The only information from the experiments is that various values of K, ranging from zero to 1, have been found. Now we assume $Q(K)$ may be written as

$$Q(K)=\begin{cases}1/(\beta-\delta) & \beta\geq K\geq\delta\\ 0 & \text{otherwise}\end{cases} \tag{5·2}$$

where we regard β and δ as parameters.

From $P(\bar{\gamma})$, $Q(K)$ and (5·1) it is easily shown (see Appendix III)

$$f(\alpha)=\frac{2}{(1+\alpha)^2}\int_R \frac{\bar{\gamma}}{\bar{\gamma}+\sqrt{\bar{\gamma}^2-1}}P(\bar{\gamma})\,d\bar{\gamma}, \tag{5·3}$$

where R is the region of $\bar{\gamma}$ defined by

$$\beta\geq(2\alpha/(1+\alpha))\cdot(\bar{\gamma}/(\bar{\gamma}+\sqrt{\bar{\gamma}^2-1}))\geq\delta. \tag{5·4}$$

Using Fig. 11, $f(\alpha)$ are calculated for various values of β and δ and the results are shown in Fig. 12. As seen from it, the position of the maximum and the shape of the tail of the curve $f(\alpha)$ are affected by the values of β and δ.

On the other hand, if we assume that the momentum distribution of the pion in proper field, $\rho(q)$, is of spherically symmetric, $f(\alpha)$ is given as (see Appendix III)

$$f(\alpha)=\frac{M}{4}\int_{(M\alpha/\mu-\mu/M\alpha)^2\mu^2/4}^{\infty}\rho(\sqrt{x})\,dx \tag{5·5}$$

where $\rho(q)$ satisfies the normalization condition

$$\int_0^\infty \rho(q)q^2\,dq=1.$$

If all events of nucleon-nucleon collisions are completely understood on the basis of our model and if we know $\rho(q)$, $f(\alpha)$ calculated by Eq. (5·3) should have the same meaning as that obtained by Eq. (5·5). In fact, as shown in § 4, the nucleon-nucleon collision, in most cases, can be reduced to the pion-nucleon collision and then we may say that $f(\alpha)$ calculated by Eq. (5·3) coincides with $f(\alpha)$ expressed as Eq. (5·5).

As shown from Eq. (5·5), $f(\alpha)$ has the general nature that the posi-

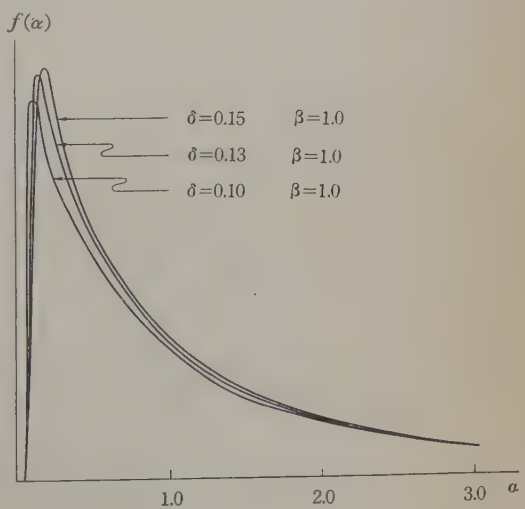


Fig. 12. The graphs of $f(\alpha)$
(a) : $f(\alpha)$ for various δ and $\beta=1.0$

tion of the maximum of $f(\alpha)$ is at $\alpha=\mu/M\approx 0.15$. This general nature should be found in $f(\alpha)$ calculated by Eq. (5·3) and then restricts the values of δ to be in the range from 0.1 to 0.15 as seen in Fig. 12.

Of course we might be able to calculate $\rho(q)$ from $f(\alpha)$, but it would be meaningless to discuss the detailed form of $\rho(q)$ because of the poor accuracy of experiments. Here we only show $f(\alpha)$ corresponding to the Gaussian type of $\rho(q)$,

$$\rho(q)=\sqrt{2/\pi}\cdot q_0^{-3}e^{-q^2/q_0^2}, \tag{5·6}$$

in Fig. 12. It is seen from Fig. 12 that the spread of the momentum distribution, which is $\sim q_0$, varies according to the value of β.

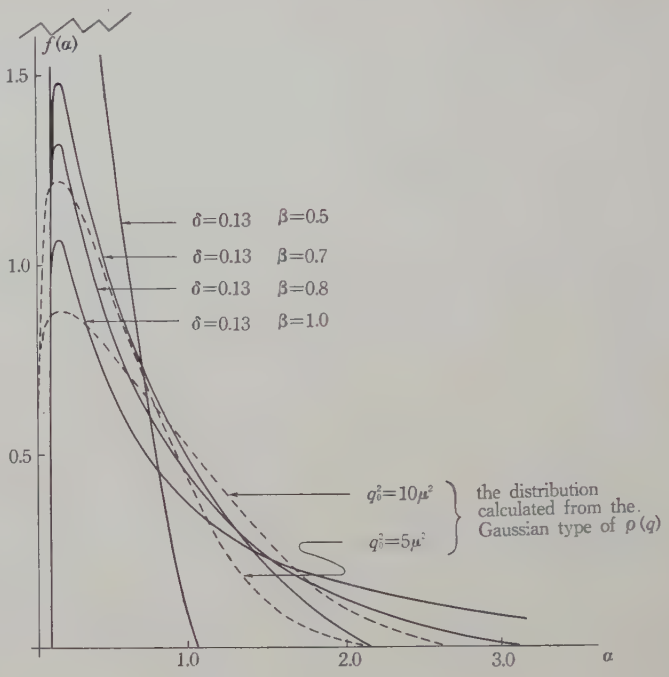


Fig. 12. The graphs of $f(\alpha)$
(b): $f(\alpha)$ for fixed $\delta(=0.13)$ and various β

If we are interested only in the spread of the momentum of pion in proper field, we can obtain it without any detailed knowledge of $\rho(q)$. For this purpose it is enough only to note the relation

$$\langle q^2\rangle=(3/4)\langle(M\alpha/\mu-\mu/M\alpha)^2\rangle\mu^2 \tag{5·7}$$

which is verified in Appendix III. With $f(\alpha)$ shown in Fig. 12 the values of $\sqrt{\langle q^2\rangle}$ for $\delta=0.13$ and $\beta=0.5$, 0.7, 0.8, 0.9 and 1.0 are shown in Table III.

Table III. The value of $\sqrt{\langle q^2\rangle}$

β	0.5	0.7	0.8	0.9	1.0
$\sqrt{\langle q^2\rangle}/M$	0.3	0.6	0.7	1.0	1.6

From Fig. 12 and Table III we may say that the spread of the momentum distribution of pions in proper field is finite and has the magnitude of order M and its distribution function decreases rapidly as q increases.

According to the experiments on the elastic scattering of electrons or pions with the nucleon, it has been found that the nucleon has a core with a finite

radius $\sim 4M^{-1}$. Roughly speaking, there is no contradiction between the value of $\sqrt{\langle q^2 \rangle}$ in Table III and that of the experiments on the structure of nucleon. Of course, the values in Table III should not be rigorously accepted since $Q(K)$ used in calculating $f(\alpha)$ has not been ascertained experimentally.

§ 6. Conclusions and discussions

We have tried in this paper to investigate to what extent we could get information about the interaction between the pion and the nucleon and about the structure of the nucleon from the analysis of the experiments on the multiple production of pions.

We assumed that two nucleons interact with the proper fields of their partners respectively and then two pion-nucleon systems (π-*N.S.*) are symmetrically produced in the center of mass system (*C.M.S.*). It may not always be realized that only these symmetric collisions in *C.M.S.* occur, and actually there are a few asymmetric collisions. The experimental events analyzed in this paper, however, are almost of symmetric feature and then this assumption seems to be satisfied here.

The second assumption which we have made is that the nucleon keeps its direction of motion and changes its momentum only along the shower axis. Since the available experimental data do not show any azimuthal asymmetry, this assumption may be a plausible one.

We treated all events as those caused by collisions of a nucleon with a nucleon. In order to pick out the nucleon-nucleon collisions we selected the events in which the number of grey and black tracks is as small as possible.

From the results in § 3 and § 4, we may conclude that in most cases the nucleon-nucleon collision can be reduced to the pion-nucleon collision on the basis of our model stated in § 2; in other words, the peripheral part of the nucleon plays the main rôle in nucleon-nucleon collisions.

In our analyses, we could not get any indication on the nature of the core-core interaction from the present experimental knowledges. This may be partly due to the poor statistics of the experiments and partly to the unconscious selection of the events. Consequently, in order that we can precisely analyse the nucleon-nucleon collision and closely study the nature of the nuclear interaction, it is absolutely necessary to investigate the interaction between pion and nucleon at high energies.

The discussions in § 5 show a possibility to investigate the structure of the nucleon from the high energy nuclear events. We may say qualitatively that the nucleon has a core with a finite radius which is roughly equal to its Compton wave length.

As seen in Fig. 7, the mean energy of emitted pions in M-system is nearly constant for any M-system. On the other hand, we see in Table II that the inelasticity of nucleon in π-*N.S.*, $K'(\approx K)$, is generally small and constant ($\approx 0.1 \sim$

0.4). These circumstances, as well as the fact that most of the nucleon-nucleon collisions can be reduced to the pion-nucleon collisions, may suggest that the pion-pion interaction is considerably stronger than the pion-core interactions.

Finally, we consider a method from which we separate the peripheral collisions from the whole events. In peripheral collisions the momentum of the pion responsible for producing one π-$N.S.$ is about 2μ or less than it. Thus ω^* is restricted in the range $(\sqrt{5}-2)\mu \leq \omega^* \leq (\sqrt{5}+2)\mu$, when we remember $\omega^* = \omega - q_{\parallel}$. The corresponding values of $\bar{\gamma}'$ to these ω^* are found in $1.03 \lesssim \bar{\gamma}' \lesssim 2.8$. Using Eq. (4·10), we obtain the relation $K \approx \bar{\gamma}(\bar{\gamma} - \sqrt{\bar{\gamma}^2-1})(\bar{\gamma}' - \sqrt{\bar{\gamma}'^2-1})/\bar{\gamma}'$. If $\bar{\gamma}$ is known from the angular distribution of shower, K must satisfy the relation

$$0.8 \gtrsim K/\bar{\gamma}(\bar{\gamma} - \sqrt{\bar{\gamma}^2-1}) \gtrsim 0.1.$$

When K and $\bar{\gamma}$ in an event do not satisfy the above condition, we may say this event is not the peripheral collision and we should take into account the core-core interaction.

Acknowledgements

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Appendix I

In this section we derive the relation among the anisotropy factor σ, the Lorentz factor of M-system $\bar{\gamma}$ and the parameter characterizing the angular distribution, m, $(f(\vartheta) \propto \cos^m \vartheta)$.

Ia. σ for the isotropic angular distribution

We consider the case that the particles move isotropically in $C.M.S.$ Let us use $x = \log \tan(\vartheta/2)$ as the variable instead of the angle of particle motion in $C.M.S.$, ϑ. Then the angular distribution is written as

$$f(x) = 2\alpha \frac{e^{2\alpha x}}{(1+e^{2\alpha x})^2} \qquad (\text{I}\cdot 1)$$

$$\alpha^{-1} = \log e \approx 0.43.$$

From Eq. (I·1), we easily obtain the anisotropy factor σ,

$$\sigma^2 \equiv \int_{-\infty}^{+\infty} x^2 f(x)\, dx = \frac{1}{4\alpha^2} \int_{-\infty}^{+\infty} \frac{z^2\, dz}{(e^{z/2}+e^{-z/2})^2} = \frac{1}{4\alpha^2}\frac{\pi^2}{3} = 0.39^2. \qquad (\text{I}\cdot 2)$$

This value, 0.39, slightly differs from the one employed by Lindern and by Čiok et al., $\sigma \approx 0.36$. But, if we are interested only in the order of magnitude of σ this difference is not essential.

Ib. σ for the anisotropic angular distribution ($=((m+1)/2)\cos^m\vartheta$, m=even)

In this case the distribution of x is easily shown as

$$f(x)=2\alpha\left(\frac{1-e^{2\alpha x}}{1+e^{2\alpha x}}\right)^m\frac{e^{2\alpha x}}{(1+e^{2\alpha x})^2}. \tag{I·3}$$

Since this $f(x)$ has the maxima at $x=\pm\log(\sqrt{m/2}+\sqrt{m/2+1})$ we may express $f(x)$ as follows:

$$f(x)=\frac{1}{2\sqrt{2\pi}\sigma'}\{e^{-(x-x_1)^2/2\sigma'^2}+e^{-(x-x_2)^2/2\sigma'^2}\}, \tag{I·4}$$

$$1/\sigma'^2=-\frac{1}{2}\left[\frac{d^2f}{dx^2}/f\right]_{x=x_1 \text{ or } x_2}, \tag{I·5}$$

where x_1 and x_2 are the positions of the maxima of $f(x)$. From Eq. (I·4) the anisotropy factor σ is given by

$$\sigma^2=\sigma'^2+\{\log(\sqrt{m/2}+\sqrt{m/2+1})\}^2.$$

On the other hand, from Eqs. (I·3) and (I·5) we can easily show that σ' is independent of m. It is quite natural to choose $\sigma'=0.39$ so that σ should be equal to the anisotropy factor at $m=0$, 0.39. Consequently, we obtain the relation between σ and m,

$$\sigma^2=0.39^2+\{\log(\sqrt{m/2}+\sqrt{m/2+1})\}^2. \tag{I·6}$$

Ic. The relation between $\bar{\gamma}$ and m.

Consider two M-systems travelling in opposite directions with the Lorentz factor $\bar{\gamma}$. For cases in which the Lorentz factor γ_c with which $C.M.S.$ moves against the laboratory system is considerably larger than 1 and the particle velocities in each M-system are near the light velocity, Čiok et al. derived the expression

$$\sigma^2=0.39^2+\{\log(\bar{\gamma}+\sqrt{\bar{\gamma}^2-1})\}^2. \tag{I·7}$$

Comparing Eq. (I·6) with Eq. (I·7) we can easily show the relation

$$\bar{\gamma}=\sqrt{m/2+1}. \tag{I·8}$$

Appendix II

We derive Eqs. (3·2) and (3·3) in this section. Let the momentum distribution in a particle system, $f(\boldsymbol{p}')d\boldsymbol{p}'$ (normalized to 1), be the Planck distribution,

$$f(\boldsymbol{p}')d\boldsymbol{p}'=\frac{d\boldsymbol{p}'}{4\pi\mu^3}\frac{z^3}{F(z)}(\exp(z\sqrt{(p'/\mu)^2+1})-1)^{-1} \tag{II·1}$$

where μ is the mass of particle, $z=\mu/T$ (T=the temperature of system) and $F(z)$ is given by

$$F(z)=z^2\sum_{m=0}^{\infty}\frac{K_2(z(1+m))}{1+m}.$$

We consider the case where the above system travels with a Lorentz factor $\bar{\gamma}$ in an observing system (in which the energy and momentum of the particle is denoted by E and $\boldsymbol{p}$, respectively). From the Lorentz transformation we get $E'=\bar{\gamma}(E-\bar{\boldsymbol{v}}\boldsymbol{p})$ and $d\boldsymbol{p}'/E'=d\boldsymbol{p}/E$ and then the momentum distribution of the particles in the observing system is given by

$$f(\boldsymbol{p})d\boldsymbol{p}=\frac{d\boldsymbol{p}}{4\pi\mu^3}\frac{z^3}{F(z)}\frac{\bar{\gamma}(E-\bar{\boldsymbol{v}}\boldsymbol{p})}{E}(\exp\{z\bar{\gamma}(E-\bar{\boldsymbol{v}}\boldsymbol{p})/\mu\}-1)^{-1}. \quad (\text{II}\cdot 2)$$

Integrating Eq. (II·2) over the angle and transforming the variable from p to E, we obtain the energy distribution, $f(E)dE$, as follows:

$$f(E)dE=\frac{dE}{2\mu}\frac{z}{F(z)}\frac{1}{\bar{\gamma}\bar{v}}\int_{y_-}^{y_+}\frac{ydy}{e^y-1}$$

$$=\frac{dE}{2\mu}\frac{z}{F(z)}\frac{1}{\bar{\gamma}\bar{v}}\left[y_+\log(1-e^{-y_+})-y_-\log(1-e^{-y_-})-\int_{y_-}^{y_+}\log(1-e^{-y})\,dy\right] \quad (\text{II}\cdot 3)$$

$$y_+=z\bar{\gamma}(E+\bar{v}p)/\mu,\quad y_-=z\bar{\gamma}(E-\bar{v}p)/\mu.$$

Carrying out the integration in (II·3) with the use of the expansion formula of $\log x$, we get Eq. (3·2) in which the factor before the square bracket is omitted.

The integration of Eq. (II·2) with p leads to the angular distribution of particles. Expanding the factor $\exp\{-z\bar{\gamma}(\bar{\boldsymbol{v}}\boldsymbol{p})/\mu\}$ in Eq. (II·2) by the powers of $(\bar{\boldsymbol{v}}\boldsymbol{p})$, this integration with p are expressed by the Kelvin function. The odd power terms of $(\bar{\boldsymbol{v}}\boldsymbol{p})$ vanish if there are two such systems including the same number of particles and travelling in opposite directions with velocity $\bar{v}$. Thus we can directly obtain Eq. (3·2), and then Eq. (3·3) from this (3·2) and Eq. (I·6).

Appendix III

IIIa. In this section we derive $f(\alpha)$ under the assumption introduced in § 5 that $P(\bar{\gamma})$ and $Q(K)$ are independent of each other. A probability with which the inelasticity of an event is found in the range $(K, K+dK)$ and the Lorentz factor of its M-system in $(\bar{\gamma}, \bar{\gamma}+d\bar{\gamma})$ is given by $P(\bar{\gamma})Q(K)d\bar{\gamma}dK$. Instead of K we use the new variable α,

$$\alpha=K\frac{\bar{\gamma}+\sqrt{\bar{\gamma}^2-1}}{2\bar{\gamma}-K(\bar{\gamma}+\sqrt{\bar{\gamma}^2-1})}$$

introduced in § 5. Integrating $P(\bar{\gamma})Q(K(\bar{\gamma},\ \alpha))\cdot\partial(\bar{\gamma},\ K)/\partial(\bar{\gamma},\ \alpha)\cdot d\bar{\gamma}d\alpha$ with $\bar{\gamma}$, we can obtain the distribution function of α, $f(\alpha)$; namely

$$f(\alpha)=\frac{2}{(1+\alpha)^2}\int P(\bar{\gamma})Q\left(\frac{2\alpha}{1+\alpha}\,\frac{\bar{\gamma}}{\bar{\gamma}+\sqrt{\bar{\gamma}^2-1}}\right)\frac{\bar{\gamma}}{\bar{\gamma}+\sqrt{\bar{\gamma}^2-1}}d\bar{\gamma}. \quad \text{(III·1)}$$

If $Q(K)$ is given by Eq. (5·2), we have only to let Q in Eq. (III·1) be $1/(\beta-\delta)$ and to carry out the integration with $\bar{\gamma}$ over the region satisfying

$$\beta\geq 2\alpha\bar{\gamma}/(1+\alpha)\,(\bar{\gamma}+\sqrt{\bar{\gamma}^2-1})\geq\delta. \quad \text{(III·2)}$$

The region of $\bar{\gamma}$ defined by Eq. (III·2) is shown in Fig. 13.

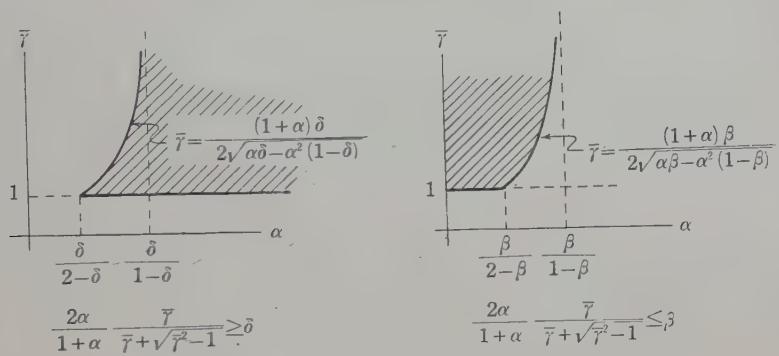

Fig. 13. Allowed region of $\bar{\gamma}$ in Eq.(III. 1)
(The shaded region satisfies Eq. (III. 2).)

It is noted that as seen from Fig. 13 the integral region is affected by whether $\delta/(1-\delta)$ is less than $\beta/(2-\beta)$ or not.

IIIb. Denote the momentum distribution of the pions in nucleon proper field as

$$(1/4\pi)\rho(q)\,d\boldsymbol{q}.$$

Transforming the variables from $(q, \cos\vartheta, \varphi)$ to (q, α, φ) where

$$\alpha=(\sqrt{q^2+\mu^2}-q\cos\vartheta)/M, \quad \text{(III·3)}$$

it is easily shown that the distribution function of α, $f(\alpha)$, is expressed as

$$f(\alpha)=\frac{M}{2}\int_{q\geq|(M\alpha/\mu)-(\mu/M\alpha)|\mu}\rho(q)\,q\,dq. \quad \text{(III·4)}$$

Putting $q^2=x$ in Eq. (III·4), we obtain

$$f(\alpha)=\frac{M}{4}\int_{\{(M\alpha/\mu)-(\mu/M\alpha)\}^2\mu^2/4}^{\infty}\rho(\sqrt{x})\,dx. \quad \text{(III·5)}$$

IIIc. We consider the quantity

$$\left\langle\left(\frac{M\alpha}{\mu}-\frac{\mu}{M\alpha}\right)^2\right\rangle\equiv\int_0^{\infty}\left(\frac{M\alpha}{\mu}-\frac{\mu}{M\alpha}\right)^2 f(\alpha)\,d\alpha. \quad \text{(III·6)}$$

From Eq. (III·5) this quantity can be expressed as

$$\frac{\mu}{4}\int_0^\infty u^2 du \int_{u^2\mu^2/4}^\infty \rho(\sqrt{z})\,dz \qquad \text{(III·7)}$$

where we use the relation $\int_0^\infty F(|x-(1/x)|)\,dx=\int_0^\infty F(u)\,du$. Carrying out the integration by parts in Eq. (III·7), we obtain

$$\left\langle\left(\frac{M\alpha}{\mu}-\frac{\mu}{M\alpha}\right)^2\right\rangle=\frac{4}{3\mu^2}\int_0^\infty q^4\rho(q)\,dq. \qquad \text{(III·8)}$$

From the definition of $\rho(q)$ the integral in Eq. (III·8) is interpreted as the average of q^2. Accordingly we can get the relation

$$\langle q^2\rangle=\frac{3}{4}\left\langle\left(\frac{M\alpha}{\mu}-\frac{\mu}{M\alpha}\right)^2\right\rangle\mu^2. \qquad \text{(III·9)}$$

References

1) H. Ezawa, Nuovo Cimento **11** (1959), 745.
2) L. B. Okun' and I. Pomerančuk, Soviet Phys. **9** (1959), 206.
D. I. Blokhintsev, *CERN symposium* **2** (1956), 155.
D. S. Černavskij, Nuovo Cimento Sup. **8** (1958), 775.
V. S. Barašenkov, V. M. Maltsev ond E. K. Mihul, Nucl, Phys **13** (1959), 583.
D. S. Černavskij, preprint communicated to Z. Koba, to whom we are obliged for informing us about it.
E. G. Bubelev and G. T. Zatsepin, report presented at Cosmic Ray Conference, Moscow 1959.
I. M. Dremin and D. S. Černavskij, J. E. T. P. **38** (1960), 229.
S. A. Slavatinskij and D. S. Černavskij, report presented at 9th Conference on High Energy Nuclear Physics, Kiev 1959.
3) D. S. Černavskij, See ref. 2).
4) E. Fermi, Prog. Theor. Phys. **5** (1950), 570.
L. D. Landau, Izv. Acad. Nauk U. S. S. R. **17** (1953), 51.
5) H. W. Lewis, J. R. Oppenheimer and S. A. Wouthuysen, Phys. Rev. **73** (1948), 127.
6) L. v. Lindern, Nuovo Cimento **5** (1958), 491.
7) P. Čiok, T. Coghen, J. Gierula, R. Hołyński, A. Jurak, M. Mięsowicz, T. Saniewska, O. Stanisz and J. Pernegr, Nuovo Cimento **8** (1958), 166.
8) J. Bartke, P. Čiok, J. Gierula, P. Hołyński, M. Mięsowicz and T. Saniewska, Nuovo Cimento **15** (1960), 18.
9) G. Maenchen, W. B. Fowler, W. M. Powell and R. W. Wright, Phys. Rev. **108** (1957), 850.
10) S. Kaneko, O. Kusumoto, S. Matsumoto and M. Takahata, preprint.
11) B. Edwards, J. Losty, D. H. Perkins, K. Pinkau and J. Reynolds, Phil. Mag. **3** (1958), 237.
12) M. Schein, R. G. Glasser and D. M. Haskin, Nuovo Cimento **5** (1955), 647.
13) K. Niu, Nuovo Cimento **10** (1958), 994.
14) G. Cocconi, Phys. Rev. **111** (1958), 1699.
15) R. Hofstadter, Ann. Rev. Nuclear Sci. **7** (1957), 231.
D. I. Blokhintsev, V. S. Barašenkov and V. G. Grišin, Nuovo Cimento **9** (1958), 241.

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On the Inelastically Scattered Proton and the Dipole γ-Absorption*

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The energy spectra and the angular distributions of the protons inelastically scattered from the levels in the dipole giant γ-resonance region are investigated by taking into account the nuclear interaction, as well as the Coulomb interaction, between the incident proton and the nucleons in the target nucleus. The calculated results for some nuclei are compared with the experimental ones. It is pointed out that, as the results of nuclear interactions, the spin flip parts of the target nucleons in the nucleus play an important role. Based on the fair agreement of the calculated results with observed ones, it is proposed that the measurements of $(p\text{-}p')$ scattering provide a powerful tool to investigate the dipole giant γ-resonance, and that the measurements give an effective way to studying the nuclear reduced matrix elements.

§ 1. Introduction

Tyrén and Maris[1] have measured the energy distributions of about 180 Mev protons inelastically scattered in the forward direction from many nuclei, and observed such peaks in the energy distributions that correspond to various excited states of the target nucleus. One of the most clearly visible peaks is the large and broad resonance-like one with several Mev width at such an energy of the proton that corresponds to the excitation energy of about 20 Mev of the residual nucleus. The existence of the peak is limited to the very small angle scattering, i. e. the angular distribution of the proton is sharply forward peaked. After examining several possible mechanisms for the reaction, they have concluded that, though the radial part of the wave function is changed by the excitation, the Coulomb interactions between the proton and the protons in the nucleus can compensate for the forward dip of the nuclear form factor in the dipole transition, and that the forward peaked angular distribution for the giant resonance peak around 20 Mev of the residual excitation in $(p\text{-}p')$ scattering should be due to the dipole excitation of the giant γ-resonance by the Coulomb interactions.[2]

Recently, Kawai and Terasawa[3] have investigated the peaks in the energy spectra of the reactions for many nuclei in terms of the Coulomb excitations. Their results are free from the ambiguity which may arise from the assumed nuclear model, because they have evaluated the cross sections for the inelastically scattered proton by means of available photo-nuclear data. They have found that the cal-

* A part of this subject has already been published in a note in Prog. Theor. Phys. **23** (1960), 183. In the right hand side of Eq. (1) in the previous note, a factor 2 should be readed as 4.

culated cross sections reproduce the shapes of the observed energy spectra for most of the investigated nuclei, and that their results are always by a factor 2 to 4 smaller than the observed ones in the range of the Coulomb excitation for most of the nuclei. Then they have concluded that the discrepancy would have to be attributed to the failure either of the Coulomb excitation hypothesis or of the Born approximation.

In view of the above situations, it seems worth while to investigate more quantitatively the high energy proton inelastically scattered from the levels of the dipole giant γ-resonance region, basing on the nuclear interactions between the nucleon and the nucleons in the target nucleus, since the nuclear interactions contribute to the excitation mode of the dipole giant γ-resonance region. In this paper we will investigate the effects of the nuclear interaction.

We use the t-matrix in order to take into account the nuclear interactions within the framework of the impulse approximation. At this point, we introduce the ratio of spin flip terms to non spin flip terms of the nucleons in the target nucleus as the results of nuclear interactions. The ratio plays an essential role for the cross section and the polarization of the proton. We use the value of ratio determined by the experimental polarization of the proton, and replace the nuclear reduced matrix element by the experimental photo-nuclear data. Therefore we can compare the calculated results with the experimental energy spectra having the characteristic peaks in $(p\text{-}p')$ scattering without referring to any special model for the mechanism of the γ-ray absorption and without the knowledge on the wave functions for the initial and final state of the nucleus. § 2 is devoted to the formulation for the energy distribution of the proton, and the calculated results are compared with the experimental ones in § 4.

Subsequently, we calculate the angular distributions of the protons inelastically scattered from the levels corresponding to the characteristic peaks with the aid of the integrated cross sections for the dipole γ-absorption, which is experimentally observed. In § 3, the formula of the differential cross section is derived. The results are contained in § 5.

Also the discussions connected with the results are done in § 4 and § 5.

In § 6 the concluding remarks are given.

§ 2. Energy distribution

In order to treat the inelastically scattered proton within the framework of the impulse approximation, we express the scattering spin matrix between particles with spin 1/2 in terms of some quantities which are invariant with respect to the space rotation and reflection and time reversal as follows, according to Wolfenstein et al.,[4]

$$M(i)=\alpha+\beta\boldsymbol{\sigma}_{1n}\boldsymbol{\sigma}_{in}+\gamma(\boldsymbol{\sigma}_{1n}+\boldsymbol{\sigma}_{in})+\delta\boldsymbol{\sigma}_{1q}\boldsymbol{\sigma}_{iq}+\varepsilon\boldsymbol{\sigma}_{1p}\boldsymbol{\sigma}_{ip}, \tag{1}$$

where 1 and i refer to the incident and target nucleons, respectively. $\boldsymbol{q}$, $\boldsymbol{n}$ and

$\boldsymbol{p}$ are three mutually perpendicular unit vectors which form a right-handed coordinate system, defined by

$$\boldsymbol{q}=\frac{\boldsymbol{k}_0'-\boldsymbol{k}_0}{|\boldsymbol{k}_0'-\boldsymbol{k}_0|} \quad \boldsymbol{n}=\frac{\boldsymbol{k}_0\times\boldsymbol{k}_0'}{|\boldsymbol{k}_0\times\boldsymbol{k}_0'|} \quad \boldsymbol{p}=\boldsymbol{q}\times\boldsymbol{n},$$

where $\boldsymbol{k}_0$ and $\boldsymbol{k}_0'$ are the momenta of the incident and scattered nucleons in the center of mass system. The scattering coefficients of spin operator α, β, γ, δ and ε depend on momentum transfer; $\boldsymbol{k}_0'-\boldsymbol{k}_0$, a sum of the energies of the incident and scattered nucleons; $|\boldsymbol{k}_0'|^2+|\boldsymbol{k}_0|^2$, and a difference of the energies of these nucleons; $|\boldsymbol{k}_0'|^2-|\boldsymbol{k}_0|^2$. Therefore, $\boldsymbol{p}=\boldsymbol{q}\times\boldsymbol{n}$ equals $(\boldsymbol{k}_0'+\boldsymbol{k}_0)/|\boldsymbol{k}_0'+\boldsymbol{k}_0|$ only on the energy shell where $|\boldsymbol{k}_0'|^2-|\boldsymbol{k}_0|^2=0$. It is important to note that the dependence of the scattering coefficients on the energy is weak compared with their variation with $\boldsymbol{k}_0'-\boldsymbol{k}_0$.

Now, in the inelastic scattering of nucleons of some handreds of Mev on light nuclei, the transition from the nuclear ground state Φ to the excited state Ψ is assumed to be brought about by a single nucleon-nucleon collision inside the nucleus, and mainly single particle levels will be excited by this collision. The single two-body collision is described by means of the t-matrix approximate to a free nucleon-nucleon collision. The approximation of replacing the t-matrix inside the nucleus by that obtained from free nucleon-nucleon scattering should be reasonable at this energy. When the mass of nucleus is assumed to be infinitely heavy, the differential cross section of the protons inelastically scattered from the nucleus, exciting the nucleus into a certain single particle state, is given by

$$\frac{d\sigma}{d\Omega}(\theta)=2\frac{k_f}{k_i}\sum_{\Psi}\mathrm{Tr}[T_\Psi(\theta)T_\Psi(\theta)^+], \tag{2}$$

where

$$T_\Psi=\langle\Psi e^{i\boldsymbol{k}_f\boldsymbol{r}_0}|\sum_i^A M(\theta)_i|e^{i\boldsymbol{k}_i\boldsymbol{r}_0}\Phi\rangle \tag{3}$$

is the scattering matrix for the nucleon involving a nuclear transition, while $\boldsymbol{k}_i$ and $\boldsymbol{k}_f$ are initial and final wave numbers of the protons, respectively, and θ is the angle between $\boldsymbol{k}_i$ and $\boldsymbol{k}_f$ where θ is $\frac{1}{2}\theta_{2\text{-}body}$, and A is the mass number of the target nucleus. Here $\sum_\Psi$ means the sum over all final sub-states with the same energies, and the factor $2=4\times(\frac{1}{2})$ arises from the transformation-coefficient 4 for small scattering angles and $\frac{1}{2}$ from the fact that one must take the trace in the two-dimensional spin space for the incident and scattered proton. Here the effect of distorted wave has been neglected since its effect is expected to be small for light nuclei in the interested angular region.

For brevity, we restrict ourselves to consider the protons inelastically scattered from the even-even target nucleus with spin 0 and isotopic spin 0 at its ground state such as C^{12} or O^{16}. Then the wave function of the final state Ψ in Eq. (3) represents a level in the group of levels which consists mainly of excited levels

with $J=1^-$ and $T=1$ in the dipole giant γ-resonance region. At this transition the parity change is $\Delta\pi=(-1)^J$ and the isotopic spin change $\Delta T=1$. Then one can write Eq. (3) as

$$T_{\Psi}=\langle\Psi|\sum^{A}e^{-i\boldsymbol{k}\boldsymbol{r}_i}\boldsymbol{\tau}_1\boldsymbol{\tau}_i\{M_0(\theta)_i+\sum_{\nu}^{1}(-1)^{\nu}\sigma(i)_{\nu}{}^{1}M(\theta)_i\}|\Phi\rangle, \qquad (3')$$

where $\boldsymbol{k}=\boldsymbol{k}_f-\boldsymbol{k}_i$, and $M_0(\theta)_i$ and $M_\nu{}^1(\theta)_i$ are the scattering spin operators of the spin independent and dependent parts of the i-th nucleon in the target nucleus as defined by Eq. (1), respectively, and $\sigma(i)_\nu{}^1$ is the spin operator for the i-th nucleon in the nucleus, and $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_i$ are the isotopic spin operators for the incident proton and the i-th nucleon in the nucleus, respectively.

Since the dependence of the scattering coefficients on the energy is weak compared with their variation with $\boldsymbol{k}$ as above mentioned, one can approximate Eq. (2) as follows, assuming that the charge exchange collisions do not take place in the present case,

$$\frac{d\sigma}{d\Omega}(\theta)=4\frac{k_f}{k_i}(|M_0|^2+\lambda|M^1|^2)|\langle\Psi|\sum^{A}e^{-i\boldsymbol{k}\boldsymbol{r}_i}\tau_{iz}|\Phi\rangle|^2, \qquad (4)$$

where

$$\lambda=|\langle\Psi|\sum^{A}e^{-i\boldsymbol{k}\boldsymbol{r}_i}\tau_{iz}\sum_{\nu}^{1}(-1)^{\nu}\sigma(i)_{\nu}{}^{1}|\Phi\rangle|^2\Big/|\langle\Psi|\sum^{A}e^{-i\boldsymbol{k}\boldsymbol{r}_i}\tau_{iz}|\Phi\rangle|^2 \qquad (5)$$

and

$$|M_0|^2=|\bar{\alpha}|^2+|\gamma|^2,\quad |M^1|^2=|\bar{\gamma}|^2+|\bar{\beta}|^2+|\bar{\varepsilon}|^2$$

while $\bar{\alpha}=\frac{1}{2}(\alpha_{pp}-\alpha_{pn})$ which is equal to $\frac{1}{4}(\alpha_1-\alpha_0)$ except for the Coulomb terms, etc.; here α_{pp} and α_{pn} are the coefficients of p-p and p-n scattering spin matrices, respectively, and α_1 and α_0 are the coefficients of the scattering spin matrices for the triplet and singlet isotopic spin states, respectively. Here it should be noted that Eq. (4) does not contain the scattering coefficient δ in Eq. (1) (See the Appendix).

In the small scattering angular region, the angular dependence of the nuclear form factor is determined by the lowest order non-vanishing multipole matrix term in expansion of $\exp[-i\boldsymbol{k}\boldsymbol{r}]$. Here, $\exp[-i\boldsymbol{k}\boldsymbol{r}]$ is contained in the nuclear form factor of the scattering amplitude for the proton. As for the proton inelastically scattered from the level in the dipole giant γ-resonance region the monopole matrix element vanishes and the dipole element does not. And the matrix element is, at very small scattering angles, proportional to the one for the electric dipole γ-excitation, since the excited state is identical with the one of the giant γ-resonance in the photo-nuclear reaction. Therefore, as $kR<1$ (R is nuclear radius) at forward scattering, one can approximate Eq. (4) as

$$\langle\Psi|\sum^{A}kz_i\tau_{iz}|\Phi\rangle=k\langle\Psi|(\sum^{Z}-\sum^{N})z_i|\Phi\rangle.$$

Since the electric dipole moment is developed only as a result of the displacement

of the center of electric charge from that of mass, i. e. since, for the dipole absorption to occur, the nuclear center of mass must be displaced from the center of charge, one must transform as follows,

$$\sum^{Z} - \sum^{N} = 2\sum^{Z} - \sum^{A},$$

where $A=N+Z$.

The term $\sum^{A}$ does not contribute the dipole moment. The term

$$|\langle \Psi | \sum^{Z} z_i | \Phi \rangle|^2$$

relates to the cross section of the dipole γ-absorption. The factor 2 in $2\sum^{Z}$ corresponds to the effective charge in the photo-nuclear reactions. Then, Eq. (2) is reduced to the form

$$\frac{d\sigma}{d\Omega}(\theta) = 4\frac{k_f}{k_i}\{|\bar{\alpha}|^2+|\bar{\gamma}|^2+\lambda(|\bar{\gamma}|^2+|\bar{\beta}|^2+|\bar{\varepsilon}|^2)\}4k^2|\langle \Psi | \sum^{Z} z_i|\Phi\rangle|^2, \qquad (6)$$

where λ is defined by Eq. (5), i. e. the ratio of the nuclear reduced matrix element of spin flip terms to the non-spin flip ones and does not depend on $\boldsymbol{k}_f-\boldsymbol{k}_i$ for the present case where the multipole order of transitions does not mix.

On the other hand, the cross section for excitation of nucleus by energy E_{exc} through the absorption of electric dipole γ-ray, $\sigma_{ph}(E_{exc})$, is connected with the nuclear reduced matrix element as follows,[5]

$$|\langle \Psi | \sum^{Z} z_i|\Phi\rangle|^2 = \frac{1}{4\pi^2 E_{exc}}\rho^{-1}(E_{exc})\left(\frac{e^2}{\hbar c}\right)^{-1}\sigma_{ph}(E_{exc}), \qquad (7)$$

where $\rho(E_{exc})$ is the number of states per unit energy interval, and E_{exc} is the energy of the state Ψ above the ground state.

From Eqs. (6) and (7), one obtains, for the energy distribution,

$$\frac{d^2\sigma}{d\Omega\, dE}(\theta) = \frac{4}{\pi^2 E_{exc}}\frac{k_f}{k_i}\{|\bar{\alpha}|^2+|\bar{\gamma}|^2+\lambda(|\bar{\gamma}|^2+|\bar{\beta}|^2+|\bar{\varepsilon}|^2)\}k^2\left(\frac{e^2}{\hbar c}\right)^{-1}\sigma_{ph}(E_{exc}), \qquad (8)$$

since

$$\frac{d\sigma}{d\Omega}\rho(E_{exc}) = \frac{d^2\sigma}{d\Omega\, dE}.$$

Now, we can investigate the energy distributions of the inelastically scattered protons from the levels considered without any knowledge on the wave functions for the ground and the excited states of target nuclei. And we need not consider mechanisms for the absorption of the dipole γ-ray.

§ 3. Differential cross section

Integration of (8) by the energy E_{exc} gives the differential cross section of

the proton inelastically scattered from the level in the dipole giant γ-resonance region,

$$\frac{d\sigma}{d\Omega}(\theta)=\frac{4}{\pi^2}\left(\frac{e^2}{\hbar c}\right)^{-1}\{|\bar{\alpha}|^2+|\bar{\gamma}|^2+\lambda(|\bar{\gamma}|^2+|\bar{\beta}|^2+|\bar{\varepsilon}|^2)\}\int\frac{k_f}{k_i}k^2\,\frac{\sigma_{ph}(E_{exc})}{E_{exc}}\,dE_{exc}, \tag{9}$$

here it is assumed that the dependences of $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$, $\bar{\varepsilon}$ and λ on the energy are very weak.

By using the relations

$$\frac{k_f}{k_i}k^2\frac{1}{E_{exc}}=\frac{2m}{\hbar^2}\,\frac{E}{E_{exc}}\sqrt{1-\frac{E_{exc}}{E}}\left(2-\frac{E_{exc}}{E}-2\sqrt{1-\frac{E_{exc}}{E}}\cos\theta\right)$$

and

$$\frac{E_{exc}}{E}\ll 1,$$

where E is the incident energy of the proton and θ is the angle between $\boldsymbol{k}_i$ and $\boldsymbol{k}_f$, and m is the mass of the proton, one will obtain

$$\frac{d\sigma}{d\Omega}(\theta)=\frac{4}{\pi^2}\left(\frac{e^2}{\hbar c}\right)^{-1}\frac{2m}{\hbar^2}\{|\bar{\alpha}|^2+|\bar{\gamma}|^2+\lambda(|\bar{\gamma}|^2+|\bar{\beta}|^2+|\bar{\varepsilon}|^2)\}$$
$$\times\left\{4\left(\frac{E}{E_H}-1\right)\sin^2\frac{\theta}{2}+\frac{\bar{E}}{4E}\right\}\int\sigma_{ph}(E_{exc})\,dE_{exc} \tag{10}$$

when one introduces the harmonic energy E_H and the mean energy $\bar{E}$ defined by Levinger-Bethe :[5]

$$E_H=\int\sigma_{ph}(E_{exc})\,dE_{exc}\Big/\int\frac{\sigma_{ph}(E_{exc})}{E_{exc}}\,dE_{exc},$$
$$\bar{E}=\int E_{exc}\,\sigma_{ph}(E_{exc})\,dE_{exc}\Big/\int\sigma_{ph}(E_{exc})\,dE_{exc}. \tag{11}$$

Within the conditions of $E_{exc}\ll E$ and small angle scattering, the differential cross section is approximated by

$$\frac{d\sigma}{d\Omega}(\theta)=\frac{4}{\pi^2}\left(\frac{e^2}{\hbar c}\right)^{-1}\left(\frac{2m}{\hbar^2}\right)\{|\bar{\alpha}|^2+|\bar{\gamma}|^2+\lambda(|\bar{\gamma}|^2+|\bar{\beta}|^2+|\bar{\varepsilon}|^2)\}\frac{E}{E_H}\theta^2\int\sigma_{ph}(E_{exc})\,dE_{exc}. \tag{12}$$

It is expected that the measurements of the differential cross section of these processes will give new information on the integrated cross section of the electric dipole γ-ray absorption.

§ 4. Results for the energy distributions

To calculate the energy distributions of the protons inelastically scattered from

the levels in the dipole giant γ-resonance regions, we need not have any special assumptions for the ground and excited states of target nuclei since the wave functions of the states are taken into account through the cross sections of the dipole γ-absorption. The ratios of spin flip to non spin flip nuclear reduced matrix elements, λ, are determined by the observed polarizations since the polarizations of the protons inelastically scattered from the levels in the dipole giant γ-resonance regions are very sensitive to the values of λ.[6] Therefore, Eq. (10) connects the experimental data of $(p\text{-}p')$ scattering with the experimental data of photo-nuclear reaction and the ratio of spin flip to non spin flip nuclear matrix element. If the quantities $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$, $\bar{\delta}$ and $\bar{\varepsilon}$ are parameterized by the analysis of the nucleon-nucleon scattering experiment,[7] the magnitude of the energy distribution of the proton inelastically scattered from the level considered is a function of the cross section of the dipole γ-absorption $\sigma_{ph}(E_{exc})$, and the value of λ. When we substitute the

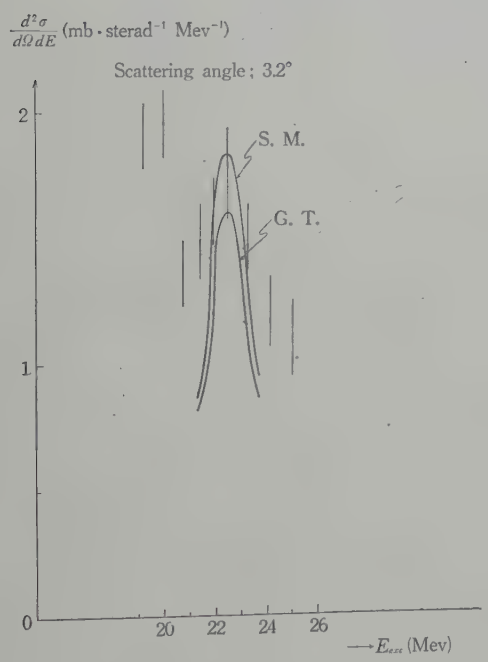

Fig. 1. Energy distribution of 182 Mev proton inelastically scattered from the level in the dipole giant γ-resonance region of C^{12}. In the calculations of $\bar{\alpha}$, $\bar{\beta}$, etc., in the text, we use the two-body scattering phase shifts derived at 180 Mev from the Gammel-Thaler (G. T.)[8] and the Signell-Marshak (S. M.)[9] potentials. For the estimation of $\sigma_{ph}(E_{exc})$, we use the sum of $\sigma(\gamma, n)$ measured by Barber et al.[14] and $\sigma(\gamma, p)$ measured by Cohen et al.[17]

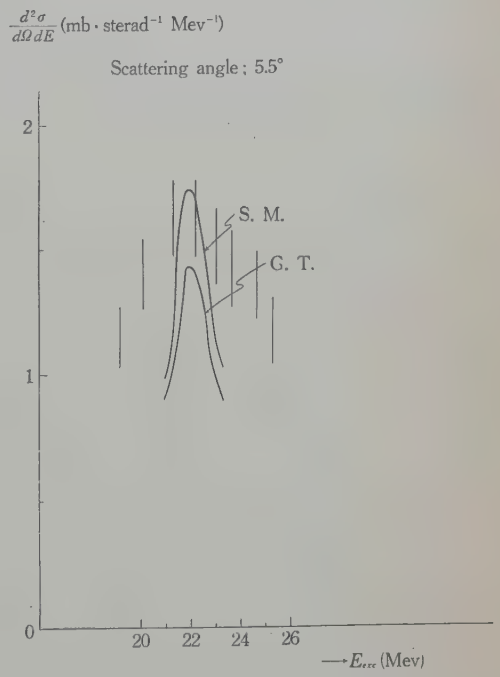

Fig. 2. Energy distribution of 177 Mev proton inelastically scattered from the level in the dipole giant γ-resonance region of O^{16}. For $\sigma_{ph}(E_{exc})$, we use the sum of $\sigma(\gamma, n)$ measured by Katz et al.[21] and $\sigma(\gamma, p)$ measured by Stephens et al.[22]

value of λ determined by the polarization experiments and the observed data of $\sigma_{ph}(E_{exc})$ into Eq. (10), we obtain fairly good agreement of the observed energy spectra with the calculated values for the proton.

The cross sections of the dipole γ-absorption and the polarizations of the inelastically scattered protons from the excited levels of C^{12} and O^{16} are measured in detail. Then, we calculate the energy spectra and differential cross sections of the protons inelastically scattered from the levels in the dipole giant γ-resonance regions of C^{12} and O^{16}. The results calculated through Eq. (10) are compared with the experimental data in Figs. 1 and 2 for C^{12} and O^{16}, respectively. It is important to note that the results agree with the experimental data in their absolute values, since the nuclear interactions as well as the Coulomb interactions between the proton and the nucleons in the nucleus are taken into account in the formula. To calculate $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$, $\bar{\delta}$ and $\bar{\varepsilon}$, we have used the two-body scattering phase shifts of energy 180 Mev derived from the Gammel-Thaler[8] and Signell-Marshak[9] potentials. And we use $\lambda=0.1$ for C^{12}, since one can reproduce the experimental polarization of the proton inelastically scattered from the level by using a small value of λ. On the other hand, we use $\lambda=0.8$ for O^{16}, since one can reproduce the observed polarization of the proton inelastically scattered from the level by using almost unity for λ. The calculated polarizations of the protons for various values of λ

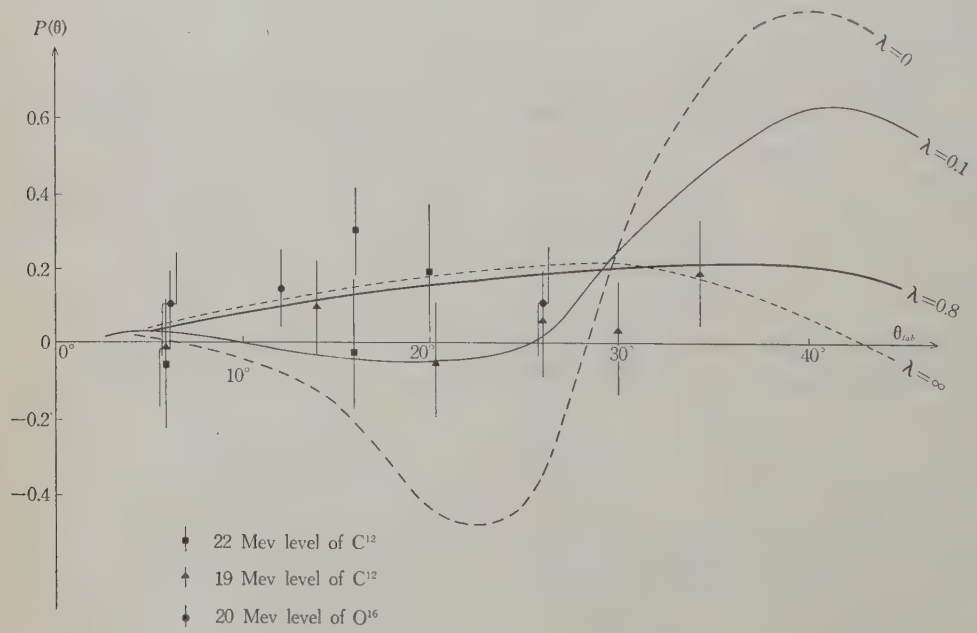

Fig. 3. Polarizations of the protons inelastically scattered from the levels in the dipole giant γ-resonance regions of C^{12} and O^{16}. Experimental points are measured by Hillman et al.[23] The curves are calculated by the use of the two-body scattering phase shifts of 180 Mev derived from the Gammel-Thaler potential.

are compared with experimental data in Fig. 3. Fig. 3 shows that the dependence of the polarizations on the small values of λ is very sensitive.

As for light nuclei the threshold energies of $\sigma(\gamma, 2p)$ and $\sigma(\gamma, np)$, etc., are, generally, very close to, or slightly higher than, the energy E_{exc} of the peak for the sum of $\sigma(\gamma, n)$ and $\sigma(\gamma, p)$ cross section.[10] For C^{12} and O^{16} investigated in this paper, the threshold energies of $\sigma(\gamma, np)$, $\sigma(\gamma, 2p)$ and $\sigma(\gamma, 2n)$ are always higher than the energy E_{exc} of the cross section peak of the dipole γ-absorption. For the sake of the clearness, we tabulate the threshold energies of the photonuclear reactions for C^{12} and O^{16} in Table I.[11]

Table I. Threshold energies and the energies of the cross section peak
The energies are in Mev.

	(γ, n)	(γ, p)	$(\gamma, 2n)$	(γ, np)	$(\gamma, 2p)$	The energy E_{exc} of the cross section peak for the sum of (γ, n) and (γ, p)
C^{12}	18.7	16.0	32	27.5	27.2	22.5
O^{16}	15.6	12.1	28.9	23.0	22.3	22

Since the contributions of $\sigma(\gamma, 2p)$, etc., to $\sigma_{ph}(E_{exc})$ are very small even for higher excitation energies, one can use the sum of $\sigma(\gamma, p)$ and $\sigma(\gamma, n)$ for $\sigma_{ph}(E_{exc})$. If one takes into account the contributions of $\sigma(\gamma, 2p)$, etc., to $\sigma_{ph}(E_{exc})$, one obtains a slightly broadened shape for the calculated result of the energy distribution of the inelastically scattered proton. Furthermore it is expected that when one takes into account the effects of the levels in the excitation energy 19～24 Mev region which are not excited by the dipole γ-ray absorption one will obtain the broadened shape of the characteristic peak of the energy spectra of $(p\text{-}p')$ scattering rather than the present results. However the energy E_{exc} which gives the characteristic peak in the energy distribution of $(p\text{-}p')$ scattering is not altered by these effects.

The measurements of the energy spectra of the inelastically scattered proton provide a procedure to determine the existence of excited levels of the nucleus and to fix the excitation energy. In the present case, since the energy E_{exc}^{max} which gives the characteristic peak of the dipole giant γ-resonance in the energy distribution of $(p\text{-}p')$ scattering must agree with the energy E_{exc} of the cross section peak for $\sigma_{ph}(E_{exc})$ in the dipole giant γ-resonance, it is expected that the observation of the energy distribution of the inelastically scattered proton provides an effective tool to fix the excitation energy E_{exc} of the cross section peak for $\sigma_{ph}(E_{exc})$. In order to evidence the considerations above mentioned, we tabulate the energies E_{peak} of the cross section peaks and the magnitudes of the peaks, σ_{max}'s of $\sigma(\gamma, n)$ and $\sigma(\gamma, p)$ for C^{12} in Tables II and III, respectively.[12)~17)]

When one uses the most recent experimental data by Barber et al.[14] and Cohen et al.[17] for $C^{12}(\gamma, n)$ and $C^{12}(\gamma, p)$ cross sections, respectively, the energy E_{exc} of

Table II. $C^{12}(\gamma, n)\ C^{11}$

Reference	E_{peak} (Mev)	σ_{max} (mb)
Montalbetti et al. (1953)	21.4	14.0
Nathans et al. (1954)	22.0	8.6
Barber et al. (1955)	22.5	8.3

Table III. $C^{12}(\gamma, p)B^{11}$

Reference	E_{peak} (Mev)	σ_{max} (mb)
Halpern et al. (1952)	21.5	18
Haslam et al. (1953)	21.2	16~33
Cohen et al. (1956)	22~23	16

the cross section peak for $\sigma_{ph}(E_{exc})=\sigma(\gamma, n)+\sigma(\gamma, p)$ agrees with E_{exc}^{max} which gives the characteristic peak of the dipole giant γ-resonance in the energy distribution of the proton inelastically scattered off C^{12}. And also when one uses the data of Barber et al. and Cohen et al. for $\sigma_{ph}(E_{exc})$, one can reproduce the magnitude of the characteristic peak in the energy distribution.

§ 5. Results for the angular distributions

For the purpose of calculating the angular distribution of the proton inelastically scattered from the level in the dipole giant γ-resonance region, one needs to estimate the integrated cross section $\int\sigma_{ph}(E_{exc})dE_{exc}$, the harmonic energy E_H and the mean energy $\bar{E}$ of the dipole γ-absorption for the target nucleus. Since the plot of $\sigma_{ph}(E_{exc})$ versus E_{exc} is a sharply peaked curve, the integrated cross section may be obtained with sufficient accuracy by integrating the experimental value up to an energy 25 Mev, and so on, i. e. by integrating the experimental one within the resonance region, and also E_H and $\bar{E}$ must be close together.[10]

Since the cross section of the dipole γ-absorption falls off rapidly enough at energies above the giant resonance, any high-threshold multipole processes are suppressed. Since the E_{peak} of the giant γ-resonance is about 20 Mev for C^{12} or O^{16}, the processes of $(\gamma, 2p)$, (γ, np), etc., set in off resonance. The contributions of $(\gamma, 2p)$, (γ, np), etc., to $\int\sigma_{ph}(E_{exc})dE_{exc}$ can be quite negligible in C^{12} and O^{16} since the absorption cross section falls off sufficiently 10 to 20 Mev above the E_{peak} of the resonance. In order to evidence the justification for the neglect of the contributions of $\sigma(\gamma, 2p)$, $\sigma(\gamma, np)$, etc., to $\int\sigma_{ph}(E_{exc})dE_{exc}$, we tabulate in Table IV the relative integrated cross sections,[11]

$$\int_{0\ \mathrm{Mev}}^{100\ \mathrm{Mev}}\sigma(\gamma, 2n)dE_{exc}\Big/\int_{0\ \mathrm{Mev}}^{100\ \mathrm{Mev}}\sigma(\gamma, n)dE_{exc}.$$

Table IV. Relative integrated cross sections

	(γ, n)	Position of the peak for (γ, n)	$(\gamma, 2n)$	Position of the peak for $(\gamma, 2n)$
C^{12}	1	23 Mev	0.003	42 Mev
O^{16}	1	22 Mev	0.002	40 Mev

In the calculation for C^{12}, we use the experimentally integrated cross section $\int\sigma_{ph}(E_{exc})\,dE_{exc}$ in order to be free from the ambiguity contained in the dipole sum rules, and $E_R \sim \bar{E}=20$ Mev. And also on the basis of the reason above mentioned, we use the sum of $\int\sigma(\gamma, p)dE_{exc}$ and $\int\sigma(\gamma, n)dE_{exc}$ for $\int\sigma_{ph}(E_{exc})dE_{exc}$; here $\int\sigma(\gamma, n)dE_{exc}$ and $\int\sigma(\gamma, p)dE_{exc}$ are measured by Strauch[18] and Halpern et al.,[15] respectively. The results for C^{12} are shown in Fig. 4 being compared with the observed value by Tyrén and Maris. Since the experimental value of them has errors of 50%, the calculated results lie within the limits of the errors. The sharp peak of the results are owing to the strong peaked matrix elements of the Coulomb interactions at forward. It seems that the agreements of the calculated results with observed value are well.

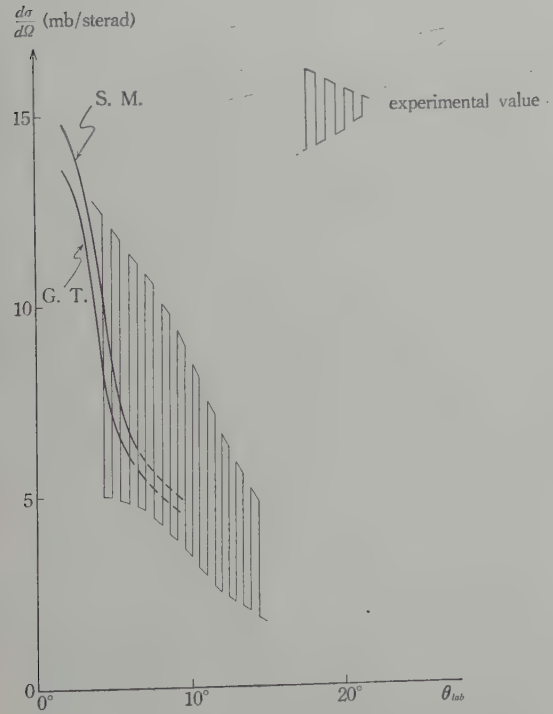

Fig. 4. Differential cross section of the proton inelastically scattered from the level in the dipole giant γ-resonance region of C^{12}.

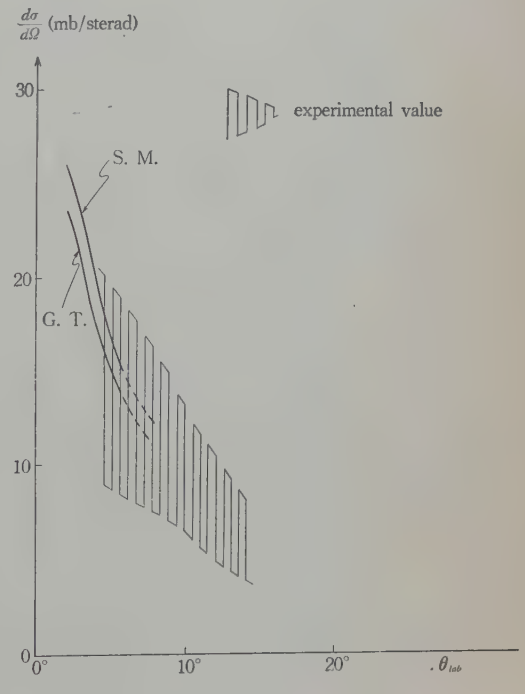

Fig. 5. Differential cross section of the proton inelastically scattered from the level in the dipole giant γ-resonance region of O^{16}.

The results for O^{16} are shown in Fig. 5. Here we use $E_{\pi}\sim\bar{E}=20$ Mev and use the integrated cross section estimated by Morinaga[19] which is the sum of $\int\sigma(\gamma, n)dE_{exc}$ and $\int\sigma(\gamma, p)dE_{exc}$.

If one uses the dipole sum rules of Levinger-Bethe[5] and Goldberger et al.[20] for $\int\sigma_{ph}(E_{exc})dE_{exc}$, one obtains the results larger than the observed ones of C^{12} and O^{16} by a factor of 3 to 4. This discrepancy is not interpreted by the errors contained in the observed $(p\text{-}p')$ scattering data. The discrepancy indicates a failure of the dipole sum rules of Levinger-Bethe and Goldberger et al. for C^{12} and O^{16}. This is consistent with the fact that the dipole sum rules of Levinger-Bethe and Goldberger et al. are not justified for α-particle nuclei since the strong correlation between nucleons inside the α-particle nuclei makes the separation of the nuclear center of mass from the center of charge very difficult and suppresses the dipole photo effect.[10]

On the basis of the quantitative agreement, it is expected that the measurements of the protons inelastically scattered from the levels considered provide a procedure to estimate the integrated cross section of the dipole γ-absorption.

For the angular distribution, the larger the value of λ, the broader the shape of the calculated results. However, the λ-dependence of the shape of the results is not so sensitive as the polarization of the proton considered is.

The effect of the distortion of the incoming and scattered waves will tend to broaden the angular distribution. However, in the large scattering angles where the distortion has a large effect, the above formula is not applied as $kR>1$. As for the estimation of the integrated cross section of the dipole γ-absorption, one needs not have the agreement of the results with the observed ones in the region of over all scattering angles, i. e. the agreement in the shapes of the angular distributions, it may be sufficient to obtain the agreement of them in their absolute values at forward directions.

§ 6. Conclusion

If one considers the proton inelastically scattered from low-lying levels of a nucleus it can be expected that the contributions of the spin flip terms of nucleon in the nucleus are small since the struck nucleon in the final state is still bound in the nucleus. In the present case, however, one must take into account the spin flip of the target nucleon as the final state of the struck nucleon is in the continuum state. The spin flip terms play important roles for the cross section and the polarization of the inelastically scattered proton.

The fact that the formulas for the cross section derived in the framework of the impulse approximation without any special assumptions reproduce the experimental data for C^{12} and O^{16} very well gives a powerful support for the justification of the impulse approximation as a method of investigating the inelastically scattered proton from the level in the dipole giant γ-resonance region for all light nuclei. Therefore one can expect that the impulse approximation is a useful method even

for the nucleon-nucleus interactions in which the final state of the struck nucleon is in the continuum state when the energies of the incident nucleons are sufficiently high; for instance, the energy is 180 Mev at the present case.

The quantitative agreement of the calculated results with the observed ones are helpful to discuss the quantitative relations among the $(p\text{-}p')$ scattering on light nuclei, the dipole giant γ-resonance of photo-reactions and the ratio of spin flip to non spin flip nuclear matrix element, where the non spin flip matrix element is proportional to the cross section of the dipole γ-absorption.

Therefore, if $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$, $\bar{\delta}$ and $\bar{\varepsilon}$ are reasonably parameterized by the two-body scattering, it is possible to determine the value of λ through the experimental cross section of the inelastically scattered proton from the level considered when $\sigma_{ph}(E_{exc})$ is measured. Studying the value of λ is to investigate the contributions of the spin flip and the non spin flip terms of the nucleons in the nucleus, i. e. the nuclear reduced matrix element. The measurements of the energy spectra for the high energy $(p\text{-}p')$ scattering give a way to investigate the nuclear matrix element when $\sigma_{ph}(E_{exc})$ is known experimentally.

The observation of the energy spectra of $(p\text{-}p')$ scattering gives a procedure to fix the energy of the cross section peak of the dipole γ-absorption. Furthermore, when the ratios of spin flip to non spin flip nuclear reduced matrix elements are determined by the observed polarizations of the protons, the measurements of $(p\text{-}p')$ scattering give a powerful tool to estimate the magnitude of the peak of $\sigma_{ph}(E_{exc})$ and the value of $\int \sigma_{ph}(E_{exc})dE_{exc}$. For example, the measurements of the protons inelastically scattered from the levels considered may remove the difficulty which any given measurement for the cross section of the dipole γ-absorption measures partial cross section for particular nuclear reactions, rather than the cross section of the γ-absorption.

We can develop the similar formulas for the nucleons inelastically scattered from the excited levels by means of the $M1$, $E2$, etc., transition matrix elements corresponding the transitions of the excited to the ground states.

The most serious error is caused probably by neglecting the elastic and inelastic scattering of the incident and scattered proton before and after the excitation. Though the problem of this effect is left for later consideration, it does not seriously alter the above results since it can be expected that the effect would be very small for the high energy proton and in the region of $kR<1$ in which we are interested.

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Appendix

In this appendix we summarize the selection rules for the proton inelastically

scattered from the nucleus in order to complete this paper.

Within the framework of the impulse approximation, the scattering matrix for the nucleon involving a nuclear transition from a state with J_0 and T_0 to a state with J and T is given by

$$T(\boldsymbol{k})=\langle JT|\sum_{i=1}^{A}\sum_{\nu'=-1}^{1}(-1)^{\nu'}\tau_{-\nu'}^{1}\tau_{\nu'}^{1}e^{-i\boldsymbol{k}\boldsymbol{r}_i}(M_0(\theta)_i+\sum_{\nu=-1}^{1}(-1)^{\nu}\sigma_{\nu}^{1}M_{\nu}^{1}(\theta)_i)|J_0T_0\rangle. \tag{A·1}$$

Here σ_{ν}^{1} and $\tau_{\nu'}^{1}$ are tensor operators of rank 1 in the spin and isotopic spin of target nucleons, respectively. $\tau_{-\nu'}^{1}$ is a tensor operator of rank 1 in the isotopic spin of the incident nucleon. The quantities M_0 and M_{ν}^{1} are given, in decomposing $M(\theta)_i$ of Eq. (1) in the text to the spin operators of the incident and target nucleons, by

$$M_0=\alpha+\gamma\sigma_n,\ M_0^1=\delta\sigma_q,\ M_1^1=-\frac{1}{\sqrt{2}}(\gamma+\beta\sigma_n-i\varepsilon\sigma_p),\ M_{-1}^{1}=\frac{1}{\sqrt{2}}(\gamma+\beta\sigma_n+i\varepsilon\sigma_p), \tag{A·2}$$

where σ_q, σ_n, σ_p are the spin operators of the incident nucleon in the $\boldsymbol{q}$, $\boldsymbol{n}$, $\boldsymbol{p}$-right-handed coordinate system.

Now, expanding the target nucleon operators $e^{-i\boldsymbol{k}\boldsymbol{r}_i}$, $e^{-i\boldsymbol{k}\boldsymbol{r}_i}\sigma_{\nu}^{1}$, we define the reduced nuclear matrix elements:

$$F_l=\langle JT\|\sum_{i}^{A}\sum_{\nu'=-1}^{1}(-1)^{\nu'}\sqrt{4\pi(2l+1)}\,(-i)^{l}j_l(kr_i)\,Y_l\tau_{-\nu'}^{1}\tau_{\nu'}^{1}M_0(\theta)_i\|J_0T_0\rangle, \tag{A·3}$$

and

$$R_{kl}=\langle JT\|\sum_{i}^{A}\sum_{\nu'=-1}^{1}(-1)^{\nu'}\sqrt{4\pi}(2l+1)\,(-i)^{l}j_l(kr_i)\,T_k(ls)\tau_{-\nu'}^{1}\tau_{\nu'}^{1}M_{\nu}^{1}(\theta)_i\|J_0T_0\rangle, \tag{A·3′}$$

since

$$\langle JT|\sum_{i}^{A}\sum_{\nu'=-1}^{1}(-1)^{\nu'}\tau_{-\nu'}^{1}\tau_{\nu'}^{1}M_{\nu}^{1}(\theta)_i e^{-i\boldsymbol{k}\boldsymbol{r}_i}\sigma_{\nu}^{1}|J_0T_0\rangle$$
$$=\sum_{lk}\frac{1}{\sqrt{2J+1}}(1l\nu\,0|k\nu)\,(kJ_0\nu M_0|JM)\,R_{kl},$$

where $T_k(ls)$ is a tensor operator of rank k composed of operators in the $\boldsymbol{l}\cdot\boldsymbol{s}$ space. If we restrict ourselves to consider the excited state levels in the dipole giant γ-resonance region, the isotopic spin flip $\Delta T=1$. The isotopic spin flip terms generally include the charge exchange collisions. However, if we disregard the charge exchanges between the incident and target nucleons, we can reduce the reduced nuclear matrix elements by putting $\sum_{\nu'=-1}^{1}(-1)^{\nu'}\tau_{-\nu'}^{1}\tau_{\nu'}^{1}$ to $\tau_{1z}\tau_{tz}$, to

$$F_l=M_0(\theta)\,\langle JT\|\sum_{i}\sqrt{4\pi(2l+1)}\,(-i)^{l}(\sum^{Z}-\sum^{N})\,j_l(kr_i)\,Y_l\|J_0T_0\rangle,$$

$$R_{lk}=M_v^1(\theta)\,\langle JT\|\sum_i \sqrt{4\pi(2l+1)}\,(-i)^l\,(\sum^Z-\sum^N)\,j_l(kr_i)\,T_k(ls)\|J_0\,T_0\rangle.$$

Taking the trace of Eq. (A·2) for the incident proton, we obtain $\mathrm{Tr}[T(\boldsymbol{k})T(\boldsymbol{k})^+]$ and $\mathrm{Tr}[T(\boldsymbol{k})\boldsymbol{\sigma}_{1n}T(\boldsymbol{k})^+]$ which correspond to the cross section and the polarized cross section of the proton, respectively, as follows,

$$\mathrm{Tr}[T(\boldsymbol{k})T(\boldsymbol{k})^+]=\sum_l \frac{1}{2l+1}\left\{(|\bar{\alpha}|^2+|\bar{\gamma}|^2)\,|F_l|^2+\frac{1}{2}\sum_k(|\bar{\beta}|^2+|\bar{\gamma}|^2+|\bar{\varepsilon}|^2)\,|R_{kl}|^2\right\}$$

$$+\sum_{ll'k}{}' \frac{1}{2k+1}\,(1l00|k0)\,(1l'00|k0)\left\{|\bar{\delta}|^2-\frac{1}{2}\,(|\bar{\beta}|^2+|\bar{\gamma}|^2+|\bar{\varepsilon}|^2)\right\}R_{kl}R_{kl'}^*, \tag{A·4}$$

$$\mathrm{Tr}[T(\boldsymbol{k})\boldsymbol{\sigma}_{1n}T(\boldsymbol{k})^+]=\sum_l{}' \frac{1}{2l+1}\{2\,\mathrm{Re}\,\alpha\gamma^*|F_l|^2+\sum_k{}'\,\mathrm{Re}\,\beta\gamma^*|R_{kl}|^2\}$$

$$-\sum_{ll'k}{}' \frac{1}{2k+1}\,(1l00|k0)\,(1l'00|k0)\,\beta\gamma^*R_{kl}R_{kl'}^*, \tag{A·4′}$$

where $T(\boldsymbol{k})$ is defined by Eq. (A·1).

The selection rules for l, l' and k are, for the non spin flip part,

$$|J-J_0|<l<J+J_0$$

and for the spin flip part,

$$|J-J_0|<k<J+J_0.$$

In both cases the parity change $\Delta\pi=(-1)^l$. Furthermore, $l+k+1$ must be even and $l'=l$, $l\pm2$, for the last term in Eqs. (A·4) and (A·4′).

For comparatively small angle scattering, the lowest allowed value of l will predominate, owing to the $\boldsymbol{k}$-dependence of the radial integrals in the reduced nuclear matrix elements. For the case where only one value of l contributes to it, the ratios of the spin flip to the non spin flip reduced matrix elements do not depend on the momentum transfer.

For the target nucleus of spin and isotopic spin 0 at its ground state such as C^{12} or O^{16}, the selection rules are $l=J$ for the non spin slip terms and $k=J$ for the spin flip terms. And in the parity change $\Delta\pi=(-1)^{J'}$, both l and k equal J for the spin flip terms, hence the last terms in Eqs. (A·4) and (A·4′) vanish.

If there is no isotopic spin change for the target nucleus, one has to put $\boldsymbol{\tau}_1\boldsymbol{\tau}_i=1$ in Eq. (A·1).

References

1) H. Tyrén and Th. A. Maris, Nuclear Physics **3** (1957), 52; **4** (1957), 277; 637; **6** (1958), 82, 446; **7** (1958), 24, 281.
2) Th. A. Maris and H. Tyrén, Nuclear Physics **3** (1957), 35; **4** (1958), 662.
3) M. Kawai and T. Terasawa, Prog. Theor. Phys. **22** (1959), 513.
4) L. Wolfenstein and J. Ashkin, Phys. Rev. **85** (1952), 947.
R. H. Dalitz, Proc. Phys. Soc. **A65** (1952), 172.
L. Wolfenstein, *Annual Review of Nuclear Science* (Annual Review, Inc., Stanford) Vol. **6** (1956), p. 43.
5) J. S. Levinger and H. A. Bethe, Phys. Rev. **78** (1950), 115.
J. S. Levinger, *Annual Review of Nuclear Science* (Annual Review, Inc., Stanford) Vol. **4** (1954), p. 13.
6) A. K. Kerman, H. McManus and R. M. Thaler, Annals of Physics **8** (1959), 551.
7) S. C. Wright, Phys. Rev. **99** (1955), 996.
H. P. Stapp, T. J. Ypsilantis and N. Metropolis, Phys. Rev. **105** (1957), 302.
8) J. L. Gammel and R. M. Thaler, Phys. Rev. **107** (1957), 291, 1337.
9) P. S. Signell and R. E. Marshak, Phys. Rev. **109** (1958), 1229.
10) K. Strauch, *Annual Review of Nuclear Science* (Annual Review, Inc., Stanford) Vol. **2** (1953), p. 105.
B. B. Kinsey, *Handbuch der Physik* (Springer Verlag, Berlin), Vol. **40** (1957), p. 202.
G. R. Bishop and R. Wilson, *Handbuch der Physik* (Springer Verlag, Berlin), Vol. **42** (1957), p. 309.
11) J. O'Connell, P. Dyal and J. Goldenberg, Phys. Rev. **116** (1959), 173.
12) R. Montalbetti, L. Katz and J. Goldenberg, Phys. Rev. **91** (1953), 659.
13) R. Nathans and J. Halpern, Phys. Rev. **93** (1954), 437.
14) W. C. Barber, W. D. George and D. D. Reagan, Phys. Rev. **98** (1955), 73.
15) J. Halpern and A. K. Mann, Phys. Rev. **83** (1951), 370.
J. Halpern, A. K. Mann and M. Rothman, Phys. Rev. **87** (1952), 164.
16) R. N. H. Haslam, R. J. Horrley, H. E. Johens and L. B. Robinson, Canad. J. Phys. **31** (1953), 636.
17) L. Cohen, A. K. Mann, B. J. Patton, K. Reibel, W. E. Stephens and E. J. Winhold, Phys. Rev. **104** (1956), 108.
18) K. Strauch, Phys. Rev. **81** (1951), 981.
L. Marshall, Phys. Rev. **83** (1951), 345.
19) H. Morinaga, Phys. Rev. **97** (1955), 1185.
20) M. Gell-Mann, M. L. Goldberger and G. W. E. Thirring, Phys. Rev. **95** (1954), 1612.
21) L. Katz, P. N. H. Haslam, R. J. Horsley, A. G. W. Cameron and R. Montalbetti, Phys. Rev. **95** (1954), 464.
22) W. E. Stephens, A. K. Mann, B. J. Patton and E. J. Winhold, Phys. Rev. **98** (1955), 839.
23) P. Hillman, A. Johansson and H. Tyrén, Nuclear Physics **4** (1957), 648.

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Composite Model and the Nature of Vector Interactions in Strangeness-Violating Decays

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In this paper we discuss decay interations based on Sakata's composite model in which neutron, n, proton, p, and Λ-particle are supposed to be "elementary". Symmetry with respect to n, p, and Λ is also assumed. Then it is shown that (πK) currents are necessarily added to the usual strangeness violating ($|\Delta S|=1$) baryon vector currents in the conventional Yukawa type interation. This is analogous to the situation that $(\pi\pi)$ and (KK) currents necessarily appear in the $|\Delta S|=0$ leptonic decay interaction in the Sakata model. By taking the (πK) currents into account, the transition probabilities for K_{e3}, $K_{\mu 3}$, and beta- and muon-decays of Λ are shown to be consistently described with the same coupling constant as that obtained from $K_{\mu 2}{}^{+}$ decay. Thus the coupling constant for strangeness-violating leptonic decays seems to be smaller than that for strangeness conserving ones. For non-leptonic decays, the Hamiltonian which guarantees the $|\Delta I|=1/2$ selection rule is easily found by virtue of the additional neutral terms with the universal coupling constant. The lifetimes and branching ratios for non-leptonic decays of Λ and K are in good agreement with experiment in terms of the normal beta-decay coupling constant. The situation is a little complicated for $K_{\pi 3}$. Admixture of $|\Delta I|=3/2$ transitions are also briefly discussed. Some comments are made on the observability of $\pi^{0\prime}$, a neutral meson having an isotopic spin 0.

§ 1. Introduction

In recent experiments the universal V-A Fermi interaction[1] is remarkably successful in explaining, at least, the strangeness-conserving leptonic processes such as beta-decay, μ-e decay, and probably μ^{-}-capture by a proton. For the non-leptonic decays of Λ, Σ, and probably Ξ, the order of magnitude of decay rates seems to be in agreement with the V-A Fermi interactions with the beta-decay coupling constant,[2] while the branching ratios and asymmetry parameters are not well explained, especially for Σ. On the other hand, the strangeness-violating leptonic decays require a smaller coupling constant by factor 5 or so.[3),4] In the meantime, it was pointed out that the vector coupling constant of μ-e decay is equal to that of beta-decay within one or two percent accuracy of recent experiments. Stimulated by this excellent agreement, Feynman and Gell-Mann[1] have introduced the additional $(\pi\pi)$ current into the vector interaction in order to explain that the strength of the coupling constant of vector interaction is not affected by renormalization. This situation is analogous to the divergenceless electro-magnetic current, but this $(\pi\pi)$ current has not yet been confirmed by experiment.[5] In this connection we must also notice that this $(\pi\pi)$ current follows[6] quite naturally from the Sakata model.[7]

For the strangeness-violating decays, Sugawara[8] proposed to add only the $(K_1^0\,\pi^\pm)$ current to the vector part from the phenomenological point of view, and thus he obtained a good result for the decay rate of K_1^0.

In the present paper, we consider the decay processes based on the composite model[7] and discuss, first of all, how the boson currents have to be added to the vector part of the Fermi interactions both for $|\Delta S|=0$[6] and for $|\Delta S|=1$ decay processes. Here we further assume the invariance under the exchange of n and Λ in addition to the usual charge independence[9]. These boson currents are obtained by reinterpreting the Sakata model in terms of the Yukawa type representation in which the pion and K-meson are also considered as elementary particles. For $|\Delta S|=1$ all of the $(K_1^0\,\pi^\pm)$, $(K_2^0\,\pi^\pm)$ and $(K^+\,\pi^0)$ currents appear and the coefficients of these terms are uniquely determined contrary to Sugawara's theory.[8]

Another reliable rule is the $|\Delta I|=1/2$ selection rule for the non-leptonic decays. This can lead to the decay branching ratios of hyperons and K-mesons,[10] but cannot give any quantitative information on the lifetime. This selection rule and the V-A Fermi interaction are, therefore, considered to be somewhat complementary, and their combined use is desirable. We introduce a neutral current for the $|\Delta S|=1$ case, following the full symmetry with respect to n, p, and Λ.[9] Then the sum of this neutral term $(\bar{n}\Lambda)(\bar{n}n)$ and the normal term $(\bar{p}\Lambda)(\bar{n}p)$ together with the boson current terms guarantees the $|\Delta I|=1/2$ selection rule.

In § 3 the strangeness-violating leptonic decays are discussed. Using the effective Hamiltonian deduced in this paper, transition rates $w(\Lambda\to p+e^-+\nu)$, $w(K^\pm\to\pi^0+e^\pm+\nu)$ and $w(K^0_{1,2}\to\pi^\pm+e^\mp+\nu)$ are calculated without any divergence and found to be consistent with experiment in terms of the coupling constant obtained from $w(K^-\to\mu^-+\nu)$.[4] § 4 is devoted to the calculation of the non-leptonic decays of Λ, K_1^0, $K^+\to 2\pi$ and $K^+\to 3\pi$, where we encounter with the same type of divergence and will replace it by a constant. Branching ratios are calculated for the both cases of the usual type $(p\Lambda)(\bar{n}p)$ interaction and the $|\Delta I|=1, 2$ type interaction which is justified by experiments. In addition, the lifetimes of these processes are in good agreement with experiment within factor 3 in terms of the beta-decay coupling constant. The transition rate for $K^-\to 3\pi$ due to the direct $(K^+\,\pi^0)(\pi^+\,\pi^0)$ terms is smaller than the experimental one by factor 20. This term turns out to contribute to the small deviation from $w(K^+\to 2\pi^++\pi^-)/w(K^+\to 2\pi^0+\pi^+)=4$. The main contribution should, therefore, come from the diagrams containing baryon closed loops, which are not calculated here.

Possibility of observing $\pi^{0\prime}$, a neutral meson having an isotopic spin 0, is also discussed in connection with $K^+\to\pi^++\pi^{0\prime}$ process which is allowed by the $|\Delta I|=1/2$ rule.

§ 2. Interaction Hamiltonians

The idea of the Sakata model[7] is to assume n, p, Λ, e, ν and μ to be fundamental particles. Other baryons, pions and K-mesons are considered to be com-

posite particles of n, p and Λ. Furthermore, " the full symmetry " is assumed in strong interactions ; that is, in the limiting case in which the basic particles p, n and Λ have an equal mass, the strong interactions are invariant under the exchange of n and Λ (or p and Λ) in addition to the usual charge independence and the conservation of electric charge and baryonic number.

The weak decay interactions are generated by one of the following fundamental Hamiltonians :

$$H_a=\sqrt{2}\,g_a\left(\bar{p}i\gamma_\mu\frac{1+\gamma_5}{\sqrt{2}}\Lambda\right)\left(\bar{n}i\gamma_\mu\frac{1+\gamma_5}{\sqrt{2}}p\right)+\text{h.c.}, \qquad (2\cdot1a)$$

$$H_b=\sqrt{2}\,g_b\left(\bar{n}i\gamma_\mu\frac{1+\gamma_5}{\sqrt{2}}\Lambda\right)\left(\bar{n}i\gamma_\mu\frac{1+\gamma_5}{\sqrt{2}}n\right)+\text{h. c.}, \qquad (2\cdot1b)$$

$$H_c=\sqrt{2}\,g_c\left(\bar{n}i\gamma_\mu\frac{1+\gamma_5}{\sqrt{2}}\Lambda\right)\left(\bar{\Lambda}i\gamma_\mu\frac{1+\gamma_5}{\sqrt{2}}\Lambda\right)+\text{h. c.}, \qquad (2\cdot1c)$$

$$H_d=\sqrt{2}\,g_d\left(\bar{p}i\gamma_\mu\frac{1+\gamma_5}{\sqrt{2}}n\right)\left(\bar{e}i\gamma_\mu\frac{1+\gamma_5}{\sqrt{2}}\nu\right)+\text{h. c.}, \qquad (2\cdot1d)$$

$$H_e=\sqrt{2}\,g_e\left(\bar{p}i\gamma_\mu\frac{1+\gamma_5}{\sqrt{2}}\Lambda\right)\left(\bar{e}i\gamma_\mu\frac{1+\gamma_5}{\sqrt{2}}\nu\right)+\text{h. c.} \qquad (2\cdot1e)$$

The combination $(\bar{\mu}\nu)$ instead of $(\bar{e}\nu)$ in (2·1d) and (2·1e), and the $(\bar{\mu}\nu)(\bar{e}\nu)$ type interaction, will not be discussed here in detail. In Eq. (2·1) p, Λ, etc., denote the wave functions of the proton, Λ-particle, etc. $\bar{p}$ is defined by $p^*\gamma_4$, where the asterisk means the hermitian conjugate. It is to be noted that the coupling constants g's are real, if the Hamiltonians are $T(=CP)$ invariant, and Λ, n, and p are transformed with the same phase under the time reversal.

The interaction Hamiltonians (2·1a) and (2·1b) consist of $|\Delta I|=1/2$ and $|\Delta I|=3/2$ parts, but if we take their sum assuming a simple relation,

Table I. Two particle configurations of mesons

Strangeness	Iso-spin	Configuration	Particle
−1	1/2	$-(\Lambda\bar{n})$	$\bar{K}^0$
		$(\Lambda\bar{p})$	$\bar{K}^+=K^-$
0	1	$(p\bar{n})$	π^+
		$(p\bar{p}-n\bar{n})/\sqrt{2}$	π^0
		$(n\bar{p})$	π^-
0	0	$(p\bar{p}+n\bar{n}-2\Lambda\bar{\Lambda})/\sqrt{6}$	$\pi^{0\prime}$
1	1/2	$(p\bar{\Lambda})$	K^+
		$(n\bar{\Lambda})$	K^0
0	0	$(p\bar{p}+n\bar{n}+\Lambda\bar{\Lambda})/\sqrt{3}$	$\pi^{0\prime\prime}$

$$g_a=g_b(=g_c), \tag{2·2}$$

the resultant Hamiltonian generates only $|\Delta I|=1/2$ transitions.

In order to reinterpret the Hamiltonians (2·1) in terms of the usual Yukawa type description introducing bosons explicitly, we list the two particle configurations of mesons in Table I.[9]

i) *Lepton-baryon weak interaction*

The fundamental "minimal" electromagnetic interaction Hamiltonian

$$H=ie\bar{p}\gamma_\mu pA_\mu \tag{2·3}$$

in the Sakata model can be rewritten in the following form in the Yukawa type description:

$$H=J_{\mu 3}A_\mu,$$

$$J_{\mu 3}=i\bar{N}\frac{1+\tau_3}{2}\gamma_\mu N+\left[\boldsymbol{\phi}\times\frac{\vec{\partial}_\mu-\overleftarrow{\partial}_\mu}{2}\boldsymbol{\phi}\right]_3+i\bar{K}\frac{1+\tau_3}{2}(\vec{\partial}_\mu-\overleftarrow{\partial}_\mu)K, \tag{2·4}*$$

where

$$N=\begin{pmatrix}p\\ n\end{pmatrix},\quad \bar{N}=(\bar{p},\bar{n}),$$

and

$$K=\begin{pmatrix}K^+\\ K^0\end{pmatrix},\quad \bar{K}=(\bar{K}^+,\ -\bar{K}^0), \tag{2·5}$$

and A_μ, ϕ_i, and K are the photon, π_i-meson, and K-meson wave functions, respectively. Next, we must replace the vector part of the strangeness conserving lepton-baryon weak interaction current, $i\bar{p}\gamma_\mu n$, in the Sakata model by $I_{\mu 1}+iI_{\mu 2}$ in the Yukawa type description, where $\boldsymbol{I}_\mu$ is the iso-vector part of $\boldsymbol{J}_\mu$ and is given by

$$\boldsymbol{I}_\mu=i\bar{N}\gamma_\mu\frac{\boldsymbol{\tau}}{2}N+\left[\boldsymbol{\phi}\times\frac{(\vec{\partial}_\mu-\overleftarrow{\partial}_\mu)}{2}\boldsymbol{\phi}\right]+i\bar{K}\frac{\boldsymbol{\tau}}{2}(\vec{\partial}_\mu-\overleftarrow{\partial}_\mu)K. \tag{2·6}$$

Thus we get the following effective Hamiltonian for the $\Delta S=0$ case:

$$H_d^v=g_d[i\bar{p}\gamma_\mu n-\sqrt{2}\,i\phi^-(\vec{\partial}_\mu-\overleftarrow{\partial}_\mu)\phi^0+i\bar{K}^+(\vec{\partial}_\mu-\overleftarrow{\partial}_\mu)K^0]\left(\bar{e}i\gamma_\mu\frac{1+\gamma_5}{\sqrt{2}}\nu\right)+\text{h. c.}, \tag{2·7}$$

where ϕ^- denotes the destruction of π^- meson or the creation of π^+ meson.

Besides the charge independence, we make use of the invariance under the exchange of n and Λ in strong interactions. Then, the vector part of $\Delta S=0$ lepton-baryon weak interaction current, $i\bar{p}\gamma_\mu n$, in the Sakata model must be replaced in the usual Yukawa type interactions by the current which is given by the

* Throughout the present paper we do not discuss the currents involving Σ and Ξ.

exchange of n and Λ in the $I_{\mu 1}+iI_{\mu 2}$ current. Thus we can write down the vector part of the effective interaction Hamiltonian in the Yukawa-type theory as follows:

$$H_e^v=g_e\Big[i\bar{p}\gamma_\mu\Lambda-i\sqrt{2}\,\bar{K}^+(\vec{\partial}_\mu-\overleftarrow{\partial}_\mu)\left(\frac{1}{2}\phi^0+\frac{\sqrt{3}}{2}\phi^{0\prime}\right)+i\bar{K}^0(\vec{\partial}_\mu-\overleftarrow{\partial}_\mu)\phi^-\Big]$$

$$\times\left(\bar{e}i\gamma_\mu\frac{1+\gamma_5}{\sqrt{2}}\nu\right)+\text{h. c.} \qquad (2\cdot 8)^*$$

It is to be noted that both in the fundamental interaction Hamiltonian (2·1e) and in the effective interaction Hamiltonian (2·8) the change of isotopic spin of baryons, $|\Delta I|$, is 1/2.

ii) *Non-leptonic weak interactions*

In a similar way, the fundamental interaction (2·1a) is reinterpreted by introducing bosons explicitly. $(\bar{p}\Lambda)$ and $(\bar{n}p)$ in (2·1a) are replaced by the first factor in (2·8) and by $I_{\mu 1}+iI_{\mu 2}$, respectively.

Noticing that (2·1a) can be rewritten by using the Fierz transformation as

$$H_a=\sqrt{2}\,g_a\left(\bar{n}i\gamma_\mu\frac{1+\gamma_5}{\sqrt{2}}\Lambda\right)\left(\bar{p}i\gamma_\mu\frac{1+\gamma_5}{\sqrt{2}}p\right)+\text{h. c.}, \qquad (2\cdot 9)$$

we get the following effective Hamiltonian for the vector part of the elementary interactions:

$$H_a=\frac{g_a}{\sqrt{2}}\Big[i\bar{p}\gamma_\mu\Lambda-\sqrt{2}\,i\bar{K}^+(\vec{\partial}_\mu-\overleftarrow{\partial}_\mu)\left(\frac{1}{2}\phi^0+\frac{\sqrt{3}}{2}\phi^{0\prime}\right)+i\bar{K}^0(\vec{\partial}_\mu-\overleftarrow{\partial}_\mu)\phi^-\Big]$$

$$\times[i\bar{n}\gamma_\mu p+\sqrt{2}\,i\phi^+(\vec{\partial}_\mu-\overleftarrow{\partial}_\mu)\phi^0-i\bar{K}^0(\vec{\partial}_\mu-\overleftarrow{\partial}_\mu)K^+]+\text{h. c.}, \qquad (2\cdot 10)$$

$$=\frac{g_a}{\sqrt{2}}\Big[i\bar{n}\gamma_\mu\Lambda-\sqrt{2}\,i\bar{K}^0(\vec{\partial}_\mu-\overleftarrow{\partial}_\mu)\left(\frac{1}{2}\phi^0-\frac{\sqrt{3}}{2}\phi^{0\prime}\right)-i\bar{K}^+(\vec{\partial}_\mu-\overleftarrow{\partial}_\mu)\phi^+\Big]$$

$$\times[i\bar{p}\gamma_\mu p+i\phi^-(\vec{\partial}_\mu-\overleftarrow{\partial}_\mu)\phi^++i\bar{K}^+(\vec{\partial}_\mu-\overleftarrow{\partial}_\mu)K^+]+\text{h. c.}, \qquad (2\cdot 11)$$

$$H_b=\frac{g_b}{\sqrt{2}}\Big[i\bar{n}\gamma_\mu\Lambda-\sqrt{2}\,i\bar{K}^0(\vec{\partial}_\mu-\overleftarrow{\partial}_\mu)\left(\frac{1}{2}\phi^0-\frac{\sqrt{3}}{2}\phi^{0\prime}\right)-i\bar{K}^+(\vec{\partial}_\mu-\overleftarrow{\partial}_\mu)\phi^+\Big]$$

$$\times[i\bar{n}\gamma_\mu n+i\phi^+(\vec{\partial}_\mu-\overleftarrow{\partial}_\mu)\phi^--i\bar{K}^0(\vec{\partial}_\mu-\overleftarrow{\partial}_\mu)K^0]+\text{h. c.}, \qquad (2\cdot 12)$$

$$H_c=\frac{g_c}{\sqrt{2}}\Big[i\bar{n}\gamma_\mu\Lambda-\sqrt{2}\,i\bar{K}^0(\vec{\partial}_\mu-\overleftarrow{\partial}_\mu)\left(\frac{1}{2}\phi^0-\frac{\sqrt{3}}{2}\phi^{0\prime}\right)-i\bar{K}^+(\vec{\partial}_\mu-\overleftarrow{\partial}_\mu)\phi^+\Big]$$

$$\times[i\bar{\Lambda}\gamma_\mu\Lambda-i\bar{K}^+(\vec{\partial}_\mu-\overleftarrow{\partial}_\mu)K^++i\bar{K}^0(\vec{\partial}_\mu-\overleftarrow{\partial}_\mu)K^0]+\text{h. c.} \qquad (2\cdot 13)$$

By the effective Hamiltonian we mean the interaction Hamiltonian of four

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* After the completion of this work the authors were informed that Y. Fujii (unpublished work) and S. Ogawa et al. (preprint April 1, 1960) had independently derived this effective Hamiltonian. S. Okubo[16] and Y. Yamaguchi[9] have discussed the possibility of conserved vector current in the Sakata model.

fermions, or in general, four external lines (A, B) (C, D), in which radiative corrections between A and B or C and D due to strong interactions are taken into account for the vector part, but radiative corrections between (A, B) and (C, D) are all neglected. This situation is shown in Fig. 1. Fig. 1(a) is included in the effective Hamiltonian, but Fig. 1(b) is not.

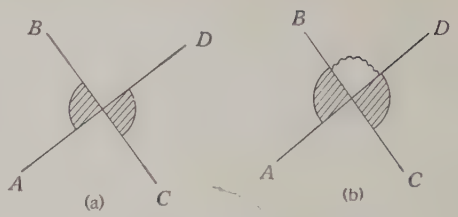

Fig. 1. a) Diagram taken into account in the effective Hamiltonian (2·10)-(2·13)
b) Diagram not included in the effective Hamiltonian

If the coupling constants g are chosen to be real, (2·10), (2·11), (2·12) and (2·13) are simplified in the final form as

$$H_a=\frac{g_a}{\sqrt{2}}\Big[i\bar{p}\gamma_\mu\Lambda-\sqrt{2}\,i\bar{K}^+(\vec{\partial}_\mu-\overleftarrow{\partial}_\mu)\Big(\frac{1}{2}\phi^0+\frac{\sqrt{3}}{2}\phi^{0\prime}\Big)+i\,\frac{K_1^0-K_2^0}{\sqrt{2}}(\vec{\partial}_\mu-\overleftarrow{\partial}_\mu)\phi^-\Big]$$
$$\times[i\bar{n}\gamma_\mu p+\sqrt{2}\,i\phi^+(\vec{\partial}_\mu-\overleftarrow{\partial}_\mu)\phi^0-i\bar{K}^0(\vec{\partial}_\mu-\overleftarrow{\partial}_\mu)K^+]$$
$$+\frac{g_a}{\sqrt{2}}\Big[i\bar{\Lambda}\gamma_\mu p+\sqrt{2}\,iK^+(\vec{\partial}_\mu-\overleftarrow{\partial}_\mu)\Big(\frac{1}{2}\phi^0+\frac{\sqrt{3}}{2}\phi^{0\prime}\Big)+i\,\frac{K_1^0+K_2^0}{\sqrt{2}}(\vec{\partial}_\mu-\overleftarrow{\partial}_\mu)\phi^+\Big]$$
$$\times[i\bar{p}\gamma_\mu n-\sqrt{2}\,i\phi^-(\vec{\partial}_\mu-\overleftarrow{\partial}_\mu)\phi^0+i\bar{K}^+(\vec{\partial}_\mu-\overleftarrow{\partial}_\mu)K^0], \qquad (2\cdot10')$$

$$H_a=\frac{g_a}{\sqrt{2}}\Big[i(\bar{n}\gamma_\mu\Lambda+\bar{\Lambda}\gamma_\mu n)-i2K_1^0(\vec{\partial}_\mu-\overleftarrow{\partial}_\mu)\Big(\frac{1}{2}\phi^0-\frac{\sqrt{3}}{2}\phi^{0\prime}\Big)-i\bar{K}^-(\vec{\partial}_\mu-\overleftarrow{\partial}_\mu)\phi^+$$
$$+iK^+(\vec{\partial}_\mu-\overleftarrow{\partial}_\mu)\phi^-\Big]\times[i\bar{p}\gamma_\mu p+i\phi^-(\vec{\partial}_\mu-\overleftarrow{\partial}_\mu)\phi^++i\bar{K}^+(\vec{\partial}_\mu-\overleftarrow{\partial}_\mu)K^-], \qquad (2\cdot11')$$

$$H_b=\frac{g_b}{\sqrt{2}}\Big[i(\bar{n}\gamma_\mu\Lambda+\bar{\Lambda}\gamma_\mu n)-i2K_1^0(\vec{\partial}_\mu-\overleftarrow{\partial}_\mu)\Big(\frac{1}{2}\phi^0-\frac{\sqrt{3}}{2}\phi^{0\prime}\Big)-i\bar{K}^+(\vec{\partial}_\mu-\overleftarrow{\partial}_\mu)\phi^+$$
$$+iK^+(\vec{\partial}_\mu-\overleftarrow{\partial}_\mu)\phi^-\Big]\times[i\bar{n}\gamma_\mu n+i\phi^+(\vec{\partial}_\mu-\overleftarrow{\partial}_\mu)\phi^--i\bar{K}^0(\vec{\partial}_\mu-\overleftarrow{\partial}_\mu)K^0], \qquad (2\cdot12')$$

$$H_c=\frac{g_c}{\sqrt{2}}\Big[i(\bar{n}\gamma_\mu\Lambda+\bar{\Lambda}\gamma_\mu n)-i2K_1^0(\vec{\partial}_\mu-\overleftarrow{\partial}_\mu)\Big(\frac{1}{2}\phi^0-\frac{\sqrt{3}}{2}\phi^{0\prime}\Big)-i\bar{K}^+(\vec{\partial}_\mu-\overleftarrow{\partial}_\mu)\phi^+$$
$$+iK^+(\vec{\partial}_\mu-\overleftarrow{\partial}_\mu)\phi^-\Big]\times[i\bar{\Lambda}\gamma_\mu\Lambda-i\bar{K}^+(\vec{\partial}_\mu-\overleftarrow{\partial}_\mu)K^++i\bar{K}^0(\vec{\partial}_\mu-\overleftarrow{\partial}_\mu)K^0], \qquad (2\cdot13')$$

where

$$K_1^0=\frac{1}{\sqrt{2}}(K^0+\bar{K}^0),\quad K_2^0=\frac{1}{\sqrt{2}}(K^0-\bar{K}^0). \qquad (2\cdot14)$$

§ 3. Strangeness-violating leptonic decays

The following leptonic decays are illustrated by the direct interactions in the effective Hamiltonian (2·8) :

$$\Lambda \to p + e^- + \nu, \tag{3·1a}$$

$$K^{\pm} \to \pi^0 + e^{\pm} + \nu, \tag{3·1b}$$

$$K_1^0 \to \pi^{\pm} + e^{\mp} + \nu, \tag{3·1c}$$

$$K_2^0 \to \pi^{\pm} + e^{\mp} + \nu. \tag{3·1d}$$

Apart from the direct terms shown in Fig. 2(a) or (b) we have diagrams involving baryon closed loops as in Fig. 2(c), which are, however, omitted in this section.

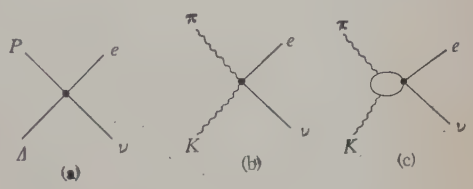

Fig. 2. a) Diagram for $\Lambda^0 \to p + e^- + \nu$
b) Diagram for $K \to \pi + e + \nu$
c) More complicated diagram omitted in our calculation

In Table II the decay rates of (3·1) are shown with both the beta-decay coupling constant,

$$g_a = 1.4 \times 10^{-49} \text{ erg cm}^3, \tag{3·2}$$

and the coupling constant g_e obtained from $K_{\mu 2}^{+}$, namely

$$g_e^2 / g_a^2 \sim 1/27. \tag{3·3}$$

Detailed calculations are given in Appendix.

Table II. Transition rates of strangeness-violating leptonic decays

Process	Experiment (sec^{-1})	Theory (3·2)	Experiment/Theory	Theory (3·3)
$w(K^+ \to \pi^0 + e^+ + \nu)$	3×10^6	8×10^7	1/27	3×10^6
$w(K_1^0 \to \pi^+ + e^- + \nu)$		8×10^7		3×10^6
$w(K_2^0 \to \pi^+ + e^- + \nu)$	$\sim 3 \times 10^6$	8×10^7	$\sim 1/27$	3×10^6
$\dfrac{w(K^+ \to \pi^0 + e^+ + \nu)}{w(K^+ \to \pi^0 + \mu^+ + \nu)}$	~ 1	1.55		1.55
$w(\Lambda^0 \to p + e^- + \nu)$	$< 10^7$	5×10^7	$< 1/5$	2×10^6
$w(\Lambda^0 \to p + \mu^- + \nu)$	$< 5 \times 10^6$	8×10^6	$< 1/2$	3×10^5

It is concluded from Table II that the coupling constant in the $|\Delta S| = 1$ leptonic decays seems to be smaller than that of the normal beta-decay by factor 5. It would be too early, however, to say anything about the beta- and muon-decays of Λ.

Here we note that[11]

$$w(K^+ \to \pi^0 + e^+ + \nu) = w(K_1^0 \to \pi^+ + e^- + \nu) = w(K_2^0 \to \pi^+ + e^- + \nu), \tag{3·4}$$

as a consequence of the $|\Delta I| = 1/2$ current rule.

Finally, it is important to point out that the Sakata model gives rise to a decay involving another neutral meson,

$$K^+ \to \pi^{0\prime} + e^+ + \nu. \qquad (3\cdot5)$$

The branching ratio

$$R = \frac{w(K^+ \to \pi^{0\prime} + e^+ + \nu)}{w(K^+ \to \pi^0 + e^+ + \nu)} \qquad (3\cdot6)$$

is plotted in Fig. 3 *vs.* the mass of $\pi^{0\prime}$. The calculation is given in the Appendix. The effects of $\pi^{0\prime}$ on the electron spectrum of $K_{e3}{}^+$ has been discussed by Sawada and Yonezawa.[12]

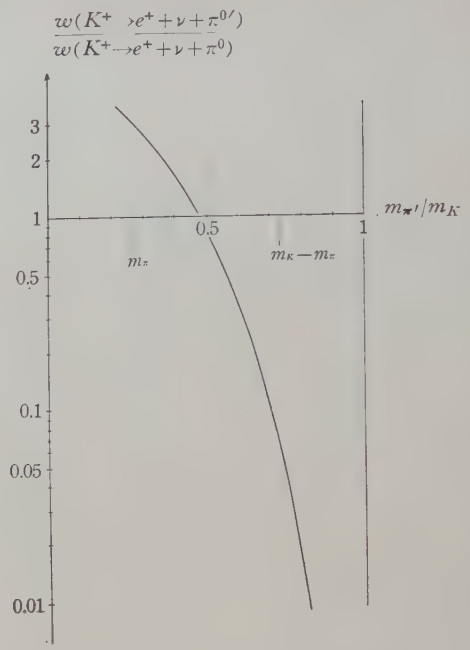

Fig. 3. Branching ratio (3·6) *vs.* the mass of $\pi^{0\prime}$

§ 4. Non-leptonic decays

The effective Hamiltonians (2.10′—12′) are responsible for the following non-leptonic processes :

$$\Lambda^0 \to p + \pi^-, \qquad (4\cdot1a)$$

$$\Lambda^0 \to n + \pi^0, \qquad (4\cdot1b)$$

$$K_1{}^0 \to \pi^0 + \pi^0, \qquad (4\cdot1c)$$

$$K_1{}^0 \to \pi^+ + \pi^-, \qquad (4\cdot1d)$$

$$K^+ \to \pi^+ + \pi^0. \qquad (4\cdot1e)$$

We are going to calculate the matrix element shown in Fig. 4(a) or (b), but more complicated diagrams like Fig. 4(c) are neglected. Although a divergent integral is unavoidable for the baryon closed loop, the ratio of, say, (4·1a) to (4·1d) is independent of the weak coupling constant and of the divergence. This ratio is, therefore, considered to be reliable.

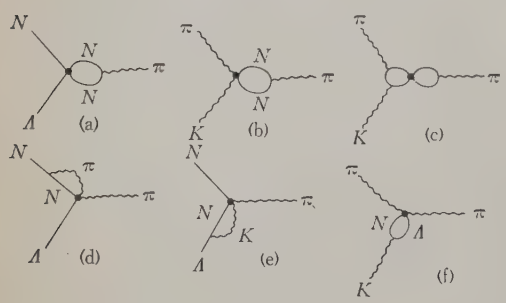

Fig. 4. a), d), and e) Diagrams for $\Lambda \to N + \pi$
b) and f) Diagrams for $K \to \pi + \pi$
c) More complicated one neglected here

It is quite difficult to estimate the contributions from Fig. 3 (d) and (e) because of badly divergent integrals. We have, however, a hope that these should be small, because these give a very small asymmetry parameter[8] while Fig. 4 (a) gives a large one in agreement with experiment. Fortunately, Fig. 4 (f) always vanishes and causes no trouble.

The results are summarized in Table III, in which the undetermined numerical

factor is chosen to explain $w(\Lambda^0 \to p+\pi^-)$.* Detailed calculation is given in the Appendix.

Table III. Decay rate of non-leptonic processes

Process	Experiment ($\sec^{-1}$)	Theory I ($\sec^{-1}$) $g_a=g_b=g$	Experiment / Theory I	Theory II ($\sec^{-1}$) $g_a=g$ $g_b=0$	Experiment / Theory II
$w(\Lambda^0 \to p+\pi^-)$	2.3×10^9	2.3×10^9	$\equiv 1$	2.3×10^9	1
$w(\Lambda^0 \to n+\pi^0)$	1.3×10^9	1.2×10^9	~ 1	1.2×10^9	~ 1
$w(K_1^0 \to \pi^+ + \pi^-)$	9×10^9	2.7×10^{10}	1/3.3	2.7×10^{10}	1/3.3
$w(K_1^0 \to \pi^0 + \pi^0)$	2×10^9	1.3×10^{10}	1/1.5	1.3×10^{10}	1/1.5
$w(K^+ \to \pi^+ + \pi^0)$	2×10^7	0	∞	2.7×10^{10}	1/1300

We should conclude that the decay rates are quite consistent with experiment.[6] It is to be emphasized that the $|\Delta I|=1/2$ selection rule is realized by choosing

$$g_a=g_b=g(=g_c). \tag{4·2}$$

Even if one chooses

$$g_a=g \quad \text{and} \quad g_b=0, \tag{4·3}$$

the branching ratios of Λ and K accidentally coincide with those in the case of (4·2)[13]; namely, in each case we have

$$w(\Lambda^0 \to p+\pi^0)/w(\Lambda^0 \to n+\pi^0)=2, \tag{4·4}$$

and

$$w(K_1^0 \to \pi^+ + \pi^-)/w(K_1^0 \to \pi^0 + \pi^0)=2. \tag{4·5}$$

The remarkable difference is, however, that

$$w(K^+ \to \pi^+ + \pi^0)=0 \quad \text{for (4·2)}, \tag{4·6}$$

and

$$w(K^+ \to \pi^+ + \pi^0)=w(K_1^0 \to \pi^+ + \pi^-) \quad \text{for (4·3)}. \tag{4·7}$$

It is likely to assume that (4·2) holds as the first approximation, and a few percent deviation from the $|\Delta I|=1/2$ rule is expected as higher order effects.

Since the decay $K^+ \to \pi^+ + \pi^{0\prime}$ is allowed by the $|\Delta I|=1/2$ rule, it is necessary to compare it with $K^+ \to \pi^+ + \pi^0$. As is shown in Fig. 5 and the Appendix, $w(K^+ \to \pi^+ + \pi^{0\prime})$ is comparable with $w(K_1^0 \to \pi^+ + \pi^-)$ even for big $\pi^{0\prime}$ mass. In order to forbid $K^+ \to \pi^+ + \pi^{0\prime}$ we have to choose

$$m_{\pi^\prime} \geq 354 \text{ Mev}. \tag{4·8}$$

* Actually this choice of the numerical factor is also consistent with $\pi \to \mu+\nu$ decay within factor 2.[2]

Incidentally, the decay rate of $K_{\pi 3}^{+}$, due to the direct term in the effective Hamiltonian (2.10′−12′), turns out to be quite small,[8]

$$w(K^+\to\pi^+ +\pi^+ +\pi^-) = w(K^+\to\pi^0+\pi^0+\pi^+) = 2.9\times 10^5 \text{ sec}^{-1}, \qquad (4\cdot 9)$$

while the experimental rate is 5×10^6 sec^{-1}. The main contribution is expected from diagrams involving baryon closed loops, which is supposed to give the experimental branching ratio

$$w(K^+\to\pi^+ +\pi^+ +\pi^-)/w(K^+\to\pi^+ +\pi^0+\pi^0)\sim 4, \qquad (4\cdot 10)$$

so that (4·9) might contribute only to a small deviation from the main terms. This is easily understood if one notices that our direct term is not totally symmetric in three pions. Such an asymmetric term is theoretically predicted to contribute by about 1/20 of the symmetric part.[14]

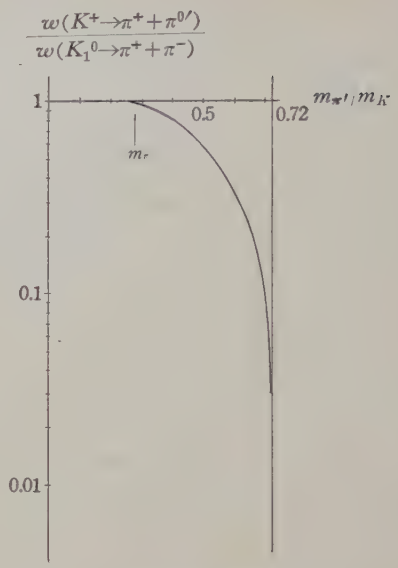

Fig. 5. Branching ratio $w(K^+\to\pi^+ +\pi^{0\prime})/w(K_1^0\to\pi^+ +\pi^-)$ *vs.* the mass of $\pi^{0\prime}$

§ 5. Discussion

We have so far studied what kind of boson current has to be added to the vector part of the Fermi interaction for the $|\Delta S|=1$ case. The essential point of our argument is, first of all, based on the Sakata model and on the symmetry with respect to n and Λ. So it is very important for the test of the Sakata model to compare the results obtained from these boson currents with experiments, because if these results contradict experiments we are obliged to discard this model, provided that the effect of mass difference between Λ and n is quite small.

For the non-leptonic decays we have introduced the neutral combinations like $(\bar{n}\Lambda)(\bar{n}n)$, which are not usually accepted. Then the sum of the normal terms $(\bar{p}\Lambda)(\bar{n}p)$ and this neutral term guarantees $|\Delta I|=1/2$. Both decay rates and branching ratios fit to experiments well, so that this may support the existence of the neutral terms.

Trouble is, however, how to get the decay rate of $K_{\pi 2}^{+}$. Electromagnetic correction to the $|\Delta I|=1/2$ rule seems to be too small to explain the observed deviation.[15] Another problem is a more quantitative treatment of $K_{\pi 3}^{+}$ and, in general, the estimate of the process involving baryon closed loops and divergent integrals as shown in Fig. 4(d) and (e).

It still remains open to question why the coupling constant of the strangeness-violating leptonic decays is smaller than that of non-leptonic decays, while the latter is consistent with the normal beta-decay coupling constant.

In conclusion we are tempted to say that the decays of strange particles can be explained in terms of the V-A Fermi interations together with the additional boson currents, which naturally satisfy the $|\Delta I|=1/2$ rule. So we can say that this is not in contradiction with the Sakata model, since these boson currents are natural consequence of the model.

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Appendix. Decay Probabilities

For the sake of convenience, lowest order perturbation calculations are shown in this Appendix.

i) $K\to e+\nu+\pi$ *decay*

The following part of the effective Hamiltonian is responsible for this process:

$$H=\frac{g}{\sqrt{2}}(i\bar{K}^{+}(\vec{\partial}_{\mu}-\vec{\partial}_{\mu})\phi^{0})\left(\bar{e}i\gamma_{\mu}\frac{1+\gamma_{5}}{\sqrt{2}}\nu\right)+\text{h. c.} \tag{A·1}$$

The decay rate is[2)]

$$w(K^{+}\to e^{+}+\nu+\pi^{0})=\frac{1}{2^{3}}\frac{(gM_{0}^{2})^{2}}{(2\pi)^{3}}\left(\frac{m_{K}}{M_{0}}\right)^{4}m_{K}I\left(\frac{m_{\pi}}{m_{K}}\right), \tag{A·2}$$

where

$$I(x)=\frac{1}{24}-\frac{x^{2}}{3}+\frac{x^{6}}{3}-\frac{x^{8}}{24}-x^{4}\ln x, \tag{A·3}$$

and

$$I\left(\frac{m_{\pi}}{m_{K}}\right)=0.0238. \tag{A·4}$$

M_0, m_K and m_π denote the masses of the nucleon, K-meson, and pion, respectively. One readily gets

$$w(K^{+}\to e^{+}+\nu+\pi^{0})=8.3\times10^{7}\ \text{sec}^{-1}, \tag{A·5}$$

for the beta-decay coupling constant

$$g=1.4\times10^{-49}\ \text{erg cm}^{3}. \tag{A·6}$$

$w(K^{+}\to e^{+}+\nu+\pi^{0\prime})$ can be obtained by replacing m_π by m_π', taking numerical factors in (2·8) into account, and is

$$\frac{w(K^{+}\to e^{+}+\nu+\pi^{0\prime})}{w(K^{+}\to e^{+}+\nu+\pi^{0})}=3I\left(\frac{m_{\pi}'}{m_{K}}\right)\Big/I\left(\frac{m_{\pi}}{m_{K}}\right). \tag{A·7}$$

ii) $\Lambda^{0}\to p+e^{-}+\nu$ *decay*

For the Hamiltonian

$$H=\sqrt{2}\,g\left(\bar{p}i\gamma_{\mu}\frac{1+\gamma_{5}}{\sqrt{2}}\Lambda\right)\left(\bar{e}i\gamma_{\mu}\frac{1+\gamma_{5}}{\sqrt{2}}\nu\right)+\text{h. c.} \tag{A·8}$$

the decay rate $w(\Lambda^0\to p+e^-+\nu)$ in the lowest order perturbation is[1),3)]

$$w(\Lambda^0\to p+e^-+\nu)=\frac{(gM_0^2)^2}{(2\pi)^3}\left(\frac{M_\Lambda}{M_0}\right)^4 M_\Lambda I\left(\frac{M_0}{M_\Lambda}\right), \tag{A·9}$$

where M_Λ means the mass of the Λ-particle and I is given by (A·3). One finds

$$w(\Lambda^0\to p+e^-+\nu)=5.3\times 10^7\ \mathrm{sec}^{-1} \tag{A·10}$$

for the coupling constant (A·6).

iii) $\Lambda\to N+\pi$ *decay*

We evaluate the decay probability w_Λ for $\Lambda^0\to p+\pi^-$ corresponding to Feynman diagram of Fig. 4(a), where we assume the following Hamiltonian for the weak interaction,

$$H=\sqrt{2}\,g\left(\bar{p}\,i\gamma_\mu\frac{1+\gamma_5}{\sqrt{2}}\Lambda\right)\left(\bar{n}\,i\gamma_\mu\frac{1+\gamma_5}{\sqrt{2}}p\right)+\mathrm{h.c.} \tag{A·11}$$

The divergent matrix element for the nucleon loop is replaced by a constant C which depends only on m_π^2. Then the decay rate reads

$$w_\Lambda=(gM_0^2)^2\frac{g_\pi^2}{4\pi}\frac{2}{(2\pi)^4}[(M_\Lambda-M_0)^2-m_\pi^2]^{1/2}[(M_\Lambda+M_0)^2-m_\pi^2]^{1/2}$$
$$\times\frac{[(M_\Lambda^2-M_0^2)^2-m_\pi^2(M_\Lambda^2+M_0^2)]}{M_\Lambda^2M_0^2}M_\Lambda|C|^2, \tag{A·12}$$

where $g_\pi^2/4\pi$ is the pion coupling constant.

But, in general, the Hamiltonian (2·10′−12′) must be used instead of (A·11). Then the decay probability can be written in terms of w_Λ, g_a, g_b and g as

$$w(\Lambda\to p+\pi^-)=w_\Lambda\left(\frac{g_a}{g}\right)^2,$$
$$w(\Lambda\to n+\pi^0)=w_\Lambda\left(\frac{g_a-2g_b}{\sqrt{2}\,g}\right)^2. \tag{A·13}$$

Especially for $g_a=g_b=g$, the $|\Delta I|=1/2$ rule holds, and (A·13) become

$$w(\Lambda\to p+\pi^-)=w_\Lambda,$$
$$w(\Lambda\to n+\pi^0)=(1/2)\ w_\Lambda. \tag{A·14}$$

For $g_a=g$ and $g_b=0$, we also obtain

$$w(\Lambda\to p+\pi^-)=w_\Lambda,$$
$$w(\Lambda\to n+\pi^0)=(1/2)\ w_\Lambda, \tag{A·15}$$

which coincide just with the result of the $|\Delta I|=1/2$ rule.

Comparison of our formula with the experimental transition rates gives

$$(gM_0^2)^2\frac{g_\pi^2}{4\pi}|C|^2=2.9\times 10^{-10}. \tag{A·16}$$

iv) $K\to\pi+\pi$ *decay*

We evaluate the decay probability w_K for $K^+\to\pi^++\pi^0$ corresponding to the Feynman diagram of Fig. 4(b), where we assume the following Hamiltonian:

$$H=\sqrt{2}\,g(i\bar{K}^+(\vec{\partial}_\mu-\overleftarrow{\partial}_\mu)\phi^0)\left(\bar{n}i\gamma_\mu\frac{1+\gamma_5}{\sqrt{2}}p\right)+\text{h. c.} \qquad (A\cdot17)$$

The divergent loop in the figure is also replaced by the same constant C as that of $\Lambda\to p+\pi$, and we have

$$w_K=\frac{g_\pi^2}{4\pi}\frac{(GM_0^2)^2}{(2\pi)^4}\cdot\frac{4(m_K^2-m_\pi^2)^2}{m_K^2M_0^2}(m_K^2-4m_\pi^2)^{1/2}|C|^2. \qquad (A\cdot18)^*$$

In general, the Hamiltonian $(2\cdot10'-12')$ must be used instead of (A·17). Then the decay probability can be written in terms of w_K, g_a, g_b and g as

$$\begin{aligned}
w(K^+\to\pi^++\pi^0)&=\frac{(g_a-g_b)^2}{g^2}w_K,\\
w(K_1^0\to\pi^++\pi^-)&=\left(\frac{g_a}{g}\right)^2w_K,\\
w(K_1^0\to\pi^0+\pi^0)&=\frac{(2g_b-g_a)^2}{2g^2}w_K.
\end{aligned} \qquad (A\cdot19)$$

Especially for $g_a=g_b=g$, the $|\Delta I|=1/2$ rule holds. (A·19) becomes

$$\begin{aligned}
w(K^+\to\pi^++\pi^0)&=0,\\
w(K_1^0\to\pi^++\pi^-)&=w_K,\\
w(K_1^0\to\pi^0+\pi^0)&=1/2\,w_K.
\end{aligned} \qquad (A\cdot20)$$

For $g_a=g$ and $g_b=0$ we obtain

$$\begin{aligned}
w(K^+\to\pi^++\pi^0)&=w_K,\\
w(K_1^0\to\pi^++\pi^-)&=w_K,\\
w(K_1^0\to\pi^0+\pi^0)&=1/2\,w_K.
\end{aligned} \qquad (A\cdot21)$$

Using (A·16), we get

$$w_K=2.7\times10^{10}\ \text{sec}^{-1}, \qquad (A\cdot22)$$

which is in agreement with the transition rate for K_1^0 decay. (A·21) must be ruled out because of the too rapid decay rate for the K^+ meson.

v) $K\to\pi+\pi^{0\prime}$ *decay*

In a similar way, we have,

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* In reference 2), the relation between w_K and $w(K^+\to\pi^0+e^++\nu)$ was established. Eq. (2·17) in reference 2) is satisfied by taking $G=-2F=\pi\Big/\sqrt{\frac{g_\pi^2}{4\pi}\cdot\frac{g_K^2}{4\pi}}$. The factor 2 in (A·1) and (A·17) must be adjusted.

$$w'(K^+\to\pi^+ +\pi^{0\prime})=\frac{g_\pi^{\,2}}{4\pi}\,\frac{(gM_0^{\,2})^2}{(2\pi)^4}\,\frac{4(m_K^{\,2}-m_{\pi'}^{\,2})^2}{M_0^{\,2}m_K^{\,2}}\,m_K$$
$$\times\frac{\sqrt{\{(m_K-m_{\pi'})^2-m_\pi^{\,2}\}\{(m_K+m_{\pi'})^2-m_\pi^{\,2}\}}}{m_K^{\,2}-m_{\pi'}^{\,2}+m_\pi^{\,2}}\,|C|^2. \qquad (A\cdot 23)$$

w'_K/w_K is plotted *vs.* $m_{\pi'}/m_K$ in Fig. 4. Since a heavy neutral meson has not yet been observed, we have to require $m_{\pi'}\geq 354$ Mev.

vi) $K_{\pi 3}\to\pi+\pi+\pi$ *decay*

The effective Hamiltonian for $K^+\to\pi^0+\pi^0+\pi^+$ is given by

$$H=-\frac{1}{\sqrt{2}}\,(iK^+(\vec{\partial}_\mu-\overleftarrow{\partial}_\mu)\phi^0)\,(i\phi^+(\vec{\partial}_\mu-\overleftarrow{\partial}_\mu)\phi^0), \qquad (A\cdot 24)$$

and the corresponding decay rate is

$$w(K^+\to\pi^0+\pi^0+\pi^+)=\frac{4}{(2\pi)^3}\,(gM_0^{\,2})^2\,\frac{m_\pi^{\,8}}{m_K^{\,4}M_0^{\,4}}\,m_K\times 0.070. \qquad (A\cdot 25)$$

References

1) For example, R. P. Feynman and M. Gell-Mann, Phys. Rev. **109** (1958), 193.
2) A. Fujii and M. Kawaguchi, Phys. Rev. **113** (1959), 1156.
3) C. H. Albright, Phys. Rev. **114** (1959), 1648.
4) B. Sakita, Phys. Rev. **114** (1959), 1650.
Z. Maki, Prog. Theor. Phys. **23** (1960), 853.
5) M. Gell-Mann, Phys. Rev. **111** (1958), 362.
6) L. B. Okun', *1958 Annual International Conference on High Energy Physics at CERN.*
C. Iso, Prog. Theor. Phys. **22** (1959), 62.
7) S. Sakata, Prog. Theor. Phys. **16** (1956), 686.
8) M. Sugawara, Phys. Rev. **113** (1959), 1361.
9) S. Ogawa, Prog. Theor. Phys. **21** (1959), 209.
M. Ikeda, S. Ogawa and Y. Ohnuki, Prog. Theor. Phys. **22** (1959), 715.
Y. Yamaguchi, Prog. Theor. Phys. Supplement No. 11 (1959), 1.
10) For example, M. Kawaguchi and K. Nishijima, Prog. Theor. Phys. **15** (1956), 180.
C. Iso and M. Kawaguchi, Prog. Theor. Phys. **16** (1956), 177.
11) S. Okubo, R. E. Marshak, E. C. G. Sudarshan, W. B. Teutsch and S. Weinberg, Phys. Rev. **112** (1958), 665.
12) S. Sawada and M. Yonezawa, Prog. Theor. Phys. **22** (1959), 610.
13) S. Okubo, R. E. Marshak and E. C. G. Sudarshan, Phys. Rev. **113** (1959), 944.
14) For example, M. Gell-Mann and A. H. Rosenfeld, *Annual Review of Nuclear Science*, Vol. **7**, p. 407, especially, Section 6.6.
16) C. Iso, Soryushiron Kenkyu (mimeographed circular in Japanese) **11** (1956), 267.
M. Kato and G. Takeda, Prog. Theor. Phys. Supplement No. 7 (1959), 35.
16) S. Okubo, Nuovo Cimento **8** (1959), 292.

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A Relativistic Model for a Shell of Flowing Radiation in a Homogeneous Universe

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Rigorous solutions of Einstein's field equations of general relativity are presented. They are found to represent the gravitational field within a shell of radiation flowing outwards in a homogeneous universe. These models suggest that the effect of complete conversion of matter into radiation appears to decrease the curvature of the universe.

§ 1. Introduction

Following a method initiated by Vaidya,[1] Raychaudhuri[2] considered the case of a mass particle being completely annihilated and converted into radiation. He found that such a process could not be adequately described by the field equations of relativity. However, Israel[3] found that the relativity equations are adequate to describe such a process of complete conversion of a mass particle into radiation and he has given a model for the same.

In all these attempts, the mass particle was embedded in Schwarzschild space-time and after the complete conversion into radiation, the hollow created was empty (flat). It would be of interest to see how these conclusions are modified when the same process is studied, not in the flat background, but in the background of a homogeneous universe.

We begin with the Lemaitre universe, given by the line-element

$$ds^2 = -e^{G(T)} Z_0^2 [d\chi^2 + \sin^2\chi (d\theta^2 + \sin^2\theta\, d\phi^2)] + dT^2 \qquad (1\cdot1)$$

and choose a convenient radial coordinate r by the substitution

$$\frac{r}{1+P_0 r/2} = Z_0 \tan\chi, \qquad (1\cdot2)$$

where P_0 is a constant. Then (1·1) reduces to

$$ds^2 = -e^{G(T)}\left[\frac{dr^2}{(1+P_0 r+Q_0 r^2)^2} + \frac{r^2}{1+P_0 r+Q_0 r^2}(d\theta^2+\sin^2\theta\, d\phi^2)\right] + dT^2, \qquad (1\cdot3)$$

where

$$Q_0=P_0^2/4+1/Z_0^2. \tag{1·4}$$

The radius of curvature of this universe will be $Z=Z_0e^{G/2}$. The advantage of this particular choice of radial coordinate is twofold: (i) The radius Z_0 of the universe is reached in (1·1) at $\chi=\pi/2$ and therefore in (1·3) at $r\to\infty$. (ii) Different closed as well as open, static or non-static, models can be included by suitably restricting the values of G, P_0 and Q_0.

The pressure p_0 and density ρ_0 of the cosmic fluid in (1·3) are

$$8\pi p_0=-\frac{d^2G}{dT^2}-\frac{3}{4}\left(\frac{dG}{dT}\right)^2-e^{-G}\left(Q_0-\frac{P_0^2}{4}\right)+\Lambda,$$

$$8\pi\rho_0=\frac{3}{4}\left(\frac{dG}{dT}\right)^2+3e^{-G}\left(Q_0-\frac{P_0^2}{4}\right)-\Lambda. \tag{1·5}$$

For the process of complete conversion of matter into radiation, the following idealized situation is visualized. At the instant $t=0$, the matter at the origin of (1·3) starts emitting radiation. At $t=t_1$, the last material speck at the origin is converted into radiation. Then at much later time t, we have a shell of radiation moving outwards from the origin. Let $r=R_0(t)$ and $r=R_1(t)$ be respectively the outer and the inner boundary of this radiation shell. Both the boundaries are null-surfaces, being respectively the first and the last wavefront of the radiation emitted.

Outside the shell, in the region $r>R_0$, is the original cosmological space-time (1·3). With the help of the well-known boundary conditions of O'Brien and Synge[4] it has already been shown by us[5] that a pure radiation field can never be connected with a homogeneous universe with non-zero density of matter. We therefore take the shell region $R_1\leq r\leq R_0$ to be occupied by a mixture of matter and flowing radiation. The application of the O'Brien-Synge conditions at the outer boundary $r=R_0$ will detemine completely the field within the shell in terms of g_{ij} of the outer cosmological model. When the same conditions are applied at the inner boundary $r=R_1$, it is found that the field in the region $0\leq r\leq R_1$ must be a cosmological one but with a curvature different from that of the cosmological model in the region $r>R_0$.

In § 2 we shall set up Einstein's field equations of general relativity appropriate to the problem and in § 3 present two rigorous solutions which represent the gravitational field within a shell of a mixture of matter and flowing radiation. In § 4 we shall study the application of the O'Brien-Synge conditions and show that the shell must be bounded by two different cosmological models. In view of the idealized situation mentioned earlier, we can interpret the solutions as representing the field after the complete conversion of a mass particle into radiation in a homogeneous universe.

§ 2. The field equations

The energy tensor for a mixture of matter and flowing radiation is

$$T_i^j=(p+\rho)v_iv^j-pg_i^j+\sigma w_iw^j \tag{2·1}$$

where p is the pressure and ρ is the density of matter while σ is the density of flowing radiation.[6] In a co-moving frame of reference we can take

$$v^1=v^2=v^3=0, \quad v_4v^4=1, \tag{2·2}$$

$$w^2=w^3=0, \quad w_iw^4=0, \quad w^i{}_{;j}w^j=0. \tag{2·3}$$

(2·1) then leads to

$$T_1^1=-p+\sigma w_1w^1, \quad T_2^2=T_3^3=-p,$$
$$T_4^4=\rho+\sigma w_4w^4, \quad T_4^1=\sigma w_4w^1. \tag{2·4}$$

There is one equation satisfied by the components of T_i^j which is

$$T_1^1-T_2^2=\frac{w_1}{w_4}T_4^1. \tag{2·5}$$

The Einstein field equations

$$R_i^j-\frac{1}{2}Rg_i^j+\Lambda g_i^j=G_i^j=-8\pi T_i^j$$

then lead to

$$-G_1^1+G_2^2=-\frac{w_1}{w_4}G_4^1, \tag{2·6}$$

$$8\pi p=G_2^2, \quad 8\pi\rho=-G_1^1+G_2^2-G_4^4, \quad 8\pi\sigma w_4w^1=-G_4^1. \tag{2·7}$$

§ 3. Rigorous solutions of the field equations

A line-element, exhibiting spherical symmetry in a co-moving reference frame can be taken as

$$ds^2=-e^\alpha dr^2-e^\beta r^2(d\theta^2+\sin^2\theta\, d\phi^2)+e^\gamma dt^2, \tag{3·1}$$
$$\alpha=\alpha(r,t), \quad \beta=\beta(r,t), \quad \gamma=\gamma(r,t).$$

Once g_{ij} in (3·1) are determined as functions of r and t, (2·7) determine the march of the physical variables p, ρ and σ. But (2·6) is the only differential equation for the determination of the three unknown functions α, β and γ. Two more relations are therefore necessary. These relations are not supplied by the theory of gravitation but are given by the physical nature of the field under consideration. They are: (i) the equation of state of the fluid of the distribution usually expressed as a relation between p and ρ, (ii) the law of radiation absorption of the medium. Given these two relations, (2·6) can be used to determine the three unknown functions α, β and γ.

But even in the static case (where $\sigma=0$) the problem treated in the above direct way becomes mathematically difficult. We therefore adopt the indirect procedure suggested by Tolman[7] of assuming two more relations between g_{ij} and/or their derivatives, along with Eq. (2·6) and then finding whether the resulting solution is physically interesting or not.

The components of the tensor $G_i^j=R_i^j-(1/2)Rg_i^j+\Lambda g_i^j$ for (3·1) are

$$-G_1^1=-e^{-\alpha}\left[\frac{\beta'^2}{4}+\frac{\beta'y'}{2}+\frac{\beta'+y'}{r}+\frac{1}{r^2}\right]+\frac{e^{-\beta}}{r^2}$$
$$+e^{-y}\left[\ddot{\beta}+\frac{3}{4}\dot{\beta}^2-\frac{\dot{\beta}\dot{y}}{2}\right]-\Lambda,$$

$$-G_2^2=-G_3^3=-e^{-\alpha}\left[\frac{\beta''}{2}+\frac{y''}{2}+\frac{\beta'^2}{4}+\frac{y'^2}{4}-\frac{\alpha'\beta'}{4}+\frac{\beta'y'}{4}\right.$$
$$\left.-\frac{\alpha'y'}{4}-\frac{\alpha'}{2r}+\frac{\beta'}{r}+\frac{y'}{2r}\right]+e^{-y}\left[\frac{\ddot{\alpha}}{2}+\frac{\ddot{\beta}}{2}+\frac{\dot{\alpha}^2}{4}+\frac{\dot{\beta}^2}{4}+\frac{\dot{\alpha}\dot{\beta}}{4}\right.$$
$$\left.-\frac{\dot{\alpha}\dot{y}}{4}-\frac{\dot{\beta}\dot{y}}{4}\right]-\Lambda,$$

$$-G_4^4=-e^{-\alpha}\left[\beta''+\frac{3}{4}\beta'^2-\frac{\alpha'\beta'}{2}+\frac{3\beta'}{r}-\frac{\alpha'}{r}+\frac{1}{r^2}\right]+\frac{e^{-\beta}}{r^2}$$
$$+e^{-y}\left[\frac{\dot{\alpha}\dot{\beta}}{2}+\frac{\dot{\beta}^2}{4}\right]-\Lambda,$$

$$-G_1^4=-e^{-y}\left[\dot{\beta}'-\frac{\dot{\beta}y'}{2}+(\dot{\beta}-\dot{\alpha})\frac{\beta'}{2}+(\dot{\beta}-\dot{\alpha})\frac{1}{r}\right]. \qquad (3\cdot2)$$

Here a dash indicates differentiation with regard to r and a dot that with regard to t. Eq. (2·6) in coordinates of (3·1) becomes

$$-G_1^1+G_2^2=-\frac{w^1}{w^4}G_1^4. \qquad (3\cdot3)$$

In the case of outward flow of radiation, we have

$$\frac{w^1}{w^4}=+e^{(y-\alpha)/2}. \qquad (3\cdot4)$$

Eq. (3·3), after substitution from (3·2) and (3·4), becomes

$$\frac{\beta''}{2}+\frac{y''}{2}+\frac{y'^2}{4}-\frac{\alpha'\beta'}{4}-\frac{\beta'y'}{4}-\frac{\alpha'y'}{4}-\frac{\alpha'}{2r}-\frac{y'}{2r}-\frac{1}{r^2}+\frac{e^{\alpha-\beta}}{r^2}$$
$$+e^{\alpha-y}\left[\frac{\ddot{\beta}}{2}-\frac{\ddot{\alpha}}{2}-\frac{\dot{\alpha}^2}{4}+\frac{\dot{\beta}^2}{2}-\frac{\dot{\alpha}\dot{\beta}}{4}-\frac{\dot{\beta}\dot{y}}{4}+\frac{\dot{\alpha}\dot{y}}{4}\right]$$

$$+e^{(\alpha-y)/2}\left[\dot{\beta}'-\frac{\dot{\beta}y'}{2}+(\dot{\beta}-\dot{\alpha})\frac{\beta'}{2}+(\dot{\beta}-\dot{\alpha})\frac{1}{r}\right]=0. \tag{3·5}$$

The form of the solution for a radiating star in empty space[1] suggests that a function $m(r, t)$ conserved along the world lines of flow of radiation will become very useful in discussing the solution of (3·5). A conserved function m satisfies the equation

$$m'w^1+\dot{m}w^4=0,$$

i. e.

$$m'e^{y/2}+\dot{m}e^{\alpha/2}=0. \tag{3·6}$$

We now put

$$\alpha=\psi(t)+\lambda(r, m),\quad \beta=\psi(t)+\mu(r, m),\quad y=y(r, m). \tag{3·7}$$

With these substitutions (3·6) for m is immediately integrable, giving

$$\int e^{(\lambda-y)/2}dr-\int e^{-\psi/2}dt=\Phi(m) \tag{3·8}$$

where Φ is an arbitrary function of m and in evaluating $\int e^{(\lambda-y)/2}\,dr$, m is to be regarded as a constant. To find m', (3·8) is differentiated with regard to r, giving

$$e^{(\lambda-y)/2}+\frac{\partial}{\partial m}\left[\int e^{(\lambda-y)/2}\,dr\right]m'=\frac{d\Phi}{dm}m'. \tag{3·9}$$

Thus in our further work, m' will occur as a function of r and m given by (3·9). Differentiating (3·9) partially with regard to r treating m as constant, it is easy to derive

$$\lambda'-y'=\frac{2}{m'}\frac{\partial m'}{\partial r}. \tag{3·10}$$

In (3·10) we have two types of derivatives with regard to r. We shall follow the convention that a dash over a function will denote partial differentiation with regard to r when it is a function of r and t, while the operator $\partial/\partial r$ will indicate the corresponding differentiation when it is a function of r and m.

Substitution of (3·6), (3·7) and (3·10) in (3·5) leads to

$$\begin{aligned}&\frac{1}{2}\frac{\partial^2\mu}{\partial r^2}-\frac{1}{4}\frac{\partial\lambda}{\partial r}\frac{\partial\mu}{\partial r}-\frac{1}{2r}\frac{\partial\lambda}{\partial r}+\frac{\partial\lambda}{\partial m}\frac{\partial\mu}{\partial r}\frac{m'}{4}+\frac{\partial\lambda}{\partial m}\frac{m'}{2r}\\&-\frac{\partial\mu}{\partial r}\frac{\partial\mu}{\partial m}\frac{m'}{2}-\frac{\partial\mu}{\partial m}\frac{m'}{r}-\frac{\partial^2\lambda}{\partial m^2}\frac{m'^2}{2}-\frac{\partial\lambda}{\partial m}\frac{\partial m'}{\partial m}\frac{m'}{2}\\&-\frac{1}{r^2}+\frac{e^{\lambda-\mu}}{r^2}+\frac{y''}{2}-\frac{y'}{2m'}\frac{\partial m'}{\partial r}-\frac{y'}{4}\frac{\partial\mu}{\partial r}-\frac{y'}{2r}\\&+\frac{\dot{\psi}}{2}\frac{m'}{\dot{m}}\left[\frac{\partial\mu}{\partial m}m'-\frac{\partial\lambda}{\partial m}m'+y'\right]=0.\end{aligned} \tag{3·11}$$

In order that λ, μ and y may be determined as functions of r and m only, independent of the function $\psi(t)$, we equate the coefficient of $\dot{\psi}$ in (3·11) to zero and get

Assumption I: $$y'=\left(\frac{\partial\lambda}{\partial m}-\frac{\partial\mu}{\partial m}\right)m'. \tag{3·12}$$

As mentioned before, we have to assume one more relation, which we take as

Assumption II: $$m'e^{\mu-\lambda}=1, \tag{3·13}$$

such that m' is always positive. (3·6) will then suggest that $\dot{m}$ will be negative for outward flow of radiation (w^1/w^4 positive). Eliminating λ, μ and y from (3·11) with the help of (3·10), (3·12) and (3·13), we get the following differential equation for m'

$$-\frac{1}{(m')^2}\left(\frac{\partial m'}{\partial r}\right)^2+\frac{1}{2m'}\frac{\partial^2 m'}{\partial r^2}-\frac{1}{rm'}\frac{\partial m'}{\partial r}+\frac{m'}{r^2}-\frac{1}{r^2}=0. \tag{3·14}$$

The most general solution of (3·14) is

$$m'=[1+Pr+Qr^2]^{-1}, \tag{3·15}$$

where P and Q are arbitrary functions of m only. From (3·10), (3·12) and (3·13) we have

$$\frac{\partial\mu}{\partial m}m'+\frac{\partial\mu}{\partial r}=\frac{1}{m'}\frac{\partial m'}{\partial r}$$

which, after substitution from (3·15), can be written as

$$\frac{\partial\mu}{\partial m}+\frac{\partial\mu}{\partial r}(1+Pr+Qr^2)=-(P+2Qr). \tag{3·16}$$

Solution I: $Q=0$

Equation (3·15) for m is immediately solvable in the case $Q=0$, the solution being

$$re^{-\int P\,dm}-\int e^{-\int P\,dm}\,dm=\phi, \tag{3·17}$$

ϕ being an arbitrary function of t only. Differentiating (3·17) with regard to t, we get

$$\dot{m}=-\dot{\phi}e^{\int P\,dm}(1+Pr)^{-1}. \tag{3·18}$$

The solution of (3·16) with $Q=0$ is

$$\mu=-\int P\,dm+F(t)$$

where F is an arbitrary function of t. Since $F(t)$ can be absorbed in $\psi(t)$, we take

$$e^{\mu}=e^{-\int P\bar{d}m}. \tag{3·19}$$

(3·13) then gives

$$e^{\lambda}=e^{-\int P\bar{d}m}(1+Pr)^{-1}, \tag{3·20}$$

while (3·6), (3·18) and (3·20) lead to

$$e^{\nu}=e^{\psi}\dot{\phi}^2 e^{\int P\bar{d}m}(1+Pr)^{-1}. \tag{3·21}$$

If we introduce a new time coordinate τ in place of t by putting

$$e^{\psi(t)}\dot{\phi}dt=d\tau, \tag{3·22}$$

the final solution in the region $R_1\leq r\leq R_0$ is given by

$$ds^2=-\frac{\bar{r}^2}{r^2}\frac{dr^2}{1+Pr}-\bar{r}^2(d\theta^2+\sin^2\theta\, d\phi^2)+\frac{r^2}{\bar{r}^2}\frac{d\tau^2}{1+Pr},$$

$$\bar{r}^2=e^{\psi-\int P\bar{d}m}r^2. \tag{3·23}$$

Equations (2·7), after substitution for G_i^j lead to

$$\begin{aligned}
8\pi p&=\frac{\bar{r}^2}{r^2}(1+Pr)\left[-\overset{**}{\phi}-\frac{5}{4}\overset{*}{\phi}^2\right]-\frac{3}{2}\overset{*}{\phi}P,\\
8\pi\rho&=\frac{\bar{r}^2}{r^2}(1+Pr)\frac{3}{4}\overset{*}{\phi}^2+\frac{3}{2}\overset{*}{\phi}P, \qquad (3·24)\\
8\pi\sigma&=\frac{1}{2(w^1)^2}\frac{r^3}{\bar{r}^2}\overset{*}{\phi}\frac{dP}{dm}.
\end{aligned}$$

Here an overhead star indicates differentiation with regard to τ.

Solution II: $Q\neq 0$

In the case when $Q\neq 0$, the general solution of (3·15) is

$$r=-\frac{1}{Q}\frac{\phi\frac{dA}{dm}+\frac{dB}{dm}}{\phi A+B}. \tag{3·25}$$

Here ϕ is an arbitrary function of t only, while A and B are undetermined functions of m. Differentiating (3·25) with regard to r and eliminating ϕ with the help of (3·25), we get

$$m'=\left[\frac{\frac{dA}{dm}\frac{d^2B}{dm^2}-\frac{d^2A}{dm^2}\frac{dB}{dm}}{Q\left(A\frac{dB}{dm}-\frac{dA}{dm}B\right)}+\left\{\frac{A\frac{d^2B}{dm^2}-\frac{d^2A}{dm^2}B}{A\frac{dB}{dm}-\frac{dA}{dm}B}-\frac{1}{Q}\frac{dQ}{dm}\right\}r+Qr^2\right]^{-1}. \tag{3·26}$$

Comparing (3·26) with (3·15), we get

$$P+\frac{1}{Q}\frac{dQ}{dm}=\frac{A\frac{d^2B}{dm^2}-\frac{d^2A}{dm^2}B}{A\frac{dB}{dm}-\frac{dA}{dm}B}, \tag{3·27}$$

$$Q=\frac{\frac{dA}{dm}\frac{d^2B}{dm^2}-\frac{d^2A}{dm^2}\frac{dB}{dm}}{A\frac{dB}{dm}-\frac{dA}{dm}B}. \tag{3·28}$$

Integrating (3·27) with regard to m, we get

$$Qe^{\int P dm}=\epsilon\left[A\frac{dB}{dm}-\frac{dA}{dm}B\right],$$

where $\epsilon=\pm 1$ and the arbitrary constant of integration is put equal to 1. Integrating this once again with respect to m, we have

$$B=\epsilon A\int\frac{Q}{A^2}e^{\int P dm}dm, \tag{3·29}$$

where the arbitrary constant of integration is taken to be zero.

Substitution of (3·29) in (3·28) leads to

$$\frac{d^2A}{dm^2}-\frac{dA}{dm}\left(P+\frac{1}{Q}\frac{dQ}{dm}\right)+AQ=0. \tag{3·30}$$

Differentiation of (3·25) with regard to t, gives

$$\dot{m}=-\frac{\dot{\phi}}{\epsilon}\frac{\left(AQr+\frac{dA}{dm}\right)^2}{Q^2e^{\int P dm}}(1+Pr+Qr^2)^{-1}. \tag{3·31}$$

Since $\dot{m}$ is negative, $\dot{\phi}$ and ϵ should be of the same sign.

The solution of (3·16) is

$$\mu=2\log(\phi A+B)-\int P dm+F(t)$$

where F is an arbitrary function of t alone. Since $F(t)$ can be absorbed in $\psi(t)$, we take

$$e^{\mu/2+(1/2)\int P dm}=\phi A+B.$$

Substituting ϕ from (3·25) in the above equation, we get

$$e^{\mu/2}=-\frac{\epsilon Qe^{1/2\int P dm}}{AQr+\frac{dA}{dm}}. \tag{3·32}$$

(3·13) gives, noting that $\epsilon^2=1$,

$$e^{\lambda}=\frac{Q^2 e^{\int P dm}}{\left(AQr+\frac{dA}{dm}\right)^2}(1+Pr+Qr^2)^{-1} \tag{3·33}$$

while (3·6), (3·31) and (3·33) give

$$e^{\nu}=\frac{e^{\psi}\dot{\phi}^2\left(AQr+\frac{dA}{dm}\right)^2}{Q^2 e^{\int P dm}}(1+Pr+Qr^2)^{-1}. \tag{3·34}$$

If we introduce a new time coordinate τ in place of t by putting

$$d\tau=e^{\psi}\dot{\phi}\epsilon\, dt, \tag{3·35}$$

the general solution can be written as

$$ds^2=-\frac{\bar{r}^2}{r^2}\frac{dr^2}{1+Pr+Qr^2}-\bar{r}^2(d\theta^2+\sin^2\theta\, d\phi^2)+\frac{r^2}{\bar{r}^2}\frac{d\tau^2}{1+Pr+Qr^2},$$

$$\bar{r}^2=\frac{e^{\psi}Q^2 e^{\int P dm}}{\left(AQr+\frac{dA}{dm}\right)^2}r^2, \tag{3·36}$$

where A is given by (3·30). Equations (2·7), after substitution for G_i^j lead to

$$8\pi p=\frac{\bar{r}^2}{r^2}(1+Pr+Qr^2)\left\{-\overset{**}{\phi}-\frac{5}{4}\overset{*}{\phi}^2\right\}-\frac{3\overset{*}{\phi}}{AQr+\frac{dA}{dm}}\left\{Q\left(\frac{dA}{dm}r-A\right)\right.$$

$$\left.-\frac{P}{2}\left(AQr-\frac{dA}{dm}\right)\right\}-\frac{3}{e^{\psi}Qe^{\int P dm}}\left\{\left(\frac{dA}{dm}\right)^2+QA^2-A\frac{dA}{dm}P\right\}+\Lambda,$$

$$8\pi\rho=\frac{\bar{r}^2}{r^2}(1+Pr+Qr^2)\frac{3}{4}\overset{*}{\phi}^2+\frac{3\overset{*}{\phi}}{AQr+\frac{dA}{dm}}\left\{Q\left(\frac{dA}{dm}r-A\right)\right.$$

$$\left.-\frac{P}{2}\left(AQr-\frac{dA}{dm}\right)\right\}+\frac{3}{e^{\psi}Qe^{\int P dm}}\left\{\left(\frac{dA}{dm}\right)^2+QA^2-A\frac{dA}{dm}P\right\}-\Lambda,$$

$$8\pi\sigma(w')^2=\frac{r^4}{\bar{r}^4}\left[\left\{\frac{\frac{dA}{dm}}{AQr+\frac{dA}{dm}}+\frac{\overset{*}{\phi}}{2}\frac{\bar{r}^2}{r}\right\}\frac{dQ}{dm}r\right.$$

$$\left.+\left\{\frac{\overset{*}{\phi}}{2}\frac{\bar{r}^2}{r^2}-\frac{AQ}{AQr+\frac{dA}{dm}}\right\}\frac{dP}{dm}r\right] \tag{3·37}$$

where an overhead star indicates differentiation with regard to τ.

§ 4. Jump conditions at the boundaries of the radiation shell

The jump conditions at discontinuities in general relativity have been given by O'Brien and Synge.[4] We have transformed the O'Brien-Synge conditions at a null-boundary to suit the system of coordinates used in the present work. Accordingly at a null-surface $r=R(t)$ in (3·1) e^{α}, e^{β}, e^{γ}, β', $T_1^1-\dot{R}T_1^4$ and $T_4^4+\dot{R}T_1^4$ must be continuous. Since $\dot{R}=w^1/w^4$ at $r=R(t)$, with the help of (2·6) and (2·7), one can say that the O'Brien-Synge conditions at a null-boundry $r=R(t)$ in (3·1) are equivalent to the continuity of g_{ij}, β', p and ρ at $r=R(t)$.

The both boundaries of the radiation shell are wavefronts of the radiation emitted. All the conserved functions retain constant values on a wavefront. We shall adopt a general convention that the constant value of a conserved function Z of the field over a null-boundary $r=R_n$ will be indicated by writing Z in brackets with an inferior n, as $(Z)_n$.

Solution I: *A radiation shell flowing in an open universe with imaginary curvature*

We shall now see that our solution I can be fitted on at $r=R_0$ and at $r=R_1$ to two different open homogeneous models.

Let us take, for $r\geq R_0$, the line-element

$$ds^2=-\frac{\bar{r}_0^2}{r^2}\frac{dr^2}{1+P_0 r}-\bar{r}_0^2(d\theta^2+\sin^2\theta\, d\phi^2)+dt^2,$$

$$\bar{r}_0^2=\frac{e^{g(t)}}{1+P_0 r}r^2, \tag{4·1}$$

which is (1·3) with $Q_0=0$. Since, from (1·4),

$$-Z_0^2=\frac{4}{P_0^2}, \tag{4·2}$$

the radius of curvature $Z=Z_0 e^{g/2}$ of this universe will be imaginary.

If the radiation shell in the region $R_1\leq r\leq R_0$ is given by (3·23) which represents our solution I, it can be fitted on the open universe (4·1) at the null-boundary $r=R_0$. The continuity of g_{ij}, at $r=R_0$, gives

$$(P)_0=P_0,\quad e^{\psi}=\frac{e^{g(t)+(\int P dm)_0}}{1+P_0 R_0},\quad d\tau=e^{g/2}dt. \tag{4·3}$$

The conditions of continuity of β', p and ρ are then automatically satisfied at $r=R_0$.

At the inner boundary $r=R_1$ of the radiation shell, it is found that, the field (3·23) can be fitted on to the Einstein-de Sitter universe with zero curvature

$$ds^2=-\frac{\bar{r}_1^2}{r^2}dr^2-\bar{r}_1^2(d\theta^2+\sin^2\theta\, d\phi^2)+dT^2,$$

$$\bar{r}_1^2=e^{G(T)}r^2, \tag{4·4}$$

which is again (1·3) with $P_0=Q_0=0$. The continuity of g_{ij}, at $r=R_1$, gives

$$(P)_1=0,\quad e^{\psi}=e^{G(T)+(\int P dm)_1},\quad d\tau=e^{G/2}dt. \tag{4·5}$$

The continuity of β', p and ρ is then satisfied at $r=R_1$. (4·3) and (4·5) together lead to

$$e^{G(T)}=e^{g(t)+(\int P dm)_0-(\int P dm)_1}[1+P_0R_0]^{-1},$$
$$dT=e^{(g-G)/2}dt. \tag{4·6}$$

The density σ of flowing radiation in the region $R_1\leq r\leq R_0$ can now be written as

$$8\pi\sigma=\frac{1}{2(w^1)^2}e^{-(3/2)G-(\int P dm)_1+\int P dm}\frac{dP}{dm}\frac{dG}{dT}r. \tag{4·7}$$

Since σ has to be positive, (4·7) suggests that if the background universe is expanding (i.e. $dG/dT>0$), dP/dm should be positive in the radiation zone $R_1\leq r\leq R_0$. Since m' is essentially positive, this indicates that dP/dr should be positive in this region. As P is zero on the inner boundary, it follows that P must be positive all through the shell and on the outer boundary, $P_0>0$.

Thus the boundary conditions at the two boundaries $r=R_0$ and $r=R_1$ determine the nature of the field in the region $0\leq r<R_1$ and also indicate the march of the arbitrary function P within the radiation shell.

The radial velocity of the radiation in the region $R_1\leq r\leq R_0$ is, from (3·23), given by

$$\frac{dr}{d\tau}=e^{-\psi+\int P dm}$$

so that

$$\frac{d}{dr}\left(\log\frac{dr}{d\tau}\right)=Pm'>0,$$

since m' is positive and P is positive throughout the shell. Thus $dr/d\tau$ is an increasing function of r, hence the outer boundary $r=R_0$ is moving faster than the inner boundary $r=R_1$. This precludes the possibility of a collision between the two boundaries. As the radiation shell moves outwards, the observable universe round the origin will be the Einstein-de Sitter universe with zero curvature.

Solution II: *A radiation shell flowing in a closed universe*

Here it will be shown that a particular case of our solution II representing the field within a radiation shell in the region $R_1\leq r\leq R_0$ can be fitted at $r=R_0$ and at $r=R_1$ to two different closed homogeneous models.

Let us begin with the closed static spherically symmetric Einstein universe, given by the line-element

$$ds^2=-\frac{\bar{r}_0^2}{r^2}\frac{dr^2}{1+P_0r+Q_0r^2}-\bar{r}_0^2(d\theta^2+\sin^2\theta\,d\phi^2)+dt^2,$$

$$\bar{r}_0^2=\frac{r^2}{1+P_0 r+Q_0 r^2} \tag{4·8}$$

which is the static form of (1·3). The radius of curvature Z_0 is given by

$$1/Z_0^2=Q_0-P_0^2/4. \tag{4·9}$$

Here P_0 and Q_0 are suitable positive constants and Q_0 is sufficiently greater than $P_0^2/4$ so that the curvature $1/Z_0^2$ is positive.

Let the matter at the origin of (4·8) be completely converted into radiation. Let $r=R_0$ and $r=R_1$ be respectively the first and the last wavefront of the emitted radiation at a much later time t. The gravitational field within the radiation shell in the region $R_1\leq r\leq R_0$ can be given by the solution (3·36). But we take a particular case of (3·36) and write for the arbitrary function $P(m)$,

$$P=P_0=\text{the positive constant in (4·9)},$$

while retaining the arbitrariness of the function $Q(m)$. We can then write the line-element, giving the field within the radiation shell in the region $R_1\leq r\leq R_0$, as

$$ds^2=-\frac{\bar{r}^2}{r^2}\frac{dr^2}{1+P_0 r+Qr^2}-\bar{r}^2(d\theta^2+\sin^2\theta\, d\phi^2)+\frac{r^2}{\bar{r}^2}\frac{d\tau^2}{1+P_0 r+Qr^2},$$

$$\bar{r}^2=\frac{e^{\psi}Q^2e^{P_0 m}}{\left(AQr+\dfrac{dA}{dm}\right)^2}r^2. \tag{4·10}$$

The differential equation for A, in this case, is

$$\frac{d^2A}{dm^2}-\frac{dA}{dm}\left(P_0+\frac{1}{Q}\frac{dQ}{dm}\right)+AQ=0. \tag{4·11}$$

The field (4·10) can be fitted into the static Einstein universe (4·8) at the null-boundary $r=R_0$. The continuity of g_{ij} at $r=R_0$ gives

$$(Q)_0=Q_0,$$

$$e^{\psi}=\frac{\left\{(A)_0Q_0R_0+\left(\dfrac{dA}{dm}\right)_0\right\}^2}{Q_0^2e^{P_0(m)_0}\{1+P_0R_0+Q_0R_0^2\}},$$

$$d\tau=dt. \tag{4·12}$$

β', p and ρ are then found to be continuous at $r=R_0$.

At the inner boundary $r=R_1$ of the radiation shell, the field (4·10) can be fitted on to an expanding cosmological model, given as

$$ds^2=-\frac{\bar{r}_1^2}{r^2}\frac{dr^2}{1+P_0 r+Q_1 r^2}-\bar{r}_1^2(d\theta^2+\sin^2\theta\, d\phi^2)+dT^2,$$

$$\bar{r}_1^2=\frac{e^{G(T)}}{1+P_0 r+Q_1 r^2}r^2, \tag{4·13}$$

where Q_1 is a constant. The radius of curvature of this universe is given by $Z=Z_1 e^{G/2}$ where Z_1 is a constant given by

$$\frac{1}{Z_1^2}=Q_1-\frac{P_0^2}{4}\,. \tag{4·14}$$

The continuity of g_{ij} at $r=R_1$ gives

$$\begin{aligned}
&(Q)_1=Q_1,\\
&e^{\psi}=\frac{\left\{(A)_1 Q_1 R_1+\left(\frac{dA}{dm}\right)_1\right\}^2 e^{G(T)}}{Q_1^2 e^{P_0(m)_1}\{1+P_0R_1+Q_1R_1^2\}},\\
&d\tau=e^{G/2}dT.
\end{aligned} \tag{4·15}$$

Equations (4·15) are sufficient for the continuity of other junction conditions. (4·12) and (4·15) together lead to

$$e^{G(T)}=\frac{\left\{(A)_0 Q_0 R_0+\left(\frac{dA}{dm}\right)_0\right\}^2}{\left\{(A)_1 Q_1 R_1+\left(\frac{dA}{dm}\right)_1\right\}^2}\,\frac{Q_1^2}{Q_0^2}\,\frac{e^{P_0(m)_0}}{e^{P_0(m)_1}}\,\frac{\{1+P_0R_1+Q_1R_1^2\}}{\{1+P_0R_0+Q_0R_0^2\}}$$

$$dT=e^{-G/2}dt. \tag{4·16}$$

The equation giving the density σ of flowing radiation in the region $R_1\leq r\leq R_0$ can be written as

$$8\pi\sigma(w^1)^2=\frac{r^4}{\bar{r}^4}\left[\frac{\frac{dA}{dm}}{AQr+\frac{dA}{dm}}+\frac{\overset{*}{\phi}}{2}\,\frac{\bar{r}^2}{r^2}\right]\frac{dQ}{dm}\,r. \tag{4·17}$$

(4·17) at the boundary $r=R_0$ will give

$$8\pi\sigma(w^1)^2=(1+P_0R_0+Q_0R_0^2)\left[1+\frac{1}{2}P_0R_0\right]\left(\frac{dQ}{dm}\right)_0 R_0 \tag{4·18}$$

while at the boundary $r=R_1$ (4·17) gives

$$8\pi\sigma(w^1)^2=e^{-2G}(1+P_0R_1+Q_1R_1^2)\left[1+\frac{1}{2}P_0R_1+\frac{1}{2}\frac{dG}{dT}e^{G/2}R_1\right]\left(\frac{dQ}{dm}\right)_1 R_1. \tag{4·19}$$

Since σ has to be positive everywhere, (4·18) and (4·19) both suggest that dQ/dm should be positive all throughout the region $R_1\leq r\leq R_0$. But m' is also positive and therefore Q is an increasing function of r in $R_1\leq r\leq R_0$, so that $Q_0>Q_1>0$, hence from (4·9) and (4·14)

$$\frac{1}{Z_0^2} > \frac{1}{Z_1^2}.$$

The radial velocity of radiation in the region $R_1 \leq r \leq R_0$ is

$$\frac{dr}{d\tau} = \frac{\left(AQr + \frac{dA}{dm}\right)^2}{e^{\psi} Q^2 e^{P_0 m}}.$$

Using (4·11), it can be shown that

$$\frac{d}{dr}\left(\log \frac{dr}{d\tau}\right) = \frac{P_0 + 2Qr}{1 + P_0 r + Qr^2} > 0,$$

since P_0 and Q are both positive. This means that $dr/d\tau$ is an increasing function of r and therefore the outer boundary $r = R_0$ is moving faster than the inner boundary $r = R_1$. This precludes the possibility of a collision b tween the two boundaries. When the radiation shell has receded to an infinite distance, the observable universe around the origin will be one with reduced curvature and given by (4·13). Thus the effect of complete conversion of matter at the origin of a closed universe appears to be to reduce the curvature of the universe.

The process can be repeated. The complete conversion of matter at the origin of (4·13) will give rise to another universe in place of (4·13) with a still reduced curvature. If we imagine successive conversions of this type, the value of $1/Z_0^2$ goes on decreasing. Q takes up smaller and smaller value in the curvature relation

$$\frac{1}{Z_0^2} = Q - \frac{P_0^2}{4}$$

where P_0 is a constant. As $Q \to P_0^2/4$, $1/Z_0^2 \to 0$. Thus by considering the accumulated effects of a series of such conversions, the above solution II presents a model which implies that a closed universe will ultimately tend to become the Einstein-de Sitter universe with zero curvature.

§ 5. Conclusions

Two rigorous solutions of the field equations of general relativity are presented in this paper. They have the following characteristics:

(i) They are regular everywhere.

(ii) For $R_1 \leq r \leq R_0$, they represent the field within a radiation shell travelling radially away from the origin through a distribution of matter of non-zero density and pressure.

(iii) The boundaries $r = R_0(t)$ and $r = R_1(t)$ are the wavefronts of flowing radiation.

(iv) The field within the radiation shell passes off continuously at the two boundaries of the shell to different cosmological models, and the pressure and the

density of matter within the shell pass over continuously at the two boundaries of the shell to the pressure and the density of cosmic fluids on the two sides.

(v) We cannot have flat space-time on either side of the shell. As soon as one switches over to the flat universe on either side of the radiation shell, the radiation density in the shell disappears.

Both the solutions presented in this paper may be interpreted as describing the gravitational situation after the complete conversion of matter into radiation at the origin of a homogeneous universe—an open universe with imaginary curvature in the case of the first solution while a closed static or expanding universe in the case of the second solution. In the both cases the radiation shell can be fitted at the inner boundary on to an expanding universe, the curvature of which is less than the universe outside the shell.

When the last wavefront of the radiation emitted has receded to the limits of observation, the observable universe round the origin is the one which was on the inner side of the shell, having a curvature smaller than that of the universe in which the complete conversion took place. Thus an effect of the total conversion of matter into radiation is to reduce the curvature of the observable universe. On the strength of the model presented here one may stretch this conclusion a little further and say that as a result of such conversions of matter into radiation either an open universe with imaginary curvature or a closed universe, after the lapse of considerable time, will tend to become the Einstein-de Sitter universe with zero curvature.

References

1) P. C. Vaidya, Proc. Indian Acad. Sci. **A33** (1951), 264.
2) A. K. Raychaudhuri, Z. Phys. **135** (1953), 225.
3) W. Israel, Proc. Roy. Soc. **248** (1958), 404.
4) S. O'Brien and J. L. Synge, Commun. Dublin Inst. Adv. Stud. A, No. 9 (1952).
5) P. C. Vaidya and K. B. Shah, Proc. Nat. Inst. Sci. India, **23A** (1957), 534.
6) P. C. Vaidya, Phys. Rev. **83** (1951), 10.
7) R. C. Tolman, Phys. Rev. **55** (1939), 364.
8) P. C. Vaidya and K. B. Shah, Curr. Sci. **28** (1959), 399.

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A Study on the Low Energy *n-p* System

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The deuteron properties and the zero energy *n-p* scattering parameters are computed for a number of potentials. These potentials are characterized by the one-pion-exchange tail and the semi-phenomenological inner interaction. The numerical results reported will provide a basis on which one can estimate the consequence of any reasonable potential consistent with the pion theory as far as the low energy *n-p* system is concerned.

§ 1. Introduction

It is by now generally agreed that the nucleon-nucleon interaction has, at least when the energy of the system is not too high, a unique tail of the one-pion-exchange potential (OPEP). Unfortunately, the available extensive calculations on the low energy *n-p* system, for example those of Feshbach-Schwinger[1] and Kalos et al.,[2] do not take proper account of this nature of the nuclear forces. The purpose of this paper is to present the results of the analysis of the low energy *n-p* system for a variety of potentials all with the OPEP tail.

The present work is closely related to the one by Iwadare et al.[3] Those authors have studied the consequence of the OPEP tail properly taking into account the reliability of the pion theory of nuclear forces. The actual numerical results presented in the following sections confirm various conclusions drawn by them. Also, these numerical results will provide a basis on which one can estimate the consequence of any reasonable potential with the OPEP tail as far as the low energy *n-p* system is concerned.

The conventional variation-iteration method[1] of solving the deuteron problem consists in finding the strength of the potential of given shape (including the hard core radius, if any) in order to produce the correct binding energy. As such this method does not take advantage of the fact that the tail of the potential is well established. Nor does it reflect the fact that we know less and less about the interaction as the inter-nucleon distance decreases. In § 2 we shall first describe a method of the solution of the deuteron problem which seems more appropriate under those circumstances.

The method is then applied in § 3 to some 50 combinations of various pion-nucleon coupling constants and the inner potential forms to compute the deuteron properties.

In § 4 we turn to the study of the singlet even state. With some interpolation, the results will enable one to check any reasonable potential in the state concerned against the zero energy scattering parameters.

We have not considered any linear or quadratic spin-orbital forces in the present work. There is no positive evidence at present for the necessity of these forces in the $T=0$ state.[4] Since, however, the potential must have the OPEP tail at any rate in the energy region concerned, the present work will form a basis on which the effects of these forces may be investigated should they turn out necessary.

§ 2. Solution of the deuteron problem

Let the S and D radial wave functions (multiplied by r) of the deuteron be $u(r)$ and $w(r)$, respectively. They are the solutions of the well-known coupled equations

$$\begin{aligned}
&\frac{d^2u(r)}{dr^2}+\{\gamma^2-U_C(r)\}u(r)-2\sqrt{2}\,U_T(r)w(r)=0,\\
&\frac{d^2w(r)}{dr^2}+\{\gamma^2-(6/r^2)-U_C(r)+2U_T(r)\}w(r)-2\sqrt{2}\,U_T(r)u(r)=0,
\end{aligned}\tag{1}$$

where γ^{-1} is the deuteron radius, $U_C(r)$ and $U_T(r)$ are M times the central and tensor potential, respectively (M=nucleon mass, $\hbar=c=1$).

We are concerned with the solution of (1) under the situation that the shape and the strength of the potentials are given but the hard core radius r_0 is unknown. For this purpose we first observe that (1) has two sets of linearly independent solutions. As such we choose $(u^{(1)}, w^{(1)})$ and $(u^{(2)}, w^{(2)})$ which have the asymptotic forms

$$\begin{aligned}
&\begin{pmatrix}u^{(1)}(r)\\ w^{(1)}(r)\end{pmatrix}\simeq\begin{pmatrix}e^{-\gamma r}\\ 0\end{pmatrix},\\
&\begin{pmatrix}u^{(2)}(r)\\ w^{(2)}(r)\end{pmatrix}\simeq\begin{pmatrix}0\\ e^{-\gamma r}(1+3/(\gamma r)+3/(\gamma r)^2)\end{pmatrix}.
\end{aligned}\tag{2}$$

Starting at sufficiently large r where (2) holds to a sufficient accuracy, we can integrate (1) inward to get solutions $(u^{(1)}, w^{(1)})$ and $(u^{(2)}, w^{(2)})$ at arbitrary r. The correct deuteron wave function can be expressed as the linear superposition of these two sets of solutions:

$$\begin{pmatrix}u(r)\\ w(r)\end{pmatrix}=A_S\begin{pmatrix}u^{(1)}(r)\\ w^{(1)}(r)\end{pmatrix}+A_D\begin{pmatrix}u^{(2)}(r)\\ w^{(2)}(r)\end{pmatrix}.\tag{3}$$

At $r=r_0$ (core radius) we have

$$u(r_0)=A_S u^{(1)}(r_0)+A_D u^{(2)}(r_0)=0,$$
$$w(r_0)=A_S w^{(1)}(r_0)+A_D w^{(2)}(r_0)=0. \tag{4}$$

For the existence of the non-trivial solution, this requires

$$\begin{vmatrix} u^{(1)}(r_0) & u^{(2)}(r_0) \\ w^{(1)}(r_0) & w^{(2)}(r_0) \end{vmatrix}=0. \tag{5}$$

Since the two sets of solutions are now known, (5) can be solved for r_0 to determine the necessary hard core radius in order to produce the correct deuteron binding energy. At $r=r_0$ thus determined, we find from (4)

$$A_D/A_S=-u^{(1)}(r_0)/u^{(2)}(r_0)=-w^{(1)}(r_0)/w^{(2)}(r_0). \tag{6}$$

From (2) and (3) it is clear that A_D/A_S is the asymptotic ratio of D to S wave amplitudes. This ratio plays an important role in Chew-Mendelstam's approach to the deuteron problem.[5]

After finding r_0 and A_D/A_S in this way, we can construct the deuteron wave function from the two sets of solutions already known. The properties of the deuteron and the low energy triplet n-p scattering system can be calculated in the usual way.

Since the method described above does not involve any variational procedure, the accuracy in the final results is determined by that of the integration method employed. Some interpolation procedure is necessary to solve (5) for r_0. However, the accuracy in r_0 of one tenth of the mesh size can be easily attained even with a simple interpolation method. This small uncertainty in r_0 hardly affects the deuteron properties. We also note that from the coding point of view for computors, the present method is considerably simpler than the variation-iteration method. The inclusion of forces other than the central and tensor presents no additional difficulty.

It is interesting to consider the applicability of the present method to the scattering problem at higher energies. Suppose that a set of phase shifts, determined from the analysis of the experimental data, is given. This enables one to construct the wave function in the asymptotic region. Our method can then be applied *mutatis mutandis* to see whether a particular potential is consistent with the given phase shifts. The practicability of such an approach depends on whether the unique (or at least small number of) set of phase shifts is available.

§ 3. Triplet even state

Now we turn to the application of the method described above for the potentials with the OPEP tail. We consider the following type of potentials:

$$V_C(x)=-\mu(g^2/4\pi)(e^{-x}/x)[1+a_C(e^{-x}/x)], \tag{7·1}$$
$$V_T(x)=-\mu(g^2/4\pi)(e^{-x}/x)(1+3/x+3/x^2)[1+b_T(e^{-2x}/x^2)], \tag{7·2}$$

where x is the internucleon distance measured in the pion Compton wavelength $\mu^{-1}=1.415\times10^{-13}$ cm. (7) represents a particular class of potentials which reduce to the OPEP (with the effective coupling constant $g^2/4\pi$) for large x. The factors within square brackets specify the modification of the OPEP at small distances.

At present we have no quantitatively reliable theory concerning this modification. However, we know that $V_T(x)$ in the triplet even state around the pion range is very nearly that of OPEP.[3] This is why no term proportional to e^{-x} is included in square brackets of (7·2). Also the actual calculation has shown that the additional term proportional to e^{-2x} in the modification factor of (7·1) makes little difference in the deuteron properties, hence it is not considered here. The effects of the terms with even shorter range such as e^{-3x}, e^{-4x}, etc., are negligible particularly because of the presence of the hard core. With these reservations the modification factors in (7) may be considered phenomenological. Figs. 1 and 2 indicate the class of potentials which (7) can represent. It is hoped that sufficiently wide variety of potentials can be at least approximately expressed in the form (7) and the results described below be useful in finding the consequence of such potentials.

The numerical calculations have been performed for $g^2/4\pi=0.065$ (0.005) 0.085 and

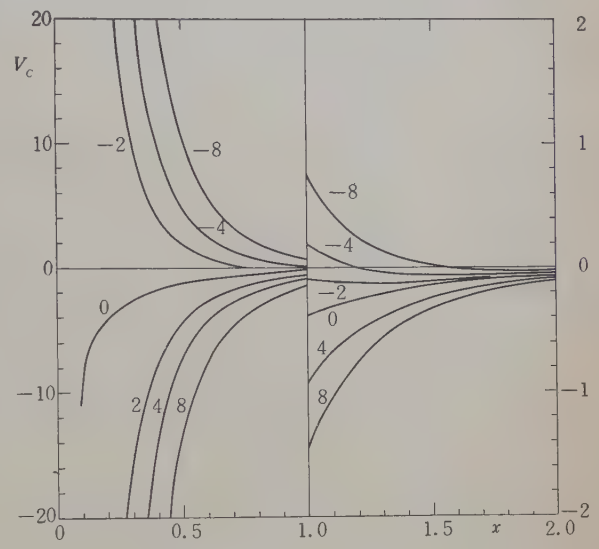

Fig. 1. Central force in the triplet even state. $V_C(x)$ [(7·1) of text] is plotted in the unit $\mu(g^2/4\pi)$. Numbers on curves are values of a_C in (7·1).

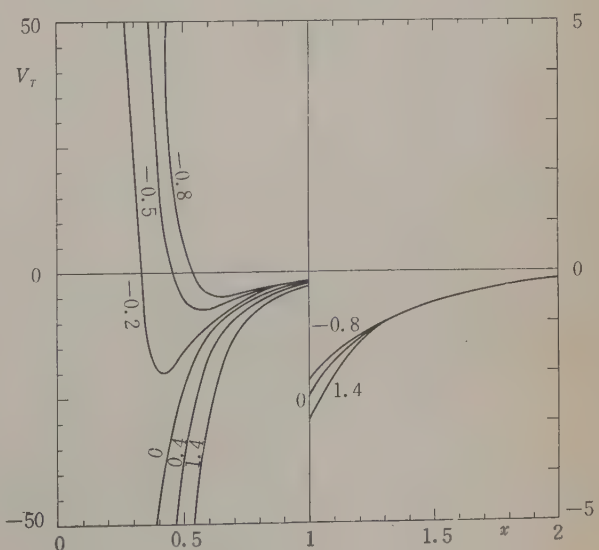

Fig. 2. Tensor force in the triplet even state. $V_T(x)$ [(7·2) of text] is plotted in the unit $\mu(g^2/4\pi)$. Numbers on curves are values of b_T in (7·2).

Table I. Deuteron properties and the triplet effective range calculated from the potential (7). x_0=hard core radius in units of the pion Compton wavelength (1.415×10^{-13} cm); Q=electric quadrupole moment in 10^{-27} cm^2; ${}^3r_e$=triplet effective range in 10^{-13} cm (rigorously speaking, this is $\rho(-\epsilon, -\epsilon)$ of Bethe[7]); P_D =D-state probability; A_D/A_S=asymptotic ratio of D to S wave amplitudes. For comparison the relevant experimental data are $Q=2.738\pm0.014$ and ${}^3r_e=1.704\pm0.030$.

$g^2/4\pi$	a_C	b_T	x_0	Q	${}^3r_e$	P_D	A_D/A_S
0.065	0	0	0.288	2.48	1.57	0.0671	0.02375
	10	0	0.358	2.58	1.74	0.0596	0.02355
	14	0	0.382	2.61	1.80	0.0573	0.02347
	0	1.4	0.417	2.66	1.65	0.0766	0.02474
	8	1.4	0.440	2.73	1.76	0.0707	0.02456
	0	2	0.442	2.72	1.67	0.0790	0.02505
0.070	0	0	0.303	2.61	1.62	0.0696	0.02493
	4	0	0.336	2.66	1.70	0.0659	0.02484
	6	0	0.351	2.68	1.73	0.0647	0.02480
	8	0	0.364	2.70	1.76	0.0636	0.02477
	10	0	0.378	2.72	1.79	0.0623	0.02473
	0	0.4	0.363	2.68	1.65	0.0736	0.02528
	0	0.6	0.381	2.71	1.66	0.0749	0.02541
	2	0.6	0.390	2.73	1.69	0.0732	0.02537
	0	0.8	0.395	2.73	1.67	0.0757	0.02553
	4	0.8	0.411	2.77	1.73	0.0727	0.02545
	0	1	0.409	2.75	1.68	0.0772	0.02567
0.075	0	0	0.325	2.74	1.67	0.0718	0.02606
	−4	0	0.287	2.68	1.58	0.0756	0.02614
	−3	0	0.298	2.70	1.61	0.0749	0.02613
	−2	0	0.307	2.72	1.63	0.0738	0.02610
	−1	0	0.315	2.73	1.65	0.0730	0.02609
	1	0	0.330	2.76	1.69	0.0709	0.02604
	2	0	0.339	2.77	1.71	0.0702	0.02602
	3	0	0.347	2.79	1.73	0.0696	0.02601
	0	−0.2	0.256	2.70	1.65	0.0679	0.02582
	2	−0.2	0.291	2.73	1.70	0.0670	0.02580
	3	−0.2	0.304	2.75	1.72	0.0662	0.02578
	0	−0.1	0.299	2.72	1.66	0.0705	0.02596
	0	0.1	0.341	2.76	1.68	0.0732	0.02616
	−1	0.1	0.334	2.75	1.66	0.0739	0.02617
	−2	0.2	0.343	2.75	1.65	0.0759	0.02628
	0	0.2	0.355	2.78	1.69	0.0740	0.02623
	−3	0.3	0.352	2.76	1.63	0.0778	0.02638
	0	0.3	0.367	2.80	1.69	0.0753	0.02632
	1	0.3	0.373	2.81	1.71	0.0740	0.02629

$g^2/4\pi$	a_C	b_T	x_0	Q	${}^3r_e$	P_D	A_D/A_S
0.080	0	0	0.340	2.88	1.72	0.0742	0.02716
	−4	0	0.305	2.81	1.63	0.0779	0.02724
	−3	0	0.314	2.83	1.66	0.0768	0.02722
	−2	0	0.323	2.85	1.68	0.0758	0.02719
	−4	−0.1	0.270	2.79	1.62	0.0766	0.02714
	0	−0.1	0.320	2.86	1.71	0.0730	0.02706
	*−2	−0.2	0.243	2.80	1.66	0.0721	0.02697
	0	−0.2	0.287	2.84	1.70	0.0712	0.02695
0.085	0	0	0.357	3.01	1.76	0.0762	0.02821
	−4	0	0.321	2.94	1.68	0.0800	0.02829
	−8	0	0.283	2.86	1.58	0.0846	0.02838
	−10	0	0.261	2.82	1.52	0.0872	0.02842
	*−4	−0.3	0.221	2.85	1.68	0.0639	0.02762
	* 0	−0.3	0.260	2.94	1.74	0.0697	0.02785

* The entries with * are subject to error of several percent. Errors in all other entries should be less than one percent.

Table II. A typical deuteron wave function. The radial wave function (multiplied by x) are shown for the potential (7) with $g^2/4\pi=0.075$, $a_C=2$, $b_T=-0.2$, and $x_0=0.291$. This wave function gives $Q=2.73$, ${}^3r_e=1.70$, $P_D=0.0670$, $A_D/A_S=0.02580$, and the normalization integral $\int_0^\infty (u^2+w^2)dx=0.92656$.

x	$u(x)$	$w(x)$
0.3	0.0156	0.0063
0.4	0.1853	0.0782
0.5	0.3284	0.1414
0.6	0.4335	0.1839
0.7	0.5048	0.2077
0.8	0.5505	0.2180
0.9	0.5781	0.2194
1.0	0.5926	0.2153
1.1	0.5981	0.2077
1.2	0.5972	0.1983
1.3	0.5918	0.1879
1.4	0.5833	0.1772
1.5	0.5727	0.1665
1.6	0.5606	0.1561
1.7	0.5475	0.1461
1.8	0.5337	0.1366
1.9	0.5196	0.1276
2.0	0.5053	0.1192
2.2	0.4768	0.1040
2.4	0.4488	0.0909

x	$u(x)$	$w(x)$
2.6	0.4218	0.0796
2.8	0.3961	0.0698
3.0	0.3717	0.0614
3.2	0.3485	0.0541
3.4	0.3268	0.0478
3.6	0.3062	0.0424
3.8	0.2870	0.0377
4.0	0.2689	0.0335
4.2	0.2519	0.0300
4.4	0.2360	0.0268
4.6	0.2210	0.0240
4.8	0.2070	0.0216
5.0	0.1939	0.0195
5.5	0.1647	0.0151
6.0	0.1397	0.0118
6.5	0.1186	0.0094
7.0	0.1007	0.0075
7.5	0.0854	0.0060
8.0	0.0725	0.0048
8.5	0.0615	0.0039
9.0	0.0522	0.0032
9.5	0.0443	0.0026
10.0	0.0376	0.0021

somewhat arbitrarily chosen a_C and b_T. Results are listed in Table I. For the pure OPEP ($a_C=b_T=0$) with $g^2/4\pi=0.0755$, Blatt and Kalos[6] give, in the same units as in Table I, $x_0=0.322$, $Q=2.74$, ${}^3r_e=1.66$, and $P_D=0.0718$, which are consistent with the present results. A typical deuteron wave function is shown in Table II.

In view of the wide variety of inner interactions covered by (7) [see Figs. 1 and 2], it is remarkable that the deuteron properties are relatively stable against the change of a_C and b_T for a given $g^2/4\pi$. This, indeed, is the reason why the deuteron problem may be used to determine $g^2/4\pi$ in spite of large uncertainties in the inner interaction.[3] This situation is a consequence of the facts that $V_T(x)$ plays a much more important role than $V_C(x)$ in the deuteron problem and that $V_T(x)$ is not very much different from the OPEP around the pion range [no term proportional to e^{-x} in the modification factor of (7·2)].

In their study of the deuteron problem, Iwadare et al.[3] have averted the uncertainties concerning the inner interacion by assuming certain quadratic forms of the wave function itself for $x<1$. Their conclusion on the pion-nucleon coupling constant, $g^2/4\pi=0.075\pm0.010$, is in good agreement with our results of Table I. We note, however, that the accurate determination of the coupling constant from the deuteron problem requires precise knowledge on the mesonic and other cor-

rections to be applied to the quantities given in Table I. We also have to remember that (7) represents only a special class of potentials. Even with these reservations, it is perhaps significant that $P_D=6\sim8\%$ and $A_D/A_S=0.024\sim0.027$ for reasonable potentials.

§ 4. Singlet even state

All the available pion theoretical calculations indicate that the potential in the singlet even state should be much more attractive than the OPEP around and inside the pion range. This feature is indeed required to reproduce the experimental zero energy scattering parameters in the state concerned.[3]

We then have chosen the following type of potential in the singlet even state:

$$V(x)=-\mu(g^2/4\pi)(e^{-x}/x)[1+a_C(e^{-x}/x)+b_C(e^{-2x}/x^2)]. \qquad (8)$$

Again the factor in square brackets specifies the modification of the OPEP at small distances. The hard core radius is not specified beforehand.

At zero energy, the experimentally known singlet *n-p* scattering length, -23.7×10^{-13} cm, determines the asymptotic behaviour of the wave function. Given a potential (8), then starting at sufficiently large x we can integrate the Schrödinger equation inward down to x_0 such that $u(x_0)=0$. The effective range may be calculated in the usual way from the solution.

Table III shows some results obtained in this way. All entries are consistent with the experimental indication ${}^1r_e=2.1\sim2.8\times10^{-13}$ cm.

Table III. Singlet *n-p* effective range ${}^1r_e$ (in 10^{-13} cm) calculated from the potential (8). The hard core radius x_0 (given in the pion Compton wavelength) is adjusted to give the scattering length ${}^1a(np)=-23.7\times10^{-13}$ cm. The experimental evidence is ${}^1r_e(np)=2.1\sim2.8$.

$g^2/4\pi$	a_C	b_C	x_0	${}^1r_e$
0.065	6	0	0.135	2.12
	10	0	0.211	2.39
	14	0	0.269	2.59
	10	4	0.264	2.46
	10	8	0.299	2.52
	10	12	0.325	2.56
0.070	6	0	0.147	2.21
	10	0	0.225	2.48
	14	0	0.284	2.68
	10	4	0.277	2.55
	10	8	0.311	2.60
	10	12	0.337	2.65
	14	8	0.347	2.76

$g^2/4\pi$	a_C	b_C	x_0	^1r_e
0.075	6	0	0.159	2.30
	8	0	0.202	2.44
	10	0	0.239	2.56
	12	0	0.271	2.67
	14	0	0.299	2.76
	10	4	0.290	2.63
	10	8	0.323	2.68
	10	12	0.349	2.72
	10	14	0.361	2.75
	12	12	0.366	2.80
	14	4	0.334	2.81
0.080	6	0	0.170	2.38
	10	0	0.252	2.64
	14	0	0.314	2.84
	10	4	0.302	2.70
	10	8	0.335	2.76
	10	10	0.349	2.78
	12	4	0.325	2.80
0.085	4	0	0.125	2.27
	6	0	0.181	2.45
	10	0	0.265	2.72
	12	0	0.298	2.82
	10	4	0.313	2.78
	10	8	0.346	2.83
	8	10	0.340	2.76

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References

1) H. Feshbach and J. Schwinger, Phys. Rev. **84** (1951), 194.
2) M. H. Kalos, L. C. Biedenharn and J. M. Blatt, Nucl. Phys. **1** (1956), 233.
L. C. Biedenharn, J. M. Blatt and M. H. Kalos, Nucl. Phys. **6** (1958), 359.
3) J. Iwadare, S. Otsuki, R. Tamagaki and W. Watari, Prog. Theor. Phys. **16** (1956), 455; ibid. **16** (1956), 472; also Suppl. Prog. Theor. Phys. No. 3 (1956), Part II.
4) R. J. N. Phillips, *Report on Progress in Physics*, **XXII** (1959), 562 (Physical Society, London).
5) G. F. Chew, *9th Conference on High Energy Nuclear Physics*, Kiev, 1959.
6) J. M. Blatt and M. H. Kalos, Phys. Rev. **92** (1953), 1563.
7) H. A. Bethe, Phys. Rev. **76** (1949), 41.

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The Proton-Neutron Mass Difference and the Inner Structure of the Nucleon

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A method is proposed which makes it possible to take into account the contribution from the pair state in the calculation of the proton-neutron mass difference without regard to the form of the matrix elements related to the pair state. It is shown that there exists a considerable difference between the result of our method and that of the Feynman-Speisman method in the internal region of the nucleon cloud. The model of form factors is considered in which the core exists only in the charge iso-scalar part. The result derived from this model is not satisfactory owing to the wrong sign of the mass difference. In order to get a good agreement with the empirical data it would be necessary to introduce a core at least into one of the form factors of the charge iso-vector type and of any kind of the magnetic moment type. The magnitude and the sign of the core is briefly estimated for each type of form factors.

§ 1. Introduction

Recently, several authors [1)–5)] have investigated whether or not we can interpret the proton-neutron mass difference as that originating from the electromagnetic interaction, by using the information on the nucleon structure obtained by the experiments on high energy electron scattering at Stanford University.[6)] The conclusions obtained by them are rather pessimistic. However, it seems to us that in their investigations there exist some unsatisfactory points which will be discussed in the following.

In the Feynman-Speisman method,[1)] the mass difference is calculated by inserting the experimental form factor into the vertex part in the Feynman diagram. This vertex part has one real and one virtual nucleon lines. Therefore, it is necessary to find the expression of the form factor for the virtual state, while the experiment only furnishes information on the real state. In order to avoid this difficulty, Wick and Sorensen[3),4)] represented the intermediate states in terms of the physical states and calculated only the contribution from one-nucleon states. This approximation gives the following expression for δm (the electromagnetic self-mass of the nucleon) :

$$\delta m\,\bar{u}(\boldsymbol{p})\,u(\boldsymbol{p}) = \sum_{k} \frac{1}{2|\boldsymbol{k}|} \cdot \frac{\langle \boldsymbol{p}|\, \boldsymbol{j}_{\mu}(0)\,|\boldsymbol{p}-\boldsymbol{k}\rangle \langle \boldsymbol{p}-\boldsymbol{k}|\boldsymbol{j}_{\mu}(0)\,|\boldsymbol{p}\rangle}{|\boldsymbol{k}|-E_{p}+E_{p-k}}, \qquad (1\cdot1)$$

where $\langle \boldsymbol{p}|\boldsymbol{j}_\mu(0)|\boldsymbol{p}-\boldsymbol{k}\rangle$ is just the matrix element whose behavior as a function of the momentum transfer can be obtained by experiments.

In the perturbation theory, the intermediate state with a single nucleon and the state involving the nucleon-antinucleon pair play equally important roles for δm. From this reason, Sunakawa and Tanaka (S.T.)[5] discussed the contribution coming from the pair state and obtained

$$-\sum_{\boldsymbol{k}}{}' \frac{1}{2|\boldsymbol{k}|}\cdot\frac{\langle 0|\boldsymbol{j}_\mu(0)|\boldsymbol{p},\ -\boldsymbol{p}+\boldsymbol{k}\rangle\langle \boldsymbol{p},\ -\boldsymbol{p}+\boldsymbol{k}|\boldsymbol{j}_\mu(0)|0\rangle}{|\boldsymbol{k}|+E_p+E_{p-k}} \qquad (1\cdot 2)$$

for $\delta m\bar{u}\cdot u$, where $|0\rangle$, $|\boldsymbol{p},\ -\boldsymbol{p}+\boldsymbol{k}\rangle$ are the state vectors describing the vacuum and the pair state respectively. They assumed some expression for $\langle 0|\boldsymbol{j}_\mu(0)|\boldsymbol{p},\ -\boldsymbol{p}+\boldsymbol{k}\rangle$, since we have no experimental information on this matrix element. In § 2, we shall propose a new method to take into account such a pair state effect without regard to the concrete expression of $\langle 0|\boldsymbol{j}_\mu(0)|\boldsymbol{p},\ -\boldsymbol{p}+\boldsymbol{k}\rangle$.

Next, we shall discuss the other unsatisfactory point. In the previous investigations, the spatial distribution of charge and magnetic moment of nucleons has been assumed to have a simple exponential type except for the charge distribution of neutron which was put equal to zero as suggested by the experiments. In the present stage, we have considerable information about the nucleon cloud in the outside region but no definite information in the inner region. The mass difference may be seriously affected by the inner structure of the nucleon. Last year, some authors investigated[7),8)] this effect by assuming the special forms of the distribution functions. However, their methods of calculation are essentially the same as the Feynman-Speisman method which is unsatisfactory as mentioned in the above. We shall see, in § 2, that in the inner region there occurs a considerable difference between the result of the Feynman-Speisman method and that of the new method which will be proposed in § 2. In § 3, the effect of the inner structure will be investigated by using the new method and assuming a simple inner structure of the form factors.

§ 2. Contribution from nucleon pair

In the second order approximation of the electromagnetic interaction, δm is given by the expression

$$\delta m\,\bar{u}(\boldsymbol{p})u(\boldsymbol{p}) = \sum_{n,k}\frac{1}{2|\boldsymbol{k}|}\cdot\frac{\langle \boldsymbol{p}|\boldsymbol{j}_\mu(0)|n\,;\,\boldsymbol{p}-\boldsymbol{k}\rangle\langle n\,;\,\boldsymbol{p}-\boldsymbol{k}|\boldsymbol{j}_\mu(0)|\boldsymbol{p}\rangle}{|\boldsymbol{k}|-E_p+E_n}$$
$$-\sum_{m,k}{}^{p}\frac{1}{2|\boldsymbol{k}|}\cdot\frac{\langle 0|\boldsymbol{j}_\mu(0)|m\,;\,\boldsymbol{k}\rangle\langle m\,;\,\boldsymbol{k}|\boldsymbol{j}_\mu(0)|0\rangle}{|\boldsymbol{k}|+E_m}, \qquad (2\cdot 1)$$

where $|n\,;\,\boldsymbol{p}\rangle$ is the state with a momentum $\boldsymbol{p}$ and n stands for the set of the residual quantum numbers and $\sum^{p}$ means that the summation should be carried out only over the states involving the nucleon with momentum $\boldsymbol{p}$. In the intermediate

states appearing in the above expression, many kinds of states can be considered, among which the one-nucleon state for the first term and the one-nucleon-pair state for the second are considered as having the predominant contributions to $\delta m_p - \delta m_n$ as frequently pointed out by several authors. For example, it was shown by (S.T.) and Sorensen[4] that the contribution from the (one nucleon + one meson) state is about ten percent of the experimental value.

Now if the contribution of the all possible intermediate states are taken into account, the result is naturally to be invariant under Lorentz transformations. Then, it is convenient to choose a special Lorentz frame in which the nucleon has an extremely large momentum. In the limit $|\boldsymbol{p}|\to\infty$, the denominator of the 1st term in Eq. (2·1) has a finite value; on the other hand, that of the 2nd term becomes to diverge. Therefore in our Lorentz system our task is to evaluate the first term of Eq. (2·1) in the case of $|\boldsymbol{p}|\to\infty$. This clever choice of the Lorentz-frame makes it very easy to calculate δm in comparison with the method of (S.T.) where they have made use of the rest system of the nucleon and consequently they had to calculate the unwelcome second term.

We take the conventional form for the matrix element of the current density of the following type:

$$\langle \boldsymbol{p}|\boldsymbol{j}_\mu(0)|\boldsymbol{q}\rangle = ie\,\bar{u}(\boldsymbol{p})\left\{\gamma_\mu F_1(p-q)+\left(\frac{i\mu}{4M}\right)[\gamma_\mu,\gamma_\nu](p-q)_\nu F_2(p-q)\right\}u(\boldsymbol{q}),$$

$$p-q\equiv(\boldsymbol{p}-\boldsymbol{q},\ E_p-E_q). \tag{2·2}$$

In order to compare the result of the present method with that of previous ones, we shall, in this section, take the following form factors which have been used in the previous investigations:

$$\left.\begin{aligned} &F_1^p(k)=F_2^p(k)=\left(\frac{\Lambda^2}{k^2+\Lambda^2}\right)^2 \quad \text{for proton,}\\ &F_1^n(k)=0,\ F_2^n(k)=\left(\frac{\Lambda^2}{k^2+\Lambda^2}\right)^2 \quad \text{for neutron.}\end{aligned}\right\} \tag{2·3}*$$

Then the calculation of Eq (1·1) for $|\boldsymbol{p}|\to\infty$ leads to**

$$\delta m=\frac{e^2}{16\pi^2}\left[\left(-\frac{1}{2}\mu-\frac{3}{8}\mu^2\right)\Lambda^2-(4+7\mu+3\mu^2)+\left(\frac{11}{2}+10\mu+\frac{9}{2}\mu^2\right)\frac{1}{1-\Lambda^2/4}\right.$$
$$-\left(\frac{3}{2}+3\mu+\frac{3}{2}\mu^2\right)\frac{1}{(1-\Lambda^2/4)^2}+\left\{\left(3+3\mu+\frac{5}{4}\mu^2\right)-\left(\frac{5}{2}+\frac{9}{2}\mu+2\mu^2\right)\right.$$
$$\left.\left.\times\frac{1}{1-\Lambda^2/4}+\left(\frac{3}{4}+\frac{3}{2}\mu+\frac{3}{4}\mu^2\right)\frac{1}{(1-\Lambda^2/4)^2}\right\}A\right], \tag{2·4}$$

---

* In what follows, we use the unit $M=1$.

** After the integration with respect to the azimuthal angle of $\mathbf{k}$, we change the integration variables $d\boldsymbol{k}$, $d\cos\theta$ into du, dv, where $u=E_{p-k}+|\mathbf{k}|$, $v=E_{p-k}-|\mathbf{k}|$ (see W. Heitler. "*The Quantum Theory of Radiation*" (The Clarendon Press, Oxford) 3rd ed., p. 297-298.

where

$$A=\begin{cases}\dfrac{\Lambda}{\sqrt{1-\Lambda^2/4}}\cdot\left[\dfrac{\pi}{2}-\tan^{-1}\dfrac{\Lambda/2}{\sqrt{1-\Lambda^2/4}}\right] & \text{for } \Lambda^2<4,\\ -\dfrac{\Lambda/2}{\sqrt{(\Lambda^2/4)-1}}\log\dfrac{\Lambda-2\sqrt{(\Lambda^2/4)-1}}{\Lambda+2\sqrt{(\Lambda^2/4)-1}} & \text{for } \Lambda^2>4.\end{cases} \tag{2·5}$$

On the other hand, in the nucleon rest system, the value of Eq. (1·1) becomes

$$\frac{e^2}{16\pi^2}\left[\left\{\frac{\mu^2}{48}\Lambda^4-\left(\frac{1}{3}+\mu+\frac{3}{8}\mu^2\right)\Lambda^2-(4+7\mu+3\mu^2)+\left(\frac{11}{2}+10\mu+\frac{9}{2}\mu^2\right)\frac{1}{1-\Lambda^2/4}\right.\right.$$
$$\left.-\left(\frac{3}{2}+3\mu+\frac{3}{2}\mu^2\right)\frac{1}{(1-\Lambda^2/4)^2}\right\}+\left\{\left(3+3\mu+\frac{5}{4}\mu^2\right)\right.$$
$$\left.\left.-\left(\frac{5}{2}+\frac{7}{2}\mu+2\mu^2\right)\frac{1}{1-\Lambda^2/4}+\left(\frac{3}{4}+\frac{3}{2}\mu+\frac{3}{4}\mu^2\right)\frac{1}{(1-\Lambda^2/4)^2}\right\}\cdot A\right]. \tag{2·6}$$

In the limit $\mu=0$ and $\Lambda\to\infty$, the above expressions should give the self-energy of a point charge. In fact we see that the expression (2·4) tends to diverge in a form $(3\alpha^2/2\pi)\log\Lambda$ for this limit in agreement with the well-known expression. On the other hand, the limiting expression of (2·6) diverges quadratically with $\Lambda\to\infty$. From this fact, it may be said that the contribution from the nucleon-pair has been correctly taken into account in the present method.

The result of the proton-neutron mass difference, $\Delta m\equiv\delta m_p-\delta m_n$, is shown in Fig. 1 and Table I as a function of root-mean-square (*rms*) radius "$a=\sqrt{12/\Lambda^2\bar{M}^2}$" with $\mu(proton)=1.79$, $\mu(neutron)=-1.91$.

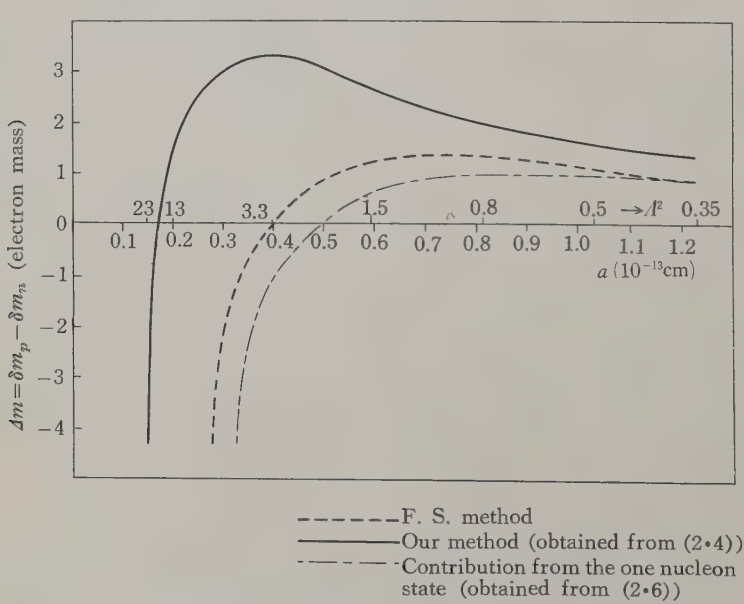

Fig. 1. Comparison between our method and F. S. method.

For the experimental value of "a" ($=0.8\times10^{-13}\,cm$), we get

$$\Delta m \fallingdotseq 2 \text{ electron mass.}$$

This result seems to be quite unfavorable at least for the above taken value of "a", an appreciable difference can be expected to occur in our treatment as compared with the result of Feynman-Speisman method.

Table I

Λ^2	$a\equiv\langle r^2\rangle^{1/2}$ (10^{-13} cm)	$\Delta m\equiv\delta m_p-\delta m_n$ obtained from (2·4) (electron mass)	$\Delta m\equiv\delta m_p-\delta m_n$ obtained from (2·6) (electron mass)
0.35	1.23	1.38	0.92
0.5	1.03	1.65	0.99
0.8	0.815	2.05	0.99
1.5	0.595	2.65	0.66
3.3	0.40	3.38	−1.06
13.0	0.20	1.47	−17.2
23.0	0.152	−3.37	−38.8

§ 3. The effect of the nucleon form factor

As stated in the Introduction, at the present stage of experiment, we can only determine the forms of the distribution of charge and magnetic moment in the peripheral region of the nucleon cloud ($r\gtrsim0.5\times10^{-13}\,cm$), while the inner structure of the nucleon remains still open to investigation. Under such circumstances, it may be reasonable to consider that the nucleon has a very complicated inner structure which is usually called "*nucleon core*". However, before giving the concrete expression of the form factors, we shall derive the expression of δm for an arbitrary form factor. Since the arbitrary form factor is always expressed as the superposition of the exponential form factors by means of the Laplace transformation, let us generalize the expression (2·4) into the one corresponding to the graph, the two vertices of which have different values of Λ. This generalization can be easily achieved by applying the parameter integral method to Eq. (2·4), the detailed explanation of which will be given in the Appendix.

Now let us return to the main problem. In the first place, in order to make the neutron charge distribution as simple as possible, we shall assume the following form factor which is a linear combination of the simple exponentials. Since the form factor gives rise to $F_1^n(0)=0$ and $\langle r^2\rangle_1^n\simeq0$, it should be composed of at least three different terms of the exponential type. Then we have

$$\left.\begin{aligned} F_1^n(q) &= C_1'\left[\frac{\Lambda_1^2}{q^2+\Lambda_1^2}\right]^2+C_2'\left[\frac{\Lambda_2^2}{q^2+\Lambda_2^2}\right]^2+C_3'\left[\frac{\Lambda_3^2}{q^2+\Lambda_3^2}\right]^2, \\ &\Lambda_1<\Lambda_2<\Lambda_3, \end{aligned}\right\} \qquad (3\cdot1)$$

or in the ordinary space

$$\rho(\boldsymbol{r}) \equiv \frac{1}{(2\pi)^3}\int d\boldsymbol{q}\exp(i\boldsymbol{q}\cdot\boldsymbol{r})F(|\boldsymbol{q}|),$$

$$= C_1'\frac{\Lambda_1^3}{8\pi}\exp[-\Lambda_1 r] + C_2'\frac{\Lambda_2^3}{8\pi}\exp[-\Lambda_2 r] + C_3'\frac{\Lambda_3^3}{8\pi}\exp[-\Lambda_3 r]. \qquad (3\cdot 2)$$

Let us separate the form factor into the iso-scalar and iso-vector parts, i.e.

$$\left.\begin{aligned} F_{1,2}^{s} &= (F_{1,2}^{p} + F_{1,2}^{n})/2, \\ F_{1,2}^{v} &= (F_{1,2}^{p} - F_{1,2}^{n})/2. \end{aligned}\right\} \qquad (3\cdot 3)^*$$

From the general consideration based on field theory, we have known that in the region of large "r" the iso-vector part is predominant as compared with the iso-scalar one.[9] Accordingly, we see that the first term of (3·1) corresponds to the iso-vector part while the remainder may be regarded as representing the iso-scalar one, as the first term becomes predominant for a large value of "r" owing to the assumption on Λ's. Thus we put

$$\left.\begin{aligned} F_1^{v}(q) &= \frac{1}{2}\cdot C_1\left(\frac{\Lambda_1^2}{q^2+\Lambda_1^2}\right)^2, \\ F_1^{s}(q) &= \frac{1}{2}\left[C_2\left(\frac{\Lambda_2^2}{q^2+\Lambda_2^2}\right)^2 + C_3\left(\frac{\Lambda_3^2}{q^2+\Lambda_3^2}\right)^2\right], \\ & C_2>0 \quad \text{and} \quad C_3<0. \end{aligned}\right\} \qquad (3\cdot 4a)$$

Furthermore,

$$\left.\begin{aligned} F_2^{s}(q) &= \left(\frac{\Lambda_1^2}{q^2+\Lambda_1^2}\right)^2, \\ F_2^{v}(q) &= 0. \end{aligned}\right\} \qquad (3\cdot 4b)$$

This model in which the "*core*" is represented by the second term of F_1^s may be too simple to reflect the real situation of the internal structure of the nucleon. The reason why we have chosen such a simple model as (3·4a) and (3·4b) is, on the one hand, to get a clear insight of the relationship of the mass difference to the internal structure of the nucleon and, on the other hand, the expression of F_1^s is consistent with the result given by the perturbation method in the usual meson theory.[10]

The unknown parameters Λ's and C's in (3·4a) and (3·4b) are restricted by the requirements $F_1^v(0) = F_1^s(0) = 1/2$, and $\langle r^2\rangle_1^v = \langle r^2\rangle_1^s = (0.8\times 10^{-13}\ cm)^2$, that is,

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* It should be noticed that, since the magnetic form factor is normalized as $F_2^{n,p}(0)=1$, this definition of the iso-scalar and iso-vector part of magnetic form factor differs from the conventional ones. The conventional form factors, denoted by $\tilde{F}_2^p$, $\tilde{F}_2^n$, are equal to $\mu_p F_2^p$, $\mu_n F_2^n$ in our notation, respectively, and we have $\tilde{F}_2 = (\mu_s+\tau_3\mu_v)(F_2^s+\tau_3 F_2^v)$.

$$C_1=1,\ C_2+C_3=1, \tag{3·5}$$

$$\Lambda_1^2=0.8,$$

$$\frac{1}{\Lambda_1^2}=\frac{1}{C_2+C_3}\left(\frac{C_2}{\Lambda_2^2}+\frac{C_3}{\Lambda_3^2}\right), \tag{3·6}$$

hence we get

$$C_2=\frac{1/\Lambda_1^2-1/\Lambda_3^2}{1/\Lambda_2^2-1/\Lambda_3^2}, \tag{3·7}$$

$$C_3=1-C_2.$$

Let us separate δm into the following three parts,

$$\delta m=\delta^{(1)}m+\delta^{(2)}m+\delta^{(3)}m, \tag{3·8}$$

where the first term is independent of μ and the second and the third term are proportional to μ and μ^2, respectively. The calculation of δm can be performed by using the expressions of the functions $\Delta^{(1,1)}(\Lambda, \overline{\Lambda})$, $\Delta^{(1,2)}(\Lambda,\overline{\Lambda})$ and $\Delta^{(2,2)}(\Lambda, \overline{\Lambda})$, which will be stated in the Appendix. The result is shown in Table II, Table III and Fig. 2. It is seen that the self-energy δm_n of the neutron is considerably large as compared with that obtained by previous investigations, for the reasonable values of Λ_2 and Λ_3. Here we have assumed that Λ_3 is about 23 (the corresponding core radius $\approx 1/M$).

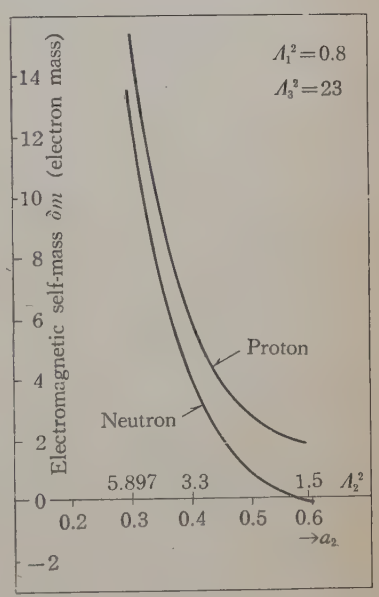

Fig. 2. Electromagnetic self-mass of nucleons in our model.

Unfortunately, the proton also has a large δm-value in our model and is still heavier than the neutron. This fact is due to the following reasons :

In our model, the "*core*" gives a nearly equal amount of contributions to the both kinds of nucleons as the "*core*" appears only in the iso-scalar part. Now let us consider the $\delta^{(1)}m$ term. Since, for this, only the interaction between the charge iso-scalar and iso-vector parts gives rise to the mass difference, we get, by using Eq. (A·3) in the Appendix, the expression

$$\delta^{(1)}m_p-\delta^{(1)}m_n=\frac{1}{2}\left[|C_2|\cdot|C_1|\Delta^{(1,1)}(\Lambda_2\Lambda_1)-|C_3||C_1|\Delta^{(1,1)}(\Lambda_3\Lambda_1)\right]. \tag{3·9}$$

On the other hand, we have the restriction $|C_3|<|C_2|$ owing to the positive charge of proton and the fact that the positive definite function $\Delta^{(1,1)}(\Lambda_2\Lambda_1)$ is nearly equal to $\Delta^{(1,1)}(\Lambda_3\Lambda_1)$ in magnitude for the small values of Λ_1 such as $(\Lambda_1=0.8)$ (see Fig. 3

Table II. The values of C_1, C_2 in the form factor

Λ_3^2	Λ_2^2	C_2	C_3
18	1.5	1.96	−0.96
	3.3	4.83	−3.83
	5.9	10.5	−9.5
20	1.5	1.95	−0.95
	3.3	4.74	−3.74
	5.9	10.0	−9.0
23	1.5	1.94	−0.94
	3.3	4.65	−3.65
	5.9	9.6	−8.6
26	1.5	1.93	−0.93
	3.3	4.58	−3.58
	5.9	9.2	−8.2

Table III. A. The electromagnetic self-mass of a neutron

Λ_3^2	Λ_2^2	$\delta^{(1)} m_n$ $(M\times10^{-2})$	$\delta^{(2)} m_n$ $(M\times10^{-2})$	δm_n $(M\times10^{-2})$	δm_n *(electron mass)*
18	1.5	0.021	−0.007	−0.008	−0.15
	3.3	0.20	−0.02	0.16	2.9
	5.9	0.64	−0.04	0.58	10.6
20	1.5	0.023	−0.007	−0.007	−0.12
	3.3	0.22	−0.02	0.18	3.2
	5.9	0.69	−0.04	0.63	11.6
23	1.5	0.025	−0.007	−0.004	−0.08
	3.3	0.24	−0.02	0.20	3.7
	5.9	0.77	−0.04	0.71	13.0
26	1.5	0.027	−0.008	−0.003	−0.05
	3.3	0.27	−0.03	0.22	4.0
	5.9	0.84	−0.04	0.78	14.3

$(\delta^{(3)} m_n = -0.022\times10^{-2} M)$

B. The electromagnetic self-mass of a proton

Λ_3^2	Λ_2^2	$\delta^{(1)} m_p$ $(M\times10^{-2})$	$\delta^{(2)} m_p$ $(M\times10^{-2})$	δm_p $(M\times10^{-2})$	δm_p *(electron mass)*
18	1.5	0.10	0.02	0.10	1.9
	3.3	0.27	0.03	0.28	5.1
	5.9	0.69	0.04	0.71	13.0
20	1.5	0.10	0.02	0.10	1.9
	3.3	0.28	0.03	0.29	5.4
	5.9	0.73	0.05	0.76	14.0
23	1.5	0.11	0.02	0.11	1.9
	3.3	0.31	0.03	0.32	5.9
	5.9	0.81	0.05	0.84	15.4
26	1.5	0.11	0.02	0.11	2.0
	3.3	0.33	0.04	0.35	6.3
	5.9	0.89	0.05	0.92	16.9

$(\delta^{(3)} m_p = -0.02\times10^{-2} M)$

in the Appendix). Consequently, the right-hand side of Eq. (3·9) becomes positive contrary to our desire.

The $\delta^{(2)}m$-term also has a contribution to the mass difference which is given by

$$\delta^{(2)}m/\mu=\frac{1}{2}[\pm|C_1|\Delta^{(1,2)}(\Lambda_1\Lambda_1)+|C_2|\Delta^{(1,2)}(\Lambda_2\Lambda_1)-|C_3|\Delta^{(1,2)}(\Lambda_3\Lambda_1)], \qquad (3\cdot10)$$

where the signs $(\pm)$ correspond to the case of proton and neutron respectively. In the right-hand side of Eq. (3·10), the only negative contribution comes from the first term for the case of neutron, as we shall see from Fig. 4 in the Appendix, but this negative contribution cannot make the expression $\delta^{(2)}m/\mu$ negative by virtue of the fact that $|C_1|\gtrsim|C_2|, |C_3|$. Accordingly, this kind of electromagnetic effect gives a further unpreferable contribution to the mass difference. From these results, we reach the following conclusion:

"*The internal sharp core assumed for the iso-scalar part is not sufficient for giving a theoretical explanation to the well-known mass difference. In other words, the same kind of core should be introduced in the iso-vector part of the charge and/or magnetic form factors in addition to that of the charge-iso-scalar part.*"

Let us make a brief discussion on the effects of this additionally introduced core with the assumption that the core radius is of the order of about the Compton wavelength $(1/M)$ of nucleons.

(1) *The effect of the internal core in the charge iso-vector part*

As the positive core in the iso-vector part of charge distribution makes the core of the neutron deeper than that of the proton, such a core is favorable for obtaining the empirical mass difference, because the deeper the core becomes, the larger the mass we get. This positive core also contributes to the $\delta^{(2)}m$-term and gives rise to a positive contribution to $(\delta m_p-\delta m_n)$ which is, however, expected to be small in so far as the magnetic form factor of the simple exponential type is adopted. Table IV shows the agreement between the empirical and the theoretical results obtained by using the following form factors which has a positive core in the charge iso-vector part:

$$\left.\begin{aligned} &F_1^v(q)=\frac{1}{2}\left[D_1\left(\frac{\Lambda_0^2}{q^2+\Lambda_0^2}\right)^2+D_2\left(\frac{\Lambda_3^2}{q^2+\Lambda_3^2}\right)^2\right], \\ &\text{other form factors being given in expressions} \\ &(3\cdot4a) \text{ and } (3\cdot4b), \end{aligned}\right\} \qquad (3\cdot11)$$

where we have put $\Lambda_0^2=0.5$, $\Lambda_3^2=23$ tentatively. These values of Λ's lead to $D_1=0.617$, and $D_2=0.383$, following the same line of arguments in the previous case. This result means that about 1/3 of the total charge of iso-vector part is contained in this core.

Table IV

		$\delta^{(1)}m$	$\delta^{(2)}m$	$\delta^{(3)}m$	δm		$\Delta m=\delta m_p-\delta m_n$
		$(M\times10^{-2})$	$(M\times10^{-2})$	$(M\times10^{-2})$	$(M\times10^{-2})$	$\left(\frac{electron}{mass}\right)$	(electron mass)
$\Lambda_2^2=3.3$	neutron	0.32	−0.03	−0.02	0.27	5.0	−0.2
	proton	0.25	0.03	−0.02	0.26	4.8	
$\Lambda_2^2=5.9$	neutron	0.90	−0.04	−0.02	0.83	15.3	−3.2
	proton	0.64	0.04	−0.02	0.66	12.1	

(2) *The effect of the internal core in the magnetic form factors*

As mentioned above, the negative value of $(\delta^{(2)}m/\mu)$ is preferable to obtain the empirical mass difference. If we assume the existence of the negative inner core in the magnetic form factor of both the proton and the neutron, i.e. in the magnetic iso-scalar part "F_2^s", it is possible to make $(\delta^{(2)}m/\mu)$ negative, because the cross-term relating to the core in the charge iso-scalar part and this magnetic core gives the largest contribution to $\delta^{(2)}m$ and has the negative sign owing to negative $\Delta^{(1,2)}(\Lambda\bar{\Lambda})$ for large value of Λ and $\bar{\Lambda}$.

Next we shall consider the positive core in the iso-vector part of magnetic form factor. The contribution to the $\delta^{(2)}m$-term from this core has a positive sign for both the proton and the neutron, and is proportional to $|\mu|$, namely the contribution to the neutron is larger than that to the proton. Furthermore, in this case, $\delta^{(3)}m$ also has the favorable sign.

Then, in our simple model, we see that in order to get a good agreement of the theoretical result with the empirical mass difference, there are the following several choices in introducing the core into the form factors:

kinds of form factor	notation in our paper	sign of the core
charge iso-vector part	F_1^v	+
magnetic iso-scalar part	F_2^s	−
magnetic iso-vector part	F_2^v	+

Acknowledgments

The authors would like to thank Prof. R. Utiyama for valuable discussions. One of us (T.K.) is obliged to Prof. Y. Asano for his encouragement.

Appendix

We shall try to replace the form factor appearing in Eq. (1·1) with the more general one defined as

$$F_{(\alpha)}(q)=\sum_i C_i^{(\alpha)}F(\Lambda_i^{(\alpha)}),\ (\alpha=1,2),\ F(\Lambda_i^{(\alpha)})\equiv\left[\frac{\Lambda_i^{(\alpha)2}}{q^2+\Lambda_i^{(\alpha)2}}\right]^2,\quad (A\cdot1)$$

which forces us to calculate the expression of the self-energy with a couple of form factors with the same functional form but with a pair of different Λ's. Namely, the expression of the self-energy becomes

$$\delta m = \sum_{ij} C_i^{(1)} C_j^{(1)} \Delta^{(1,1)}(\Lambda_i^{(1)} \Lambda_j^{(1)}) + \mu \sum_{ij} C_i^{(1)} C_j^{(2)} \Delta^{(1,2)}(\Lambda_i^{(1)} \Lambda_j^{(2)}) + \mu^2 \sum_{ij} C_i^{(2)} C_j^{(2)} \times \Delta^{(2,2)}(\Lambda_i^{(2)} \Lambda_j^{(2)}), \qquad (A\cdot 2)$$

where $\Delta^{(1,1)}$, $\Delta^{(1,2)}$ and $\Delta^{(2,2)}$ correspond respectively to (*charge-charge*), (*charge-m.m.*) and (*m.m.-m.m.*) interaction.

The very convenient integral relation

$$\frac{1}{a^2 b^2} = 6 \int_0^1 dy \, \frac{y - y^2}{[b + (a-b)y]^4}$$

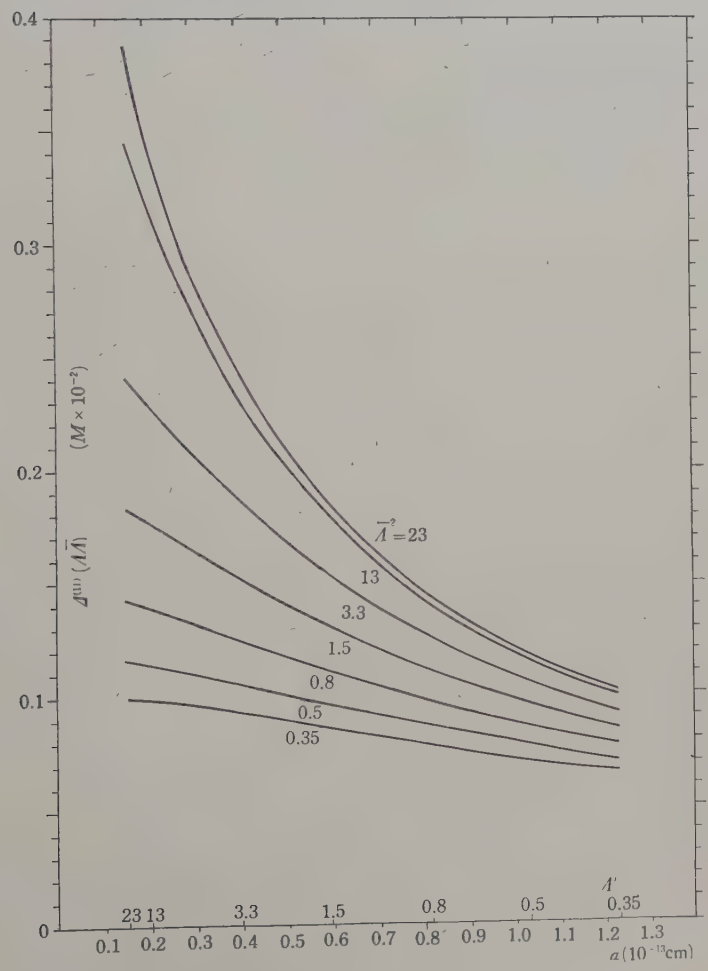

Fig. 3. Behavior of the function $\Delta^{(1,1)}(\Lambda\bar{\Lambda})$.

allows us to rewrite the expression "$F(\Lambda)\times F(\bar{\Lambda})$" appearing in $\Delta^{(i,j)}$ in the following way:*

$$\left[\frac{F(\Lambda)}{\Lambda^4}\right]\cdot\left[\frac{F(\bar{\Lambda})}{\bar{\Lambda}^4}\right]\equiv\left[\frac{1}{q^2+\Lambda^2}\right]^2\cdot\left[\frac{1}{q^2+\bar{\Lambda}^2}\right]^2$$

$$=6\int_0^1 dy\,(y-y^2)\left[\frac{1}{q^2+[\bar{\Lambda}^2+(\Lambda^2-\bar{\Lambda}^2)y]}\right]^4$$

$$=6\int_0^1 dy\,(y-y^2)[F(a)/a^4]^2_{a^2=[\bar{\Lambda}^2+(\Lambda^2-\bar{\Lambda}^2)y]}. \qquad (A\cdot 3)$$

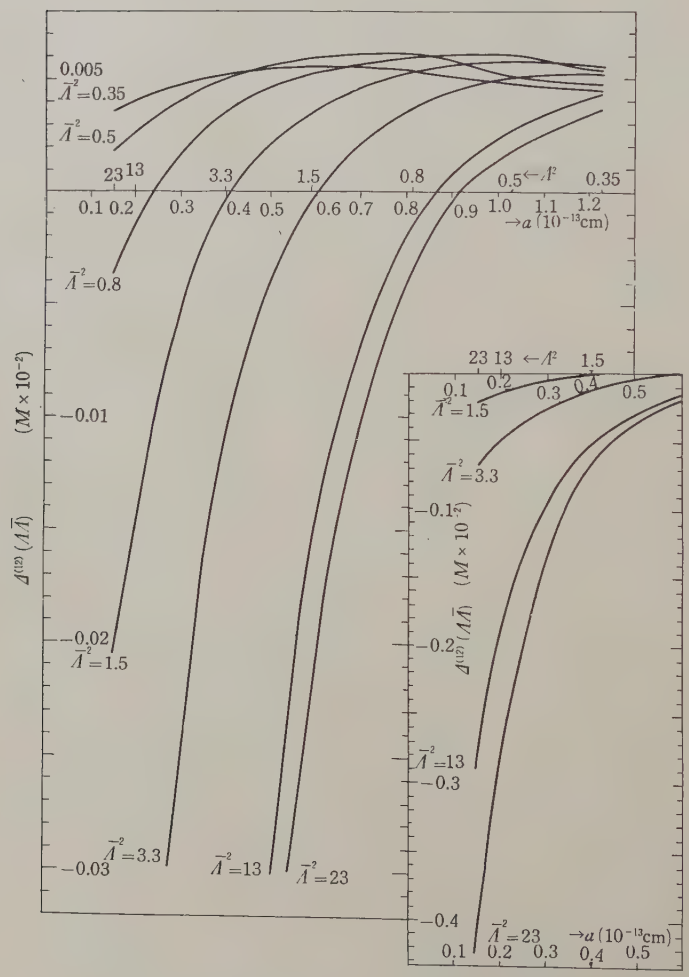

Fig. 4. Behavior of the function $\Delta^{(1,2)}(\Lambda\bar{\Lambda})$.

* In order to make the notation simple, Λ, $\bar{\Lambda}$ are used instead of $\Lambda^{(\alpha)}{}_i$, $\Lambda^{(\beta)}{}_j$.

Inserting this relation into Eq. (1·1), and taking the limit $|\boldsymbol{p}|\to\infty$, we obtain

$$\Delta^{(1,1)}(\Lambda\bar{\Lambda})+\mu\Delta^{(1,2)}(\Lambda\bar{\Lambda})+\mu^2\Delta^{(2,2)}(\Lambda\bar{\Lambda})$$
$$=6(\Lambda^4\bar{\Lambda}^4)\int_0^1 dy\,(y-y^2)\left[\frac{\text{expression (2·4)}}{\Lambda^8}\right]_{\Lambda^2=[\bar{\Lambda}^2+(\Lambda^2-\bar{\Lambda}^2)y]}.$$

By carrying out the y-integration, the result becomes

$$\Delta^{(1,1)}(\Lambda\bar{\Lambda})=\beta\cdot\frac{\Lambda^4\bar{\Lambda}^4}{(\Lambda^2-\bar{\Lambda}^2)^3}\left\{\left[\left(\frac{5}{32}\left(\frac{\Lambda^2+\bar{\Lambda}^2}{\Lambda^2\bar{\Lambda}^2}\right)+\frac{1}{64}\right)(\Lambda^2-\bar{\Lambda}^2)-\frac{1}{256}(\Lambda^2\bar{\Lambda}^2)\left(\frac{4-\Lambda^2}{\Lambda^2}\right)^2\right.\right.$$
$$\left.\left.-\frac{1}{128}\left(\Lambda^2\bar{\Lambda}^2-\frac{20}{3}(\Lambda^2+\bar{\Lambda}^2)\right)\left(\frac{4-\Lambda^2}{\Lambda^2}\right)-\frac{1}{128}(4-\bar{\Lambda}^2)\Lambda^2+\frac{1}{12}(\Lambda^2+\bar{\Lambda}^2)\log\frac{\Lambda^2}{4}\right]+\right.$$

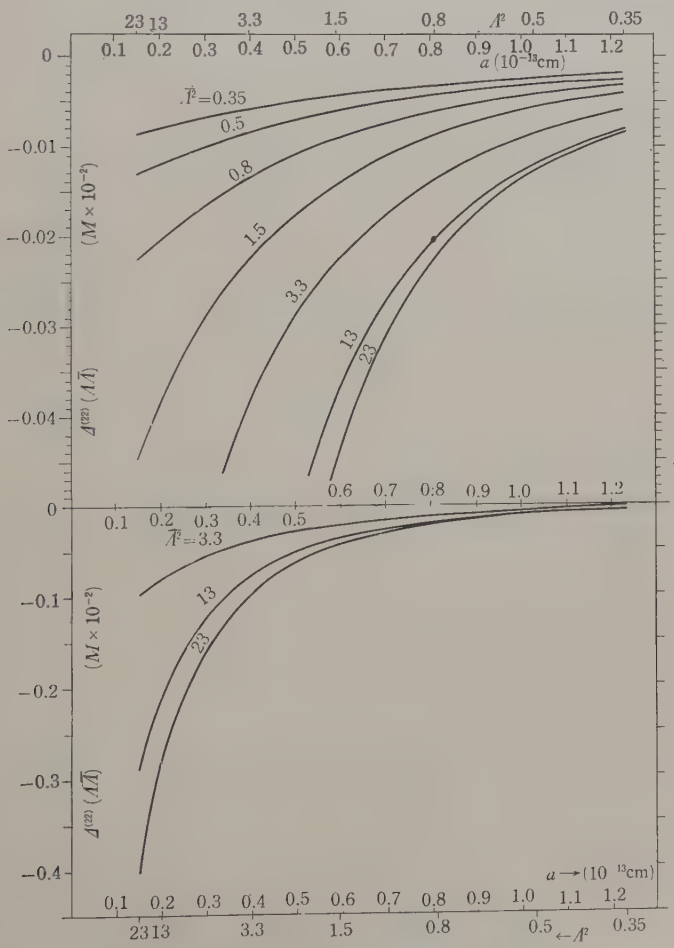

Fig. 5. Behavior of the function $\Delta^{(2,2)}(\Lambda\bar{\Lambda})$.

$$+\left[-\frac{1}{32}(\Lambda^2+\bar{\Lambda}^2-20)\left(\frac{4-\Lambda^2}{\Lambda^2}\right)+\frac{1}{64}(\Lambda^2\bar{\Lambda}^2-\frac{10}{3}(\Lambda^2+\bar{\Lambda}^2))\left(\frac{4-\Lambda^2}{\Lambda^2}\right)^2\right.$$

$$\left.+\frac{1}{128}(\Lambda^2\bar{\Lambda}^2)\left(\frac{4-\Lambda^2}{\Lambda^2}\right)^3-\frac{1}{128}(\Lambda^2\bar{\Lambda}^2+8\Lambda^2+4\bar{\Lambda}^2-64)\right]A(\Lambda)\Big\}+[\Lambda\rightleftarrows\bar{\Lambda}], \tag{A·4}$$

$$\Delta^{(1,2)}(\Lambda\bar{\Lambda})=\beta\cdot\frac{\Lambda^4\bar{\Lambda}^4}{(\Lambda^2-\bar{\Lambda}^2)^3}\left\{\left[\frac{1}{8}\cdot\frac{\Lambda^4-\bar{\Lambda}^4}{\Lambda^2\bar{\Lambda}^2}+\frac{1}{4}(\Lambda^2+\bar{\Lambda}^2-2)\log\frac{\Lambda^2}{4}\right]\right.$$

$$\left.-\left[\frac{1}{32}(\Lambda^2\bar{\Lambda}^2-6\Lambda^2-4\bar{\Lambda}^2)\left(\frac{4-\Lambda^2}{\Lambda^2}\right)+\frac{1}{32}(\Lambda^2\bar{\Lambda}^2+2\Lambda^2-24)\right]A(\Lambda)\right\}+[\Lambda\rightleftarrows\bar{\Lambda}], \tag{A·5}$$

$$\Delta^{(2,2)}(\Lambda\bar{\Lambda})=\beta\cdot\frac{\Lambda^4\bar{\Lambda}^4}{(\Lambda^2-\bar{\Lambda}^2)^3}\left\{\left[\frac{3}{64}(\Lambda^2-\bar{\Lambda}^2)+\frac{1}{48}(\Lambda^2\bar{\Lambda}^2+3(\Lambda^2+\bar{\Lambda}^2))\log\frac{\Lambda^2}{4}\right.\right.$$

$$\left.+\frac{1}{192}\bar{\Lambda}^2(4-\Lambda^2)-\frac{1}{128}(4-\bar{\Lambda}^2)\Lambda^2\right]$$

$$\left.-\left[\frac{1}{384}(\Lambda^2\bar{\Lambda}^2-24\Lambda^2-16\bar{\Lambda}^2)\left(\frac{4-\Lambda^2}{\Lambda^2}\right)+\frac{1}{128}(3\Lambda^2\bar{\Lambda}^2-4\bar{\Lambda}^2-32)\right]A(\Lambda)\right\}$$

$$+[\Lambda\rightleftarrows\bar{\Lambda}], \tag{A·6}$$

where $\beta=6e^2/4(2\pi)^2$ and the $A(\Lambda)$ is given by Eqs. (2·5) in § 2. The behavior of these functions are shown in Figs. (3), (4) and (5).

References

1) R. P. Feynman and G. Speisman, Phys. Rev. **94** (1954), 500.
2) K. Hiida and M. Sawamura, Prog. Theor. Phys. **18** (1957), 451.
3) G. C. Wick, *Proceedings of the Seventh Annual Rochester Conference on High Energy Nuclear Physics* (1957), (Interscience Publishers, Inc., New York, 1957).
4) R. A. Sorensen, preprint.
5) S. Sunakawa and K. Tanaka, Phys. Rev. **115** (1959), 754.
6) R. Hofstadter, F. Bumiller and M. R. Yearian, Rev. Mod. Phys. **30** (1958), 482.
7) M. Cini, E. Ferrari and R. Gatto, Phys. Rev. Letters **2** (1959), 7.
8) Y. Katayama, M. Taketani, S. Ragusa and D. R. de Oliveira, Prog. Theor. Phys. **23** (1960), 328.
9) G. F. Chew, R. Karplus, S. Gasiorowicz and F. Zachariasen, Phys. Rev. **110** (1958), 265.
10) K. Hiida, N. Nakanishi, Y. Nogami and M. Uehara, Prog. Theor. Phys. **22** (1959), 247.

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Neutrino and Cosmology*

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The elementary relativistic cosmological model is refined by taking into account the proper motions of particles. It is found that the latter were in the past much faster than now, and it is thereby inferred that there must actually exist, in our Universe, a large population of low energy neutrinos, due to previous high energy nuclear reactions. It is shown how these neutrinos can be detected through their gravitational influence. In case they fail to be detected, one must conclude that there exists a possibility of annihilation of neutrino-antineutrino pairs.

§ 1. Introduction

The cosmological influence of neutrinos has recently been the subject of some speculations.[1)-3)] The purpose of the present paper is to show that neutrinos indeed must have an important role in relativistic cosmology, unless they can annihilate in pairs. (Throughout this paper, the term "neutrino" also includes antineutrinos, unless otherwise explicitly stated).

The recent revision of the extragalactic distance scale[4)] has cleared up the contradiction previously encountered between the geological (radioactive) and cosmical ages of the Universe. According to the most recent analysis of astronomical data,[5)] our Universe is closed (hyperspherical) and can be represented by the metric[6)]

$$ds^2=dt^2-a^2[d\chi^2+\sin^2\chi(d\theta^2+\sin^2\theta\, d\varphi^2)], \tag{1}$$

where $a=a(t)$ is called the "Universe radius".

The Einstein gravitational equations can then be written as[7)]

$$d(\rho a^3)+p\,d(a^3)=0, \tag{2}$$

$$\dot{a}^2=\frac{k}{3}\rho a^2-1, \tag{3}$$

where $\rho=T_0{}^0$ and $p=-T_1{}^1=-T_2{}^2=-T_3{}^3$. We have here omitted the so-called cosmological constant, as it has no experimental justification and only leads to theoretical difficulties.[8)]

From (2) and (3) it follows

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$$\ddot{a}=-\frac{k}{6}(\rho+3p)a, \tag{4}$$

which shows how the expansion is continuously *decelerated* by the gravitational attraction of matter density and *pressure*. Note that the attraction of radiative energy is twice as large as that of static energy,[9] because, for radiation, $\rho+3p=2\rho$.

§ 2. A cosmological model

In elementary cosmological models, the Universe is considered as homogeneous and isotropic, and no account is taken of the proper motions of its constituting particles. This oversimplification has been criticized by Nariai,[10] who has shown that these proper motions were much faster in the past, when the Universe was much more concentrated than now. Nariai and Ueno[10),11] have also investigated the conceptual smoothing-out process which leads to a homogeneous universe, and shown that it is not free from ambiguities. We here intend to clarify some of these facts and to discuss their consequences.

For the sake of simplicity, we shall consider the Universe as being constituted of a large number N of pole particles of rest mass m_A, located at $\xi_A^k=\xi_A^k(t)$, where $A=1, 2, \cdots, N$. (These may be galaxies, stars, atoms or elementary particles, according to which phase of the cosmic evolution is under consideration.) Their energy-momentum tensor density can then be written as[12]

$$T^{\mu\nu}=\sum_A m_A u_A^\mu u_A^\nu (ds_A/dt)\delta^3(x^k-\xi_A^k), \tag{5}$$

where u_A^μ is the four-velocity of particle A.

We now introduce some statistical considerations. Let n be the mean number of particles per unit volume. As the total volume of the Universe is

$$V=2\pi^2 a^3, \tag{6}$$

one has

$$n=N/2\pi^2 a^3. \tag{7}$$

If we consider a domain sufficiently large so that the mean velocity in it vanishes, the only non-vanishing components of T_ν^μ are given by

$$\rho=\overline{T_0^0}=nm\overline{(1-v^2)^{-1/2}}, \tag{8}$$

and

$$p=\frac{1}{3}(\rho-\overline{T_\mu^\mu})=\frac{1}{3}nm\left[\overline{(1-v^2)^{-1/2}}-\overline{(1-v^2)^{1/2}}\right], \tag{9}$$

where m is the mean mass of the particles

$$m=(\sum m_A)/N,$$

and the upper bar indicates a *weighted* average

$$\overline{(1-v^2)^{\pm 1/2}} = \sum m_A (1-v_A^2)^{\pm 1/2} / \sum m_A .$$

Here v_A is the magnitude of the three-velocity of particle A.

§ 3. Thermodynamics

Introducing (7), (8) and (9) into (2), one gets

$$d[Nm\overline{(1-v^2)^{-1/2}}] + Nm[\overline{(1-v^2)^{-1/2}} - \overline{(1-v^2)^{1/2}}](da/a) = 0. \qquad (10)$$

Let us first examine the case of slow proper motions. High energy phenomena, such as the creation of nucleon-antinucleon pairs, can then be neglected, so that the total rest mass

$$M = Nm$$

is constant.[12] One may further write, with a good approximation,

$$\overline{(1-v^2)^{1/2}} \cdot \overline{(1-v^2)^{-1/2}} = 1 + O(v^4) \sim 1.$$

The solution of (10) then is

$$\overline{(1-v^2)^{-1/2}} = [1 + (b/Ma)^2]^{1/2} \qquad (11)$$

where b is a constant. This solution is valid as long as $Ma \gg b$. One may then also write $\rho \sim \rho_0 + 3p$, 2, where $\rho_0 = nm$ is the rest density and is proportional to a^{-3}. Introducing this relation into (2) one gets $pV^{5/3} = \text{const.}$, i. e. the Universe adiabatically expands (or contracts) as a monoatomic gas.[10]

The other extreme is given by

$$\overline{(1-v^2)^{1/2}} \ll \overline{(1-v^2)^{-1/2}}.$$

The solution of (10) then is

$$Ma\overline{(1-v^2)^{-1/2}} = \text{const.}$$

It follows that Eq. (11) is still correct, because now $Ma \ll b$. However, the constant b need not have the same value as previously. One may summarize the situation by saying that (11) is everywhere valid, and b is some function of a, which is asymptotically constant both for $b \ll Ma$ and $b \gg Ma$.

Note that if $b \gg Ma$, one has $3p \sim \rho \gg \rho_0$, and it follows from (2) that $pV^{4/3} = \text{const.}$, i. e. the Universe adiabatically expands as pure radiation.[10]

These results are in agreement with the well-known fact[13] that the linear momentum of a *free* particle is inversely proportional to a. Thus, if there were no collisions, and if all particles had the same velocity, Eq. (11) would be accurate. The present proof has the advantage of deriving this result statistically, so that collisions need not be taken into account.

§ 4. Radiation energy and pressure

The frequency of photons or of neutrinos is proportional to their linear mo-

mentum. The fact that the latter is inversely proportional to a[13] thus corresponds to the well-known Doppler red shift law.[14]

The actual density of electromagnetic radiation is extremely low. It seems that a much larger contribution to radiation pressure is due to neutrinos. Indeed, when the Universe was much smaller ($a \lesssim b/M$), there should have been an enormous number of high energy nuclear collisions, resulting in a large production of pions. However, almost each pion decay produces three neutrinos, which are practically indestructible. Their energy, which has decreased inversely to the Universe radius, is now very low: as now $b/Ma \sim 10^{-4}$ to 10^{-3}, the energy of these neutrinos, which were produced when $a \lesssim b/M$, is now at most a few kevs, so that their presence cannot be directly detected, e. g. by induced inverse beta decays.[15] Their total mass may nevertheless be high enough to influence cosmical evolution.[2] (Further important sources of neutrinos at the present time are beta decays and electron bremsstrahlung in stars.[16])

The gravitational effect of radiation can be taken into account by adding to ρ and p terms w and $(w/3)$ respectively, where w is the energy density of radiation.

§ 5. Global evolution

Eqs. (2) and (3) now read

$$d[M\overline{(1-v^2)^{-1/2}}+2\pi^2 wa^3]+[M\overline{(1-v^2)^{-1/2}}-\overline{(1-v^2)^{1/2}}+2\pi^2 wa^3](da/a)=0, \tag{12}$$

$$\dot{a}^2=\frac{k}{3}\left[\frac{M}{2\pi^2 a^3}\overline{(1-v^2)^{-1/2}}+w\right]a^2-1. \tag{13}$$

We further need three equations of state relating M, w, $\overline{(1-v^2)^{-1/2}}$, $\overline{(1-v^2)^{1/2}}$ and a. These can be obtained only through a complete phenomenological theory of matter, relating the mean number of antinucleons to density, pressure, etc.

For the sake of simplicity, we shall here neglect all these factors and suppose that M is a constant, and that there is no energy exchange between the matter and radiation fields, so that

$$2\pi^2 a^4 w = W \tag{14}$$

is also a constant. Both approximations are approximately correct for a not too concentrated Universe, because the radiation energy density w is mostly due to neutrinos, the interaction of which with matter is negligible.

We shall further assume that one can write, with a good approximation,

$$\overline{(1-v^2)^{1/2}}^{-1} \sim \overline{(1-v^2)^{-1/2}} = [1+(b/Ma)^2]^{1/2}.$$

It then follows from (12) and (14) that b is actually constant.

The integration of (13) is now straightforward, and its result is most con-

veniently expressed as a function of a parameter x defined in the domain

$$0 < x < 2\pi - 2 \operatorname{arctg}(q/(m-A)),$$

where $A=b/M$, $m=kM/12\pi^2$ and $q=[k(b+W)/6\pi^2]^{1/2}$. One has

$$t=(m-A)\sin x+mx+q(1-\cos x)+sT, \tag{16}$$

$$(a^2+A^2)^{1/2}=A+q\sin x+(m-A)(1-\cos x). \tag{17}$$

Here, s is any integer, and T represents the period

$$T=2m\left(\pi-\operatorname{arctg}\frac{q}{m-A}\right)+2q\frac{q^2-(m-A)^2}{q^2+(m-A)^2}.$$

One sees that the evolution has singular points at $t=sT$. These singular points can be removed only by introducing strong short range *attractive* forces. Indeed, Eq.(4) shows that in order to get a smooth minimum of $a(t)$, one must have $\rho+3p<0$, i.e. p must be strongly negative. This phenomenon is a characteristic property of general relativity theory: Repulsive forces cause gravitational attraction, while attractive forces cause gravitational repulsion.

Although nuclear fields may have a preponderant role when the Universe is very concentrated, it seems that the solution of the singular state problem should rather be sought in pure gravitational theory. Indeed, the amplitude of adiabatically compressed gravitational waves should behave in the same fashion as the amplitude of electromagnetic waves, i.e. increase beyond any bounds when the volume goes to zero.[17),18)] This means that the metric (1) may then be unstable, as suggested by Power and Wheeler.[19)] Thus, it no longer represents an approximation to world geometry, and probably even not to world topology.[20)-22)] We shall not further discuss this problem, however, as this would lead us onto too speculative grounds.

§ 6. Influence of neutrinos

It seems extremely difficult, within our present knowledge, to estimate the number of neutrinos that have been produced during the explosive phase of the expansion (see Sec. 4). However, one may reasonably infer from Eq. (11) that the Universe was then a kind of ultrarelativistic nuclear plasma, so that most nucleons have been involved in many high energy collisions, and thus have produced many nucleon-antinucleon pairs. It thus seems that the total number of nucleons + antinucleons was then much higher than the present number of nucleons. Of course, the total number of high energy collisions that occurred during this period is still much larger. Each such collision produced several pions, which immediately decayed into three times as many neutrinos. Thus the total number of neutrinos which have been produced should be much higher (probably by several orders of magnitude) than the actual number of nucleons in our Universe. The total energy of these neutrinos has since decreased (as it is inversely

proportional to the Universe radius). It should nevertheless be still larger than the rest mass of the nucleons, or at least of the same order of magnitude.

Let us once more consider Eq. (4), which we now write as

$$-6\dot{a}/ka=\rho_m+3p_m+2w.$$

The index m here refers to matter. The quantity on the left-hand side, as well as ρ_m and p_m, are directly observable. Unfortunately, the actual precision of astronomical data is much too low in order to get a reasonable estimate of the neutrino energy density w.

We may hope, however, that such an estimate will be possible in not too distant a future. If it be found that w is negligible with respect to ρ_m, it seems that there will be no escape from the conclusion that neutrino-antineutrino pairs can annihilate,[23] somewhat like negaton-positon pairs. It may be that this process is actually very difficult to observe (e. g. $\mu\to e+\gamma$ has not yet been observed, but only $\mu\to e+\nu+\tilde{\nu}$) only because we are usually dealing with high energy neutrinos, while the probability of annihilation increases with decreasing energy.[24]

Thus, some information on elementary particles might perhaps be obtained from cosmological data.

References

1) D. R. Brill and J. A. Wheeler, Rev. Mod. Phys. **29** (1957), 465.
2) W. H. McCrea, Endeavour **17** (1958), 5.
3) S. Hayakawa, Prog. Theor. Phys. **21** (1959), 324.
4) A. Sandage, Astrophys. Journ. **127** (1958), 513.
5) H. Hönl and H. Dehnen, Zs. Phys. **156** (1959), 382.
6) L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison Wesley, Cambridge, 1951), p. 337.
7) R. C. Tolman, *Relativity, Thermodynamics and Cosmology* (Oxford, 1934), p. 395.
8) A. Peres, Nuovo Cimento **8** (1958), 533.
9) R. C. Tolman, loc. cit. pp. 211, 272, 278, 285.
10) H. Nariai, Pub. Astr. Soc. Japan **7** (1955), 144.
11) H. Nariai and Y. Ueno, Pub. Astr. Soc. Japan **9** (1957), 179.
12) W. Tulczyjew, Bull. Acad. Polon. Sc. **5** (1957), 279.
13) R. C. Tolman, loc. cit. p. 384.
L. Landau and E. Lifshitz, loc. cit. p. 350.
14) R. C. Tolman, loc. cit. p. 391.
L. Landau and E. Lifshitz, loc. cit. p. 346.
15) F. Reines and C. L. Cowan, Nature, **178** (1956), 446.
16) G. M. Gandelman and V. S. Pinayev, JETP **37** (1959), 1072.
17) E. Schrödinger, Physica **6** (1939), 899.
18) N. Rosen and G. Tauber, preprint.
19) E. A. Power and J. A. Wheeler, Rev. Mod. Phys. **29** (1957), 480.
20) J. A. Wheeler, Ann. of Phys. **2** (1957), 604.
21) D. Finkelstein and C. W. Misner, Ann. of Phys. **6** (1959), 230.
22) A. Peres, Quantum limitations of Mach's principle (to be published).
23) N. Rosen and P. Singer, Bull. Research Council Israel, **8F** (1959), 51.
24) J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley, Cambridge, 1955), p. 263.

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Propagation of Shock Waves in Inhomogeneous Gases. II

——*Hydromagnetic Shock*——

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Using the previous method, the propagation of a hydromagnetic shock wave in inhomogeneous magnetic field is considered. The existence of magnetic field has the effect of restraining the growth of shock strength. When the magnetic pressure is large compared with gas pressure, even the diminution of shock strength can occur. As an application, the propagation of shock wave in solar chromosphere is discussed.

§ 1. Introduction

In the previous paper,[1]* we considered the shock propagation in an inhomogeneous medium as the first approximation. The same method can be applied to the case of a hydromagnetic shock, i.e. a shock wave propagating through a plasma of infinite conductivity in an inhomogeneous magnetic field. This will be important in treating interstellar gasdynamics and stellar surface phenomena.

Although the method can be applied to general hydromagnetic shocks, including oblique or parallel shocks, it would be very cumbersome to deal with the general cases because of the necessity of taking a suitable coordinate system for each layer of matter. So we here treat only the case of the normal shock, where the direction of magnetic lines of force is perpendicular to that of the propagation velocity.

In § 2, we derive the general formulae for the normal shock, which are applied to special cases in § 3. Especially, the propagation in the solar chromosphere is discussed.

§ 2. General formulae

Corresponding to the formulae (2·5)～(2·8) in "I", we have the following relations for the case of the normal hydromagnetic shock:[2]**

$$z_{12}=\frac{1-\lambda^2 y_{12}-(1-3\lambda^2)(1-1/y_{12})p_{m1}/p_1^*}{y_{12}-\lambda^2}\,, \qquad (2\cdot1)$$

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* This paper is hereafter referred to as Paper "I".

** Notation is the same as that in "I", except for those newly introduced.

$$u_2=u_1\pm\phi(z_{12},\ p_1^*,\ p_{m1},\ \tau_1), \tag{2·2}$$

and

$$U=u_1\pm\psi(z_{12},\ p_1^*,\ p_{m1},\ \tau_1), \tag{2·3}$$

where

$$\phi(z_{12},\ p_1^*,\ p_{m1},\ \tau_1)=\sqrt{p_1^*\tau_1}\sqrt{(z_{12}-1)(1-y_{12})}, \tag{2·4}$$

$$\begin{aligned}\psi(z_{12},\ p_1^*,\ p_{m1},\ \tau_1)&=\phi(z_{12},\ p_1^*,\ p_{m1},\ \tau_1)/(1-y_{12})\\&=\sqrt{p_1^*\tau_1}\sqrt{(z_{12}-1)/(1-y_{12})}.\end{aligned} \tag{2·5}$$

τ is the specific volume, i.e. $\tau_1=1/\rho_1$, and $y_{12}=\tau_2/\tau_1=\rho_1/\rho_2$. p^* is the total pressure, which is the sum of gas pressure p_g and magnetic pressure $p_m=H^2/8\pi$ (H is the magnetic field strength, whose direction is always parallel to the shock front) :

$$p^*=p_g+p_m=p_g+H^2/8\pi. \tag{2·6}$$

z_{12} here means the ratio of total pressures ahead and behind the shock wave :

$$z_{12}=p_2^*/p_1^*.$$

In the function ϕ and ψ, y_{12} is related implicitly to z_{12} by the Rankine-Hugoniot relation (2·1) for this case.*

Finally, in the relation (2·1), it is assumed that the magnetic flux is "frozen" in the material throughout the process, because of the infinite conductivity of the plasma concerned. This condition is expressed as

$$H_1/\rho_1=H_2/\rho_2,$$

or

$$p_{m1}\tau_1^2=p_{m2}\tau_2^2. \tag{2·7}$$

Using this relation and the definition (2·6), we obtain the restriction imposed on y_{12} and z_{12} as

$$z_{12}y_{12}^2\geq p_{m1}/p_1^*. \tag{2·8}$$

Of course, the following inequality is valid in general :

$$p_m/p^*\leq 1. \tag{2·9}$$

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* Solving the relation (2·1) for y_{12}, we get

$$y_{12}=\frac{1+\lambda^2 z_{12}-(1-3\lambda^2)p_{m1}/p_1^*+\sqrt{\{1+\lambda^2 z_{12}-(1-3\lambda^2)p_{m1}/p_1^*\}^2+4(z_{12}+\lambda^2)(1-3\lambda^2)p_{m1}/p_1^*}}{2(z_{12}+\lambda^2)} \tag{2·1′}$$

which reduces to (2·1) of "I" when $p_{m1}\to 0$. From this equation we get the limiting relations

$$y=\lambda^2+\{(1-\lambda^4)+[(1-\lambda^2)(1-3\lambda^2)/\lambda^2]p_m/p^*\}\cdot 1/z+O(1/z^2),\ \text{for}\ z\gg 1/\lambda^2$$

and

$$y=1-\frac{1-\lambda^2}{1+\lambda^2+(1-3\lambda^2)(p_m/p^*)}(z-1)+O((z-1)^2),\ \text{for}\ z-1\ll 1.$$

Now we treat the problem of shock propagation in the plasma of variable density or inhomogeneous magentic field strength. The conditions (2·10) and (3·4) in "I" hold, except that p^* is taken instead of p. Following the same procedure as in "I", we finally get the equation of the first approximation,

$$\frac{d\ln\rho}{dz}-\frac{d\ln p^*}{dz}\left(1+2\sqrt{\frac{(z-1)y(1-\lambda^2)}{z(1-y)\{1+\lambda^2+(1-3\lambda^2)p_m/y^2p^*z\}}}\right)$$
$$+\frac{(1-3\lambda^2)}{y\{z+\lambda^2+(1-3\lambda^2)p_m/y^2p^*\}}\frac{d}{dz}\left(\frac{p_m}{p^*}\right)$$
$$=\frac{1}{z-1}+\frac{y-\lambda^2}{(1-y)\{z+\lambda^2+(1-3\lambda^2)p_m/y^2p^*\}}$$
$$+2\sqrt{\frac{y(1-\lambda^2)}{z(z-1)(1-y)\{1+\lambda^2+(1-3\lambda^2)p_m/y^2p^*z\}}}. \qquad (2\cdot10)$$

To this equation are added the following conditions:

$$zy-1+\lambda^2(y-z)+(1-3\lambda^2)(1-1/y)p_m/p^*=0,$$
$$zy^2\geq p_m/p^*, \qquad (2\cdot11)$$

and

$$p_m/p^*\leq 1.$$

To solve Eq. (2·10) in general, the initial distribution of the magnetic pressure p_m should be a function of density ρ. For example, $p_m\propto\rho^{k_m}$, where k_m can be called "magnetic polytropic index", though it is not always necessary to take this form (Case i)). The initial distribution of p_g is assumed always as

$$p_g\propto\rho^k.$$

When the magnetic field is absent, (2·10) reduces to (3·7) in "I".

In order to see the effect of the magnetic field explicitly, we take the limiting cases. In the limit of strong shock, i.e. $z\gg 1/\lambda^2$, we get

$$z\propto(p_g/p^*)p_g^{-a(\gamma,k)}, \qquad (2\cdot12)$$

where $a(\gamma,k)$ is defined by (4·5) in "I". We see the increase of the shock strength as the decrease of gas pressure is restrained by the factor of $p_g/p^*<1$ due to the existence of a magnetic field. Conversely, in the limit of weak shock, i.e. $z-1\ll 1$, we get

$$z-1\propto(p_g/p^*)^{3/4}\{1+[(1-3\lambda^2)/(1+\lambda^2)]p_m/p^*\}^{1/4}\cdot p_g^{-b(k)}, \qquad (2\cdot13)$$

where $b(k)$ is defined by (4·7) in "I". Since the factor multiplied by $p_g^{-b(k)}$ is always less than unity, here also exists the restraining effect of the magnetic field.*

* Note that the restraining factors of (2·12) and (2·13) do not depend on k, but only on the ratio of gas and magnetic pressures. These factors would perhaps be valid also in the second approximation.

Thus, the existence of the magnetic field has in general the effect of restraining the growth of shock strength as the decrease of gas pressure. Moreover, it is noted that this so to speak "*magneto-restraining effect*" is more remarkable for strong shocks, and the shock strength can even be diminished when the magnetic field is sufficiently large compared with gas pressure. This latter effect will be seen in the solar chromosphere as will be shown in the next section.

§ 3. Special cases

Now we apply the formulae obtained in § 2 to the following two special cases:

Case i). *Total pressure p^* is initially constant everywhere*

This corresponds to both a laboratory system and interstellar space, since the gravity is neglected.*

Noting that

$$dp^*=0 \quad \text{and} \quad dp_g=-dp_m$$

for this case, we can reduce the left-hand side of (2·10) to

$$\frac{d(p_m/p^*)}{dz}\left[-\frac{1}{k(1-p_m/p^*)}+\frac{(1-3\lambda^2)}{y\{z+\lambda^2+(1-3\lambda^2)\,p_m/y^2p^*\}}\right]. \qquad (3\cdot1)$$

In Fig. 1, the calculated relations between shock strength and magnetic pressure are shown for $\lambda^2=1/4(\gamma=5/3)$ and $k=1$ (isothermal). Initial shock strength $z_0=2.0$ is taken at $p_m/p^*=0.9$ and 0.98 respectively. We see the shock strength approaches to a finite value when the magnetic pressure becomes negligible.

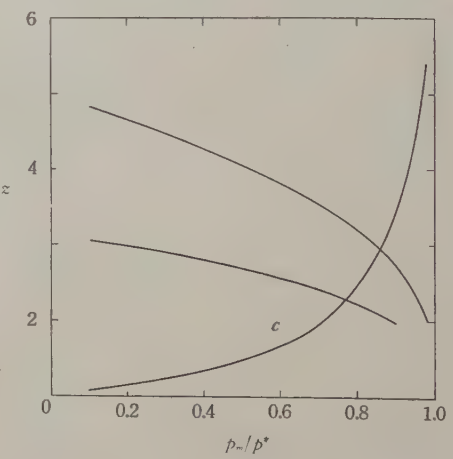

Fig. 1. The variation of shock strength z versus the magnetic pressure when the total pressure is constant. The variation of sound velocity c in appropriate unit is also plotted.

This behaviour of shock strength is due to that of sound velocity. As is known, the square of sound velocity is expressed as the sum of squares:

$$c^2=\gamma p_g/\rho+H^2/4\pi\rho. \qquad (3\cdot2)$$

In our case, this can be reduced to

$$c^2\propto p^{*(k-1)/k} \times\{1+(2-\gamma)/\gamma\cdot p_m/p^*\}/(1-p_m/p^*)^{1/k}$$

or

$$c^2\propto(1+p_m/5p^*)/(1-p_m/p^*) \quad (\text{for } \gamma=5/3 \text{ and } k=1).$$

* Intersteller matter is usually in violent motion. Thus it would be necessary to treat the propagation in moving gases.

This is also plotted in Fig. 1. The rapid increase of sound speed at large p_m/p^* corresponds to the rapid decrease of shock strength. For the limit $p_m/p^* \to 0$, the sound speed approaches a finite value owing to the sound velocity $\gamma p_g/\rho$, and the shock strength is also saturated to a finite value. This tendency is a special representation of the effects of the magnetic field restraining the shock from growing up, as mentioned in § 2.

Case ii). *Constant magnetic field p_m=constant in the gravitational field*

This is applied to the phenomena in the solar chromosphere. In this case, the lefthand side of (2·10) is reduced to

$$\frac{d\ln\rho}{dz}\left[1-kp_g/p^*\left\{1+\frac{(1-3\lambda^2)\,p_m/p^*}{y\{z+\lambda^2+(1-3\lambda^2)\,p_m/y^2p^*\}}\right.\right.$$
$$\left.\left.+2\sqrt{\frac{(z-1)\,y(1-\lambda^2)}{z(1-y)\{1+\lambda^2+(1-3\lambda^2)\,p_m/zy^2p^*\}}}\right\}\right]. \qquad (3\cdot3)$$

Now we investigate the propagation of a shock wave which is generated at the bottom of the solar chromosphere and propagates outwards. For simplicity, we assume that the chromosphere is in the isothermal state and so the parameter k, the polytropic index of gas, is taken equal to unity. For the other physical parameter p_m, we take as p_m=0.4 dyne/cm², i. e. $H=\sqrt{10}$ gauss which corresponds to the general solar magnetic field. The magnetic field is assumed to be constant throughout the whole chromosphere. Taking the initial value of the total pressure and shock strength respectively as $p_0^*=10^3$ dyne/cm² and $z_0=2.2$, Eq. (2·10) supplemented by (3·3) and (2·11) is integrated by the numerical method. The result of integration, the variation of shock strength versus the variation of total pressure, is shown in Fig. 2.

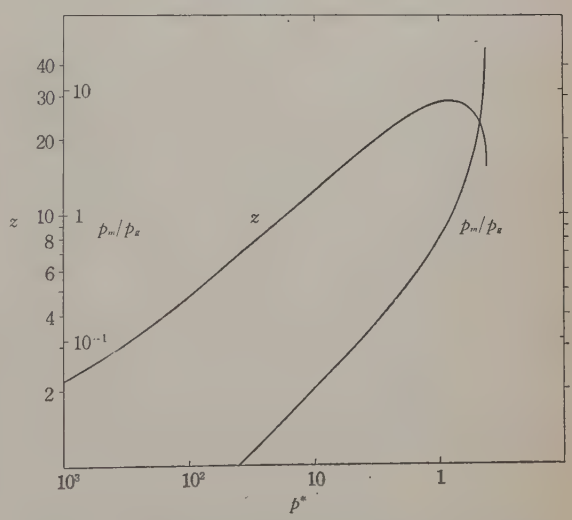

Fig. 2. The variation of shock strength versus the total pressure when a constant magnetic field is present.

It shows a remarkable feature of shock strength coming up to a maximum value at the point where the magnetic pressure is nearly equal to gas pressure. Though the height which corresponds to the point of maximum shock depends on the physical parameter of the chromosphere, it is about 2000 km above the photosphere for the adopted value of p_m. Beyond the maximum point, the mag-

netic pressure exceeds the gas pressure and the shock strength decreases rapidly. This tendency is also a special representation of " magneto-restraining effect " mentioned in § 2.

Note here that we treat the propagation of a shock wave of constant power-supply, which corresponds to the continuous pushing of a piston. Thus the growing or decreasing of shock strength comes from the reflection of rarefactive or compressive wavelets by the inhomogeneities. When the shock weakens by a magnetic field, the compressive reflection-wavelets grow up to a shock wave of reflection. Thus the magnetic field takes the rôle of a reflecting wall.*

Although the shock waves really considered in general are not of constant power-supply, and so the propagation of a shock-pulse must be treated by taking account of the energy dissipation, it is still interesting to speculate that the whole feature of propagation, i.e. the increase of shock strength in early phase and rapid decrease (reflection) in later phase, has some connection with the formation and the disappearence of solar fine mottles and the spicules.[3] As for the contribution of shock waves to the energy balance and the heating of chromosphere or corona, it is further necessary to estimate the dissipation of the energy by shock waves, which will be treated in another paper.

The authors express their heartfelt thanks to Prof. W. Unno for his valuable suggestions.

References

1) Y. Ôno, S. Sakashita and H. Yamazaki, Prog. Theor. Phys. **23** (1960), 294.
2) See for example, Y. Kato, Prog. Theor. Phys. **21** (1959), 409.
3) For example, C. de Jager, Supplement to Nuovo Cimento Vol. XIII Ser. X (1959), 291.

* In this sense, it would be better to say the "*magnetic-wall effect*" rather than the "*magneto-restraining effect*" in this case.

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Spin-Lattice Relaxation of Donor Electrons in Silicon

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The spin-lattice relaxation of donor electrons in silicon is investigated. The relaxation rate is calculated using the adiabatic approximation, in which the electronic wave function is determined instantaneously according to the Hamiltonian which is perturbed by the lattice deformation. The rate thus obtained was found to be equal to that obtained when we take a simple product of the electronic and the lattice wave functions as a starting wave function and take the electron-lattice interaction as a perturbation. We considered the modulation of the spin-orbit coupling due to the lattice vibration as well as deformation of the donor wave function due to the external magnetic field. When we consider one-phonon process, the relaxation rate was found to be proportional to H^4T, as is observed. The absolute magnitude of the rate also agrees with the observation in orders of magnitude.

§ 1. Introduction

The recent works[1)–9)] on the electron spin resonance of donor states in silicon gave much information on their wave functions, and the hyperfine splitting of the donor electron resonance was accounted for by the effective-mass formalism of Kohn and Luttinger for impurity states.[10)]

On the other hand, the situation is more complicated as to the spin relaxation. There are two types of relaxation process. In one of them, the electron spin and the nuclear spin of the donor atom flip simultaneously by virtue of the hyperfine interaction. The relaxation time for this process has been calculated by Pines, Bardeen and Slichter[11)] and gives good agreement with experiment. In the second type of relaxation process only the electron spin flips. PBS and Abrahams[12)] have discussed several mechanisms which lead to this process. The relaxation time as calculated by them is much longer than what is observed and they concluded that the observed relaxation time must be due to a concentration-dependent process of unknown origin.

Recently, however, Feher and Gere[9)] found that the spin relaxation time of phosphorous doners in silicon does not depend on the donor concentration, when this is less than 10^{16}/cc, and is proportional to the absolute temperature between 1.25°K and 2°K. More recently, Honig and Stupp[7)] found that there are three distinct relaxation mechanisms in phosphorous-doped silicon: (a) The first of them is concentration independent and has an H^4 and T dependence, (b) the second

is independent of concentration and magnetic field, and has a T^7 dependence, and (c) the third depends linearly on concentration for concentration below 10^{16}P/cc. We may regard the first two mechanisms as characteristic of the relaxation process of an isolated donor electron, which is caused by the interaction with the lattice vibration, and these may be devoted to direct test with theory. Our theory presented in this paper is not at variance with PBS' and Abrahams' in that we consider the modulation of the spin-orbit coupling by the lattice vibration, but we have carried out a more detailed calculation and, in particular, taken account of the deformation of the donor orbit due to the external magnetic field, which was found to be effective enough to give a correct order of magnitude of the relaxation time of the first kind (a), when one-phonon process is considered.

§ 2. Adiabatic approximation

We take the following as the total Hamiltonian of the system of a donor electron in a crystal:

$$H=H_e+H_L+H_{eL}, \tag{1}$$

where

$$H_e=H_0+H_{ls}+H_z, \tag{2}$$

$$H_0=-(\hbar^2/2m)\Delta+U(\boldsymbol{r}), \tag{3}$$

$$H_{ls}=(\hbar^2/2m^2c^2)(\nabla U\times\boldsymbol{p})\cdot\boldsymbol{S}, \tag{4}$$

$$H_z=\beta H(2S_{z'}+l_{z'}), \tag{5}$$

$$H_L=-\Sigma_g(\hbar^2/2M_g)\Delta_g+\Sigma_{g\neq g'}V(\boldsymbol{R}_g,\ \boldsymbol{R}_{g'}). \tag{6}$$

Here $U(\boldsymbol{r})$ is the periodic potential of the perfect lattice plus the attractive potential from the excess charge of the donor atom. H_{ls} is the spin-orbit coupling energy and H_z is the orbital and spin Zeeman energies. We shall denote the coordinates fixed to the crystal by x, y and z and the $x'y'z'$ system is chosen so that the z' direction coincides with that of the external magnetic field. The direction cosines of x', y' and z' directions should be denoted by (l_1, m_1, n_1), (l_2, m_2, n_2) and (l_3, m_3, n_3), respectively. H_L is the lattice energy of the crystal. $V(\boldsymbol{R}_g, \boldsymbol{R}_{g'})$ is the potential energy between the g-th and g'-th ions, $\boldsymbol{R}_g$ being the displacement of the g-th ion from its equilibrium position. H_{eL} is an interaction energy between the electron and the lattice, which we shall specify later.

It may be appropriate in the present case to solve the time-dependent Schrödinger equation,

$$i\hbar\partial\Psi/\partial t=H\Psi,$$

by the adiabatic approximation. When the following two equations are solved,

$$(H_e+H_{eL})\psi_j(\boldsymbol{r}\,;\,\boldsymbol{R})=E_j(\boldsymbol{R})\psi_j(\boldsymbol{r}\,;\,\boldsymbol{R}), \tag{7}$$

$$[H_L+E_j(\boldsymbol{R})]\varphi_{jn}(\boldsymbol{R})=E_{jn}\varphi_{jn}(\boldsymbol{R}), \tag{8}$$

where $\boldsymbol{R}$ is the abbreviation of $\boldsymbol{R}_1, \boldsymbol{R}_2, \cdots, \boldsymbol{R}_N$, our starting wave function is

$$\Psi_{jn}=\psi_j(\boldsymbol{r}\,;\,\boldsymbol{R})\varphi_{jn}(\boldsymbol{R}). \tag{9}$$

Then we assume that Ψ is expanded as

$$\Psi=\Sigma_{jn}a_{jn}(t)\Psi_{jn}\exp(-iE_{jn}t/\hbar). \tag{10}$$

Since we see

$$(\Psi_{im}|H|\Psi_{jn})=E_{jn}\delta_{ij}\delta_{mn}+(im|H'|jn), \tag{11}$$

where

$$\begin{aligned}(im|H'|jn)=&-\Sigma_g(\hbar^2/2M_g)(\psi_i\varphi_{im}|(\Delta_g\psi_j)\varphi_{jn})\\&-\Sigma_g(\hbar^2/M_g)(\psi_i\varphi_{im}|(\nabla_g\psi_j)(\nabla_g\varphi_{jn})),\end{aligned} \tag{12}$$

we can see that the usual time-dependent perturbation method holds with the perturbing Hamiltonian H'. Then the probability amplitude $|a_{jn}(t)|^2$ that the system is found in the state Ψ_{jn} at time t when it is in the state Ψ_{im} with probability of unity at $t=0$ is given by

$$|a_{jn}(t)|^2=(2\pi t/\hbar)\delta(E_{jn}-E_{im})|(im|H'|jn)|^2. \tag{13}$$

We now make two assumptions in order to solve Eqs. (7) and (8).

1. The eigenfunctions and the eigenvalues of the lattice are not so affected by the motion of the donor electron that φ_{jn} and E_{jn} are the same as those of the perfect lattice. Thus we drop the j-dependence of them. This assumption may be approximately satisfied in our case, because a donor electron is rather loosely coupled to the center and its orbit is large compared with the atomic spacing.

2. We use a perturbation procedure to solve Eq. (7) regarding H_{eL} as a small perturbation proportional to $\boldsymbol{R}_g$. Let ϕ_j be a solution of the following unperturbed equation,

$$H_e\phi_j=E_j\phi_j. \tag{14}$$

Then the solution of Eq. (7) is given by

$$\psi_j=\phi_j+\Sigma_{i\neq j}\phi_i(\phi_i|H_{eL}|\phi_j)/(E_j-E_i), \tag{15}$$

which gives

$$(\psi_i|\nabla_g|\psi_j)=(\phi_i|\nabla_g H_{eL}|\phi_j)/(E_j-E_i), \tag{16}$$

where we have retained only the lowest order term of $\boldsymbol{R}_g$, that is, one which is independent of $\boldsymbol{R}_g$.

From the above two assumptions, we obtain

$$(im|H'|jn)=-\Sigma_g(\hbar^2/M_g)(\psi_i|\nabla_g|\psi_j)(\varphi_m|\nabla_g|\varphi_n). \tag{17}$$

Using the relation

$$(\varphi_m|(\hbar^2/M_g)\nabla_g|\varphi_n)=(E_n-E_m)(\varphi_m|\boldsymbol{R}_g|\varphi_n) \tag{18}$$

leads to

$$(im|H'|jn) = -\Sigma_q (\phi_i|\nabla_q H_{eL}|\phi_j)(\varphi_m|\boldsymbol{R}_q|\varphi_n)(E_n - E_m)/(E_j - E_i)$$
$$= (\phi_i\varphi_m|H_{eL}|\phi_j\varphi_n)[(E_n - E_m)/(E_i - E_j)]. \tag{19}$$

Since the above matrix element is multiplied by $\delta(E_{im} - E_{jn})$ in Eq. (13), the last factor can be replaced by unity when used in (13), and we finally have

$$(im|H'|jn) = (\phi_i\varphi_m|H_{eL}|\phi_j\varphi_n). \tag{20}$$

Thus we see that the transition probability (13) is the same as that which is obtained when we regard $H_e + H_L$ as an unperturbed Hamiltonian and treat H_{eL} as a small perturbation. This result may be ascribed to our early two assumptions. Similar results have been pointed out by several authors.[13)-16)]

§ 3. The second-order perturbation

We shall now consider about ϕ_i and ϕ_j, which are in our case the ground states of a donor electron with spin up and spin down, respectively. They satisfy

$$(H_0 + H_{ls} + H_z)\phi = E\phi. \tag{21}$$

As a result of the work by Kohn and Luttinger,[10)] we now have much knowledge about the wave function of the donor electron, which is the solution of the equation

$$H_0\phi_n = E_n\phi_n. \tag{22}$$

Regarding H_{ls} as a small perturbation and neglecting the orbital Zeeman energy for a while, the ground solution of Eq. (21) is expressed in terms of the solutions of Eq. (22) as

$$\phi_j = \phi_0\beta - \Sigma_n{}'\phi_n\alpha(\phi_n\beta|H_{ls}|\phi_0\alpha)/(E_n - E_0 + 2\beta H)$$
$$= \phi_0\beta + \frac{1}{2}\Sigma_n{}'\phi_n\alpha(\phi_n|A_-|\phi_0)/(E_n - E_0 + 2\beta H), \tag{23}$$

where $$\boldsymbol{A} = (\hbar^2/2m^2c^2)\nabla U \times \boldsymbol{p}, \quad A_- = A_{x'} - iA_{y'}, \tag{24}$$

and similarly for ϕ_i. Then we have

$$(\phi_i|H_{eL}|\phi_j) = \frac{1}{2}\Sigma_n{}'[(\phi_0|A_-|\phi_n)(\phi_n|H_{eL}|\phi_0)/(E_n - E_0 - 2\beta H)$$
$$+ (\phi_0|H_{eL}|\phi_n)(\phi_n|A_-|\phi_0)/(E_n - E_0 + 2\beta H)]. \tag{25}$$

Here we meet with the so-called Van Vleck cancellation,[17)] that is, the following equation holds:

$$[(\phi_0|A_-|\phi_n)(\phi_n|H_{eL}|\phi_0) + (\phi_0|H_{eL}|\phi_n)(\phi_n|A_-|\phi_0)]/(E_n - E_0) = 0, \tag{26}$$

which can be proved by the theorem that the average of angular momentum by a non-degenerate wave function vanishes. When a donor electron is perturbed by the lattice deformation, the wave function is determined by the equation

$$(H_0+H_{eL})\phi'=E'\phi',$$

the solution of which is given by

$$\phi'=\phi_0-\Sigma_n'\phi_n(\phi_n|H_{eL}|\phi_0)/(E_n-E_0).$$

Since ϕ' may also be non-degenerate if so is ϕ_0, we obtain

$$(\phi'|A_-|\phi')=0,$$

which leads to Eq. (26). Then we have from (25) and (26)

$$(\phi_i|H_{eL}|\phi_j)=\frac{1}{2}\Sigma_n'[(\phi_0|A_-|\phi_n)(\phi_n|H_{eL}|\phi_0)$$
$$-(\phi_0|H_{eL}|\phi_n)(\phi_n|A_-|\phi_0)][2\beta H/(E_n-E_0)^2]. \tag{27}$$

According to the idea of the deformable ion, we take as H_{eL} the following expression,

$$H_{eL}=-\boldsymbol{R}(\boldsymbol{r})\cdot\nabla U(\boldsymbol{r}), \tag{28}$$

where
$$\boldsymbol{R}(\boldsymbol{r})=(2MN)^{-1/2}\Sigma_{sk}[2\omega_s(\boldsymbol{k})]^{-1/2}$$
$$\times[a_s(\boldsymbol{k})\boldsymbol{e}_s\exp(i\boldsymbol{kr})+a_s(\boldsymbol{k})^*\boldsymbol{e}_s\exp(-i\boldsymbol{kr})]. \tag{29}$$

Here $a_s(\boldsymbol{k})$ and $a_s(\boldsymbol{k})^*$ are the destruction and the creation operators of a phonon with wave vector $\boldsymbol{k}$ and polarization s, $\boldsymbol{e}_s$ being a unit vector specifying the direction of polarization. They have the following matrix elements,

$$(N_s(\boldsymbol{k})-1|a_s(\boldsymbol{k})|N_s(\boldsymbol{k}))=(N_s(\boldsymbol{k})|a_s(\boldsymbol{k})^*|N_s(\boldsymbol{k})-1)=[\hbar N_s(\boldsymbol{k})]^{-1/2}. \tag{30}$$

M is the mass of a silicon atom and N is the total number of unit cell. From (27), (28) and (29) we have

$$(\phi_i|H_{eL}|\phi_j)=-\beta H(2MN)^{-1/2}\Sigma_{sk}[2\omega_s(\boldsymbol{k})]^{-1/2}[I_{ks}a_s(\boldsymbol{k})+I_{-ks}a_s(\boldsymbol{k})^*], \tag{31}$$

where
$$I_{ks}=\Sigma_n'[(\phi_0|A_-|\phi_n)(\phi_n|\exp(i\boldsymbol{kr})\partial U/\partial s|\phi_0)$$
$$-(\phi_0|\exp(i\boldsymbol{kr})\partial U/\partial s|\phi_n)(\phi_n|A_-|\phi_0)]/(E_n-E_0)^2, \tag{32}$$

$\partial/\partial s$ meaning the differentiation along the direction of polarization. When H is 3000 oe, the magnitude of $\boldsymbol{k}$ is about $10^5\,\mathrm{cm}^{-1}$, from the conservation law of energy, $2\beta H=\hbar\omega_s(\boldsymbol{k})=\hbar u_s k$.

Since the spacial extent of the wave function is about $20\times10^{-8}\,\mathrm{cm}$, $\exp(i\boldsymbol{kr})$ in (32) may be expanded as $1+i\boldsymbol{kr}+\cdots$. Now since A_- belongs to the representation T_2 of T_d, while $\partial U/\partial s$ belongs to T_1, the first contribution of the above expansion vanishes. The second term gives

$$I_{ks}=\Sigma_n'[(\phi_0|A_-|\phi_n)(\phi_n|(i\boldsymbol{kr})\partial U/\partial s|\phi_0)$$
$$-(\phi_0|(i\boldsymbol{kr})\partial U/\partial s|\phi_n)(\phi_n|A_-|\phi_0)]/(E_n-E_0)^2. \tag{33}$$

Using the relations

$$\partial U/\partial s=[\partial/\partial s,\ H_0],$$
$$[(i\boldsymbol{kr}),\ H_0]=(\hbar^2/m)(i\boldsymbol{k}\Gamma),$$

we have

$$(\phi_n|(i\boldsymbol{kr})\partial U/\partial s|\phi_0)=(E_0-E_n)(\phi_n|(i\boldsymbol{kr})\partial/\partial s|\phi_0)-(\hbar^2/m)(\phi_n|(i\boldsymbol{k}\Gamma)\partial/\partial s|\phi_0), \tag{34}$$

the second term of which vanishes because of symmetry. Then we see

$$\begin{aligned}I_{ks}&=-\Sigma_n'[(\phi_0|\Lambda_-|\phi_n)(\phi_n|(i\boldsymbol{kr})\partial/\partial s|\phi_0)+(\phi_0|(i\boldsymbol{kr})\partial/\partial s|\phi_n)(\phi_n|\Lambda_-|\phi_0)]\cdot(E_n-E_0)\\&=-i\hbar\cdot(\boldsymbol{k}\times\boldsymbol{e}_s)\Sigma_n'[(\phi_0|\Lambda_z|\phi_n)(\phi_n|x\partial/\partial y|\phi_0)+(\phi_0|x\partial/\partial y|\phi_n)(\phi_n|\Lambda_z|\phi_0)]\cdot(E_n-E_0),\end{aligned} \tag{35}$$

where $\boldsymbol{h}=(l_1-il_2,\ m_1-im_2,\ n_1-in_2)$.

On the other hand, the shift of spectroscopic splitting factor of the donor electron is given by

$$\begin{aligned}\Delta g&=-\Sigma_n'[(\phi_0|\Lambda_{z'}|\phi_n)(\phi_n|l_{z'}|\phi_0)+(\phi_0|l_{z'}|\phi_n)(\phi_n|\Lambda_{z'}|\phi_0)]/(E_n-E_0)\\&=2i\Sigma_n'[(\phi_0|\Lambda_z|\phi_n)(\phi_n|x\partial/\partial y|\phi_0)+(\phi_0|x\partial/\partial y|\phi_n)(\phi_n|\Lambda_z|\phi_0)]/(E_n-E_0),\end{aligned} \tag{36}$$

hence we have

$$I_{ks}=-\frac{1}{2}\boldsymbol{h}\cdot(\boldsymbol{k}\times\boldsymbol{e}_s)\Delta g. \tag{37}$$

Since only the vector product between $\boldsymbol{k}$ and $\boldsymbol{e}_s$ appears in (37), only the transverse waves are effective in the relaxation process. From (30), (31) and (37) the matrix element of the transition process in which the spin of the donor electron flips, simultaneously emitting a phonon with $\boldsymbol{k}$ and s, is given by

$$(\phi_i N_s(\boldsymbol{k})|H_{eL}|\phi_j N_s(\boldsymbol{k})+1)=\frac{1}{2}\beta H\boldsymbol{h}(\boldsymbol{k}\times\boldsymbol{e}_s)\Delta g[\hbar(N_s(\boldsymbol{k})+1)/2MN2\omega_s(\boldsymbol{k})]^{1/2}. \tag{38}$$

The transition probability per unit time of the process in which the spin direction changes from up to down is given by

$$\begin{aligned}W_1&=(2\pi/\hbar)\Sigma_{sk}|(\phi_i N_s(\boldsymbol{k})|H_{eL}|\phi_j N_s(\boldsymbol{k})+1)|^2\delta(2\beta H-\hbar\omega_s(\boldsymbol{k}))\\&=\hbar\omega_0{}^5\Delta g^2(N_0+1)/24\pi u_t{}^5\rho,\end{aligned} \tag{39}$$

where ρ is the density of the crystal, u_t is the velocity of the transverse wave, $\hbar\omega_0$ is the Zeeman energy, $2\beta H$, and $N_0=|\exp(\hbar\omega_0/kT)-1|^{-1}$. Similarly, the transition probability per unit time from spin down to up is given by

$$W_2=\hbar\omega_0{}^5\Delta g^2 N_0/24\pi u_t{}^5\rho. \tag{40}$$

Then the difference in population between the two Zeeman levels approaches its

equilibrium value according to an exponential function $\exp(-t/T_1)$, where

$$1/T_1=\hbar\omega_0{}^5\Delta g^2(2N_0+1)/24\pi u_t{}^5\rho. \tag{41}$$

We shall estimate the numerical value of $1/T_1$, (41). We have the following values of the parameters which appear in (41).

$$\Delta g=3.4\times10^{-3,8)},\ u_t=(c_{44}/\rho)^{1/2}=5.9\times10^5,\ \rho=2.35. \tag{42}$$

Since $2N_0+1$ (not $2N_0$) is well approximated by $2kT/\hbar\omega_0$, when $kT/\hbar\omega_0\geqq1.5$, the substitution of (42) into (41) leads to

$$1/T_1=2.3\times10^{-11}H^4T\ \text{sec}^{-1}, \tag{43}$$

where H is measured in unit of koe. The temperature and field dependence of $1/T_1$ is in agreement with that of Honig and Stupp's first mechanism. However, they found that the numerical factor before H^4T was 2.6×10^{-7} in contrast with 2.3×10^{-11} of (43). Thus our relaxation time (43) is about 10^4 times longer than the experimental one, and the second order perturbation is insufficient to give a quantitative agreement with experiment.

§ 4. The third-order perturbation

Before going to the third-order process, we shall mention about the g-shift, (36). Hitherto, it has not been necessary to know the detailed form of the wave functions ϕ_0 and ϕ_n, because the quantities which depend upon these have been combined to a simple expression which can in turn be replaced by Δg. The excited states which appear in (27) or (36), can be classified into two groups, one associated with the lowest conduction band, and the other higher excited bands, the former of which we shall denote by ϕ_{0j} and the latter by ϕ_n. Since the excitation energy and the matrix element of A for the first group may be approximately 0.04 ev and $e^2\hbar^2/\kappa m^2c^2a^3\simeq10^{-5}\,\text{cm}^{-1}$, respectively, the contribution to Δg from this group amounts to 10^{-7}, which is quite negligible. Since the effective-mass formalism breaks near the impurity atom, where $\partial U/\partial r$ predominantly contributes to the average of A, the above estimate may be somewhat in error. But there are some evidences that the g-shift arises mainly through the excited conduction bands. Feher[8] measured the shifts of the electron spin resonance of P, As and Sb donors and found that they were the same and nearly equal to the Δg observed for the conduction electron resonance in silicon. We shall assume in the following that the main contribution to (33) comes solely from states ϕ_n associated with the excited conduction bands.

Including the orbital Zeeman energy as a perturbation, we go on to the third order process. This amounts to taking account of the deformation of the wave function due to the external field. Since the orbital Zeeman energy involves i in its operator form, the wave function is no longer real and the Van Vleck cancellation does not occur. This makes the third order matrix element comparable to (or

larger than) the second order one (27). The matrix elements in question involves A, H_{eL} and $\beta Hl_{z'}$ once and the following may be the largest ones,

$$\frac{1}{2}\sum_{nj}[(\phi_0|A_-|\phi_n)(\phi_n|H_{eL}|\phi_{0j})(\phi_{0j}|\beta Hl_{z'}|\phi_0)+(\phi_0|H_{eL}|\phi_n)(\phi_n|A_-|\phi_{0j})(\phi_{0j}|\beta Hl_{z'}|\phi_0)$$
$$+(\phi_0|\beta Hl_{z'}|\phi_{0j})(\phi_{0j}|H_{eL}|\phi_n)(\phi_n|A_-|\phi_0)$$
$$+(\phi_0|\beta Hl_{z'}|\phi_{0j})(\phi_{0j}|A_-|\phi_n)(\phi_n|H_{eL}|\phi_0)]/(E_n-E_0)(E_{0j}-E_0), \qquad (44)$$

where both of wave functions of two groups, ϕ_n and ϕ_{0j}, appear, ϕ_{0j} making the denominator small and ϕ_n making the matrix element of A large. We shall not attempt to evaluate these matrix elements, but only estimate their orders of magnitude. First we note that the first term of the expansion of the matrix elements such as $(\phi_0|\exp(i\boldsymbol{kr})\partial U/\partial s|\phi_n)$ in terms of $\boldsymbol{kr}$ vanishes, because $\partial U/\partial s$ is an odd function whereas the wave functions associated with the matrix element are even functions.* Then roughly speaking, the ratio of (44) to (27) is of order of $(E_n-E_0)/(E_{0j}-E_0)$. We may take E_n-E_0 equal to the separation between the lowest and the first excited conduction bands at the location in $\boldsymbol{k}$ space where the minimum of the lowest conduction band occurs. According to Morita,[18] this amounts to 9 ev. On the other hand, ϕ_{0j} in (44) must form the bases of the representation T_2, because so do the angular momentum operators, l_x, l_y and l_z. According to Kohn and Luttinger,[10] the lowest excited states with T_2 symmetry lie at 0.04 ev above the ground state. Then the ratio $(E_n-E_0)/(E_{0j}-E_0)$ is about 2×10^2. Since the matrix element must be squared to obtain the transition probability, $1/T_1$ from the third order perturbation (44) may be about 4×10^4 times as large as that given by (41), resulting in a quantitative agreement with the observation.

§ 5. Summary

The relaxation process of a donor electron spin due to lattice vibration was treated by the adiabatic approximation. So far as the first order term in H_{eL} is concerned, the relaxation rate was found to be equal to that which is obtained in a usual perturbation procedure in which H_e+H_L is regarded as an unperturbed Hamiltonian and the H_{eL} is introduced as a perturbation. This result is applied immediately to the one-phonon process and was found to give an H^4T term of the relaxation rate. When the deformation of the wave function due to the external field is taken into account, the magnitude of the rate agrees with that of Honig and Stupp's first mechanism in orders of magnitude.

* Strictly speaking, the true symmetry of the system is T_d and the parity is not a good quantum number. However, the potential may be dominantly of symmetry of O_h and wave functions may be classified in terms of g (gerade) and u (ungerade). This is very important, because the first term of the expansion would give a contribution which is proportional to H^2T to the transition probability.

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References

1) R. C. Fletcher, W. A. Yager, G. L. Pearson, A. N. Holden, W. T. Read and F. R. Merritt, Phys. Rev. **94** (1954), 1392.
2) R. C. Fletcher, W. A. Yager, G. L. Pearson and F. R. Merritt, Phys. Rev. **95** (1954), 844.
3) G. Feher, R. C. Fletcher and E. A. Gere, Phys. Rev. **100** (1955), 1784.
4) A. Honig, Phys. Rev. **96** (1954), 234.
5) A. Honig and J. Combrisson, Phys. Rev. **102** (1956), 917.
6) A. Abragam and J. Combrisson, Comp. rend. **243** (1956), 576.
7) A. Honig and E. Stupp, Phys. Rev. Letters **1** (1958), 275; Phys. Rev. (to appear).
8) G. Feher, Phys. Rev. **114** (1959), 1219.
9) G. Feher and E. A. Gere, Phys. Rev. **114** (1959), 1245.
10) W. Kohn, *Solid State Physics*, edited by F. Seitz and D. Turnbull (Academic Press, New York, 1957), Vol. 5.
11) D. Pines, J. Bardeen and C. P. Slichter, Phys. Rev. **106** (1957), 489.
12) E. Abrahams, Phys. Rev. **107** (1957), 491.
13) J. R. Ziman, Proc. Cambridge Phil. Soc. **51** (1955), 707.
14) A. Haug, Z. Physik **146** (1956), 75.
15) H. Stumpf, Z. Naturforsch. **119** (1956), 259.
16) B. Goodman, Phys. Rev. **110** (1958), 888.
17) J. H. Van Vleck, Phys. Rev. **57** (1940), 426.
18) A. Morita, J. Phys. Chem. Solids **8** (1959), 363.

Note added in proof: Recently, Hasegawa (private communication) carried out a detailed theoretical investigation of the donor-electron relaxation, taking the deformation potential as the electron-lattice interaction. We think it is necessary to carry our third order perturbation a little further. Now, we found that (44) of the text became very small. So we calculated the relaxation rate caused by the following process:

$$1/2\,\Sigma_{nj}\langle\phi_0|A_-|\phi_n\rangle\langle\phi_n|\beta H l_{z'}|\phi_{nj}\rangle\langle\phi_{oj}|H_{eL}|\phi_o\rangle/(E_n-E_o)(E_{oj}-E_o)$$

+other three terms in which the order of the matrix elements interchanged.

We took as ϕ_{oj} the two- and three-fold degenerate states of E and T_1 symmetry, which are the linear combinations of the six s-type solutions associated with each conduction-band minimum. As to ϕ_n's, we may approximate them by the excited Bloch functions. Then we obtained the following result,

$$1/T_1=(2/15\pi\rho)\{(2/u_t^5)+(3/u_t^5)\}(2\beta H/\hbar)^4 kT\cdot P,$$
$$P=Q_e^2(l^2m^2+m^2n^2+n^2l^2)+2Q_{t1}^2\{1-2(l^2m^2+m^2n^2+n^2l^2)\},$$
$$Q_e=\langle l_y A_y\rangle\{\langle\bar{x}(\partial U/\partial x)\rangle-\langle\bar{y}(\partial U/\partial y)\rangle\}/3(E_n-E_o)(E_e-E_o),$$
$$Q_{t1}=\langle l_y A_z\rangle\langle\bar{y}(\partial U/\partial z)\rangle/3(E_n-E_o)(E_{t1}-E_o),$$

where l, m and n are the direction cosines of the external field and $\langle\ \rangle$ means the average by the Bloch function at the location on the $+x$ axis in k-space where the minimum of the conduction band occurs. In the above matrix elements, $\bar{x}$ and $\bar{y}$ are the coordinates measured from the centre of the unit cell in which the integration is being carried out. We may expect $\langle(\bar{x}\partial U/\partial x)\rangle-\langle\bar{y}(\partial U/\partial y)\rangle\simeq$ 1 ev. Q_e and Q_{t1}, which are dimensionless quantities, come from the excited states of E and T_1

symmetry, respectively. We note that Q_{t1} would vanish if the symmetry around an atom were O_h, so we may expect Q_{t1} is smaller than Q_e, presumably $Q_{t1}=1/3\sim 1/5\,Q_e$. When the external field is in the [100] direction, the relaxation occurs only through the T_1 states. Since the measurements on the anisotropy of the relaxation time is not available, we shall only estimate the magnitude of the above relaxation rate. If we take $P\cong Q_e^2$ and compare our result with Honig and Stupp's experimental one, we have $Q_e=0.16$. Then from $\Delta g=-(4/3)\langle l_y A_y\rangle/(E_{\pi}-E_{\sigma})=3.4\times 10^{-3}$ and $E_e-E_o=0.01$ ev[19], we have $\langle \bar{x}(\partial U/\partial x)\rangle-\langle \bar{y}(\partial U/\partial y)\rangle=2$ ev. Thus our result seems to give a reasonable order of magnitude of the relaxation time. Finally, we note that since u_t is larger than u_l, the transverse wave is much more effective (about 10 times) than the longitudinal wave to cause the spin flipping.

Recently, G. Feher, E. A. Gere and D. K. Wilson (Bull. Am. Phys. Soc. II, **5** (1960), 264) have measured the angular dependence of the relaxation time of phosphorous donors (10^{15}/cc) in silicon. Their result can be fitted by the choice of $Q_e=0.245$ and $Q_{t1}=0.09$. Then we have $Q_{t1}/Q_e=1/2.7$ and $\langle \bar{x}(\partial U/\partial x)\rangle-\langle \bar{y}(\partial U/\partial y)\rangle=2.9$ ev.

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Theory of Matter with Super Light Velocity

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The matter with a super light velocity is treated as one possible model in the course of looking for the physical concept that will essentially govern the future theory of elementary particles. At first, it is investigated according to the canonical quantization to what extent this matter could be reconciled to the usual particle aspect of elementary particles. Further, taking into consideration the interaction of this matter with other particles, it is attempted to quantize this matter so as to derive the Lorentz-invariant, but quasi-causal S-matrix and to remove the free state of this matter from possible physical states.

§ 1. Introduction

Recent successive researches for the field theory of elementary particles have made clear the general features of the theory apart from the detailed structure of interactions.[1] That is, they start only from the several fundamental postulates concerning the special theory of relativity and quantum mechanics, such as the Lorentz-invariance, the micro-causality, the existence of the vacuum state, the assumption on the asymptotic behavior of field and so on.

However, these conditions are so rigid that it seems nearly impossible to seek for the new possibility of the unified and consistent future theory in these framework, although we have yet no definite evidence against it. At present time, many authors have serious doubts as to whether the above framework will remain valid in the future theory. So far, several attempts have been made to change the present field theory, for instance, the denial of the point model for elementary particles in the non-local field theory,[2] or the introduction of the indefinite metric into the Hilbert space in Heisenberg's non-linear field theory.[3]

On the other hand, experimental evidences for strange particles as shown in Nishijima-Gell-Mann's rule seem strongly to suggest that the future theory should necessarily be able to give rise to new degrees of freedom, such as the iso-spin, strangeness, etc.

Now, in order to get a clue to the modification of the present theory, it seems quite important to discover the new physical concept which will probably characterize the future theory. From such a viewpoint, it is worthwhile to pay our attention to the concept of "*B*-matter" recently introduced by Sakata and his coworkers.* They have proposed a new baryon model (named "Nagoya Model",)

* Their works appeared in Prog. Theor. Phys. **23** (1960), 1174.

in which N, P and Λ are considered to be composed of B-matter and leptons, based on Kiev symmetry[4] in weak interactions. According to this model, B-matter could exist only by accompanying leptons, and not by itself. Further, this matter would probably violate principles of the usual quantum mechanics. While such a matter is unfamiliar to us from the customary point of view of the field quantization, it may offer some image for the future theory.

From the viewpoint on the theoretical side, it is important to seek for the new possibility, by focusing our attention on such an image as symbolized by the above model.

In this paper we shall investigate the matter with a super light velocity (hereafter we shall describe it in terms of "S-field") as one possible model in the course of looking for the above image. The relativistic invariant S-field is defined by the following wave equation,

$$(\Box+m^2)\phi(x)=0, \quad \left(\Box\equiv\Delta-\frac{\partial^2}{\partial x_0{}^2}\right) \tag{1·1}$$

with m real or by its linearized equation

$$(\gamma_\mu\partial_\mu+im)\psi(x)=0. \tag{1·2}^{*,\,**}$$

From (1·1), we easily find that the four momentum $(\boldsymbol{p}, iE)$ satisfies the modified Einstein relation

$$E^2=\boldsymbol{p}^2-m^2. \tag{1·3}$$

Therefore, the group velocity of this wave with $\boldsymbol{p}^2>m^2$ becomes a super light velocity :

$$v_g=c\,\partial E/\partial p=cp/E>c. \tag{1·4}$$

Although Pais and Uhlenbeck[5] once investigated a multi-mass field with complex conjugate pair of masses, μ and μ^*, they abandoned such a possibility as an unacceptable field by reason of occurrence of the solution with unbounded character, to which we shall refer in § 3. Further, Schmidt[6] recently investigated in detail the invariant solution of (1·1) and its source problem from the viewpoint of the causality. However, the essential character of the S-field will be revealed rather through the quantization procedure. In fact, it is found in § 3 and § 4, that S-field is intimately connected with the limitation of the fundamental concept of the present field theory such as the localizability of the field, the indefinite metric, the existence of the vacuum state, the micro-causality, parity indefiniteness, etc.

---

* More generally, the linearized equation of (1·1) is given by $(\gamma_\mu\partial_\mu+i\alpha+\beta\gamma_5)\psi(x)=0$, where α and β are arbitrary real constants. However, this equation is transformed into (1·2) by suitable linear transformation of ψ. See Appendix A.

** $a_\mu b_\mu=ab=\boldsymbol{a}\boldsymbol{b}-a_0b_0$

In § 2, we shall study the propagation character of the classical S-field in connection with the results obtained by Schmidt.[6] In § 3, the quantization of the S-field is given by the orthodox canonical formalism, in order to see to what extent this field could be reconciled with the usual particle aspect for elementary particles. Further in § 4, we shall consider the possibility of the alternative quantization procedure for the S-field, taking into consideration its interaction with other particles.

§ 2. The invariant function and the propagation character of the classical wave

Let us consider the invariant solution of the source problem of (1·1),

$$(\Box+m^2)G(x)=\delta^4(x). \tag{2·1}$$

This problem has already been investigated in detail by Schmidt.[6] Therefore, in the present paper, we shall refer mainly to the problem concerning the solution with the *abnormal part*, which the above author did not take into consideration as an unphysical one.

The general solution of (2·1) is given by

$$G(x)=-\frac{1}{(2\pi)^4}\int_C \frac{e^{ikx}}{k^2-m^2}d^4k, \tag{2·2}$$

where C denotes the integration path in complex-k_0 plane. The poles of the integrand in the above integration in the complex k_0-plane are shown in Fig. 1 by bold lines. There exist four invariant solutions with the integration paths indicated in Fig. 1, and they are written explicitly as follows,

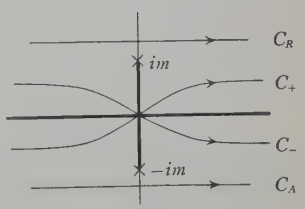

Fig. 1. Integration paths in complex-k_0 plane

$$G_\pm(x)=-\frac{1}{(2\pi)^4}\int\frac{e^{ikx}}{k^2-m^2\mp i\varepsilon}d^4k, \tag{2·3}$$

$$G_{R,A}(x)=-\frac{1}{(2\pi)^4}\int\frac{e^{i(k\pm i\eta\xi)x}}{(k\pm i\eta\xi)^2-m^2}d^4k, \tag{2·4}*$$

where $\eta>m$, ε is an infinitesimal positive number and ξ_μ a time-like unit vector in a future light cone. After the integration over k_0, or $\boldsymbol{k}$, they are rewritten as

$$G_\pm(x)=\frac{-1}{(2\pi)^2}\int_0^\infty du\,\frac{e^{\pm ir\sqrt{m^2+u^2}}}{r}\cos x_0 u, \tag{2·3′}$$

* See Appendix C.

$$
\begin{aligned}
G_R(x) &= G_A(-x) \\
&= -\frac{1}{(2\pi)^3}\left\{\int_{k^2>m^2} d^3k \frac{1}{\sqrt{k^2-m^2}} \sin\sqrt{k^2-m^2}\, x_0 e^{ikx}\right. \\
&\left. + \int_{k^2<m^2} d^3k \frac{1}{\sqrt{m^2-k^2}} \sinh\sqrt{m^2-k^2}\, x_0 e^{ikx}\right\}, \quad x_0>0 \\
&= 0, \qquad\qquad x_0<0.
\end{aligned} \tag{2·4'}
$$

From the above expressions, it is found that $G_{R,A}$ involves the *abnormal* part which implies the second term of (2·4') and is an exponentially increasing function of x_0. The separation of this part from the *normal* part which means the first term of (2·4') and is a usual oscillating function of x_0 is, of course, not Lorentz-invariant.

However, it is possible to divide both parts Lorentz-covariantly, by making use of the above time-like unit vector ξ_μ:

$$
G_{R,A}(x) = G^{(n)}_{R,A}(x, \xi) + G^{(ab)}_{R,A}(x, \xi), \tag{2·5}
$$

where

$$
G^{(n)}_{R,A}(x, \xi) = -\frac{1}{(2\pi)^4}\left\{P\int \frac{e^{ikx}}{k^2-m^2} d^4k \pm i\pi \int \epsilon(k\xi) e^{ikx} \delta(k^2-m^2) d^4k\right\}. \tag{2·6}*
$$

In fact, we find that $G^{(n)}_R(x, \xi)$ with $\xi_\mu = n_\mu = (0, 0, 0, i)$, which corresponds to the integration path in Fig. 2, is just the first term of (2·4') and identical with the quasi-retarded function defined by Schmidt,[6] that is,

$$
G^{(n)}_{R,A}(x, n) = -\frac{1}{(2\pi)^2}\frac{1}{r}\int_0^\infty du \cos(r\sqrt{m^2+u^2} \mp x_0 u). \tag{2·7}
$$

As seen from (2·7) and (2·3'), there exists the following important relations between the above functions,

$$
\left.
\begin{aligned}
G^{(+)}_+(x) &= G^{(n)(+)}_R(x, n), \\
G^{(-)}_+(x) &= G^{(n)(-)}_A(x, n),
\end{aligned}
\right\} \tag{2·8}
$$

where $(\pm)$'s mean the positive or negative time frequency parts. In the usual theory of Klein-Gordon particle with a real mass, the above relations hold for the exact retarded or advanced function and express a causality of theory. The physical interpretation of the above relation will be referred to in § 4.

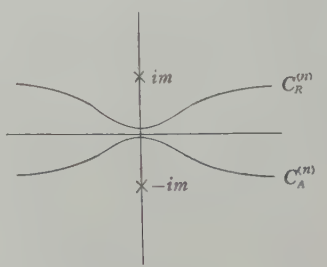

Fig. 2.

* $\epsilon(a)$ is the sign factor defined by

$$
\begin{aligned}
\epsilon(a) &= 1, \quad a>0, \\
&= -1, \quad a<0.
\end{aligned}
$$

Secondly, we shall study the invariant solution of (1·1). These solutions are given by the difference of solutions of the inhomogeneous equation (2·1), that is,

$$D^{(1)}(x) \equiv \{G_+(x) - G_-(x)\}/2i = \frac{-1}{2(2\pi)^3}\int d^4k\, e^{ikx}\,\delta(k^2-m^2)$$

$$= \frac{-1}{2(2\pi)^3}\int_{-\infty}^{\infty} d\tau \int d^4k\, e^{i(k+\tau\xi)x}\,\delta(k^2-m^2-\tau^2)\,\delta(k\xi)$$

$$= \frac{-1}{2(2\pi)^3}\int d^3p\,\frac{1}{2\eta\xi_0}\,(e^{i(p+\eta\xi)x} + e^{i(p-\eta\xi)x}), \tag{2·9}$$

with

$$p\xi = 0, \quad \eta = \sqrt{p^2-m^2} = \sqrt{\mathbf{p}^2 - (\mathbf{p}\cdot\boldsymbol{\xi})^2/\xi_0^2 - m^2},$$

and

$$D(x) \equiv G_A(x) - G_R(x). \tag{2·10}$$

The latter D-function involves again both the normal and the abnormal part, and is also separated covariantly in a similar way as in the $G_{R,A}$-functions:

$$D(x) = D^{(n)}(x,\ \xi) + D^{(ab)}(x,\ \xi), \tag{2·11}$$

where

$$D^{(n)}(x,\ \xi) = G_A^{(n)}(x,\ \xi) - G_R^{(n)}(x,\ \xi) = \frac{-i}{(2\pi)^3}\int \epsilon(k\xi)\, e^{-ikx}\,\delta(k^2-m^2)\,d^4k,$$

or

$$= \frac{-i}{(2\pi)^3}\int_{-\infty}^{\infty} d\tau \int d^4k\,\epsilon(\tau)\, e^{i(k+\tau\xi)x}\,\delta(k^2-m^2-\tau^2)\,\delta(k\xi), \tag{2·12}$$

$$= \frac{-i}{(2\pi)^3}\int d^3p\,\frac{1}{2\eta\xi_0}\,\{e^{i(p+\eta\xi)x} - e^{i(p-\eta\xi)x}\}, \tag{2·13}$$

with $p\xi = 0$, $p^2 \geq m^2$, $\eta = \sqrt{p^2-m^2} = \sqrt{\mathbf{p}^2 - (\mathbf{p}\cdot\boldsymbol{\xi})^2/\xi_0^2 - m^2}$,

$$D^{(ab)}(x,\ \xi) = G_A^{(ab)}(x,\ \xi) - G_R^{(ab)}(x,\ \xi)$$

$$= \frac{+1}{(2\pi)^3}\int_{-\infty}^{\infty} d\tau \int d^4k\,\epsilon(\tau)\, e^{i(k-i\tau\xi)x}\,\delta(k^2-m^2+\tau^2)\,\delta(k\xi) \tag{2·14}$$

$$= \frac{+1}{(2\pi)^3}\int \frac{d^3p}{2\rho\xi_0}\,\{e^{i(p-i\rho\xi)x} - e^{i(p+i\rho\xi)x}\}, \tag{2·15}$$

with $p\xi = 0$ and $\rho = \sqrt{m^2-p^2} = \sqrt{m^2 + (\mathbf{p}\cdot\boldsymbol{\xi})^2/\xi_0^2 - \mathbf{p}^2}$.

Especially, for $\xi_\mu = n_\mu$

$$D^{(n)}(x,\ n) = \frac{-1}{(2\pi)^3}\int_{k^2>m^2} \frac{d^3k}{\sqrt{k^2-m^2}}\,\sin\sqrt{k^2-m^2}\,x_0\, e^{ik\cdot x}, \tag{2·13′}$$

$$D^{(ab)}(x,\ n) = \frac{-1}{(2\pi)^3}\int_{k^2<m^2} \frac{d^3k}{\sqrt{m^2-k^2}}\,\sinh\sqrt{m^2-k^2}\,x_0\, e^{ik\cdot x}. \tag{2·14′}$$

Of course, the normal and the abnormal parts of the invariant function are not Lorentz-invariant, while they have the following covariant property,

$$F(x,\ \xi)=F(x',\ \xi'),$$

where

$$x_\mu'=\Lambda_{\mu\nu}x_\nu,\quad \xi_\mu'=\Lambda_{\mu\nu}\xi_\nu,$$

and $\Lambda_{\mu\nu}$'s are Lorentz transformation coefficients.

The above defined function $D(x)$ satisfies the following familiar relations:

$$D(x)=0,\quad x_\mu^{\,2}>0 \tag{2·16}$$

$$\left.\begin{aligned}&\frac{\partial}{\partial x_0}D(\boldsymbol{x},\ x_0)|_{x_0=0}=-\delta^3(\boldsymbol{x}),\\ \text{and}\quad&\frac{\partial}{\partial x_l}D(\boldsymbol{x},\ x_0)|_{x_0=0}=0.\end{aligned}\right\} \tag{2·17}$$

Of course, $D^{(n)}(x,\xi)$-function has no such a property. In fact, it is in general non-vanishing in the space-like region except the special hypersurface with $\xi x=0$.

By means of this property of D-function, we can solve the initial value problem in the usual way. The solutions of (1·2) and (1·1) are given by

$$\phi(x)=-\int_\sigma D(x-x')\overleftrightarrow{\partial}_\mu'\phi(x')\,d\sigma_\mu' \tag{2·18}$$

$$(\ A\overleftrightarrow{\partial}_\mu B\equiv(\partial_\mu A)B-A\partial_\mu B\ )$$

and

$$\psi(x)=\int_\sigma(\gamma_\mu\partial_\mu+m\gamma_5)D(x-x')\gamma_\mu\psi(x')\,d\sigma_\mu', \tag{2·19}$$

where the hypersurface σ is indicated in Fig. 3. Since the D-function vanishes in the space-like region, S-field obviously has a hyperbolic propagation character, namely, the behavior of the wave in an arbitrary world point is completely determined only by the information on the wave on a hypersurface in a past light cone. This is obvious also from the fact that the propagation character is in general determined only from the form of terms which involve the highest degrees of derivative with respect to space-time.[5]

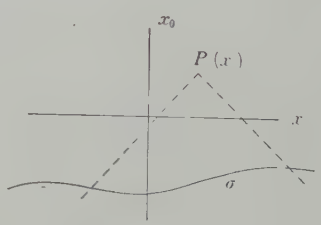

Fig. 3.

Although the above statement apparently seems to contradict the fact that S-field has a group-velocity larger than the light velocity, this situation is easily understood as follows. First of all, it is noted that the propagation character essentially indicates the propagation behavior of the spatial discontinuity of the wave. On the contrary, the wave with which the group velocity is concerned is

composed of the plane waves with an infinitesimal interval of nearly adjacent wave-lengths and thus has no spatial discontinuity. Furthermore in such a wave, the field quantities at two different spatial regions at a given time are no longer independent from each other. Therefore although the group velocity of S-field seems to imply the propagation of information with a super light velocity, this is not the case.

In fact, the above statement is concretely confirmed in the following way. For simplicity, let us take a flat σ-surface perpendicular to the time axis as in Fig. 4. In general, the right-hand side of (2·18) is divided into four components, which come from the devision of the D-function into the normal and the abnormal parts as in (2·11) and from the separation of the integral region over the σ-plane into the inside and the outside regions of the light cone as shown in Fig. 4. This devision is written as

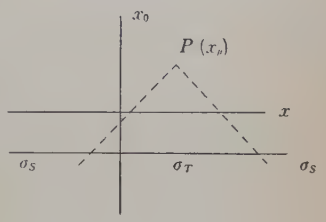

Fig. 4.

$$\phi(x)=\phi_S^{(n)}(x)+\phi_T^{(n)}(x)+\phi_S^{(ab)}(x)+\phi_T^{(ab)}(x), \tag{2·20}$$

where

$$\left.\begin{aligned}\phi_{S(T)}^{(n)}(x)&=\int_{\sigma_S(\sigma_T)} D^{(n)}(x-x',\ n)\overleftrightarrow{\partial}_0\phi(x')d^3x',\\ \phi_{S(T)}^{(ab)}(x)&=\int_{\sigma_S(\sigma_T)} D^{(ab)}(x-x',\ n)\overleftrightarrow{\partial}_0\phi(x')d^3x',\end{aligned}\right\} \tag{2·21}$$

and $\sigma_S(\sigma_T)$ denotes the space-like (the time-like) region of the σ-surface. From the relation of (2·16),

$$\phi_S^{(n)}(x)+\phi_S^{(ab)}(x)=0. \tag{2·22}$$

Now, the wave of which the group velocity can be defined is composed only of the plane waves whose wave vectors satisfy $\boldsymbol{p}^2>m^2$. On the other hand, $D^{(ab)}(x, n)$ involves only the plane wave with $\boldsymbol{p}^2<m^2$ as shown in (2·14′). Thus, for the wave under consideration, one finds that the following relation is satisfied,

$$\phi_S^{(ab)}(x)+\phi_T^{(ab)}(x)=0. \tag{2·23}$$

Then, from (2·22) and (2·23),

$$\phi_S^{(n)}(x)=\phi_T^{(ab)}(x),$$

or

$$\int_{\sigma_S} D^{(n)}(x-x',\ n)\overleftrightarrow{\partial}_0'\phi(x')d^3x'=\int_{\sigma_T} D^{(ab)}(x-x',\ n)\overleftrightarrow{\partial}_0'\phi(x')d^3x'. \tag{2·24}$$

Thus, making use of (2·22) and (2·23), we obtain two different expressions for $\phi(x)$,

$$\phi(x)=\phi_T^{(n)}(x)+\phi_T^{(adv)}(x)=\int_{\sigma_T} D(x-x')\overleftrightarrow{\partial_0'}\phi(x')d^3x', \qquad (2\cdot25)$$

and

$$\phi(x)=\phi_T^{(n)}(x)+\phi_S^{(n)}(x)=\int_{\sigma(\sigma_S+\sigma_T)} D^{(n)}(x-x',\ n)\overleftrightarrow{\partial_0'}\phi(x')d^3x'. \qquad (2\cdot26)$$

In the first expression, the information only in the past time-like region contributes to $\phi(x)$, contrary to the second expression in which the information in the space-like region is necessary in addition to the past time-like region. Namely, if we describe the propagation of S-field only in terms of the normal D-function, it becomes necessary to take into account the information in the space-like region, which is however completely determined by $\phi_T^{(adv)}(x)$, namely the information in the time-like region, as shown in (2·24). Thus, the apparently acausal behavior of the S-field is explained by the fact that the behavior of the wave under consideration is not independent in a space-like region, but is imposed by the condition such as (2·24)*.

We have hitherto confined ourselves only to the propagation character of classical wave. In the quantized theory, the causal property for the energy propagation is not completely determined only by the field equation, but connected essentially with the definition of the vacuum state. We shall deal with this problem in the next section.

§ 3. The quantization of S-field ——*The canonical formalism*——

The Lagrange functions, which lead to the fundamental equation of S-field

$$(\Box+m^2)\phi(x)=0, \qquad (1\cdot1)$$

or

$$(\gamma_\mu\partial_\mu+im)\psi(x)=0 \qquad (1\cdot2)$$

are given by

$$\mathcal{L}(x)=\phi^*(x)(\Box+m^2)\phi(x), \qquad (3\cdot1)\dagger$$

or

$$\mathcal{L}(x)=\psi^\dagger(x)(\gamma_\mu\partial_\mu+im)\psi(x), \qquad (3\cdot2)$$

where

$$\psi^\dagger(x)=\psi^*(x)\gamma_4\gamma_5=\overline{\psi}(x)\gamma_5. \qquad (3\cdot3)$$

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* It fact, the well-known fact that the phase velocity in the usual Klein-Gordon field becomes larger than the light velocity is essentially based on the same reason. In this case, it is enough to separate the D-function into two parts so that one of them covers completely the interval of wave-lengths of the wave under consideration.

† * denotes the complex conjugate.

The hermiticity of the above Lagrangian is obvious. According to the usual canonical formalism, we obtain the following commutation relations for $\phi(x)$ and $\psi(x)$,

$$\left[\phi(\boldsymbol{x}, x_0), \frac{\partial}{\partial x_0}\phi^*(\boldsymbol{x}', x_0)\right]=\delta^3(\boldsymbol{x}-\boldsymbol{x}'), \tag{3·4}$$

$$\{\psi(\boldsymbol{x}, x_0), \psi^*(\boldsymbol{x}', x_0)\}=\gamma_5\delta(\boldsymbol{x}-\boldsymbol{x}'), \tag{3·5}$$

and their covariant expressions

$$[\phi(x), \phi^*(x')]=iD(x-x'), \tag{3·4′}$$

$$\{\psi(x), \bar{\psi}(x')\}=-i\gamma_5(\gamma_\mu\partial_\mu-m\gamma_5)D(x-x'), \tag{3·5′}*$$

by making use of the equation of motion and the property of D-function. D-function in the above commutation relation involves the normal and the abnormal parts, as shown in the preceding section, so the canonical formalism based on the local independence of fields makes necessary to introduce the abnormal component in the field operators themselves.

In order to give the explicit representation for field operators, it is necessary to remark several characters concerning this abnormal component of field operators. Let us assume the following form for the solution of (1·1):

$$\exp(ip_\mu x_\mu). \tag{3·6}$$

Then, the four vector p_μ satisfies

$$p_\mu p_\mu=\boldsymbol{p}^2-p_0^2=m^2. \tag{3·7}$$

From (2·2), one finds that the solutions are divided into two classes with distinct characters. The first class of the solutions is characterized by the real four vector p_μ which necessarily leads to $\boldsymbol{p}^2\geq m^2$. In this case the condition $\boldsymbol{p}^2\geq m^2$ is Lorentz-invariant,** but the sign of p_0 is not, because the (real) four vector p_μ is a space-like vector. The second class of the solution is characterized by the four vector p_μ, where $\boldsymbol{p}^2<m^2$ and thus p_0 is pure imaginary. However, this statement is not Lorentz-invariant, because this four vector p_μ is in general transformed into complex four vector in other reference system. Thus, it is necessary to specify the reference system, in which the second class of solution is defined by the conditions $\boldsymbol{p}^2<m^2$ and p_0 to be pure imaginary. It is obvious that the first and the second classes in the above division correspond to the normal and the abnormal parts of the invariant functions in the previous section, because the latter part involves the

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* It should be noted that the above quantization of a spinor field in connection with the equation of motion violates the invariance under the space inversion. The detailed discussion is left for Appendix A.

** In fact, if p_μ forms a real four vector, then $\boldsymbol{p}^2\geq m^2$. Taking account of the property of the Lorentz transformation, the real property of p_μ remains invariant in any other reference system, so the relation $\boldsymbol{p}^2\geq m^2$ also holds invariantly.

factor which is in general exponentially increasing both in space-like and time-like directions, and so has no physical reality.

Accordingly, in order to deal with the normal and the abnormal parts in a unified and Lorentz-covariant way, it is convenient to introduce a time like unit vector ξ_μ in a future light cone: $\xi_\mu\xi_\mu=-1$, $\xi^0>0$. Then the four vector p_μ in (3·7) for both parts is described in the following form, respectively:

N) *Normal part*:

$$p_\mu=k_\mu+\tau\xi_\mu, \tag{3·8}$$

where we assume

$$k_\mu\xi_\mu=0 \quad \text{or} \quad k_0=\boldsymbol{k}\cdot\boldsymbol{\xi}/\xi_0. \tag{3·9}$$

Then, τ, which is assumed to be real, is determined from (3·7) as follows,

$$\tau=\pm\sqrt{k_\mu^2-m^2}=\pm\sqrt{\boldsymbol{k}^2-(\boldsymbol{k}\cdot\boldsymbol{\xi})^2/\xi_0^2-m^2}, \tag{3·10}$$

with the condition

$$k_\mu k_\mu\geq m^2. \tag{3·11}$$

Therefore, the normal part is characterized by $\boldsymbol{k}$ and the sign of τ which is Lorentz-invariant.

Ab) *Abnormal part*:

$$p_\mu=k_\mu+i\tau\xi_\mu, \tag{3·8′}$$

under the conditions (3·9). Then, τ, which is assumed to be real, is determined as

$$\tau=\pm\sqrt{m^2-k_\mu^2}=\pm\sqrt{m^2-\boldsymbol{k}^2-(\boldsymbol{k}\cdot\boldsymbol{\xi})^2/\xi_0^2}, \tag{3·12}$$

with the condition

$$0\leq k_\mu^2<m^2. \tag{3·13}$$

Therefore, the abnormal part is also characterized by the $\boldsymbol{k}$ and the invariant sign of τ.

Thus, the field operator subject to the commutation relation

$$[\phi(x),\ \phi^*(x')]=iD(x-x'), \tag{3·4′}$$

is explicitly represented by

$$\phi(x,\ \xi)=\phi^{(n)}(x,\ \xi)+\phi^{(ab)}(x,\ \xi), \tag{3·14}$$

$$\left.\begin{aligned}\phi^{(n)}(x,\ \xi)&=\sum_{\substack{p\xi=0\\ p^2\geq m^2}}\frac{1}{\sqrt{2\eta\xi_0 V}}\{a(p,\xi)e^{i(p+\eta\xi)x}+b^*(p,\xi)e^{i(p-\eta\xi)x}\},\\ \phi^{(ab)}(x,\ \xi)&=\sum_{\substack{p\xi=0\\ m^2>p^2\geq 0}}\frac{1}{\sqrt{2\rho\xi_0 V}}\{c(p,\xi)e^{i(p-i\rho\xi)x}+d(p,\xi)e^{i(p+i\rho\xi)x}\},\end{aligned}\right\} \tag{3·15}$$

where

$$\left.\begin{aligned}\eta&=\sqrt{p_\mu^2-m^2}=\sqrt{\boldsymbol{p}^2-(\boldsymbol{p}\cdot\boldsymbol{\xi})^2/\xi_0^2-m^2},\\ \rho&=\sqrt{m^2-p_\mu^2}=\sqrt{m^2-\boldsymbol{p}^2+(\boldsymbol{p}\cdot\boldsymbol{\xi})^2/\xi_0^2}.\end{aligned}\right\} \tag{3·16}$$

The above commutation relation (3·4′) is reduced to the following two relations,

$$[\phi^{(n)}(x,\xi),\ \phi^{(n)*}(x',\xi)]=iD^{(n)}(x-x',\xi), \tag{3·17}$$

and

$$[\phi^{(ab)}(x,\xi),\ \phi^{(ab)*}(x',\xi)]=iD^{(ab)}(x-x',\xi), \tag{3·18}$$

assuming the commutativity between the normal and the abnormal components. Substituting the expression (3·14) and (3·15) into the relation (3·17) and (3·18), respectively, and taking into consideration the explicit form of $D^{(n)}$ and $D^{(ab)}$-functions, we arrive at the final relations;

$$[a(p,\xi),\ a^*(p,\xi)]=[b(p,\xi),\ b^*(p',\xi)]=\delta_{pp'}, \tag{3·19}$$

under the conditions $p_\mu^2\geq m^2$, $p\xi=0$,

and

$$[c(p,\xi),\ d^*(p',\xi)]=i\delta_{pp'}, \tag{3·20}$$

under the conditions $0\leq p_\mu^2<m^2$, $p\xi=0$, and all other commutators vanish.

It should be worth-while to note that the total field $\phi(x,\xi)$ has also an explicit dependence on the time-like unit vector ξ_μ, as well as its component fields $\phi^{(n)}(x,\xi)$ and $\phi^{(ab)}(x,\xi)$, and exhibits the oscillatory behavior only in the hypersurface indicated by $\xi x=\text{const.}$ Thus, we find that there exist continuously many numbers of the unitary non-equivalent representations, corresponding to ξ_μ, each of which is subject to the same commutation relation (3·4′). This fact is approved later in this section directly from the spectrum of the displacement operator for $\phi(x,\xi)$.

However, each representation specified by ξ_μ is not Lorentz-invariant. In fact, the Lorentz transformation is represented by the unitary operator $U(\Lambda)$, which satisfies

$$U(\Lambda)\phi(x,\xi)U^{-1}(\Lambda)=\phi'(x,\xi)=\phi(\Lambda^{-1}x,\Lambda^{-1}\xi). \tag{3·21}$$

This relation implies the following equations,

$$\left.\begin{aligned}U(\Lambda)a(p,\xi)U^{-1}(\Lambda)&=a(\Lambda^{-1}p,\Lambda^{-1}\xi)\quad(=a'(p,\xi)),\\ U(\Lambda)b(p,\xi)U^{-1}(\Lambda)&=b(\Lambda^{-1}p,\Lambda^{-1}\xi)\quad(=b'(p,\xi)),\end{aligned}\right\} \tag{3·22}$$

under the conditions $p\xi=0$ and $p_\mu^2\geq m^2$, and similar relations for the $c(p,\xi)$ and $d(p,\xi)$'s. Thus, we find that the Lorentz-transformation in general gives rise to the mixing of each irreducible representation (corresponding to ξ_μ) of the canonical commutation relation (3·4′). The group theoretical structure under the Lorentz transformation is given by Appendix B.

The displacement operators $P_\mu(\xi)$'s for the respective field $\phi(x,\xi)$ are easily obtained from the following relation,

$$-i\frac{\partial\phi(x,\xi)}{\partial x_\mu}=[\phi(x,\xi),\ P_\mu(\xi)]. \tag{3·23}$$

By making use of the commutation relations (3·19) and (3·20), the explicit form of $P_\mu(\xi)$'s is determined as follows:

$$P_\mu(\xi)=P_\mu^{(n)}(\xi)+P_\mu^{(ab)}(\xi), \tag{3·24}$$

$$P_\mu^{(n)}(\xi)=\sum_{\substack{p\xi=0\\ p^2\geq m^2}}\{(\eta+\xi p)_\mu a^*(p,\xi)a(p,\xi)+(\eta\xi-p)_\mu b^*(p,\xi)b(p,\xi)\}, \tag{3·25}$$

$$P_\mu^{(ab)}(\xi)=\sum_{\substack{p\xi=0\\ m^2>p^2\geq 0}}\{(\rho\xi+ip)_\mu d^*(p,\xi)c(p,\xi)+(\rho\xi-ip)_\mu c^*(p,\xi)d(p,\xi)\}, \tag{3·26}$$

where η and ρ are given by (3·16). Further, the projection of $P_\mu(\xi)$ in the ξ_μ-direction, $W(\xi)$, has a simple form, namely

$$W(\xi)=-\xi_\mu P_\mu(\xi)=W^{(n)}(\xi)+W^{(ab)}(\xi), \tag{3·24′}$$

$$W^{(n)}(\xi)=\sum\eta\{a^*(p,\xi)a(p,\xi)+b^*(p,\xi)b(p,\xi)\}, \tag{3·25′}$$

$$W^{(ab)}(\xi)=\sum\rho\{d^*(p,\xi)c(p,\xi)+c^*(p,\xi)d(p,\xi)\}. \tag{3·26′}$$

The behavior of $P_\mu(\xi)$ under the Lorentz-transformation is given by

$$P_\mu'(\xi')=\Lambda_{\mu\nu}P_\nu(\xi)=U(\Lambda)P_\mu(\xi')U^{-1}(\Lambda), \tag{3·27}$$

making use of the relation (3·22).

In order to obtain the spectrum of the above displacement operators, we must determine the explicit representation of operators, a's, b's, c's and d's. For its purpose, we must assume the existence of the "vacuum" state besides the commutation relations (3·20).

First, let us focus our attention on the normal part, and require for the vacuum state $|0\rangle$,

$$a(p,\xi)|0\rangle=0,\quad b(p,\xi)|0\rangle=0,\quad \langle 0|0\rangle=1, \tag{3·28}$$

for all p_μ and ξ_μ under the conditions $p\xi=0$, $p_\mu^2\geq m^2$.*

This choice of vacuum state, of course, is Lorentz invariant as seen from (3·22), and guarantees the positive integer eigenvalue for the number operators $a^*(p,\xi)a(p,\xi)=n^+(p,\xi)$ and $b^*(p,\xi)b(p,\xi)=n^-(p,\xi)$. Although, as easily seen from expressions (3·25) and (3·25′), this vacuum state is the lowest eigenstate of $W^{(n)}(\xi)$, it is, in general, not the case for $P^{(n)}(\xi)$. Further, the spectrum of the $P^{(n)}(\xi)$ has an explicit dependence on ξ_μ's. The latter fact indicates directly the unitary non-equivalence of representations $\phi(x,\xi)$ with different ξ_μ's, mentioned early in this section.

* If we take $a(p,\xi)|0\rangle=0$, $b^*(p,\xi)|0\rangle=0$ instead of (3·28), we must encounter with the indefinite metric in Hilbert space.

For the abnormal component, the more complicated situation occurs. The commutation relation (3·20) under the consideration takes the following form,

$$[c,\ d^*]=i. \tag{3·29}$$

For the definition of the vacuum state several possibilities are conceivable. At first, let us consider the following definition of the vacuum state:

$$\left.\begin{aligned} c|0\rangle&=0,\\ d|0\rangle&=0,\end{aligned}\right\} \tag{3·30}*$$

and

$$\langle 0|0\rangle=1.$$

This definition of the vacuum state, however, necessarily gives rise to the indefinite metric in Hilbert space. In order to deal with this case, it is convenient to introduce the following notation:

$$N^+=-id^*c,\quad N^-=ic^*d, \tag{3·31}$$

and

$$|n+,\ m-\rangle=\frac{1}{\sqrt{n!\,m!}}d^{*n}c^{*m}|0\rangle. \tag{3·32}$$

Then, making use of (3·29) and (3·30), we obtain

$$\left.\begin{aligned} N^+|n+,\ m-\rangle&=n|n+,\ m-\rangle,\\ N^-|n+,\ m-\rangle&=m|n+,\ m-\rangle,\end{aligned}\right\} \tag{3·33}$$

and

$$\langle n'+,\ m'-|n+,\ m-\rangle=(-i)^m(+i)^n\delta_{m'n}\delta_{n'm}. \tag{3·34}$$

Namely operators N^+ and N^-, which are adjoint operators to each other, take the positive integer eigenvalue associated with the eigenvector $|n+,\ m-\rangle$. Further, as seen from (3·34), the eigenstate $|n+,\ m-\rangle$, which has a vanishing norm by itself, has a finite scalar product with other states with different indices. This is, however, a familiar feature in theories with an indefinite metric.

On the other hand, the displacement operator $P_\mu^{(ab)}(\xi)$ or its projection $W^{(ab)}(\xi)$ in the direction ξ_μ can be expressed by the operators N^+ and N^-:

$$P_\mu^{(ab)}(\xi)=-\sum_{\substack{p\xi=0\\ m^2<p^2\geqq 0}}\{(p-i\rho\xi)_\mu N^+(p,\xi)+(p+i\rho\xi)_\mu N^-(p,\xi)\}, \tag{3·35}$$

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* Let us consider the following cases instead of (3·30),

$$\left.\begin{aligned} c|0\rangle&=0,\\ d^*|0\rangle&=0,\end{aligned}\right\}\quad\text{or}\quad\left.\begin{aligned} c^*|0\rangle&=0,\\ d|0\rangle&=0.\end{aligned}\right\}$$

Since, as seen from the above commutation relation, either case leads to the vanishing norm of the vacuum state, we shall take such a possibility out of consideration.

$$W^{(ab)}(\xi)=\sum_{\substack{p\xi=0\\ m^2>p^2\geq 0}} i\rho\{N^+(p,\xi)-N^-(p,\xi)\}. \tag{3·36}$$

Therefore $P_\mu^{(ab)}(\xi)$ has, in general, complex eigenvalues and $W^{(ab)}(\xi)$ has pure imaginary eigenvalues associated with the eigenstate $|n+, m-\rangle$ with a vanishing norm.

Another quantization of the abnormal part is also conceivable, in which the indefinite metric no longer occurs. For this purpose, it is convenient to transform the commutation relation (3·29) into the diagonal form. This can be done through the linear transformation of operators c and d:

$$\left.\begin{aligned}\bar{c}&=\alpha c+\beta d,\\ \bar{d}^*&=\gamma c+\delta d,\end{aligned}\right\} \tag{3·37}$$

under the condition

$$\alpha\delta^*=\beta\gamma^*. \tag{3·38}$$

For instance, if we take $\alpha=\gamma=1/\sqrt{2}$, $\beta=-\delta=i/\sqrt{2}$, namely

$$\left.\begin{aligned}\bar{c}&=(c+id)/\sqrt{2},\\ \bar{d}^*&=(c-id)/\sqrt{2},\end{aligned}\right\} \tag{3·37′}$$

then

$$\left.\begin{aligned}[\bar{c},\ \bar{c}^*]&=1,\\ [\bar{d},\ \bar{d}^*]&=1,\end{aligned}\right\} \tag{3·39}$$

and all other commutators vanish.

Thus, as in the normal case, under the following definition of the vacuum state

$$\left.\begin{aligned}\bar{c}|0\rangle(=(c+id)|0\rangle)&=0,\\ \bar{d}|0\rangle(=(c^*+id^*)|0\rangle)&=0,\end{aligned}\right\} \tag{3·30′}*$$

the basic states $|n+, m-\rangle$, defined by

$$|n+, m-\rangle=\frac{1}{\sqrt{n!\,m!}}\bar{c}^{*n}\bar{d}^{*m}|0\rangle, \tag{3·40}$$

span the Hilbert space with a definite metric and the operators $N^+=\bar{c}^*\bar{c}$ and $N^-=\bar{d}^*\bar{d}$ become number operators with positive integer eigenvalues. However, in this case, the displacement operators $P_\mu^{(ab)}(\xi)$'s or its projection $W^{(ab)}(\xi)$ in the

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* If we take the vacuum state as

$$\bar{c}|0\rangle(=(c+id)|0\rangle)=0,$$
$$\bar{d}^*|0\rangle(=(c-id)|0\rangle)=0.$$

instead of (3·30′), this again leads to the indefinite metric. In fact, this vacuum state is just identical with the preceding one in (3·30).

direction ξ_μ can no longer be commutable with the number operator. In fact, as easily proved, both $P_\mu^{(ab)}(\xi)$ (or $W^{(ab)}(\xi)$) and the commutation relation can never be diagonalized at the same time. In terms of $\bar{c}$ and $\bar{d}$, $P_\mu^{(ab)}(\xi)$ and $W^{(ab)}(\xi)$ are expressed as

$$P_\mu^{(ab)}(\xi)=\sum\{p_\mu(\bar{d}\bar{d}^*-\bar{c}^*\bar{c})-i\rho\xi_\mu(\bar{d}\bar{c}-\bar{c}^*\bar{d}^*)\}, \tag{3·41}$$

$$W^{(ab)}(\xi)=\sum i\rho(\bar{c}^*\bar{d}^*-\bar{d}\bar{c}). \tag{3·42}$$

Thus, in this case the vacuum state is no longer an eigenstate of $P_\mu^{(ab)}(\xi)$ and $W^{(ab)}(\xi)$. Further, we find that the expectation values of $P_\mu^{(ab)}(\xi)$ and $W^{(ab)}(\xi)$ associated with the state $|n+, m-\rangle$, (3·40), take the following values, respectively,

$$\langle n+, m-|P_\mu^{(ab)}(\xi)|n+, m-\rangle=(m-n+1)p_\mu, \tag{3·43}$$

$$\langle n+, m-|W^{(ab)}(\xi)|n-, m-\rangle=0. \tag{3·44}$$

Finally, we shall state about the vacuum expectation value of the anti-commutation relations of $\phi^{(n)}(x, \xi)$ or $\phi^{(ab)}(x, \xi)$. For the normal part, we can easily show

$$\langle 0|\{\phi^{(n)}(x, \xi), \phi^{(n)*}(x', \xi)\}|0\rangle=D^{(1)}(x-x'), \tag{3·45}$$

on account of the second expression of $D^{(1)}(x)$-function in (2·9). Namely the vacuum expectation value of the anticommutator of $\phi^{(n)}(x, \xi)$ has no ξ_μ-dependence and is Lorentz-invariant. On the other hand, concerning the abnormal component, the following relation holds:

$$\langle 0|\{\phi^{(ab)}(x, \xi), \phi^{(ab)*}(x', \xi)\}|0\rangle=iD^{(ab)}(x-x', \xi), \tag{3·46}$$

with respect to the vacuum state associated with the first definition (3·30). On the other hand, the vacuum state concerning the second definition (3·30′), as mentioned above, is not displacement invariant, therefore the corresponding expression of the anticommutator is no longer displacement invariant, that is,

$$\langle 0|\{\phi^{(ab)}(x, \xi), \phi^{(ab)*}(x', \xi)\}|0\rangle=\sum_{\substack{p\xi=0\\ m^2>p^2\geq 0}}\frac{1}{\rho\xi_0}e^{ip(x-x')}\cosh\rho\xi(x+x'). \tag{3·47}$$

At the end of this section, we shall mention some remark on the problem of causality. In the previous section, we have defined the "quasi" causal function (2·3) or (2·3′), which satisfies an important relation (2·8) associated with a causality. In the case of the usual Klein-Gordon-type field, the corresponding causal function is connected with the vacuum expectation value of the chronological product of the field operators and guarantees the causality of the theory. Namely

$$\begin{aligned}\Delta_F(x-x')&=2\langle 0|P(\phi(x), \phi^*(x'))|0\rangle=\langle 0|\{\phi(x), \phi^*(x')\}|0\rangle\\ &+\epsilon(x_0-x_0')\langle 0|[\phi(x), \phi^*(x')]|0\rangle=\Delta^{(1)}(x-x')+i\epsilon(x_0-x_0')\Delta(x-x').\end{aligned} \tag{3·48}$$

Here, with respect to the Lorentz-invariance of the causal function, it is important that the second term in (3·48) is a function vanishing in a space-like region. On the other hand, the "quasi" causal function G_+, (2·3) under our consideration can be written, in a form similar to (3·48), by

$$G_+(x)=\frac{-i}{2(2\pi)^3}\int e^{ipx}\delta(p^2-m^2)\,d^4p-\frac{1}{(2\pi)^4}P\int\frac{e^{ipx}}{p^2-m^2}\,d^4p=iD^{(1)}(x)+\bar{G}(x), \tag{3·49}$$

where $\bar{G}(x)$ is given by

$$\begin{aligned}\bar{G}(x)&=\frac{-1}{(2\pi)^4}P\int d^4p\frac{e^{ipx}}{p^2-m^2}\\&=-\frac{1}{2(2\pi)^2}\frac{1}{r}\int_0^\infty\{\cos(r\sqrt{m^2+u^2}-x_0u)+\cos(r\sqrt{m^2+u^2}+x_0u)\}\,du\\&=-\frac{1}{4\pi}\delta(r^2-x_0^2)+\begin{cases}0, & (r^2<x_0^2)\\ \dfrac{m}{8\pi}\dfrac{I_1(m\sqrt{r^2-x_0^2})}{\sqrt{r^2-x_0^2}}, & (r^2>x_0^2)\end{cases}\end{aligned} \tag{3·50}$$

as shown by Schmidt.[6] The first term in (3·49) could be connected to the vacuum expectation value as in (3·45). However, since the above $\bar{G}$ function is Lorentz-invariant, but non-vanishing function in a space-like region, it is impossible to factorize the chronological sign factor $\epsilon(x_0)$ from this function. Therefore, we can say that it is impossible to connect the quasi-causal function directly to the vacuum expectation value of the chronological product of the field operators. The latter situation, in connection with the failure in the definition of the relativistic vacuum state, makes it difficult to decide whether the quantized S-field has a rigorous causal property for the energy propagation, so long as we pay attention only to S-field without any interaction. This problem will be discussed from another viewpoint in the next section.

As the conclusion of this section, we can state the following facts. Under the formalism of the canonical quantization and the requirement of the Lorentz-invariance, S-field cannot be described in terms of the simple local field and has no familiar particle aspect. In fact, it is impossible to define the Lorentz-invariant and lowest energy state; that is, the vacuum state. In other words this implies that the concept of "energy" or "momentum" of the S-field by itself has no objectivity.

§ 4. Interaction of S-field

As the result of the preceding investigation, we find that the quantized S-field under the canonical formalism cannot have any familiar particle aspect. Therefore, it is impossible to grasp S-field in the same level as the usual elementary particle.

However, it may be a matter of interest for us to ask whether one can successfully confine the curious behavior of S-field only within an inner region without any appreciable effect in an asymptotic outer region, in a loose sense of the word.

In order to deal with such a problem, we have to take into consideration the interaction of S-field with other elementary particles. However, S-field quantized through the procedure in the previous section is no longer any simple local field, but rather depends on the time-like four vector ξ_μ's as if it were a bilocal field. At present, we have no legitimate means to deal with the interaction of such a field. Meanwhile, it is to be noted that the method of quantization of S-field given in the previous section does not mean the one and only way. For instance, in the familiar quantization method of a field in the Heisenberg representation, a different choice of the asymptotic field associated with the definition of the vacuum, in general, may lead to a unitary non-equivalent representation for the same system, as originally pointed out by Haag.[7]

Anticipating such a possibility, let us consider an interaction system described by the following field equations and Schrödinger equation:

$$(\Box-\kappa^2)\phi(x)=0, \tag{4·1}$$

$$(\gamma_\mu\partial_\mu+\kappa')\psi_{a,b}(x)=0, \tag{4·2}$$

$$i\frac{\delta\Psi(\sigma)}{\delta\sigma(x)}=(H_{\mathrm{I}}(x)+H_{\mathrm{II}}(x))\Psi(\sigma), \tag{4·3}$$

$$[\phi(x),\ \phi^*(x')]=i\Delta(x-x',\kappa), \tag{4·4}$$

$$\{\psi_a(x),\ \bar{\psi}_b(x')\}=(\gamma_\mu\partial_\mu-\kappa')\Delta(x-x',\kappa')\delta_{ab}, \tag{4·5}$$

where $\psi_{a,b}(x)$'s indicate usual spinor fields and

$$H_{\mathrm{I}}(x)=g\bar{\psi}_a(x)\psi_b(x)\phi(x)+\mathrm{h.\,c.}, \tag{4·6}$$

$$H_{\mathrm{II}}(x)=-\delta\kappa^2\phi^*(x)\phi(x), \tag{4·7}$$

and further Δ-functions in the commutation relations (4·4) and (4·5) are the usual invariant Δ-functions with the real mass κ or κ'.

Now, if we take the formal "unitary" transformation $U(\sigma)$, for the above system, defined by

$$i\delta U(\sigma)/\delta\sigma(x)=H_{\mathrm{II}}(x)U(\sigma),$$

so as to eliminate H_{II} from the interaction terms, then in the above Eqs. (4·1)–(4·6) $\phi(x)$ is replaced by $\tilde{\phi}(x)$ in a new representation and κ^2 by $\kappa^2-\delta\kappa^2$. Therefore, if we assume

$$\kappa^2<\delta\kappa^2, \tag{4·8}$$

the original system may become equivalent to the interaction system of the quantized S-field, with a spinor field, $\psi_{a,b}(x)$.

However, as will be shown later, both systems are actually not unitary equiva-

lent, so far as we assume the existence of the vacuum state for each of $\phi(x)$ and $\tilde{\phi}(x)$ in the same time. In spite of this non-equivalence of both systems, it is still interesting to investigate the original system, assuming the existence of the vacuum state with respect to $\phi(x)$.

The perturbation expansion of the S-matrix of the original system becomes as follows,

$$S=\sum_{l,m=0}^{\infty}(-i)^{l+m}\frac{1}{l!\,m!}\int dx_1\cdots dx_{l+m}$$
$$\times\langle f|P(j(x_1),\cdots,j(x_l),j^*(x_{l+1}),\cdots,j^*(x_{l+m}))|i\rangle\sum_{n=0}^{\infty}(-i)^n\frac{1}{n!}\int dy_1\cdots dy_n$$
$$\times\langle f'|P(\phi(x_1),\cdots,\phi(x_l),\phi^*(x_{l+1}),\cdots,\phi^*(x_{l+m}),H_{\mathrm{II}}(y_1),\cdots,H_{\mathrm{II}}(y_n))|i'\rangle, \tag{4·9}$$

where $j(x)=g\bar{\psi}_a(x)\psi_b(x)$ and $|i\rangle$ ($|f\rangle$) and $|i'\rangle$ ($|f'\rangle$) denotes the initial (final) states associated with the spinor and the Bose field, respectively. Now, the above expression (4·9) indicates that the S-matrix of our system is equivalent to that of the system which involves only the interaction term H_{I}, provided that we replace the free propagation function of the Bose field, $\Delta_F(x,\kappa^2)$, in the latter case by the function modified by the interaction $H_{\mathrm{II}}(x)$. This modified propagator, which we shall describe with $\Delta_F^S(x)$, can be expressed by the perturbation series:

$$\Delta_F^S(x-y)=2\sum_{n=0}^{\infty}(-i)^n\frac{1}{n!}\int dx_1\cdots dx_n\langle 0|P(\phi(x),\phi^*(y),H_{\mathrm{II}}(x_1),\cdots,H_{\mathrm{II}}(x_n))|0\rangle, \tag{4·10}$$

though it is not a convergent series as will be seen from (4·12). However, there holds the well-known relation which connects both $\Delta_F(x,\kappa^2)$ and $\Delta_F^S(x)$ in terms of the proper self-energy part Π^* due to the interaction $H_{\mathrm{II}}(x)$, that is,

$$\Delta_F^S(p)=\Delta_F(p,\kappa^2)+\Delta_F^S(p)\,\Pi^*\,\Delta_F(p,\kappa^2), \tag{4·11}$$

or equivalently

$$\Delta_F^{S-1}(p)=\Delta_F^{-1}(p,\kappa^2)-\Pi^*, \tag{4·11′}$$

where

$$\Delta_F(p,\kappa^2)=1/2\pi i(p^2+\kappa^2-i\varepsilon),$$

and Π^* is given, after the simple consideration, by

$$\Pi^*=2\pi i\delta\kappa^2,$$

hence $\Delta_F^S(x)$ becomes as

$$\Delta_F^S(x)=\frac{-2i}{(2\pi)^4}\int\frac{e^{ipx}}{p^2+\kappa^2-\delta\kappa^2-i\varepsilon}d^4p. \tag{4·12}$$

The above function is, except a constant factor, nothing but the quasi-causal func-

tion $G_+(x, m^2)$ with $m^2=\delta\kappa^2-\kappa^2(>0)$ which was defined by (2·3) or (3·49), and then it was impossible to connect this function directly with the vacuum expectation value of the field operator, $\phi(x, \xi)$.

Now, the S-matrix of the Möller scattering in the lowest order with respect to the coupling constant g is given by

$$S^{(2)}=-i\iint dx\,dy\,j(x)G_+(x-y)\,j^*(y). \tag{4·13}$$

This can be written in terms of the quasi-retarded function according to the procedure given by Ma,[8] namely

$$S^{(2)}=-i\iint dx\,dy\,\{j^{(+)}(x)G_R(x-y,\,n)j^{*(-)}(y)+j^{(-)}(x)G_R(y-x,\,n)j^{*(+)}(y)\}, \tag{4·14}$$

where use is made of the relation (2·8) and $j^{(\pm)}(x)$ means the positive or negative time frequency part of the source function $j(x)$.

From the above expression, we just find that the quantized S-field propagates the energy with a super light velocity, because $G_R(x, \xi=n)$ is in general a non-vanishing function in a space-like region.

It should be worth-while to note that even though the original form of our system apparently belongs to a familiar theory of the local fields with a local interaction, hence each term of the perturbation expansion of the S-matrix with respect to $\delta\kappa^2$ and g, i.e. (4·9) is, of course, Lorentz-invariant and rigorously causal, the resultant expression after the summation of terms of the perturbation expansion with respect to $\delta\kappa^2$ shows Lorentz-invariant, but an explicitly acausal feature. This strange situation is, of course, caused mathematically by the non-convergence of the series of the perturbation expansion with respect to $\delta\kappa^2$. In terms of the physical word, however, we believe that this situation is due to the non-existence of the true vacuum state with regard to the interaction $H_{\rm II}(x)$. In fact, if the latter statement is true, the appearance of the acausal property is reasonably conceived from the general framework according to Lehmann.[1] In order to support such a conjecture, let us consider the relation between the field $\phi(x, \xi)$ in the previous section and the field $\phi(x)$ in the present section, identifying both fields and their space-time derivatives at the hypersurface $\xi x=$const., more simply, $\xi x=0$:

$$\left.\begin{aligned}&\phi(x, \xi)|_{\xi x=0}=\phi(x)|_{\xi x=0},\\ &\xi_\mu\partial_\mu\phi(x, \xi)|_{\xi x=0}=\xi_\mu\partial_\mu\phi(x)|_{\xi x=0}.\end{aligned}\right\} \tag{4·15}$$

It is convenient to expand the field $\phi(x)$ in the following form in accordance with that of the field $\phi(x, \xi)$,

$$\phi(x)=\sum_{p\xi=0}\frac{1}{\sqrt{2\gamma\xi_0 V}}\{u(\boldsymbol{p}+\gamma\boldsymbol{\xi})e^{i(p+\gamma\xi)x}+v^*(\boldsymbol{p}-\gamma\boldsymbol{\xi})e^{i(p-\gamma\xi)x}\}, \tag{4·16}$$

where

$$\gamma=\sqrt{p^2+m^2}=\sqrt{\boldsymbol{p}^2-(\boldsymbol{p}\cdot\boldsymbol{\xi})^2/\xi_0^2+m^2}, \tag{4·17}$$

and $u(\boldsymbol{p})$ and $v(\boldsymbol{p})$ satisfy the usual canonical commutation relations. From (4·16), we obtain

$$\begin{aligned}\phi(x)|_{\xi x=0}&=\sum\frac{1}{\sqrt{2\gamma\xi_0 V}}\{u(\boldsymbol{p}+\gamma\boldsymbol{\xi})+v^*(\boldsymbol{p}-\gamma\boldsymbol{\xi})\}e^{i(\boldsymbol{p}-(\boldsymbol{p}\cdot\boldsymbol{\xi})\boldsymbol{\xi}/\xi_0^2)\cdot\boldsymbol{x}},\\ \xi_\mu\partial_\mu\phi(x)|_{\xi x=0}&=\sum(-i)\sqrt{\frac{\gamma}{2\xi_0 V}}\{u(\boldsymbol{p}+\gamma\boldsymbol{\xi})-v^*(\boldsymbol{p}-\gamma\boldsymbol{\xi})\}e^{i(\boldsymbol{p}-(\boldsymbol{p}\cdot\boldsymbol{\xi})\boldsymbol{\xi}/\xi_0^2)\cdot\boldsymbol{x}}.\end{aligned} \tag{4·18}$$

Substituting the above expressions and the corresponding ones for $\phi(x,\xi)$ derived from (3·14) and (3·15) into (4·15) and comparing the both sides, one finds the following relations,

$$\begin{pmatrix}a(p,\xi)\\ b(p,\xi)\end{pmatrix}=\frac{1}{2\sqrt{\eta\gamma}}\{(\eta\pm\gamma)u(\boldsymbol{p}+\gamma\boldsymbol{\xi})+(\eta\mp\gamma)v^*(\boldsymbol{p}-\gamma\boldsymbol{\xi})\}, \tag{4·19}$$

with the conditions $p^2\geq m^2$, $p\xi=0$, and

$$\begin{pmatrix}c(p,\xi)\\ d(p,\xi)\end{pmatrix}=\frac{1}{2\sqrt{\rho\gamma}}\{(\rho\pm i\gamma)(u(\boldsymbol{p}+\gamma\boldsymbol{\xi})+(\rho\mp i\gamma)v^*(\boldsymbol{p}-\gamma\boldsymbol{\xi})\}, \tag{4·20}$$

with the conditions $m^2>p^2\geq 0$, $p\xi=0$.

Now, as originally pointed out by Haag[7] for the relation between the two neutral Bose fields with different masses, we can easily verify the following fact, taking account of the relations (4·19) and (4·20). If there is the vacuum state associated with the field $\phi(x)$, namely

$$\left.\begin{aligned}u(\boldsymbol{p})|0)&=0\\ v(\boldsymbol{p})|0)&=0\end{aligned}\right\}\quad\text{for all } p, \tag{4·21}$$

there is no normalizable vacuum state $|0\rangle$ associated with the field $\phi(x,\xi)$, which satisfies

$$\left.\begin{aligned}a(p,\xi)|0\rangle&=b(p,\xi)|0\rangle=0,\\ c(p,\xi)|0\rangle&=d(p,\xi)|0\rangle=0,\end{aligned}\right\} \tag{4·22}$$

over all p's under the conditions in (4·19) and (4·20). This is, of course, due to the infinite degrees of freedom of the field operator.

Thus, the quantization of S-field in this section makes it possible to remove the eigenstates of the number operators of S-field from the Hilbert space. Speaking more physically, we can say that it is possible to quantize S-field so that S-field no longer exists in the *free* state with the modified Einstein relation (1·3), but takes a role only in the interactions with other particles, bringing the acausal effect into them.

§ 5. Summary and conclusion

We have investigated the matter with a super light velocity as one possible model in the course of looking for the physical concept which will essentially govern the future theory, but may be hidden behind the present field theory. In such an attempt, it is important to study to what extent this matter could be reconciled with the framework of the present theory or contradicts the present physical concept of elementary particles. With such a purpose, we investigated at first in § 2 the propagation character of the classical wave and explained the apparent contradiction between the group velocity over the light velocity and the hyperbolic propagation character. In § 3, it was shown that in the framework of the usual canonical quantization, this matter is no longer represented by a simple local field, but rather has an infinite number of unitary non-equivalent representations, each of which is characterized by the time-like unit four-vector as if being bilocal field and is transformed into each other by the Lorentz transformation. There is no vacuum state which satisfies the Lorentz-invariance and the lowest energy condition, while the displacement operator of the field does exist. Further, the abnormal component, which is inherent in S-field and characterized by its unbounded behavior in space and time, brings the indefinite metric into the Hilbert space. Therefore, we have to say as a whole that S-field, free from any interactions, has no physical reality. In § 4, we considered the interactions of the S-field with other fields attempting to confine the above curious behavior of S-field only within the structure of interaction with other particles. The result of this section shows that the suitable quantization of S-field provides the Lorentz-invariant, but quasi-causal S-matrix and further makes it possible to eliminate the free state of S-field from the Hilbert space by means of the infinite degrees of freedom of the field operators. These features of S-field seem to have similarity somewhat to those of B-matter referred to in the introduction of this paper.

In conclusion, the author would like to express his sincere thanks to Prof. T. Inoue for his kind interest in this work and to Prof. Y. Munakata and Dr. M. Ida for valuable discussions.

Appendix A

α) The field equation and the transformation property of spinor field

Let us consider the general spinor field equation which is invariant under the proper Lorentz-transformation:

$$(\gamma_\mu \partial_\mu + \alpha + \beta\gamma_5)\psi(x) = 0, \tag{A·1}$$

where α and β are complex numbers:

$$\alpha = \alpha_1 + i\alpha_2, \quad \beta = \beta_1 + i\beta_2. \tag{A·2}$$

The Lagrange function associated with the above equation takes the following form:

$$\mathcal{L}(x)=\psi^*\gamma_4(a+b\gamma_5)(\gamma_\mu\partial_\mu+\alpha+\beta\gamma_5)\psi(x). \tag{A·3}$$

From the reality condition of $\mathcal{L}(x)$,* we obtain

$$a=a^*,\ b=b^*, \tag{A·4}$$

$$\beta_1/\alpha_1=\alpha_2/\beta_2=-b/a. \tag{A·5}$$

On the other hand, multiplying the operator $(\gamma_\mu\partial_\mu-\alpha+\beta\gamma_5)$ from the left to Eq. (A·1) gives

$$\{\Box-(\alpha_1^2+\beta_2^2-\beta_1^2-\alpha_2^2)\}\psi(x)=0. \tag{A·6}$$

Now, applying the usual canonical quantization to the above field, we obtain the following commutation relation,

$$\{\psi(\boldsymbol{x}, x_0),\ \psi^*(\boldsymbol{x}', x_0)\}=-\frac{a+b\gamma_5}{a^2-b^2}\delta^3(\boldsymbol{x}-\boldsymbol{x}'). \tag{A·7}$$

Replacing $\psi(x)$ by $\tilde{\psi}(x)$:

$$\tilde{\psi}(x)=\exp(-f\gamma_5)\psi(x),$$

with a real constant, f, (A·7) is transformed into

$$\{\tilde{\psi}(\boldsymbol{x}, x_0),\ \tilde{\psi}^*(\boldsymbol{x}', x_0)\}=-\frac{1}{a^2-b^2}$$

$$\times\{(a\cosh 2f+b\sinh 2f)+(b\cosh 2f+a\sinh 2f)\gamma_5\}\delta^3(\boldsymbol{x}-\boldsymbol{x}'). \tag{A·8}$$

Therefore, choosing the magnitude of f as

$$\frac{b}{a}=-\tanh 2f,\quad \text{or}\quad \frac{a}{b}=-\tanh 2f, \tag{A·9}$$

according to $|a|>|b|$ or $|a|<|b|$, makes it possible to drop the term involving γ_5 or the constant term in the right-hand side of (A·8), respectively.

Consequently, we can confine ourselves, without loss of generality, into the following two cases:

$$\begin{aligned} &\text{i)}\quad a\neq 0,\ b=0,\\ &\text{ii)}\quad a=0,\ b\neq 0. \end{aligned} \tag{A·10}$$

Let us now consider the above two cases, separately.

i) $a\neq 0,\ b=0.$

In this case, from (A·5)

$$\alpha_2=\beta_1=0,$$

thus the equation of motion becomes

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* This condition implies that the Euler equations derived from the variation of ψ and ψ^*, respectively, are compatible with each other.

$$(\gamma_\mu\partial_\mu+\alpha_1+i\beta_2\gamma_5)\psi=0. \tag{A·11}$$

By making use of the transformation

$$\psi(x)=\exp i\theta\gamma_5\tilde{\psi}(x), \tag{A·12}$$

with $\beta_2/\alpha_1=-\tan 2\theta$, this equation is led to the usual Dirac field equation

$$(\gamma_\mu\partial_\mu+\sqrt{\alpha_1^2+\beta_2^2})\tilde{\psi}(x)=0, \tag{A·13}$$

and the commutation relation (A·8) yields

$$\{\tilde{\psi}(\boldsymbol{x}, x_0),\ \tilde{\psi}^*(\boldsymbol{x}', x_0)\}=-\frac{1}{a}\delta^3(\boldsymbol{x}-\boldsymbol{x}'). \tag{A·14}$$

ii) $a=0,\ b\neq 0$

From (A·5), one has $\alpha_1=\beta_2=0$, and the equation of motion becomes

$$(\gamma_\mu\partial_\mu+i\alpha_2+\beta_1\gamma_5)\psi=0. \tag{A·15}$$

Making use of the transformation of (A·12) with $\alpha_2/\beta_1=-\tan 2\theta$ or $\beta_1/\alpha_2=\tan 2\theta$, the equation of motion, respectively, becomes

$$\left.\begin{aligned}&(\gamma_\mu\partial_\mu+\sqrt{\alpha_2^2+\beta_1^2}\gamma_5)\tilde{\psi}=0,\\ \text{or}\quad&(\gamma_\mu\partial_\mu+i\sqrt{\alpha_2^2+\beta_1^2})\tilde{\psi}=0,\end{aligned}\right\} \tag{A·13′}$$

and the commutation relation takes the following form,

$$\{\tilde{\psi}(\boldsymbol{x}, x_0),\ \tilde{\psi}^*(\boldsymbol{x}', x_0)\}=\frac{\gamma_5}{b}\delta^3(\boldsymbol{x}-\boldsymbol{x}'). \tag{A·14′}$$

Summarizing the above result, we can say that the general spinor fields subject to (A·1) are classified into two cases, one of which is a well-known Dirac field and the other is the field with a super light velocity and described by

$$(\gamma_\mu\partial_\mu+m\gamma_5)\psi(x)=0 \tag{A·16}$$

$$\{\psi(x),\ \bar{\psi}(x')\}=-i\gamma_5(\gamma_\mu\partial_\mu-m\gamma_5)D(x-x'), \tag{A·17}$$

making use of the invariant D-function defined by (2·11).

The equation of motion (A·16) has several invariances under the transformation such as space or time reflections, and charge conjugation. They are given by

$$\begin{aligned}\psi(x)&\to\gamma_5\gamma_4\psi(x),\quad\text{(space reflection, } P)\\ &\to\gamma_4\psi(x),\quad\text{(time reflection, } T)\\ &\to\gamma_5 C\bar{\psi}(x),\quad\text{(charge conjugation, } C)^*.\end{aligned}$$

Under each of these transformation the Lagrangian is also invariant, but the commutation relation (A·18) is no longer invariant and changes the sign of the right-

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* The notation C follows Schwinger's:

$$C\gamma_\mu C^{-1}=-\gamma_\mu{}^T.$$

hand side. However, this is naturally invariant under the combined transformations CP or CT.

Thus, from the above argument we can say that the S-field with spin one half has the unique correspondence with the parity violation character.

β) *The quantization*

Let us divide the spinor field into the normal and the abnormal parts, in a similar way to (3·14) :

$$\psi(x,\xi)=\psi^{(n)}(x,\xi)+\psi^{(ab)}(x,\xi), \tag{A·18}$$

where

$$\psi^{(n)}(x,\xi)=\sum_{\substack{p^\xi=0\\ p^2\geqq m^2}}\frac{1}{\sqrt{2\xi_0 V}}\{a^r(p,\xi)\psi_r^{+}(p,\xi)e^{i(p+\eta\xi)x}+b^{r*}(p,\xi)\psi_r^{-}(p,\xi)e^{i(p-\eta\xi)x}\}, \tag{A·19}$$

$$\psi^{(ab)}(x,\xi)=\sum_{\substack{p^\xi=0\\ m^2>p^2\geqq 0}}\frac{1}{\sqrt{2\xi_0 V}}\{c^r(p,\xi)\varphi_r^{+}(p,\xi)e^{i(p+i\rho\xi)x}+d^r(p,\xi)\varphi_r^{-}(p,\xi)e^{i(p-i\rho\xi)x}\}, \tag{A·20}$$

with the same notation η, ρ as (3·16) in the text. In the above expression the constant spinors $\psi_r^{\pm}$, $\varphi_r^{\pm}$ are subject to the following equation,

$$\{i(p_\mu\pm\eta\xi_\mu)\gamma_\mu+m\gamma_5\}\psi_r^{\pm}(p,\xi)=0, \tag{A·21}$$

$$\{i(p_\mu\pm i\rho\xi_\mu)\gamma_\mu+m\gamma_5\}\varphi_r^{\pm}(p,\xi)=0. \tag{A·22}$$

The solution for $\psi_r^{\pm}(p,\xi=n)$ is explicitly given by

$$\psi^{+}_{r=1,2}=\frac{m}{\sqrt{2(p\mp\epsilon)\epsilon}}\begin{pmatrix} u^{\pm} \\ (1/m)(\pm p-\epsilon)u^{\pm}\end{pmatrix}, \tag{A·23}*$$

$$\psi^{-}_{r=1,2}=\frac{m}{\sqrt{2(p\mp\epsilon)\epsilon}}\begin{pmatrix} u^{\mp} \\ (1/m)(\mp p+\epsilon)u^{\mp}\end{pmatrix}, \tag{A·24}$$

where

$$\epsilon=\sqrt{\boldsymbol{p}^2-m^2},\quad p=|\boldsymbol{p}|,$$

and

$$\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{p}u^{\pm}=\pm u^{\pm},$$

with the obvious notation. The above solution satisfies the orthogonal conditions,

$$\left.\begin{aligned}&\psi_r^{+*}(p,n)\gamma_5\psi_s^{+}(p,n)=\epsilon_r\delta_{rs},\\ &\psi_r^{-*}(p,n)\gamma_5\psi_s^{-}(p,n)=\epsilon_r\delta_{rs},\\ &\psi_r^{+*}(p,n)\gamma_5\psi_s^{-}(p,n)=0,\end{aligned}\right\} \tag{A·25}$$

* Hereafter, we shall use the representation, where $\gamma_5=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$.

where

$$\epsilon_r = \pm 1 \quad \text{for} \quad r = \pm 1.$$

From (A·25), we obtain

$$\sum_r \epsilon_r \{\psi_r^+(p, n)\psi_r^{\pm *}(p, n) + \psi_r^-(p, n)\psi_r^{-*}(p, n)\} = \gamma_5. \tag{A·26}$$

Introducing the operators

$$\Lambda^{\pm}(p, n) = \{\pm (i\boldsymbol{p}\gamma_4\boldsymbol{\gamma} + m\gamma_4\gamma_5) + \epsilon\}/2\epsilon \tag{A·27}$$

and making use of (A·26), one finds

$$\sum_r \epsilon_r \psi_r^{\pm}(p, n)\psi_r^{\pm *}(p, n) = \Lambda^{\pm}(p, n)\gamma_5. \tag{A·28}$$

Now, let us consider the Lorentz-transformation which transforms n_μ into a given ξ_μ, namely

$$\xi_\mu = \Lambda_{\mu\nu} n_\nu \tag{A·29}$$

and take the matrix S,

$$S\gamma_\mu S^{-1} = \Lambda_{\nu\mu}\gamma_\nu. \tag{A·30}$$

Then, since the functions $S\psi_r^{\pm}(\Lambda^{-1}p, n)$ satisfy Eq. (A·21), we can identify them with $\psi_r^{\pm}(p, \xi)$, namely

$$\psi_r^{\pm}(p, \xi) = S\psi_r^{\pm}(\Lambda^{-1}p, n). \tag{A·31}$$

In this case, one can easily verify that

$$\mathfrak{S}(p, \xi)\psi_r^{\pm}(p, \xi) = \pm \epsilon_r \psi_r^{\pm}(p, \xi), \tag{A·32}$$

where $\mathfrak{S}(p, \xi)$ is the generalized spin operator,

$$\mathfrak{S}(p, \xi) = -i\gamma_5\gamma_{[\mu}\gamma_{\nu]}p_\mu\xi_\nu/\sqrt{p_\sigma^2} \tag{A·33}$$

and especially $\mathfrak{S}(p, n) = \boldsymbol{\sigma}\cdot\boldsymbol{p}/p$.

Further, corresponding to (A·28),

$$\sum_r \epsilon_r \psi_r^{\pm}(p, \xi)\bar{\psi}_r^{\pm}(p, \xi) = \mp \{i(p_\mu \pm \eta\xi_\mu)\gamma_5\gamma_\mu - m\}/2\eta. \tag{A·34}$$

Also, concerning the abnormal part $\varphi_r^{\pm}(p, \xi)$, the similar treatment is applied. The solutions $\varphi_r^{\pm}(p, \xi = n)$'s are given by

$$\begin{aligned}
\varphi_r^+(p, n) &= \frac{1}{2\sqrt{\lambda}} \begin{pmatrix} (i\sqrt{m+\lambda} \pm \sqrt{m-\lambda})u^{\pm} \\ (\sqrt{m+\lambda} \pm i\sqrt{m-\lambda})u^{\pm} \end{pmatrix}, \\
\varphi_r^-(p, n) &= \frac{1}{2\sqrt{\lambda}} \begin{pmatrix} (\sqrt{m-\lambda} \mp i\sqrt{m+\lambda})u^{\pm} \\ (-i\sqrt{m-\lambda} \pm \sqrt{m+\lambda})u^{\pm} \end{pmatrix},
\end{aligned} \tag{A·35}$$

where

$$\lambda = \sqrt{m^2 - \boldsymbol{p}^2}.$$

Taking

$$\varphi_r^{\pm}(p,\xi)=S\varphi_r^{\pm}(\Lambda^{-1}p,n),$$

we obtain

$$\mathfrak{S}(p,\xi)\varphi_r^{\pm}(p,\xi)=\epsilon_r\varphi_r^{\pm}(p,\xi). \tag{A·36}$$

The final relations corresponding to (A·34) are

$$\left.\begin{aligned}\sum_r \epsilon_r\varphi_r^{+}(p,\xi)\bar{\varphi}_r^{-}(p,\xi)&=\{i(p_\mu+i\rho\xi_\mu)\gamma_5\gamma_\mu-m\}/2i\rho,\\ \sum_r \epsilon_r\varphi_r^{-}(p,\xi)\bar{\varphi}_r^{+}(p,\xi)&=-\{i(p_\mu-i\rho\xi_\mu)\gamma_5\gamma_\mu-m\}/2i\rho.\end{aligned}\right\} \tag{A·37}$$

On the other hand, the right-hand side of the commutation relation (A·17) is also divided into the normal and the abnormal parts. Making use of the expressions (2·13), (2·15), they are given by

$$\begin{aligned}&-i\gamma_5(\gamma_\mu\partial_\mu-m\gamma_5)D^{(n)}(x,\xi)\\&=\frac{-1}{(2\pi)^3}\int_{\substack{p\xi=0\\ p^2\geq m^2}}\frac{d^3p}{2\eta\xi_0}[\{i(p_\mu+\eta\xi_\mu)\gamma_5\gamma_\mu-m\}\exp\{i(p+\eta\xi)x\}\\&-\{i(p_\mu-\eta\xi_\mu)\gamma_5\gamma_\mu-m\}\exp\{i(p-\eta\xi)x\}],\end{aligned} \tag{A·38}$$

and

$$\begin{aligned}&-i\gamma_5(\gamma_\mu\partial_\mu-m\gamma_5)D^{(ab)}(x,\xi)\\&=\frac{-1}{(2\pi)^3}\int_{\substack{p\xi=0\\ m^2>p^2\geq 0}}\frac{d^3p}{2i\rho\xi_0}[\{i(p_\mu-i\rho\xi_\mu)\gamma_5\gamma_\mu-m\}\exp\{i(p-i\rho\xi)x\}\\&-\{i(p_\mu+i\rho\xi_\mu)\gamma_5\gamma_\mu-m\}\exp\{i(p+i\rho\xi)x\}].\end{aligned} \tag{A·39}$$

Substituting the expressions (A·19), (A·20) and (A·38), (A·39) into (A·17) and comparing both sides, we obtain

$$\{a^r(p,\xi),\ a^{s*}(p',\xi)\}=\{b^r(p,\xi),\ b^{s*}(p',\xi)\}=\epsilon^r\delta_{rs}\delta_{pp'}, \tag{A·40}$$

under the condition

$$p^2\geq m^2,\quad p\xi=0,$$

$$\{c^r(p,\xi),\ d^{s*}(p',\xi)\}=\epsilon_r\delta_{rs}\delta_{pp'}, \tag{A·41}$$

under the condition

$$m^2>p^2\geq 0,\quad p\xi=0,$$

and all other commutators vanish.

The several problems concerning the transformation property under the Lorentz-transformation, the displacement operator or the vacuum state can be carried through in essentially the same way as in § 3 for the Bose field, hence we do not enter further in detail.

However, it is to be noted that, as seen from (A·40), there appears a new indefinite metric in the Hilbert space with a different origin than that due to the abnormal part. In fact, the appearance of the sign factor ϵ_r in the right-hand side of (A·40) implies that the states with a different spin direction must have an inverse sign of metric in a Hilbert space. Further, this situation cannot be avoided by the modification of the vacuum state, because of the anticommutator. Rather, this is a reasonable result since S-field with spin one half becomes necessarily parity indefinite mentioned early.

Appendix B

The group theoretical structure of S-field under the Lorentz-transformation

In the usual Klein-Gordon field, there exist for any pair of annihilation operators, $a(\boldsymbol{p})$ and $a(\boldsymbol{p}')$, a large number of the Lorentz-transformations, which transform the pair in each other and are ∞^3 in number for each pair. Further, since the total number of operators $a(\boldsymbol{p})$ is obviously ∞^3, this field is complete to represent the Lorentz-group which has ∞^6 elements. On the contrary, the situation is different for the field under consideration. There do not necessarily exist the Lorentz-transformations for some pair of operators $a(p, \hat{\xi})$ and $a(p', \hat{\xi}')$.* Let us consider the pair, $a(p, n)$ and $a(p', n)$. Then, the fourth components of p_μ and p_μ' are vanishing because of the condition in (3·19), namely $p_\mu=(\boldsymbol{p}, 0)$ and $p_\mu'=(\boldsymbol{p}', 0)$. Lorentz-transformations which connect both operators must, at least, belong to the spatial rotation, so they exist when and only when $\boldsymbol{p}^2=\boldsymbol{p}'^2$ and are ∞^1 in number. Further, one finds that the number of operators $a(p', n)$'s which can be transformed by the Lorentz-transformation into the specified operator $a(p, n)$ is ∞^2, taking account of the condition $\boldsymbol{p}^2=\boldsymbol{p}'^2$. In order to clarify the above situation more generally, it is convenient to define the "invariant class" of operators $a(p, \hat{\xi})$'s in such a way that operators which are connected by a Lorentz-transformation to each other belong to the same invariant class.** (Of course, the operators in the usual Klein-Gordon field has one and only one invariant class.) Taking into consideration the above argument confined to $\hat{\xi}_\mu=n_\mu$, we can say, without rigorous proof here, the following properties about the structure of classes.

i) The number of operators with the common $\hat{\xi}_\mu$ in one invariant class is ∞^2, and thus the total number of operators in one invariant class becomes $\infty^{2+3}=\infty^5$, taking account of the degrees of freedom of $\hat{\xi}_\mu$, namely ∞^3.

ii) The number of Lorentz-transformations which connect a given pair of operators in one invariant class is ∞^1. Thus, we find that each invariant class

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* For simplicity, we confine ourselves only to the discussion about operators $a(p, \hat{\xi})$'s. Of course, the situation is just same for operators b's, c's and d's.

** The author is indebted to Dr. M. Ida for the following group theoretical consideration.

is complete to represent the Lorentz-group, taking into consideration the number of operators, ∞^5, in each invariant class as stated in i) and the degeneracy in number of ∞^1 above-mentioned.

iii) The number of class is ∞^1, which yields the total number of operators, $a(p, \xi)$'s over all classes, ∞^6, in connection with the latter statement in i).

Appendix C

Some remarks on the invariant functions in § 2

In § 2, we defined the retarded and the advanced functions as follows,

$$G_{R,A}(x) = -\frac{1}{(2\pi)^4}\int \frac{e^{i(k\pm i\eta\xi)x}}{(k\pm i\eta\xi)^2 - m^2} d^4k, \tag{2·4}$$

under the condition $\eta > m$. The latter condition guarantees that the integrand in the above integration has no poles on the real k_μ's axes in the complex k_μ-planes and thus the integration has a definite meaning. Now we shall show that $G_{R,A}(x)$'s have no dependence on the direction of the time-like unit vector, ξ_μ. For this purpose, it is enough to show the invariance of $G_{R,A}(x)$'s under the infinitesimal Lorentz-transformation for ξ_μ:

$$\xi_\mu' = \xi_\mu + \omega_{\mu\nu}\xi_\nu, \tag{C·1}$$

where $\omega_{\mu\nu}$'s are infinitesimals satisfying

$$\omega_{\mu\nu} = -\omega_{\nu\mu}. \tag{C·2}$$

Making use of the following relations,

$$\begin{aligned}\xi_\mu' x_\mu &= \xi_\mu(x_\mu - \omega_{\mu\nu}x_\nu)\\ &= \xi_\mu x_\mu', \quad (x_\mu' \equiv x_\mu - \omega_{\mu\nu}x_\nu),\end{aligned} \tag{C·3}$$

and

$$k_\mu x_\mu = k_\mu' x_\mu', \quad (k_\mu' \equiv k_\mu - \omega_{\mu\nu}k_\nu), \tag{C·4}$$

we obtain

$$\begin{aligned}\Delta G_{R,A}(x,\xi) &\equiv G_{R,A}(x,\xi') - G_{R,A}(x,\xi)\\ &= \frac{1}{(2\pi)^4}\int\left\{\frac{e^{(ik\mp\eta\xi')x}}{(ik\mp\eta\xi')^2+m^2} - \frac{e^{(ik\mp\eta\xi)x}}{(ik\mp\eta\xi)^2+m^2}\right\} d^4k\\ &= \frac{1}{(2\pi)^4}\int \frac{e^{(ik\mp\eta\xi)x'} - e^{(ik\mp\eta\xi)x}}{(ik\mp\eta\xi)^2+m^2}\, d^4k\\ &= \frac{-i}{(2\pi)^4}\int \frac{(ik\mp\eta\xi)_\mu\omega_{\mu\nu}}{(ik\mp\eta\xi)^2+m^2}\frac{\partial}{\partial k_\nu} e^{(ik\mp\eta\xi)x} d^4k.\end{aligned} \tag{C·5}$$

Further, performing the partial integration with respect to variables k_μ's which is permissible by virtue of the condition $\eta > m$, we get

$$\Delta G_{R,A}(x,\xi)$$

$$= \frac{-i}{(2\pi)^4}\sum_{\nu=1}^{4}\left\{\int \frac{(ik\mp\eta\xi)_\mu\omega_{\mu\nu}}{(ik\mp\eta\xi)^2+m^2}\, e^{(ik\mp\eta\xi)x}(dk)_\nu \Big|_{k_\nu=-\infty}^{k_\nu=+\infty}\right\}$$

$$-i\int e^{(ik\mp\eta\xi)x}\frac{\partial}{\partial k_\mu}\left\{\frac{\omega_{\mu\nu}(ik\mp\eta\xi)_\nu}{(ik\mp\eta\xi)^2+m^2}\right\}, \qquad (C\cdot 6)$$

where

$$(dk)_\nu(=d^4k/dk_\nu)=(dk_2dk_3dk_0,\ dk_1dk_3dk_0,\ dk_1dk_2dk_0,\ dk_1dk_2dk_3/i).$$

The first term in the left side of Eq. (C·6) obviously vanishes. Further, one easily finds that the second term also vanishes, because of the fact:

$$\frac{\partial}{\partial k_\mu}\left\{\frac{\omega_{\mu\nu}(ik\mp\eta\xi)_\nu}{(ik\mp\eta\xi)^2+m^2}\right\}=-\frac{2i(ik\mp\eta\xi)_\mu\omega_{\mu\nu}(ik\mp\eta\xi)_\nu}{\{(ik\mp\eta\xi)^2+m^2\}^2}+\frac{i\omega_{\mu\mu}}{(ik\mp\eta\xi)^2+m^2}=0,$$

on account of (C·2). Thus, $G_{R,A}$'s have no dependence on ξ_μ.

Consequently, the $D(x)$-function, which was defined by (2·10) as the difference between $G_R(x)$ and $G_A(x)$, is also independent of ξ_μ. Therefore, in order to show the properties of the $D(x)$-function, (2·16) and (2·17), it is enough to investigate the $D(x)$-function expressed in terms of the special direction $\xi_\mu=n_\mu$, namely

$$D(x)=D^{(n)}(x,n)+D^{(ab)}(x,n).$$

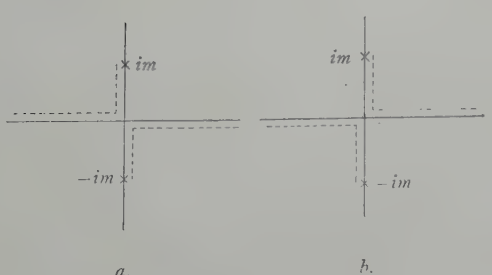
Taking into consideration that both $D^{(n)}(x,n)$ and $D^{(ab)}(x,n)$, as seen from (2·13′) and (2·14′), vanish at $x_0=0$ and that $D(x)$ is a form-invariant function with respect to the Lorentz-transformation, we easily find (2·16), that is,

$$D(x)=0, \quad \text{for} \quad x^2>0. \qquad (2\cdot 16)$$

Further, the relations in (2·17) are also seen from the expression, (2·13′) and (2·14′).

Finally, let us remark on the integral expression of the quasi-causal function defined by (2·3):

$$G_\pm(x)=-\frac{1}{(2\pi)^4}\int\frac{e^{ikx}}{k^2-m^2\mp i\epsilon}d^4k, \qquad (2\cdot 3)$$



a. *b.*

Fig. 5.

where ϵ is an infinitesimal positive number. The poles of the integrand in the complex k_0-plane are shown by the dotted lines in Fig. 5 *a* and *b*, corresponding to $G_+(x)$ and $G_-(x)$. Thus, the integration in (2·3) on the real k_0-axis has a definite meaning.

References

1) H. Lehmann, Nuovo Cimento **11** (1954), 342.
2) H. Lehmann, K. Symanzik and W. Zimmermann, Nuovo Cimento **2** (1955), 425; **6** (1957), 319.
K. Nishijima, Prog. Theor. Phys. **17** (1957), 765.
2) H. Yukawa, Phys. Rev. **76** (1949), 300.
3) For instance, W. Heisenberg, Rev. Mod. Phys. **29** (1957), 267.
4) A. Gamba, R. E. Marshak and S. Okubo, *Proceedings of the Ninth Conference on High Energy Nuclear Physics*; Proc. Nat. Acad. Sci. **45** (1959), 881.
5) A. Pais and G. E. Uhlenbeck, Phys. Rev. **79** (1950), 145.
6) H. Schmidt, Zs. für Phys. **151** (1958), 365, 408.
R. Haag, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **29** (1955), no. 12.
D. W. Hall and A. S. Wightman, Kgl. Danske Videnkab. Selskab, Mat.-fys. Medd. **31** (1957), no. 5.
A. S. Wightman and S. S. Schweber. Phys. Rev. **98** (1955), 812.
O. W. Greenberg, Phys. Rev. **115** (1959), 706.
8) S. T. Ma, Nuclear Phys. **7** (1958), 163.

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Ground States of Ca Isotopes

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The properties of the ground states, the binding energies (pairing energies) and magnetic moment of the isotopes Ca^{42}, Ca^{43}, and Ca^{44} are studied from the standpoint of configuration mixing. All the states with one- and two-particle excitations of $f_{5/2}$ or $p_{3/2}$ particle are mixed with the main configuration $f^n{}_{7/2}J$. The calculated values of the pairing energies and magnetic moment are in agreement with the experimental values if two-body interaction with the Rosenfeld spin mixture is used for the interaction between nucleons in the unfilled shell.

§ 1. Introduction

The Ca isotopes which are usually considered to be "$f_{7/2}$ shell nuclei" have been well interpreted by the *jj*-coupling shell model. Analyses of the binding energies and the excited states of these nuclei give us some interesting problems. The pairing energies of the Ca isotopes obtained from experimental values[1] of the binding energies of these isotopes are given by 3.0, 2.55 and 5.5 Mev for Ca^{42}, Ca^{43} and Ca^{44}, respectively. In the *jj*-coupling shell model, the pairing energies of the $f_{7/2}$ shell nuclei are given by the following formulae,[2]

$$\langle f^n_{7/2}J|V|f^n_{7/2}J\rangle = 2nF_0 \qquad \text{(for even } n\text{)}$$
$$= 2(n-1)F_0 \quad \text{(for odd } n\text{)},$$

in the limit of short range interaction, where F_0 is the well-known radial integral. The experimental values mentioned above cannot be explained by these formulae. The situation is almost same, even if the interaction with finite range is introduced.

According to the *jj*-coupling shell model, it is expected[3] that the excitation energy of the first excited state of the nucleus with two particles in the $f_{7/2}$ shell should be equal to that of the nucleus with four particles in the same shell irrespective of the form of two-body interactions. The experimental values of Ca^{42} and Ca^{44}, on the contrary, show a different feature,[1]

$$Ca^{42} \text{ —— } 1.51 \text{ Mev} \quad Ca^{44} \text{ —— } 1.16 \text{ Mev}.$$

For $f_{7/2}$ nuclei, the spin 7/2 level competes with the spin 5/2 level. In Ca^{43}, the 7/2 level is the ground state and the 5/2 level lies close to the ground state. Some authors[4] have tried to interpret this competition by using a long range parameter of the nuclear potential but it seems to be questionable whether such a long range value has any physical meaning.

In order to interpret the above problems more reasonably, the pairing energies and the lower levels of the isotopes, Ca^{42}, Ca^{43} and Ca^{44}, have been calculated from the standpoint of configuration mixing.

In the present paper only the pairing energies of Ca isotopes and the magnetic moment of Ca^{43} have been calculated and the two-body interaction in the unfilled shell, especially its spin exchange nature, has been discussed by comparing the calculated values of the pairing energies with experimental values. In a forthcoming paper, details of the low-lying levels of these nuclei will be reported.

§ 2. Configuration mixing

It is well known that the ground states of Ca^{42}, Ca^{43} and Ca^{44} have the spins 0, 7/2 and 0, respectively. From the analysis of experimental data for the energy levels of Ca^{41} and of the level order of Mayer's shell model,[5] it is assumed that the following configurations with the one- and two-particle excitations are mixed with the main configuration $f_{7/2}^n J$ for the Ca isotopes.

$Ca^{42}(n=2)$ $\quad f_{5/2}^2 0,\ p_{3/2}^2 0$

$Ca^{43}(n=3)$ $\quad f_{7/2}^2(J_1) f_{5/2} 7/2,\ f_{7/2}^2(J_2) p_{3/2} 7/2$

where $J_1=2, 4, 6$ and $J_2=2, 4$

$\quad f_{7/2} f_{5/2}^2(J_1) 7/2,\ f_{7/2} p_{3/2}^2(J_2) 7/2$

where $J_1=0, 2, 4$ and $J_2=0, 2$

$Ca^{44}(n=4)$ $\quad f_{7/2}^3(5/2) f_{5/2} 0,\ f_{7/2}^3(3/2) p_{3/2} 0$

$\quad f_{7/2}^2(J_1) f_{5/2}^2(J_1) 0,\ f_{7/2}^2(J_2) p_{3/2}^2(J_2) 0$

where $J_1=0, 2, 4$ and $J_2=0, 2$

These configurations are all the ones with the one- and two-particle excitations that can be possibly considered.

The two-body interaction between particles in the unfilled shell is assumed to be of the form:

$$V = V(r)(1+a\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2),\quad V(r) = V_0 \exp(-r^2/r_0^2).$$

The strength of the interaction V_0 is determined by the requirement that the pairing energy of Ca^{42} should have the value 3 Mev. For one-particle wave functions, the oscillator functions

$$\psi = r^{-1} R_{nl}(r)\, Y_{lm}(\theta, \varphi)$$

$$R_{nl}(r) = N_{nl} \exp(-\nu/2\, r^2)\, r^{l+1} L_{n+l+1/2}^{l+1/2}(\nu r^2)$$

are used where $R_{nl}(r)$, $Y_{lm}(\theta, \varphi)$, $N_{n,l}$ and $L_{n+l+1/2}^{l+1/2}(\nu r^2)$ have their usual meanings. In Ca^{41} the $f_{5/2}$ level has not yet been found and must presumably lie at a height of at least 4 Mev higher than the ground state.[6] Therefore, we cannot put a

definite value for the single particle energy $\varepsilon_{5/2}$, but rather use $\varepsilon_{5/2}$ and $\varepsilon_{3/2}$ as undetermined parameters in this paper. In numerical computations, the parameters Δ and Δ' have been introduced, for the sake of convenience,

$$\Delta = \frac{\varepsilon_{5/2}}{\langle f_{7/2}^2 0|V|f_{7/2}^2 0\rangle}$$

$$\Delta' = \frac{\varepsilon_{3/2}}{\langle f_{7/2}^2 0|V|f_{7/2}^2 0\rangle}$$

where the denominator $\langle f_{7/2}^2 0|V|f_{7/2}^3 0\rangle$ corresponds to the pairing energy of Ca^{42} of the case without configuration mixing (see the next section). Furthermore, $\Delta=2\Delta'$ is assumed. This corresponds to the fact that the $p_{3/2}$ level of Ca^{41} lies 2 Mev above the $f_{7/2}$ level and the $f_{5/2}$ level must lie 4 Mev or more above the same level. Therefore, the value of Δ varies in the neighbourhood of $\Delta=1.6$. If the value of pairing energy of Ca^{42} is taken as 3 Mev from the analysis of the binding energies of Ca^{41} and Ca^{42}, $\langle f_{7/2}^2 0|V|f_{7/2}^2 0\rangle$ and the single particle energy $\varepsilon_{5/2}$ are calculated. We always note that $\varepsilon_{5/2}$ is equal to or more than 4 Mev.

§ 3. Pairing energies of Ca isotopes

At first, we have obtained the values of the pairing energy of Ca^{42} by solving the three-dimensional secular equation for the configurations $f_{7/2}^2 0$, $f_{5/2}^2 0$ and $p_{3/2}^2 0$. The energies are measured in units of $\langle f_{7/2}^2 0|V|f_{7/2}^2 0\rangle$. This calculated value is taken to be equal to the experimental value, 3 Mev. The calculated value is given as a function of the range parameter ($\lambda=1/\nu r_0$). The excitation energy parameter Δ is taken to be 1.4, 1.6, 1.8 and 2.2 for the Rosenfeld mixture $a=2.3$ and to be 1.6 for $a=0.2$, 0.1, 0, -0.1, -0.2, -0.3 and -0.4. The values of $\langle f_{7/2}^2 0|V|f_{7/2}^2 0\rangle$ which is used as an energy scale in the numerical computations as well as the values of the excitation energy $\varepsilon_{5/2}$ are also determined as a function of the range parameter λ for the given values of Δ and a as is shown in Tables I and II.

Table I. Calculated values of $\langle f_{7/2}^2 0|V|f_{7/2}^2 0\rangle$ and single particle energy of $f_{5/2}$ in the case of the Rosenfeld mixture (in Mev). The pairing energy of Ca^{42} is taken to be 3 Mev.

Δ / λ	$\Delta=1.4$		$\Delta=1.6$		$\Delta=1.8$		$\Delta=2.2$	
	$\langle f_{7/2}^2 0\|V\| f_{7/2}^2 0\rangle$	$\varepsilon_{5/2}$	$\langle f_{7/2}^2 0\|V\| f_{7/2}^2 0\rangle$	$\varepsilon_{5/2}$	$\langle f_{7/2}^2 0\|V\| f_{7/2}^2 0\rangle$	$\varepsilon_{5/2}$	$\langle f_{7/2}^2 0\|V\| f_{7/2}^2 0\rangle$	$\varepsilon_{5/2}$
0	2.237	3.132	2.296	3.674	2.348	4.226	2.434	5.355
0.2	2.162	3.027	2.225	3.560	2.279	4.102	2.356	5.184
0.4	2.021	2.829	2.086	3.337	2.142	3.856	2.241	4.931
0.6	1.785	2.499	1.849	2.959	1.908	3.435	2.015	4.433
0.8	1.593	2.230	1.655	2.647	1.712	3.082	1.818	4.000
1.0	1.492	2.089	1.550	2.480	1.606	2.890	1.708	3.757

Table II. Calculated values of $\langle f_{7/2}^2 0|V|f_{7/2}^2 0\rangle$ and single particle energy of $f_{5/2}$ in the cases of spin mixture parameter $a=0.2$, 0.1, 0, -0.1, -0.2, -0.3 and -0.4 (in Mev).

a / λ	0.2		0.1		0		−0.1	
	$\langle f_{7/2}^2 0\|V\|f_{7/2}^2 0\rangle$	$\varepsilon_{5/2}$	$\langle f_{7/2}^2 0\|V\|f_{7/2}^2 0\rangle$	$\varepsilon_{5/2}$	$\langle f_{7/2}^2 0\|V\|f_{7/2}^2 0\rangle$	$\varepsilon_{5/2}$	$\langle f_{7/2}^2 0\|V\|f_{7/2}^2 0\rangle$	$\varepsilon_{5/2}$
0	2.296	3.674	2.296	3.674	2.296	3.674	2.296	3.674
0.2	2.585	4.136	2.463	3.941	2.405	3.848	2.374	3.798
0.4	2.881	4.609	2.736	4.378	2.623	4.197	2.545	4.071
0.6	2.893	4.629	2.891	4.626	2.797	4.475	2.702	4.323
0.8	2.829	4.526	2.927	4.683	2.883	4.613	2.802	4.483
1.0	2.787	4.459	2.932	4.691	2.924	4.679	2.860	4.576
1.2	2.768	4.428	2.936	4.698	2.950	4.720	2.899	4.639
1.4	2.758	4.414	2.941	4.705	2.972	4.756	2.928	4.684

a / λ	−0.2		−0.3		−0.4	
	$\langle f_{7/2}^2 0\|V\|f_{7/2}^2 0\rangle$	$\varepsilon_{5/2}$	$\langle f_{7/2}^2 0\|V\|f_{7/2}^2 0\rangle$	$\varepsilon_{5/2}$	$\langle f_{7/2}^2 0\|V\|f_{7/2}^2 0\rangle$	$\varepsilon_{5/2}$
0	2.296	3.674	2.296	3.674	2.296	3.674
0.2	2.351	3.761	2.337	3.739	2.327	3.723
0.4	2.448	3.981	2.446	3.914	2.415	3.865
0.6	2.623	4.196	2.556	4.090	2.505	4.008
0.8	2.719	4.351	2.645	4.232	2.581	4.130
1.0	2.783	4.453	2.709	4.334	2.642	4.227
1.2	2.829	4.526	2.757	4.411	2.688	4.301
1.4	2.862	4.580	2.792	4.467	2.762	4.420

The effect of configuration mixing can be expressed by the quantity ΔE defined by the relation $\langle f_{7/2}^2 0|V|f_{7/2}^2 0\rangle(1+\Delta E)=3$ Mev, and this quantity is plotted in Figs. 1 and 2 as a function of λ for various values of Δ and a. It can be seen from Figs. 1 and 2 that for $a=0.2\sim-0.4$ the effect of mixing ΔE is of the order of 10%, therefore $\langle f_{7/2}^2 0|V|f_{7/2}^2 0\rangle$ is approximately equal to 3 Mev, the pairing energy of Ca^{42}. For the Rosenfeld mixture, however, ΔE is very large and exceeds 100% in some ranges of λ. So the effect of configuration mixing is particularly important in the case of the Rosenfeld mixture. As is shown in Tables I and II, the excitation energy is greater than 4 Mev for $a=0.2\sim-0.4$ except in the neighbourhood of the short range limit, but it is smaller than 4 Mev for the Rosenfeld mixture. Therefore, in the latter case, the large value 2.2 of Δ is considered only in the computations for Ca^{43} and Ca^{44}.

Next, we proceed to the calculation for the cases of Ca^{43} and Ca^{44}. In the case of Ca^{43}, the spin 7/2 level competes with the spin 5/2 level, but the results of the calculations show that the 7/2 level is ground state except for the special

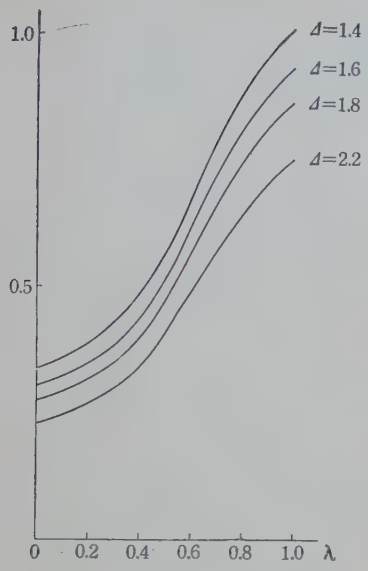

Fig. 1. The effect of configuration mixing ΔE in the case of the Rosenfeld mixture.
They are plotted as a function of the range parameter λ for various values of the excitation energy parameter Δ. The unit of the ordinate ΔE is $\langle f_{7/2}^2 0|V|f_{7/2}^2 0\rangle$.

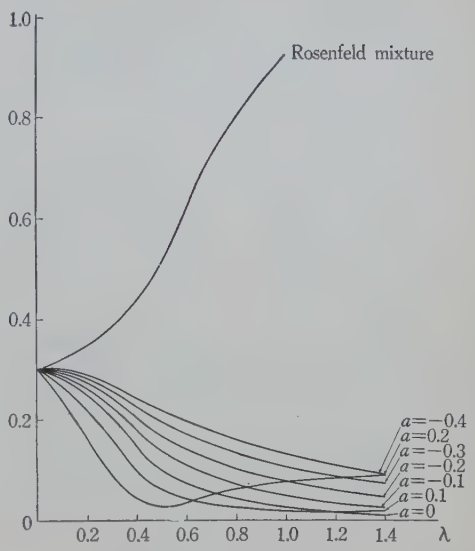

Fig. 2. The effect of configuration mixing ΔE in the cases of a=0.2, 0.1, 0, −0.1, −0.2, −0.3 and −0.4.
In these cases the value of the excitation energy parameter Δ is fixed to be 1.6.

range of a and λ and the 5/2 level lies closely above the 7/2 level. Moreover, the 3/2 level is found evidently to be the second excited state. These results are consistent with the experimental data.[1] Details of the calculations will be reported in the forthcoming paper.

The pairing energies of Ca^{43} and Ca^{44} are obtained by solving the secular equations of eleven and eight dimensions, respectively. Numerical calculations have

Table III. Calculated values of the pairing energy of Ca^{43} in the Rosenfeld mixture. The experimental value is 2.55 Mev.

λ	Ca^{42}	Ca^{43}	
		calculated values	exp
0	3.0	2.913	2.55
0.2		2.914	
0.4		2.795	
0.6		2.638	
0.8		2.462	
1.0		2.303	

Table IV. Calculated values of the pairing energy of Ca^{43} in the cases of spin mixture $a=0.2$, 0.1, 0, -0.1, -0.2, -0.3 and -0.4.

λ	Ca^{42}	Ca^{43}							
		calculated values							exp
0.2	3.0	3.327	3.148	3.072	3.034	3.006	2.990	2.997	2.55
0.4		4.356	3.850	3.571	3.442	3.344	3.275	3.225	
0.6		5.485	4.841	4.417	4.131	3.931	3.783	3.673	
0.8		6.558	5.849	5.330	4.939	4.643	4.416	4.238	
1.0		7.507	6.771	6.183	5.710	5.338	5.045	4.808	
1.2		8.344	7.563	6.914	6.377	5.945	5.596	5.310	
1.4		9.014	8.184	7.498	6.911	6.434	6.043	5.802	

been performed by an electronic computor. The results are shown in Tables III and IV as a function of λ for various values of a and for the values of Δ determined above, where the values of $\langle f_{7/2}^2 0|V|f_{7/2}^2 0\rangle$ given in Tables I and II are used.

§ 4. Comparison with experimental data

The binding energies of the last nucleon of the isotopes Ca^{41}, Ca^{42}, Ca^{43} and Ca^{44} are 8.37, 11.34, 7.95 and 11.29 Mev, respectively from the experimental data.[1] Using these values, the pairing energies of 3.0, 2.55 and 5.47 Mev are obtained respectively for Ca^{42}, Ca^{43}, and Ca^{44}, as is shown in Fig. 3.

It is shown from Table III that the experimental value of the pairing energy Ca^{43} is well reproduced for the Rosenfeld mixture but the inconsistent results are obtained for the spin mixture $a=0.2\sim-0.4$. From these results, we can get the following conclusion: The known fact that the binding energies of even-even nuclei are larger than the ones of odd nuclei is well interpreted for the

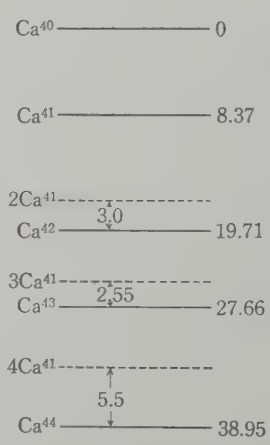

Fig. 3. Pairing energies in Ca isotopes

Table V. Calculated values of the pairing energy of Ca^{44} in the Rosenfeld mixture. $\Delta=2.2$ The experimental value is 5.47 Mev.

λ	calculated values	exp
0	5.794	5.47
0.2	5.773	
0.4	5.487	
0.6	5.070	
0.8	4.570	
1.0	4.086	

Table VI. Calculated values of the pairing energy of Ca^{44} in the cases of spin mixture $a=0.2$, 0.1, and 0. $\Delta=1.6$ The experimental value is 5.47 Mev.

λ \ a	0.2	0.1	0
0.2	6.681	6.328	6.184
0.4	8.885	7.765	7.205
0.6	11.247	9.788	8.867
0.8	13.263	11.846	10.723
1.0	14.971	13.652	12.431
1.2	16.449	15.170	13.871
1.4	17.645	16.372	15.033

Rosenfeld mixture, but not for $a=0.2\sim-0.4$. The same results are also obtained in the case of Ca^{44}. It is shown from Tables V and VI that the pairing energy of Ca^{44} are well interpreted for the Rosenfeld mixture in a suitable region of λ, but for $a=0.2\sim-0.4$ the calculated values are very large.

§ 5. Magnetic moment of Ca^{43}

The magnetic moments have so far been calculated by some authors[7),8)] from the standpoint of the configuration mixing. The effect of the configuration mixing for the magnetic moment was first investigated and the deviations from the Schmidt lines were explained successfully. These situations were well summarized together with the other moments by Noya et al.[9)] However, these calculations are based on the first order perturbation theory and, moreover, contain some approximations for simplicity.

Now the magnetic moment of Ca^{43} is calculated by diagonalizing the energy matrix for the Rosenfeld mixture which is reasonable for interpreting the pairing energies of the Ca isotopes. The wave function of the ground state of Ca^{43} can be written as

$$\psi=(\psi_0+\sum_n \alpha_n\psi_n)/(1+\sum|\alpha_n|^2)^{1/2}$$

where ψ_0 and ψ_n are the wave functions of the main configuration $f_{7/2}^3 7/2$ and of the excited configurations mentioned in § 2. The mixing amplitudes α_n have been calculated for Rosenfeld mixture. Numerical values are given in Table VII. The matrix elements of the magnetic moment operator can be readily obtained by the standard method of tensor operators.[10),9)] Only the configurations with one different single particle state give the non-vanishing contributions to the matrix elements of the magnetic moment. Moreover, according to the selection rule of the reduced matrix element $\langle j_1\|\mu\|j_2\rangle$, the principal quantum numbers and orbital angular momenta must be equal for both states. The non-vanishing elements are given in Table VIII. Owing to the smallness of the mixing amplitude α_n,

Table VII. Mixing amplitudes α_n. The wave function ϕ of the ground state is written as $\psi=[\psi_0+\sum_{n=1}^{10}\alpha_n\psi_n]/[1+|\alpha_n|^2]^{1/2}$,

where

$\psi_0=\psi(f_{7/2}{}^3;\,7/2)$, $\psi_1=\psi(f_{7/2}f_{5/2}{}^2(0):7/2)$, $\psi_2=\psi(f_{7/2}p_{3/2}{}^2(0):7/2)$, $\psi_3=\psi(f_{7/2}{}^2(2)f_{5/2}:7/2)$, $\psi_4=\psi(f_{7/2}{}^2(4)f_{5/2}:7/2)$, $\psi_5=\psi(f_{7/2}{}^2(6)f_{5/2}:7/2)$, $\psi_6=\psi(f_{7/2}{}^2(2)p_{3/2}:7/2)$, $\psi_7=\psi(f_{7/2}{}^2(4)p_{3/2}:7/2)$, $\psi_8=\psi(f_{7/2}f_{5/2}{}^2(2):7/2)$, $\psi_9=\psi(f_{7/2}f_{5/2}{}^2(4):7/2)$, and $\psi_{10}=\psi(f_{7/2}p_{5/2}{}^2(2):7/2)$.

λ \ α_n	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}
0.2	0.186	0.150	0.0145	0.0213	0.0365	0.0382	0.0177	−0.0265	−0.0141	−0.0175
0.4	0.214	0.166	0.0185	0.0264	0.0526	0.0391	0.0225	−0.0371	−0.0186	−0.0256
0.6	0.254	0.184	0.0275	0.0342	0.0761	0.0378	0.0356	−0.0591	−0.0259	−0.0092
0.8	0.282	0.193	0.0412	0.0492	0.0984	0.0277	0.0409	−0.0834	−0.0401	−0.0031
1.0	0.289	0.177	0.0624	0.0641	0.119	0.0221	0.0461	−0.110	−0.0498	−0.0092

Table VIII. Matrix elements $(m|\vec{\mu}|n)$ of the magnetic moment $\vec{\mu}$ for the wave functions ψ_m given in Table VII. Values of a and b defined by $(m|\vec{\mu}|n)=ag_s+bg_l$ are given in the Table, where $g_s=5.585$, $g_l=1$ for protons, and $g_s=-3.826$, $g_l=0$ for neutrons. For off-diagonal elements $a=-b$.

Diagonal elements $(m\|\vec{\mu}\|m)$			Off-diagonal elements $(m\|\vec{\mu}\|n)$		
m	a	b	m	n	$a(=-b)$
0	$\frac{1}{2}$	3	0	3	$-\frac{5\sqrt{5}}{63}$
1	$\frac{1}{2}$	3	0	4	$-\frac{2\sqrt{5}}{7\sqrt{3}}$
2	$\frac{1}{2}$	3	0	5	$-\frac{\sqrt{13}}{9}$
3	$-\frac{11}{126}$	$\frac{226}{63}$	1	3	$\frac{5\sqrt{5}}{63}$
4	$\frac{5}{14}$	$\frac{22}{7}$	1	4	$\frac{10}{21\sqrt{3}}$
5	$\frac{19}{18}$	$\frac{22}{9}$	1	5	$\frac{\sqrt{5}}{9}$
6	$\frac{11}{14}$	$\frac{19}{7}$	3	8	$\frac{55}{63\sqrt{6}}$
7	$\frac{185}{378}$	$\frac{569}{189}$	3	9	$-\frac{\sqrt{55}}{63\sqrt{6}}$
8	$\frac{13}{42}$	$\frac{67}{21}$	4	8	$-\frac{4\sqrt{2}}{21}$
9	$-\frac{17}{126}$	$\frac{229}{63}$	4	9	$\frac{2\sqrt{10}}{7\sqrt{11}}$
10	$\frac{79}{126}$	$\frac{181}{63}$	5	8	$\frac{\sqrt{65}}{63\sqrt{6}}$
			5	9	$\frac{13\sqrt{13}}{63\sqrt{66}}$

Table IX. Calculated values of the magnetic moment of Ca^{43}. The experimental value is -1.32 and the Schmidt value is -1.913. The unit is the nuclear magneton.

λ \ μ	μ_{cal}	μ_{sch}	μ_{exp}
0.2	$-1.735,2$		
0.4	$-1.675,2$		
0.6	$-1.586,8$	-1.913	-1.31
0.8	$-1.480,7$		
1.0	$-1.358,0$		

only matrix components concerning the states $f_{7/2}^3 7/2$, $f_{7/2}f_{5/2}^2(0)\,7/2$ and $f_{7/2}p_{3/2}^2(0)\,7/2$ are considered. These states are ones of the lowest seniority. The calculated values of the magnetic moment are shown in Table IX for the various values of the range parameter λ.

For the range $\lambda=0.6\sim0.8$ which are suitable to interpret the pairing energies of the Ca isotopes, the calculated values of the magnetic moment are a little larger (absolutely) than the experimental value -1.32 but we can well recognize that the calculated values have a good trend toward the experimental value from the Schmidt value -1.913 by the configuration mixing. For $\lambda=1.0$, the calculated value is almost in agreement with the experimental value.

From the results on the pairing energies of the Ca isotopes and the magnetic moment of Ca^{43}, it seems to be reasonable for us to see that the Rosenfeld mixture, in which the main part of the two-body interaction consists of the spin exchange force, is rather better than the ones with smaller values of the spin mixture parameter a.

We wish to thank Prof. H. Horie for his suggestion and encouragement. We are indebt to Dr. M. Katzuragi of Japan Atomic Energy Research Institute for diagonalizing many matrices.

References

1) K. Way et al., *Nuclear Level Scheme TID*-5300 (1955).
2) G. Racah ond I. Talmi, Physica **18** (1952), 1097.
3) C. Schwartz and A. de-Shalit, Phy. Rev. **94** (1954), 1257.
T. Komoda, Prog. Theor. Phys. **20** (1958), 580.
4) I. Talmi, Helv. Phys. Acta **25** (1952), 185.
D. Kurath, Phys. Rev. **80** (1950), 98.
A. R. Edmonds and B. H. Flowers, Proc. Roy. Soc. **A215** (1952), 120.
5) M. G. Mayer and J. H. D. Jensen, *Elementary Theory of Nuclear Shell Structure* (John Wiley and Sons, Inc., New York, 1955).
6) C. K. Bockelman and W. W. Buchener, Phys. Rev. **107** (1957), 1366.
7) A. Arima and H. Horie, Prog. Theor. Phys. **12** (1954), 623.
8) R. J. Blin-Stoyle and M. A. Perks, Proc. Proc. Phys. Soc. **A67** (1954), 885.
9) H. Noya, A. Arima and H. Horie, Supplement of Prog. Theor. Phys. No. 8 (1958), 33-112.
10) G. Racah, Phys. Rev. **62** (1942), 438; **63** (1943), 1366.

Letters to the Editor

The opinions expressed in these columns do not necessarily reflect those of the Board of Editors. Communications should be submitted in duplicate and should be held to within 100 lines (typica type) on standard size letter paper (approx. 21×30 cm.), so that each letter will be arranged into two pages when printed. Do not forget to count in enough space for formulas, figures and tables.

High Energy Limit of Vertex Function in Quantum Field Theory

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February 20, 1960

Several years ago, Lehmann et al.[1] proved that the vertex function in quantum field theory should damp at the high energy limit. In this note we will study the behavior of the vertex function at the high energy limit starting from more general assumptions than what they made. For the definiteness of arguments we take the same model as LSZ, namely the interacting system of neutral pseudoscalar meson of mass μ with nucleon of mass $M>2\mu$. The arguments in this note are independent of the validity of the dispersion relation for vertex function.

The Γ_5, which is our "vertex function", corresponds to the improper vertex part while the vertex function given in LSZ to the proper vertex part Γ_5^*. Γ_5 is connected with Γ_5^* by the following relation:

$$\Gamma_5(k, p, p-k) = S_F^{-1}(p) S_F'(p) \Gamma_5^*(k, p, p-k) \times \Delta_F^{-1}(k) \Delta_F'(k) S_F'(p-k) S_F^{-1}(p-k). \tag{1}$$

Here, S_F' and Δ_F' are respectively nucleon and meson renormalized propagators, and their general spectral representations are given by Bogoliubov et al.[2] For example, $\Delta_F'(k)$ can be written as follows:

$$\Delta_F'(k) = -\frac{1}{\mu^2+k^2} + \sum_{j=2}^{n} C_j (\mu^2+k^2)^{j-2} + (\mu^2+k^2)^{n-1} \int_{(3\mu)^2}^{\infty} \frac{\sigma(\lambda^2)\, d\lambda^2}{(\mu^2-\lambda^2)^{n-1}(\lambda^2+k^2)}. \tag{2}$$

Here, $\sigma(-k^2)$ is the imaginary part of $1/\pi \cdot \Delta_F'(k)$. The $\Delta_F'(k)$ given in LSZ corresponds to the case $n=1$ in the above expression.

Generally the following equation holds irrespective of the high energy behavior of Δ_F'

$$2\pi\sigma(-k^2) = |\Delta_F'(k)|^2 F(-k^2), \tag{3}$$

where $F(-k^2)$ is a complicated function consisting of τ-functions and Δ^+, S^+ and S^- functions.[1]

From Eq. (3) we see that $F(-k^2)$ is positively definite and it should damp asymptotically as $(-k^2)^{-n+2}$ or more rapidly when $|\Delta_F'(k)|$ increases as $(-k^2)^{n-2}$ for $-k^2\to\infty$. This asymptotic behavior will hereafter be indicated symbolically as $F(-k^2) \lesssim (-k^2)^{-n+2}$.

It is shown in LSZ that

$$F(-k^2) \geq \frac{1}{4\pi}\theta(-k^2-4M^2) \times\sqrt{-k^2}\sqrt{-k^2-4M^2}|f(-k^2)|^2. \tag{4}$$

The right-hand side of this inequality is the contribution of a part of $F(-k^2)$ which is taken only from a nucleon-antinucleon pair state, and $f(-k^2)$ is defined as follows:

$$\Gamma_5^*(k, p, p-k) \equiv \gamma_5 f(-k^2) \quad \text{for } i\gamma p = i\gamma(p-k) = -M. \tag{5}$$

Since $F(-k^2) \lesssim (-k^2)^{-n+2}$, it should follow from Eq. (4) that $|f(-k^2)|^2 \lesssim (-k^2)^{-n+1}$, that is, $\Gamma_5^*(-k^2) \lesssim (-k^2)^{-(n-1)/2}$. On the other hand, for nucleon momenta on the energy shell, from Eq. (1) we have

$$\Gamma_5(-k^2) = \Gamma_5^*(-k^2)\Delta_F^{-1}(k)\Delta_F'(k). \tag{6}$$

Therefore, we insist that $\Gamma_5(-k^2)$ should increase as $(-k^2)^{-(n-1)/2}\cdot(-k^2)\cdot(-k^2)^{n-2} = (-k^2)^{(n-1)/2}$ or more slowly for $-k^2 \to \infty$; in other words, the high energy limit of the vertex function $\Gamma_5(-k^2)$ is completely determined by that of $\Delta_F'(k)$.

Thus, it is concluded that if one writes down the dispersion relation for Γ_5, its form can be determined from the assumption about the behavior of Δ_F' at high energy limit. For example, when one uses Lehmann's representation for Δ_F', $n=1$ in Eq. (2), there is no necessity of subtraction in the dispersion relation for vertex function,[4] because in this case $\Gamma_5^*(-k^2)$ cannot behave as constant for $-k^2 \to \infty$, but in virtue of the necessary condition of the existence of $\Delta_F'(-k^2)$ it should damp as shown in LSZ. When $\Delta_F'(k)$ has no zero on the complex $(-k^2)$-plane it is sufficient to perform one subtraction because n is less than 3 in this case as shown in the previous paper.[5]

1) H. Lehmann, K. Symanzik and Zimmermann, Nuovo Cimento **2** (1955), 425. This article will hereafter be referred to as LSZ.
2) N. N. Bogoliubov, B. V. Medvedev and M. K. Polivanov, *Problems of the Theory of Dispersion Relations* (Moscow, 1958).
3) H. Lehmann, Nuovo Cimento **11** (1954), 342.
4) S. D. Drell and F. Zachariasen, Phys. Rev. **111** (1958), 1727.
5) S. Aramaki, Prog. Theor. Phys. **22** (1959), 485.

On the G_A/G_V Ratio in β-Decay

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Recently the G_A/G_V ratio in β-decay has been calculated by many authors.[1),2),3)] Some of them[2),3)] have investigated this problem from the standpoint of the composite model. Previously we[4)] dealt with the effective pion-nucleon interaction and the anomalous magnetic moment of the nucleon on the basis of the composite model for the pion and determined the coupling constants (and cutoff) to be consistent with the experimental results. Then the G_A/G_V ratio in β-decay is also calculated by using those coupling constants (and cutoff) from the view-

point of this model.

As it is recognized that the weak *V-A* interaction has the universality and the vector coupling constant is not affected by the renormalization on the assumption of the conserved current,[5] we may calculate the ratio of the observed axial vector coupling constant G_A to the bare one G_A^0 instead of the observed ratio G_A/G_V.

The fundamental interactions are assumed the following forms,

$$H_w=\left\{\bar{\psi}\gamma_\mu(G_V^0+G_A^0\gamma_5)\frac{\tau_1+i\tau_2}{2}\psi\right\}\times\left(\bar{e}\gamma_\mu\frac{1+\gamma_5}{\sqrt{2}}\nu\right)+\text{H. C.}, \tag{1}$$

$$H_{st}=\sum_k\sum_{i=1}^{3}\frac{g_{1k}}{M^2}(\bar{\psi}O_k\tau_i\psi)(\bar{\psi}O_k\tau_i\psi)+\sum_k\frac{g_{0k}}{M^2}(\bar{\psi}O_k\psi)(\bar{\psi}O_k\psi), \tag{2}$$

where

$$O_s=1 \qquad O_v=i\gamma_\mu$$

$$O_t=\frac{1}{2\sqrt{2}\,i}(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)$$

$$O_a=i\gamma_5\gamma_\mu \qquad O_p=\gamma_5 \tag{3}$$

and ψ represents the wave function of the nucleon. G_A/G_A^0 is calculated in the lowest order perturbation like Iso[2] (it is illustrated in Fig. 1) and the result is as follows:

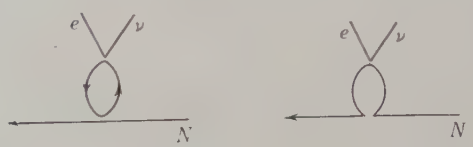

Fig. 1.

$$\frac{G_A}{G_A^0}=1+\frac{2i}{(2\pi)^4}\left[\frac{1}{2}(-g_{1p}+g_{0p})-\frac{1}{2}(-g_{1s}+g_{0s})-(-g_{1t}+g_{0t})+4g_{1a}+(-g_{1v}+g_{0v})+(-g_{1a}+g_{0a})\right]\times\int\frac{(dk)}{M^2}\frac{k^2-2M^2}{(k^2+M^2)^2}. \tag{4}$$

Table I.

	g_{1p}	g_{0p}	g_{1s}	g_{0s}	g_{1t}	g_{0t}	$(G_A/G_A^0)-1$ Case 1	Case 2	Case 3
I	7.50	7.50	−12.66	−12.66	−0.86	−0.86	0	0	0
II	10.18	10.18	−10.18	−10.18	−1.72	1.72	(−)	0.062	0.001
	1.30	1.30	−1.30	−1.30	4.14	−4.14	0.144	(−)	(−)
III	5.40	−5.40	5.40	−5.40	−0.40	0.40	(−)	0.015	0.00024
	−3.37	3.37	−3.37	3.37	8.37	−8.37	−0.29	(−)	(−)

Table II.

g_{1v}	g_{0v}	g_{1a}	g_{0a}	$(G_A/G_A^0)-1$ Case 1	Case 2	Case 3
7.5	−15	7.5	0	0	0	0

As the correction of the electric charge must be zero, the following equality must be satisfied (cf. reference 2), Appendix A),

$$\int \frac{(dk)}{M^2} \frac{k^2+2M^2}{(k^2+M^2)^2}=0. \qquad (5)$$

Then, by using (5), we have

$$\text{(Case 1)} \quad \int \frac{(dk)}{M^2} \frac{k^2-2M^2}{(k^2+M^2)^2} = \int \frac{(dk)}{M^2} \frac{-4M^2}{(k^2+M^2)^2}, \qquad (6)$$

$$\text{(Case 2)} \quad \int \frac{(dk)}{M^2} \frac{k^2-2M^2}{(k^2+M^2)^2} = \int \frac{(dk)}{M^2} \frac{2k^2}{(k^2+M^2)^2} \qquad (7)$$

and

(Case 3) the average of (6) and (7).

Next we integrate with respect to k_0 and $\boldsymbol{k}$ separately and cutoff integral at the nucleon mass M. The results are shown in Table I (scalar, tensor and pseudoscalar coupling) and Table II (vector and axial-vector coupling) and it is possible to improve the result from meson theory $(G_A/G_V<1)$.

However, it seems to us that the second order contributions (Fig. 2) are also important for reproducing the experimental value $G_A/G_V=1.2$, hence we should like to discuss them at the first opportunity.

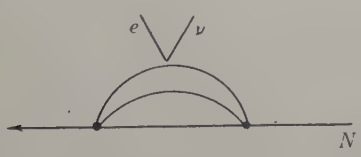

Fig. 2.

1) R. J. Finkelstein and A. Moszkovsky, Phys. Rev. **95** (1954), 1695.
M. Ross, Phys. Rev. **104** (1956), 1736.
Y. Fujii and S. Furuichi, Prog. Theor. Phys. **23** (1960), 251.
2) C. Iso, Prog. Theor. Phys. **22** (1959), 62.
3) Y. Ataka, Prog. Theor. Phys. **22** (1959), 321.
M. Nakagawa, Prog. Theor. Phys. **22** (1959), 725.
4) C. Ihara, Prog. Theor. Phys. **23** (1960), 1035.
5) S. S. Gershtein and J. B. Zeldovich, JETP, **2** (1956), 576.
R. P. Feynman and M. Gell-Mann, Phys. Rev. **109** (1958), 193.

Possible Corrections to Nuclear Magnetic Moments Due to Pairing Forces in Nuclei. I

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Recently the Bardeen-Cooper-Schrieffer (BCS) theory[1] has been applied to nuclear matter and to finite nuclei by a number of authors.[2)-6)] Gorkov[7] using the Green's function techniques rederived the results of the BCS theory, and Migdal[8] extended this model to arbitrary systems of quasiparticles. Migdal's method is convenient for calculations of perturbations of the type of the impurities in superconductors (see e.g. Refs. 1) and 8)). Following this method we can examine first the possibility of an enhancement of the magnetic energy of an even-even nucleus in the constant magnetic field $\boldsymbol{H}$ (oriented

along the z-axis) due to the BCS-type pairing force. In the usual notation the relevant perturbation operator is

$$V=-H\mu_z=-H\frac{e\hbar}{2Mc}\sum_{k=1}^{A}\{(g_s^{(k)}-g_l^{(k)})\times\frac{1}{2}\sigma_z^{(k)}+g_l^{(k)}j_z^{(k)}\}. \quad (1)$$

The correction to the single particle mixed density due to V in the presence of the energy gap Δ can be found as

$$\rho'(\boldsymbol{r},\boldsymbol{r}')=(2\pi i)^{-1}\int_C G'(\boldsymbol{r},\boldsymbol{r}',\epsilon)\,d\epsilon$$
$$=\sum_{\lambda,\lambda'}\rho'_{\lambda\lambda'}\varphi_\lambda(\boldsymbol{r})\varphi^*_{\lambda'}(\boldsymbol{r}') \quad (2)$$

where $G'(\boldsymbol{r},\boldsymbol{r}',\epsilon)$ is the time Fourier transform of the corresponding correction to the Green function, the contour C is the real axis plus the upper half plane semicircle, and the ϵ_λ^0's and φ_λ's are the single particle eigenvalues and eigenfunctions of a given model Hamiltonian H_0. Following Migdal[6] we find

$$\rho'_{\lambda\lambda'}=\rho^1_{\lambda\lambda'}+\rho^2_{\lambda\lambda'},$$
$$\rho^1_{\lambda\lambda'}=\frac{(\epsilon_\lambda\epsilon_{\lambda'}-E_\lambda E_{\lambda'})V_{\lambda\lambda'}-\Delta^2V^*_{\lambda\lambda'}}{2E_\lambda E_{\lambda'}(E_\lambda+E_{\lambda'})}, \quad (3)$$
$$\rho^2_{\lambda\lambda'}=\frac{\Delta(\epsilon_\lambda\Delta'_{\lambda\lambda'}+\epsilon_{\lambda'}\Delta'^*_{\lambda\lambda'})}{2E_\lambda E_{\lambda'}(E_\lambda+E_{\lambda'})},$$

where $\epsilon_\lambda=\epsilon_\lambda{}^0-\lambda$, $E_\lambda=\sqrt{\epsilon_\lambda{}^2+\Delta^2}$, where the chemical potential λ is to be found from the condition on the total number of particles interacting through the pairing force. The quantity Δ' is to be interpreted as a correction to the gap Δ induced by V. In our case of a magnetic interaction (compare e.g. Tables II and III of BCS[1]) or Eq. (2) of Ref. 8) we have $V^*=-V$, and consequently, $\Delta'^*=-\Delta'$ (compare Ref. 6)). The sought correction to the nuclear energy is

$$\Delta E=\langle V\rangle=\mathrm{Tr}(V\rho')^*. \quad (4)$$

In any representation in which j_z and σ_z are diagonal, V has only diagonal matrix elements. Consequently, the term $\mathrm{Tr}(V\rho^1)$ corresponding to the usual matrix elements (compare Tables II and III of BCS[1]) vanishes, and also all the odd orders in V are equal to zero.

If one uses Migdal's[6] $\int_C F'_{\lambda\lambda'}(d\epsilon/2\pi)$ of his Eq. (16), and the equation $\gamma\int_C F'_{\lambda\lambda'}\,d\epsilon/2\pi=\Delta'^*_{\lambda\lambda'}$, one sees that for the so obtained $\Delta'_{\lambda\lambda'}$ also $\mathrm{Tr}(V\rho^2)$ vanishes, and $\Delta E=0$. However, a different special solution may be obtained. Migdal[6] proposes an integral equation for $\Delta'(\boldsymbol{r})$ which is his Eq. (17). This equation results from a) the assumption of no change in the pairing interaction constant γ resulting from the presence of V, and b) the condition $\Delta'^*(\boldsymbol{r})=-\gamma\sum_{\lambda\lambda'}(\Delta'^*_{\lambda\lambda'}/2E_\lambda)\varphi_\lambda(\boldsymbol{r})\varphi^*_{\lambda'}(\boldsymbol{r})$ imposed upon $\Delta'(\boldsymbol{r})$ analogous to the integral equation (Eq. (12) of Ref. 6)) for the modified energy gap equal to $\Delta(\boldsymbol{r})+\Delta'(\boldsymbol{r})$. If we solve Migdal's Eq. (17) by equating the coefficients of $\varphi_\lambda\varphi^*_{\lambda'}$'s to zero, we obtain $(\epsilon_\lambda-\epsilon_{\lambda'})\Delta'_{\lambda\lambda'}=2\Delta V_{\lambda\lambda'}$ (i.e. $\Delta'(\boldsymbol{r})=2i\Delta V(\boldsymbol{r})$). In this case we would obtain

$$\Delta E=-\Delta^2\sum_\lambda(V_{\lambda\lambda})^2/(2E_\lambda{}^3). \quad (5)$$

ΔE vanishes for $\Delta\to0$ which means a correct correspondence to the second

* The zero order (in V) density ρ^0 gives no contribution to the trace for an even-even nucleus, as V is odd in magnetic substates.

order perturbation theory. It is also proportional to H^2, i.e. to the magnetic energy, which means a correction to the magnetic moment proportional to H.* However, such a ΔE, if existing at all, would be extremely small as it was checked numerically on the example of the Mg^{24} nucleus using Nilsson[9] wave functions for the deformation parameter $\eta=6$, and the value of Δ obtained from the observed pairing energy.[10]

In addition to the pairing forces there are still residual two-body forces present in the system which can be treated by perturbation theory. An example of a simultaneous treatment of the both types of forces is given in the recent paper by Rockmore[11] on the moments of inertia. Bell[12] has applied the Brueckner theory to calculate the magnetic moments in the presence of residual two-body forces in the case of zero energy gap Δ. Corrections to the ordinary and the exchange currents were obtained and found to be negligible. This method can be easily generalized to our case of a finite Δ. The most practical method of doing it is to introduce Fourier transforms of the mixed single particle density matrix in the fashion applied in a previous note on the effect of the energy gap on the optical model potential.[13] Similarly as for the case of no extra residual forces, the zero order (in V) density gives no contribution both to the direct and the exchange correction terms to the current density operator and/or to the magnetic energy. Again we have to go to the first order (in V) density ρ', which gives zero or quite negligible corrections as above.

In their recent paper Grin et al.[14] presented an extension of Migdal's theory to odd A-nuclei. If the state occupied by the odd nucleon is $\lambda=\pm\lambda_0$, then the component $\rho^0_{\pm\lambda_0\pm\lambda_0}$ of the zero order (in V) density is equal to one, and $\rho^0_{\mp\lambda_0\mp\lambda_0}=0$. This indicates that the usual ("nonsuperfluid") theory applies to the contribution to the magnetic moment of the odd (aligned) nucleon in an odd A-nucleus.

We have so far considered only the direct particle contributions to the magnetic moments which turn out to be practically—if not exactly—zero. However, on the uniform model, there is in general still the collective contribution of the rotational motion of the nucleus as a whole characterized by the gyromagnetic ratio g_R-the coefficient of the rotational angular momentum $\boldsymbol{R}$ in the general expression for $\boldsymbol{\mu}$. The expression for g_R on the particle picture can be given in the form (compare Bohr and Mottelson[15])

$$g_R=\frac{\hbar}{j\Omega}\,\mathrm{Tr}(\hat{\mu}_x\rho') \qquad (6)$$

where $\hat{\mu}_x=\sum_k(g_l^{(k)}l_x^{(k)}+g_s^{(k)}\frac{1}{2}\sigma_x^{(k)})$; ρ' is the density matrix of first order in the perturbation $U=-\hbar\Omega\sum_k(l_x^{(k)}+\frac{1}{2}\sigma_x^{(k)})$; Ω is the angular velocity of rotation around the axis x-perpendicular to the body axis ; j is the nuclear moment of inertia. A discussion of g_R in the presence of

* Actually the complete ΔE involves also a small term Tr $W\rho^0$ (to order H^2) where W is the correction to V quadratic in $\boldsymbol{H}$, and ρ^0 the zero order (in V) density. This term has the same property, and, essentially, magnitude as in the case of no pairing force.

the pairing forces will be the subject of a future publication.

1) J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **108** (1957), 1175.
2) A. Bohr, B. R. Mottelson, and D. Pines, Phys. Rev. **110** (1958), 936.
3) V. G. Soloviev, Nucl. Phys. 9 (1959), 655.
3) V. G. Soloviev, Nuovo Cimento **10** (1958), 1022.
5) S. T. Belyaev, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **31**, No. 11 (1958).
6) A. B. Migdal, Zh. Exptl. Teor. Fiz. (U.S.S.R.) **37** (1959), 249; see also Nucl. Phys. **13** (1959), 655.
7) L. P. Gorkov, Zh. Exptl. Teor. Fiz. (U.S.S.R.) **34** (1958), 735.
8) K. Nakamura, Prog. Theor. Phys. **22** (1959), 156.
9) S. G. Nilsson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **29**, No. 16 (1955).
10) N. K. Glendenning and J. Sawicki, to be published.
11) R. M. Rockmore, preprint.
12) J. S. Bell, Nucl. Phys. **4** (1957), 295.
13) J. Sawicki, Nuovo Cimento **15** (1960), 504.
14) Y. T. Grin, S. I. Drozdov, and D. F. Zaretskii, Zh. Exptl. Teor. Fiz. (U.S.S.R.)**38** (1960), 222.
15) A. Bohr, and B. Mottelson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **30**, No. 1, (1955).

On the Interaction in Nucleon Core

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Previously[1] the author has pointed out that the value of coupling constant in nucleon core ought to be reduced in appearance to $f=(\mu/2M)g$ in spite of the fact that its value in the neighborhood of pion cloud is g, where g is the renormalized pseudoscalar coupling constant of symmetrical pseudoscalar theory, M and μ are the masses of nucleon and pion respectively. The purpose of this paper is to examine how this situation should be interpreted from the field theoretical point of view. Let us develope our discussion on this problem by taking into account the spread of nucleon core. If the pion source density at y of nucleon (the center of mass at x) is written as $\rho_\alpha(x, y-x)\equiv\rho_\alpha(x, z)$, the pion-nucleon interaction becomes

$$-L(x)=\int\rho_\alpha(x, y-x)\phi_\alpha(y)\,dy$$
$$\equiv\int\rho_\alpha(x, z)\phi_\alpha(x+z)\,dz, \qquad (1)$$

where ϕ_α is the pion field. As was shown in the description of the multipole model of elementary particles[2] proposed previously, $\rho_\alpha(x, z)$ can generally be expressed as follows:

$$\rho_\alpha(x, z)=ig\bar{\psi}(x, z)\tau_\alpha\psi(x, z), \qquad (2)$$

$$\psi(x, z)=\sum U_\sigma^{nK}(z)B_\sigma^{nK}(x),$$

$$\bar{\psi}(x, z)=\sum\bar{B}_{\sigma'}^{n'K'}(x)\bar{U}_{\sigma'}^{n'K'}(z). \qquad (3)^*$$

Since our discussions are restricted to the pion-nucleon interaction, we here omit the superscripts n and k which indicate the baryon states.

So far as the phenomena in the neighborhood of pion cloud are concerned, the nucleon core may be treated as if it were a point. This will make it possible to see the property of the interaction through only the first term

* With regard to the notation, see reference 2).

(monopole interaction) in the multipole expansion of $\phi_\alpha(x+z)$,

$$\phi_\alpha(x+z)=\phi_\alpha(x)+z_\mu\frac{\partial\phi_\alpha(x)}{\partial x_\mu}+\cdots. \tag{4}$$

Then

$$-L_{strong}(x)=ig\bar{B}_{\sigma'}(x)\times\left(\int_{\Omega_0}\bar{U}_{\sigma'}(z)\,U_{\sigma''}(z)\,dz\right)\times\tau_\alpha B_{\sigma''}(x)\phi_\alpha(x), \tag{5}*$$

where the integral in Eq. (5) must be performed over the space-time spread of a nucleon core Ω_0. However, such a treatment as this will no longer be valid in describing the phenomena inside the nucleon core, because such an expansion as (4) may lose its meaning in the case where x is not always larger than z. And $-L(x)$ in (5) should be modified as follows;

$$-L(x)=ig\bar{B}_{\sigma'}(x)\times\left(\int_{\Omega_0}\bar{U}_{\sigma'}(z)\,U_{\sigma''}(z)\phi_\alpha(x+z)\,dz\right)\times\tau_\alpha B_{\sigma''}(x). \tag{5'}$$

If we are forced to reinterpret the expression of (5′) in the framework of the current field theory (point model), the (5′) may be formally written down as follows;

$$-L(x)=ig\xi(Mx)\bar{B}_{\sigma'}(x)\times\left(\int_{\Omega}\bar{U}_{\sigma'}(z)\,U_{\sigma''}(z)\,dz\right)\times\phi_\alpha(x)\tau_\alpha B_{\sigma''}(x). \tag{5''}$$

The integral in (5″) should not be performed over all the spread of nucleon core Ω_0, but should be done over the suitable four-dimensional volume Ω which may generally depend on x and may be smaller than Ω_0. This may correspond to the following situation in electromagnetic field. When the electric charge is distributed over the sphere with radius r_0 (where the distribution function depends neither on θ nor on φ), the field at the distance r from the center ($r\ll r_0$) can be determined by the charge contained inside the sphere with radius r. Thus it may be easily seen that the value of coupling constant in nucleon core is practically reduced to

$$g\xi(Mx)\frac{\left(\int_{\Omega}\bar{U}_{\sigma'}(z)\,U_{\sigma''}(z)\,dz\right)}{\left(\int_{\Omega_0}\bar{U}_{\sigma'}(z)\,U_{\sigma''}(z)\,dz\right)}\equiv gF(Mx). \tag{6}$$

If our conclusion mentioned in the previous paper[1] may be expressed in terms of the nonlocal field theory, it will mean that the value of factor F in (6) averaged over the spread of nucleon core turns out to be equal to $(\mu/2M)$ although the detailed form of $U_\sigma(z)$ has not yet been known.

We may also expect such an effect as this in the electromagnetic interaction, that is, the interaction strength in nucleon core may become much weaker

* In order to derive the usual pion-nucleon interaction from (5), it is of course necessary to introduce the following condition as was mentioned in the previous paper.[2] When the $\int\bar{U}_{\sigma'}(z)U_{\sigma''}(z)dz$ is expanded in γ-algebra with respect to spin suffix,

$$\int\bar{U}_{\sigma'}(z)U_{\sigma''}(z)dz=\delta_{\sigma'\sigma''}M_A+(\gamma_5)_{\sigma'\sigma''}M_B,$$

the M_A is always equal to zero.

than that in the outer region. As the result, the contribution from nucleon current to the anomalous magnetic moment of nucleon may turn out to be very small. This will make it possible to explain the experimental results for the anomalous magnetic moment of nucleon. But the reduction of the value of coupling constant in nucleon core may give rise to a troublesome question in electromagnetic interaction, that is, the so-called gauge invariance may not be satisfied in the strict sense of the word. Moreover, the description of Compton scattering by nucleon may be influenced by the existence of such an effect as this.

The author would like to express his thanks to the members of the Laboratory of Elementary Particle Study in Osaka University for their helpful comments.

1) S. Minami, Prog. Theor. Phys. **23** (1960), 887.
2) D. Ito, S. Minami and H. Tanaka, Prog. Theor. Phys. **22** (1959), 159.

Some Remarks on Fermi-Yang Model

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Meson theory with the cutoff prescription which had succeeded in the P-wave pion phenomena, has not been very satisfactory for the problem of pion-nucleon S-wave scattering and the nucleon structure. The author obtained qualitatively good results for these "core phenomena", using Fermi-Yang model,[1),2)] which is defined as follows:

(i) Pions are represented by N-$\bar{N}$ two-body bound-states.

(ii) Nucleons are represented by N themselves or N-N-$\bar{N}$ three-body bound states.

(iii) Fermi interactions between fundamental particles are replaced by the short-range static potentials.

The many-body correlations between fundamental particles are neglected in this "model", and the interactions between pions and nucleons are reduced to two-body interactions between fundamental particles. In the present letter, the pion-pion and K-particle-nucleon potentials are discussed by the application of the Fermi-Yang model and the same method as that used in the author's previous work (referred to as I in the following).

The pion-pion potential is derived as follows: One pion of the pion-pion system is composed of N_1 and $\bar{N}_2$, whose isospins are $\boldsymbol{\tau}^{(1)}$ and $\boldsymbol{\tau}^{(2)}$ respectively. The isospin of the other pion π' is denoted by $\boldsymbol{\omega}'$. Then the N_1-π' potential is given by

$$H_{\pi'\text{-}\bar{N}_2}(R_1) = -4\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\omega}' V_0 U(R), \quad (1)$$

and $\bar{N}_2$-π' by

$$H_{\pi'\text{-}\bar{N}_2}(R_2) = -4\boldsymbol{\tau}^{(2)}\cdot\boldsymbol{\omega}' V_0 U(R), \quad (2)$$

where the notation is the same as that in I, R_1 and R_2 are the distance between N_1-π' and $\bar{N}_2$-π', respectively. $U(R)$ is a short-range function defined by I (2·9). Then the pion-pion potential is

$$H_{\pi\pi'}(R) = -4V_0\langle \boldsymbol{\tau}^{(1)}+\boldsymbol{\tau}^{(2)}\rangle_{\text{pion}} \cdot \boldsymbol{\omega}' W(R) = -8V_0 \boldsymbol{\omega}\cdot\boldsymbol{\omega}' W(R), \quad (3)$$

where

$$W(R) \equiv 2\pi \int_0^\infty \rho(r)\, r^2 dr \times \int_{-1}^{1} dz\, U(\sqrt{r^2 z^2 - rRz + (r^2/4)}) \quad (4)$$

is a function having a range about $1/M$. We write the total isospin quantum number as T, then

$$\boldsymbol{\omega}\cdot\boldsymbol{\omega}' = \frac{1}{2}T(T+1) - 2 = \begin{cases} -2 & (T=0), \\ -1 & (T=1), \\ +1 & (T=2). \end{cases} \quad (5)$$

Since $V_0<0$, the pion-pion potentials are strongly attractive for $T=0$ state, attractive for $T=1$ state and repulsive for $T=2$ state.

We adopt the composite model extended to every baryon and meson by Sakata.[3] Now there are two different approaches to the Sakata model. The first approach starts from the different masses of fundamental particles (i.e. $M_\Lambda \neq M_N$) and the same order strength of the interactions between them. The second approach starts from the same mass of fundamental particles and the weaker interaction between them containing fundamental Λ than that containing no Λ. Following Okun',[4] we adopt the latter standpoint.

The interactions between N and Λ, $\bar{\Lambda}$, are given by

$$\left.\begin{aligned} H'_{N\text{-}\Lambda} &= -2V_1(r), \\ H'_{N\text{-}\bar{\Lambda}} &= 2V_1(r), \end{aligned}\right\} \quad (6)$$

where

$$V_1(r) = V_1 v(r).$$

We assume that the potential $V(r)$ and $V_1(r)$ have the same potential shape and different depths. The K-N potential is derived,

$$H_{K\text{-}N}(R) = 2(-\eta V_0 + V_1 - \boldsymbol{\tau}\cdot\boldsymbol{\tau}_K V_0)\, U(R), \quad (8)$$

where $\boldsymbol{\tau}_K = \langle \boldsymbol{\tau}_N \rangle_K$ is the isospin of the K-particle. Similarly, $\bar{K}$-N potential is obtained,

$$H_{\bar{K}\text{-}N}(R) = 2(\eta V_0 - V_1 - \boldsymbol{\tau}\cdot\boldsymbol{\tau}_K V_0)\, U(R). \quad (9)$$

Under the condition I (A·19a), $-\eta V_0 - V_0 < 0$. The force between N and $\bar{\Lambda}$ are attractive, hence $V_1<0$. From the larger binding energy of pion than K-particle,

$$-\eta V_0 - V_0 + V_1 > 0. \quad (10)$$

We write the total isospin quantum number as T, then

$$\boldsymbol{\tau}\cdot\boldsymbol{\tau}_K = 2T(T+1) - 3 = \begin{cases} -3 & (T=0), \\ 1 & (T=1). \end{cases} \quad (11)$$

Then we reach the conclusion that for the $T=1$ K-N state (for example, K^+-p state) the interaction (8) becomes repulsive, and for the $T=1$ $\bar{K}$-N state (for example, $T=1$ K^--p state) the potential (9) becomes more attractive than that for $T=1$. The $\bar{K}$-N potential may make $(N\bar{N}\Lambda)$ bound states, if the interaction is sufficiently strong, then the $T=0$ level is lower than the $T=1$ level. It may be considered that these circumstances correspond to the smaller mass of the

(physical) Λ-particle than that of the Σ-particle. The present results (8) and (9) are not inconsistent with the experiments on the K-nucleon scatterings.[5]

1) E. Fermi and C. N. Yang, Phys. Rev. **76** (1949), 1439.
2) Y. Ataka, Prog. Theor. Phys. **22** (1959), 321, referred to as I.
3) S. Sakata, Prog. Theor. Phys. **16** (1956), 686.
4) L. B. Okun', Zh. Eks. Teor. Fiz. (SSSR) **34** (1958), 469.
5) *1959 Annual International Conference on High Energy Physics at Kiev.*

A Proton-Proton Potential

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A search has been going on for the energy independent potential which is consistent with the available p-p scattering data. We report here on the preliminary results.

Based on the belief that the potential must have the one-pion-exchange tail, we have assumed the following type of potentials:

$$
\begin{aligned}
{}^1V^+(x) &= -\mu(g^2/4\pi)\,(e^{-x}/x) \\
&\times[1+{}^1a_C^+(e^{-x}/x)+{}^1b_C^+(e^{-x}/x)^2], \qquad (1)\\
{}^3V^-(x) &= (1/3)\,\mu(g^2/4\pi)\,(e^{-x}/x) \\
&\times|1+{}^3a_C^-(e^{-x}/x)+{}^3b_C^-(e^{-x}/x)^2| \\
&+(1/3)\,\mu(g^2/4\pi)\,(e^{-x}/x) \\
&\times(1+(3/x)+(3/x^2)) \\
&\times[1+{}^3a_T^-(e^{-x}/x)+{}^3b_T^-(e^{-x}/x)^2]S_{12} \\
&+\mu G_{LS}^-(e^{-x}/x)[a_{LS}^-(e^{-x}/x) \\
&+b_{LS}^-(e^{-x}/x)^2](\boldsymbol{L}\cdot\boldsymbol{S}), \qquad (2)
\end{aligned}
$$

where μ=pion mass, $x=\mu r$, and $(g^2/4\pi)$ =effective pion-ncleon coupling constant. We have also assumed a hard core of radius ${}^1x_0^+$ and ${}^3x_0^-$ for (1) and (2), respectively.

The singlet even potential (1) with the parameters given in (4) can reproduce the zero energy scattering parameters and at the same time ${}^1S_0$ phase shift of MacGregor et al.'s solution 1 (MMS 1) at 310 Mev.[1] This potential, however, gives too large ${}^1D_2$ phase shifts above 100 Mev. The difficulty has been overcome by adding to (1) a weak quadratic spin-orbit potential (Q-potential)

$${}^1V_Q^+(x)Q_{12}=-0.0054\,\mu(e^{-x}/x^3)Q_{12},$$

where*

$$Q_{12}=1/2\cdot[(\boldsymbol{L}\cdot\boldsymbol{\sigma}_1)(\boldsymbol{L}\cdot\boldsymbol{\sigma}_2)+(\boldsymbol{L}\cdot\boldsymbol{\sigma}_2)(\boldsymbol{L}\cdot\boldsymbol{\sigma}_1)].$$

There is some pion theoretical evidence for a weak, although long ranged, Q-potential.[2] We did not find such a force useful for the triplet odd potential (2).

The best over-all fit so far to the experimental data has been attained for the following values of parameters:

$$(g^2/4\pi)=0.08,\ {}^1x_0^+=0.343,\ {}^3x_0^-=0.32,$$
$${}^1a_C^+=10,\ {}^1b_C^+=9,\ {}^3a_C^-=-9,\ {}^3b_C^-=5,$$

* $Q_{12}=-l(l+1)$ for the singlet states.

$^3a_T^- = -1.2$, $^3b_T^- = 0.35$, $G_{LS}^- = -0.1474$,
$a_{LS}^- = -0.75$, $b_{LS}^- = 7.9$. (4)

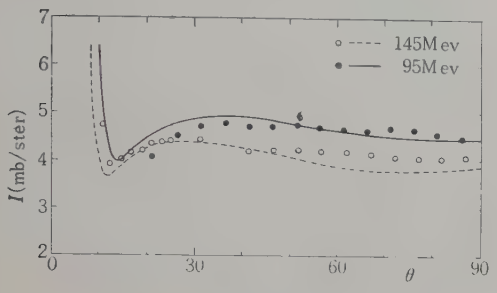

Fig. 1.

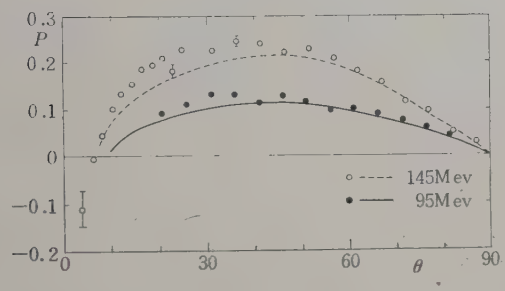

Fig. 2.

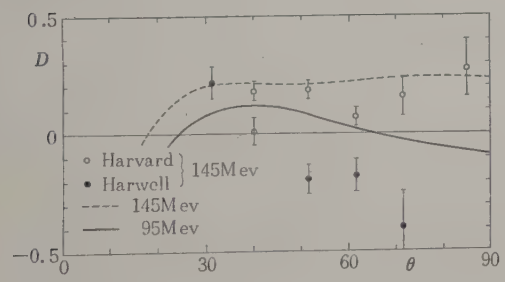

Fig. 3.

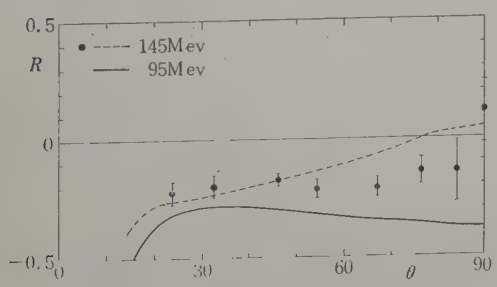

Fig. 4.

As an illustration we show in Figs. 1 to 4 the observable quantities calculated from (1) to (4) at 95 and 145 Mev. Experimental points are taken from Palmieri et al.,[3] Hwang et al.,[4] and Bird et al.[5] At lower energies the fit to the experimental data is quite satisfactory. At 310 Mev, on the other hand, the phase shifts calculated from the proposed potential are close to MMS 1 as shown in Table I.

Table I. Phase shifts at 310 Mev. Entries are the nuclear Blatt-Biedenharn phase shifts in radians.

	calculated from the potential (1) to (4)	MMS 1
$^1S_0$	−0.158	−0.156
$^1D_2$	0.244	0.207
$^1G_4$	0.026	0.013
$^1I_6$	0.007	
$^3P_0$	−0.198	−0.197
$^3P_1$	−0.459	−0.480
$^3F_3$	−0.069	−0.062
$^3H_5$	−0.022	−0.020
$^3P_2$	0.309	0.291
$^3F_2$	0.007	0.020
ϵ_2	−0.101	−0.051
$^3F_4$	0.080	0.056
$^3H_4$	0.001	0.006
ϵ_4	−0.367	−0.215
$^3H_6$	0.015	0.004
$^3K_6$	−0.006	
ϵ_6	−0.739	

A full account, with some possible refinements, will be published shortly.

1) M. H. MacGregor, M. J. Moravcsik and H. P. Stapp, Phys. Rev. **116** (1959), 1248.
2) S. Okubo and R. E. Marshak, Ann. Phys. **4** (1958), 166.
3) J. N. Palmieri, A. M. Cormack, N. F. Ramsey and R. Wilson, Ann. Phys. **5** (1958), 299.

4) C. F. Hwang, T. R. Ophel, E. H. Thorndike, R. Wilson and N. F. Ramsey, Phys. Rev. Letters **2** (1959), 310.
5) L. Bird, D. N. Edwards, B. Rose, A. E. Taylor and E. Wood, Phys. Rev. Letters **4** (1960), 302.

Non-Static Effect in the $T=0$ Two-Nucleon Interaction

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None of the solutions of the phase shift analysis[1] of the p-p scattering data at 310 Mev can be understood in terms of the potentials expected on the basis of the *static* pion theory.[2]* It seems that the non-static effects are of decisive importance at this energy as far as the $T=1$ states are concerned. The strong $(\boldsymbol{L}\cdot\boldsymbol{S})$ force found necessary by Gammel and Thaler[5] to fit the 310 Mev p-p data could be a reflection of such effects.

The situation is not clear in the $T=0$ states. Gammel and Thaler[6] found that a similar, though slightly weaker, $(\boldsymbol{L}\cdot\boldsymbol{S})$ potential was required to fit the 300 Mev n-p data. This conclusion has, however, been withdrawn later.[7] De Swart et al.[8] argue, in connection with the question raised by Feshbach[9] concerning the deuteron magnetic moment, that the $(\boldsymbol{L}\cdot\boldsymbol{S})$ potential may not be indispensable in the $T=0$ states. Their argument is based on the n-p data below 150 Mev. The pion theory, on the other hand, seems to predict the $(\boldsymbol{L}\cdot\boldsymbol{S})$ potential of the correct sign in the $T=1$ state but it fails to give even the sign of the $(\boldsymbol{L}\cdot\boldsymbol{S})$ potential uniquely in the $T=0$ state.[10]

In this note I would like to point out that the non-static effect must be important also in the $T=0$ state at 300 Mev. The argument is based on the experimental n-p total cross section at 300 Mev, $\sigma_{np}=35$ mb.[11] The contribution to this from the $T=1$ states may be estimated from the phase shifts of MacGregor et al.[1] One finds 11.7 mb and 10.4 mb for their solution 1 and 2, respectively.

In order to estimate the $T=0$ contribution, the triplet even phase shifts have been calculated for the potentials considered by the writer recently in connection with the deuteron problem.[12] These potentials consist of the one-pion-exchange tail and a variety of phenomenological inner potentials. No $(\boldsymbol{L}\cdot\boldsymbol{S})$ forces are included. For any potential which fits the deuteron properties reasonably well, it was found that the ${}^3D_2$ phase shift exceeds 0.78 at 300 Mev. This gives the contribution of at least 20.7 mb to σ_{np} leaving at most only $3\sim4$ mb for all other $T=0$ states to contribute, which is definitely too small.

* The parameter R has been measured recently by Bird et al.[3] at 142 Mev. Their results are in disagreement with those predicted from the potential considered in reference 2). This seems to indicate that the non-static effects are already appreciable at this energy. See also a recent work by Nigam[4] in this connection.

In fact the calculated σ_{np} already exceeds 35 mb even without any contribution from the singlet odd states. The same difficulty can be noticed in an earlier work by Gammel, Christian and Thaler.[13]

The argument presented above indicates the importance of non-static effects in the $T=0$ states at 300 Mev. If such effects are to be represented by the $(\boldsymbol{L}\cdot\boldsymbol{S})$ potential,* it must be repulsive in the ${}^3D_2$ state in agreement with the choice of Signell-Marshak[14] and also of Gammel-Thaler.[6] It does not agree with the prediction of the ps-ps pion theory as calculated by Sugawara and Okubo.[10]

1) M. H. MacGregor, M. J. Moravcsik and H. P. Stapp, Phys. Rev. **116** (1959), 1248.
2) T. Hamada, J. Iwadare, S. Otsuki, R. Tamagaki and W. Watari, Prog. Theor. Phys. **22** (1959), 566; **23** (1960), 366.
3) L. Bird, D. N. Edwards, B. Rose, A. E. Taylor and E. Wood, Phys. Rev. Letters **4** (1960), 302.
4) B. P. Nigam, Prog. Theor. Phys. **23** (1960), 61.
5) J. L. Gammel and R. M. Thaler, Phys. Rev. **107** (1957), 291.
6) J. L. Gammel and R. M. Thaler, Phys. Rev. **107** (1957), 1337.
7) R. J. N. Phillips, *Reports on Progress in Physics*, **XXII** (1959), 562 (Physical Society, London).
8) J. J. De Swart, R. E. Marshak and P. S. Signell, Nuovo Cimento **6** (1957), 1189.

* There is some evidence that one of the non-static effects is to reduce the effective pion-nucleon coupling constant at high energies.[10),15)] Such an effect serves to reduce the ${}^3D_2$ phase shift at high energies. It seems, however, that this effect alone is too small to account for the required reduction in the ${}^3D_2$ phase shift. I am indebted to Professor Sugawara for an interesting comment on this point.

9) H. Feshbach, Phys. Rev. **107** (1957), 1626.
10) M. Sugawara and S. Okubo, Phys. Rev. **117** (1960), 605, 611.
11) W. H. Hess, Rev. Mod. Phys. **30** (1958), 368.
12) T. Hamada, Prog. Theor. Phys. to be published.
13) J. L. Gammel, R. S. Christian and R. M. Thaler, Phys. Rev. **105** (1957), 311.
14) P. S. Signell and R. E. Marshak, Phys. Rev. **109** (1958), 1229.
15) P. S. Signell, Prog. Theor. Phys. **22** (1959), 492.

Note on the Radiative Corrections for the Electron-Electron Scattering

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The current quantum field theory is a local one which neglects the inner structure of elementary particles. In nature there are unrenormalizable interactions such as weak decay interactions. From these facts many physicists believe the existence of unapplicable regions of the theory at small distances at which the structure effects of elementary particles will appear. Much effort has been spent in order to get an experimental verification of the existence of this limitation. But there seems to have

been achieved no such experimental evidence so far.

Many physicists seem to believe that the present theory will break down at small distances, possibly about 0.1 *yukawa* (1 *yukawa*$=10^{-13}$ cm) or smaller. Therefore it is worth noting the electron-electron scattering experiment from several hundred Mev to 1 Bev in the *c.m.s.* which is scheduled to be carried out at Stanford.[1] The radiative corrections for the scattering cross section become more important when the energy resolution of the electrons gets smaller and the energy of the electrons gets larger. In this note we shall reexamine the α^3 corrections for this cross section at high energies.

Many years ago Schwinger[2] conjectured a form for the radiative corrections to the electron scattering in a Coulomb field. According to him, the total cross section, σ, is proportional to the elastic scattering cross section, σ_{el}, so that we have

$$\sigma = e^{-\delta}\sigma_{el} \tag{1}$$

with the fractional decrease δ being given asymptotically in the high energy limit by

$$\delta \simeq \frac{4\alpha}{\pi}\log\frac{2E}{m}\sin\frac{\theta}{2}\log\frac{E}{\Delta E}, \tag{2}$$

where α is the fine structure constant, m the electron mass, E the electron energy, θ the scattering angle and ΔE is the energy resolution associated with the experiment (in the *c.m.s.*). The expression (2) is the leading term of δ, so that the neglected terms which are proportional to α, are either linear in $\log(E/\Delta E)$ and $\log(E/m)$ terms or constants of order unity depending only on the scattering angle θ. σ_{el} has no $\alpha(\log(E/m))^2$ term. The ΔE dependence of this conjecture was verified by Yennie and Suura.[3]

Several years ago, Redhead[4] calculated the α^3 corrections for the electron-electron scattering in the laboratory system. His result is given asymptotically in the high-energy limit by

$$\sigma \simeq \sigma_M\left\{1+\frac{\alpha}{\pi}\left[4\left(1-2\log\frac{E}{m}\sin\theta\right)\times\log\frac{E}{\Delta E^*}+6\left(\frac{14}{9}-\log\frac{E}{2m}\sin^2\theta\right)\times\log\frac{2E}{m}\right]\right\}, \tag{3}$$

for $E\sin\theta \gg m$ and $E \gg \Delta E^*$, where σ_M is Møller's cross section, E and θ are defined in the *c.m.s.* but only ΔE^* in the laboratory system. In the expression (3) the neglected terms are constants of order α/π depending only on the scattering angle θ. Suura[5] has shown that the $\alpha(\log m)^2$ terms are contributions from soft-photon processes and must be canceled out in the *c.m.s.* The expression (3) has both $\log(E/m)\times\log(E/\Delta E^*)$ and $(\log(E/m))^2$ terms. At first glance, Redhead's result seems inconsistent with Schwinger's conjecture and Suura's arguments and many people seem to doubt his result.

The authors recalculated the α^3 corrections for the electron-electron scattering and got the same result as that of Redhead. Why does the $\alpha(\log(E/m))^2$ term appear in the expression (3)? The reason is simple. σ is the cross section

for the electron-electron scattering in the laboratory system and not that in the *c.m.s.* On the other hand, Schwinger's conjecture and Suura's arguments hold only in the *c.m.s.* If we calculate the cross section in the *c.m.s.* in fact we get

$$\sigma \simeq \sigma_M \left\{1+\frac{\alpha}{\pi}\left[4\left(1-2\log\frac{E}{m}\sin\theta\right)\times\log\frac{E}{\Delta E}+\frac{22}{3}\log\frac{2E}{m}\right]\right\}, \quad (4)$$

in which no $(\log(E/m))^2$ term appears.

As was shown by expressions (3) and (4), the radiative corrections for the electron-electron scattering are very large even at $E\simeq 1$ Bev and the main contribution for the radiative corrections comes from real and virtual soft-photon processes. It will be difficult to calculate the theoretical cross section at $E\simeq 1$ Bev with the required accuracy.[1] In order to do this, further investigations must be carried out.

All scattering processes in Q.E.D., more or less, have this difficulty associated with soft-photon processes when applied to testing the validity of Q.E.D. Bound state problems such as the energy levels of the hydrogen atom, on the other hand, do not suffer from this difficulty. It may be an interesting problem to investigate the size effect for the energy levels of the μ-mesic hydrogen. The levels depend sensitively on the structure of the proton and muon. From both the experiments on the energy levels and the electron-proton scattering at Stanford,[6] it may be possible to determine the difference between the structure of the muon and the electron.*

The authors would like to thank Prof. H. Yukawa for his many suggestions and continuous encouragement. They are also much indebted to Prof. S. Hayakawa for his helpful comments.

1) W. K. H. Panofsky, *Proceedings of the Ninth Conference on High Energy Nuclear Physics at Kiev* (1959).
W. C. Barber, B. Richter, W. K. H. Panofsky, G. K. O'Neill and B. Gittelman, unpublished paper (1959). One of the authors (K. H.) is grateful to Prof. H. Kumagai for his kindness who lended him this paper.
2) J. Schwinger, Phys. Rev. **76** (1949), 790.
3) D. R. Yennie and H. Suura, Phys. Rev. **105** (1957), 1378.
4) M. L. G. Redhead, Proc. Roy. Soc. **220** (1953), 219.
5) H. Suura, Phys. Rev. **99** (1955), 1020.
6) R. Hofstadter, F. Bumiller and M. R. Yearian, Rev. Mod. Phys. **30** (1958), 482.

* This possibility was pointed out by one of the authors (K. H.) and T. Shiozaki at the session on the muon which was held December 1959 at the Research Institute for Fundamental Physics, Kyoto University.

Note on the Radiative Correction to Electron-Electron Collision

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The radiative correction to electron-electron collision to order α was calculated by Redhead[1] and later by Polovin,[2] who confirmed the result of the former author. Their result has a very remarkable feature compared to

the Schwinger correction[3] to electron scattering by Coulomb field. Namely, besides the terms of the form

$$-\alpha/\pi \cdot \ln(Ec/m) \ln(Ec/\Delta E), \quad (1)$$

which is the leading term of the Schwinger correction, the correction to electron-electron scattering contains terms of the type

$$-\alpha/\pi \cdot (\ln(Ec/m))^2. \quad (2)$$

In (1) and (2), m is the electron mass and E_c is the electron energy in the center-of-mass system. ΔE is the maximum energy loss of the scattered electrons. The very existence of terms like (2) would make perturbation calculation of quantum electrodynamics almost meaningless, because (2) increases indefinitely with increasing energy. The correction of the type (1), on the other hand, leads to no serious problem, because including higher order corrections one obtains a bounded, well behaved expression.[4]

The term like (2) is not unfamiliar in quantum electrodynamics. In fact, in the calculation of the Schwinger correction to potential scattering, one obtains such terms for both elastic and inelastic cross sections, if one assumes a finite photon mass to avoid the infrared divergence. However, in the total cross section these terms cancel out completely. The cancellation does not occur accidentally. It happens because these terms come from contributions of real and virtual photons of very small momenta, and hence must necessarily vanish in the total cross section, just as does the infrared divergence.[5] This argument is quite general and should be applicable also for large angle electron-electron scattering. Then the existence of (2) should be due to photons of higher momenta. However, there is a strong reason to suspect that the terms like (2) in Redhead's result do come from contributions of soft quanta. For if one adds up all $(\ln m)^2$ terms in the cross section of Redhead for large angle scattering in relativistic limit, one finds that the coefficient of $(\ln m)^2$ is just the Møller cross section. This situation would happen if the term is contributed by soft quanta, since emission and absorption of soft quanta occur in the region of Coulomb tail, hence will not affect the actual scattering process which happens at shorter distance. Thus the existence of (2) is very much puzzling.

The key to solve the problem is obtained if one recognizes the fact that the inelastic cross section for producing a photon of maximum energy ΔE is not an invariant concept. Redhead evaluates $d\sigma_{inel}$ in the laboratory system although the result is expressed in terms of E_c, which is an invariant quantity. One can obtain the correct total cross section in the center-of-mass system from that in the laboratory frame by just modifying the inelastic part of the total cross section of Redhead. Thus

$$d\sigma_{tot}^{cm} = d\sigma_{tot}^{lab} + d\sigma_{inel}^{cm} - d\sigma_{inel}^{lab}. \quad (3)$$

The result of this modification is

$$d\sigma_{tot}^{cm} = \frac{e^4 \sin\theta \, d\theta}{8\pi E_c^2} \cdot \frac{1}{\chi^2(1-\chi)}$$
$$\times \left\{ 1/2(\cdot 2 - 3\chi + 3\chi^2 - \chi^3) \right.$$
$$\times [1 + 4\alpha/\pi \cdot (1 - 2w + 2u - 2v)$$

$$\times \ln(E_c/2\Delta E) + \alpha/\pi \cdot (-46/9 - 2u + (28/3)w - 6(u-w)^2 - 4(u-v)^2)]$$
$$+ \alpha/\pi \cdot [(2-\chi+\chi^2)(u-w) - \chi(u-v) + (2-\chi)(u-w)^2 - (2\chi-\chi^2)(w-v)^2 - (1/4)\pi^2\chi(2-\chi)]\}$$

+terms obtained by the interchange

$$\chi \leftrightarrow 1-\chi \text{ and } v \leftrightarrow w, \qquad (4)$$

where

$$u = \ln(Ec/m \gg 1,$$
$$w = \ln(Ec \sin\theta/m) \gg 1,$$
$$v = \ln(Ec \cos\theta/m) \gg 1,$$

and χ is given by

$$\chi = \sin^2(\theta/2).$$

θ is the scattering angle in the center-of-mass system. It should be noticed that terms quadratic in u, v, and w enter only in the form of square of their differences, which are of order unity. Thus, there appears no term like (2). Neglecting the logarithmic angular dependence, the asymptotic form of the cross section in extreme high energy limit is given by

$$d\sigma_{tot}^{cm} \sim d\sigma_{M\phi ller}(1 - 8\alpha/\pi \cdot \ln(E_c/m) \times \ln(E_c/2\Delta E)).$$

The reason for the appearance of (2) in the laboratory frame is not completely clear. However, it is not very surprising because expressing the energy of particles in the laboratory system in terms of E_c, one introduces additional factor m other than those coming from denominators of electron propagators. In any case the concept of the cross section for scattering with maximum energy loss ΔE in the laboratory system would lose its actual physical meaning in the energy region $E_c/m \gg 1$, because then the scattering occurs almost in the forward direction.

1) M. L. G. Redhead, Proc. Roy. Soc. **220A** (1953), 219.
2) R. V. Polovin, J.E.T.P. **31** (1956), 449.
3) J. Schwinger, Phys. Rev. **76** (1949), 790.
4) D. R. Yennie and H. Suura, Phys. Rev. **105** (1957), 1378.
5) H. Suura, Phys. Rev. **99** (1955), 1020.

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On the Problem of Unsaturated Helium Films

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A theoretical explanation is given for the lowering of the transition temperature, and the increase of the entropy, with decreasing film thickness as met with in unsaturated helium films. It is shown that in the ideal gas approximation the boundary conditions obeyed by the eigenfunctions assume rather an important role. There is, however, no quantitative agreement between the experimental and the theoretical results even when the dependence of the film density on the thickness of the film is taken into account. On the other hand, if one considers the non-ideal gas approximation for the liquid and takes into account (i) the slow variation of the parameters Δ and μ with the film thickness, and (ii) the contributions arising from the presence of surface tension waves on the liquid surface, the experimental data are explained in a good quantitative manner. In this case, however, the boundary conditions are found to become rather unimportant.

§ 1. Introduction

Experimental investigations into the properties of unsaturated helium films[1] have shown that the transition from helium I to helium II takes place at a temperature lower than that for the bulk and this transition temperature decreases monotonically with the decreasing film thickness. Further, the entropy of such films is found to be larger than the bulk value. Whereas the properties of liquid helium in bulk have been investigated rather in detail, the behaviour of thin films is comparatively less studied. Particularly, the monotonic variation of the film properties with its thickness has remained almost unexplained.

Recently it was suggested by the authors[2] that such a behaviour of the thin films could be understood if one enumerated the eigenfunctions in the bounded continuum more exactly than is ordinarily done (by taking into account the surface term also). In the present communication we have investigated this problem in greater detail. In Sections 2 and 3 we have studied the ideal gas problem by (i) carrying out the summation over states as such, so as to avoid the divergence difficulty met with in reference 2), and (ii) taking into account the variation of the specific volume of the fluid with film thickness. We find that if one tries to understand the experimental data by assuming the ideal gas approximation for liquid helium, the boundary conditions assume rather an important role, i.e., one is obliged to assume the boundary condition $\partial\psi/\partial n=0$. In the next section we use the non-ideal Bose gas spectrum characterised by an energy gap Δ and an effective mass μ. It is found that the experimental data can be quantitatively un-

derstood if these parameters are taken to be slowly varying functions of the density in the film. However, in this case, the variation of the parameter μ comes out to be opposite to that required for understanding the density dependent properties of the bulk liquid.[3] Next, we show that the contribution of the surface tension waves towards the entropy of such thin films is quite important and when taken into account leads to a proper density dependence of the parameters concerned.

§ 2. The ideal gas approximation

It is well known that in a bounded continuum of volume V and surface area S, the number of eigenfunctions with momentum lying in the interval p and $p+dp$ is given by[4]

$$a(p)\,dp = V\,\frac{4\pi g}{h^3}\,p^2\,dp\left[1+\theta\left(\frac{S}{V}\right)\cdot\frac{h}{8p}+\cdots\right], \tag{1}$$

where g is the relevant weight factor and θ depends upon the boundary conditions obeyed by the eigenfunction; $\theta=-1$ for ψ vanishing at the boundary and $\theta=+1$ for $\partial\psi/\partial n$ (n being the normal to the surface) vanishing at the boundary; for intermediate boundary conditions, which might arise in certain special cases, θ would assume values between the above two limits.

Working with expression (1) in order to calculate the number of particles in the excited states of a degenerate Bose gas we found in reference 2) that the integral over the correction term diverges. Making a crude approximation of replacing p in the correction term by its average value, we got for the variation of the transition temperature with film thickness the relation

$$1=\frac{v}{v_0}(r)^{3/2}\left[1+\theta\,\frac{0.352}{t}\right], \tag{2}$$

where v and v_0 are respectively the specific volumes in the film and the bulk and $r[=T/T_0]$ is the ratio of the transition temperature for the film to that for the bulk. Here, t is expressed in units of the mean thermal wavelength of the particles.

The divergence difficulty mentioned above arose due to our having replaced the summation over states by integration. To be exact, the number of particles should have been calculated by the relation

$$N=\sum_{n_x=1}^{\infty}\sum_{n_y=1}^{\infty}\sum_{n_z=1}^{\infty}\left[\exp\left\{\frac{\lambda^2}{4}\left(\frac{n_x^2}{l^2}+\frac{n_y^2}{b^2}+\frac{n_z^2}{t^2}\right)\right\}-1\right]^{-1}, \tag{3}$$

where

$$\lambda=[h^2/2mkT]^{1/2}$$

is the mean thermal wavelength and l, b and t are respectively the length, the breadth and the thickness of the cuboidal assembly. The lower limits of n_x, n_y

and n_z as taken above correspond to the case $\psi=0$ at the boundary. The upper limits, to be more exact, should have been l/l_0, b/l_0 and t/l_0, respectively, rather than infinite (where l_0 is a parameter of the order of the interatomic distance). The correction arising out of this has been calculated by S. Franchetti[5] and is found to be almost negligible. Further, whereas the summation over n_x and n_y may be replaced by integration, this is not justified for the summation over n_z (because of the smallness of t). Thus we get[6]

$$N=\frac{\pi lb}{\lambda^2}\sum_{m=1}^{\infty}\left[\sum_{n_z=1}^{\infty}\frac{1}{m}\exp\left(-\frac{mn_z^2\lambda^2}{4t^2}\right)\right], \tag{4}$$

$$=\frac{\pi lb}{\lambda^2}\ln\left[\prod_{n=1}^{\infty}\left\{1-\exp(-\lambda^2n^2/4t^2)\right\}^{-1}\right],$$

$$=V\left[\frac{2\pi mkT}{h^2}\right]^{3/2}\zeta(3/2)\left[1-\frac{\lambda}{\sqrt{\pi}\,\zeta(3/2)\,t}\cdot\ln(4\pi t/\lambda)+\cdots\right]. \tag{5}$$

As indicated above, this relation has been derived for $\theta=-1$; we may generalize it and write for the number of excited particles per gramme at the transition temperature T of the film

$$n=v\left[\frac{2\pi mkT}{h^2}\right]^{3/2}\zeta(3/2)\left[1+\theta\left\{\frac{\lambda}{\sqrt{\pi}\,\zeta(3/2)\,t}\cdot\ln(4\pi t/\lambda)\right\}\right], \tag{6}$$

whence one gets

$$1=\frac{v}{v_0}r^{3/2}\left[1+\theta\frac{\lambda}{\sqrt{\pi}\,\zeta(3/2)\cdot t}\ln(4\pi t/\lambda)\right]. \tag{7}$$

Similarly, for Helmholtz free energy per gramme we have (see Appendix I)

$$F=-v\left[\frac{2\pi mkT}{h^2}\right]^{3/2}\zeta(5/2)\,kT\left[1+\theta\,\frac{\lambda}{t}\,\frac{\zeta(2)}{2\sqrt{\pi}\,\zeta(5/2)}\right]. \tag{8}$$

Thus the specific entropy of the film is

$$S=\frac{5}{2}\,v\left[\frac{2\pi mkT}{h^2}\right]^{3/2}\zeta(5/2)\,k\left[1+\theta\,\frac{\lambda}{t}\cdot\frac{2}{5}\cdot\frac{\zeta(2)}{\sqrt{\pi}\,\zeta(5/2)}\right]. \tag{9}$$

Assuming for the present that the specific volume of the film is the same as that in the bulk, it is easily seen from Eqs. (7) and (9) that the lowering of the transition temperature and the rise of the entropy with the decrease of film thickness can only be understood if θ is taken to be positive. In the next section we investigate the contribution arising from the increased density in the film towards explaining the above properties.

§ 3. Variation of the specific volume with the film thickness

As the film is formed under the influence of the attractive forces of the solid wall, it may be considered effectively under a high pressure. Consequently, the

mean density in the film will be higher than that in bulk. This effect has been calculated by S. Franchetti[7] who gives

$$\frac{v-v_0}{v_0}=\frac{-\Delta t}{t+\Delta t}, \tag{10}$$

where Δt is the contraction which the film of thickness t has suffered under the influence of the attractive forces. As a reasonable estimate for the case under investigation we take 0.8×10^{-8} cm as a value of Δt and in order to simplify matters, we neglect its small dependence on the temperature.

Now, the mean density of the film will also be affected by the dependence of the excitation energy and the zero point energy on its thickness. Whereas the effect of the former is shown to be negligible (see Appendix II), that of the latter is given by Atkins[8] as

$$\frac{v-v_0}{v_0}=-\frac{0.133\lambda_c}{t}, \tag{11}$$

where λ_c is the cutoff wavelength of the longitudinal Debye modes in the bulk liquid. The value of λ_c may be derived from the specific heat data for bulk liquid helium below 0.6°K, where these modes are the only dominant excitations. Its value comes out to be

$$\lambda_c=5.75\times10^{-8}\text{ cm}, \tag{12}$$

whence

$$\frac{v-v_0}{v_0}=\frac{-0.765\times10^{-8}}{t}. \tag{13}$$

Adding this to Eq. (10) we have

$$\frac{v-v_0}{v_0}=-\left[\frac{0.8}{t+\Delta t}+\frac{0.765}{t}\right]\times10^{-8}. \tag{14}$$

This gives

$$\frac{1}{v_0}\frac{\partial v}{\partial t}=\left[\frac{0.8}{(t+\Delta t)^2}+\frac{0.765}{t^2}\right]\times10^{-8}\text{ cm}^{-1}, \tag{15}$$

which for a film a few atoms thick is of the order of 10^7 cm^{-1}. Now by taking into account this variation of the specific volume with film thickness, it can again be seen from Eqs. (7) and (9) that the proper dependence of the entropy and the transition temperature on the film thickness is explicable only if $\partial\psi/\partial n$ vanishes at the boundary. There is, however, no quantitative agreement between the theoretical and the experimental results. In order to expect a quantitative agreement one should include the effect of the interaction forces.

§ 4. The non-ideal Bose gas

Next, we consider the non-ideal Bose gas characterised by an energy gap Δ

between the ground state and the excited states, and the effective mass μ. In this case the energy-momentum relation is

$$\varepsilon=\frac{p^2}{2\mu}+\Delta, \tag{16}$$

where, for the case of the bulk, $\Delta_0/k=8.8°\mathrm{K}$ and $\mu_0=9.1$ times the mass of the helium atom. For these values of the parameters[9] the transition temperature, as well as the entropy at the λ-point for the bulk liquid helium agrees with those observed experimentally. Working with this spectrum and making use of Eq. (1), the number of excited particles per gramme is given by

$$n=v\left[\frac{2\pi\mu kT}{h^2}\right]^{3/2}\exp[-\Delta/kT]\left[1+\frac{\theta}{t}\left(\frac{h^2}{8\pi\mu kT}\right)^{1/2}+\cdots\right]. \tag{17}$$

Also

$$n=\frac{v}{\mu}\cdot\rho_n.$$

Hence

$$\frac{\rho_n}{\rho}=8.781\times10^{55}v\mu^{5/2}T^{3/2}\exp[-\Delta/kT]\left[1+\frac{\theta}{t}\frac{11.25\times10^{-19}}{\sqrt{\mu T}}\right]. \tag{18}$$

Further, for the Helmholtz free energy one gets

$$F=-v\left[\frac{2\pi\mu kT}{h^2}\right]^{3/2}kT\exp[-\Delta/kT]\left[1+\frac{\theta}{t}\left(\frac{h^2}{8\pi\mu kT}\right)^{1/2}+\cdots\right], \tag{19}$$

whence the specific entropy comes out to be

$$S=2.897\times10^{32}v\mu^{3/2}T^{3/2}[5/2+\Delta/kT]\exp[-\Delta/kT]$$
$$\times\left[1+\frac{\theta}{t}\frac{(2+\Delta/kT)}{(5/2+\Delta/kT)}\cdot\frac{11.25\times10^{-19}}{\sqrt{\mu T}}\right]\text{cal. gm.}^{-1}\text{ deg.}^{-1}. \tag{20}$$

Let us assume that the energy gap Δ and the effective mass μ depend upon the density in the film through the relations

$$\mu=\mu_0\left[1+\alpha\frac{(v-v_0)}{v_0}\right], \tag{21}$$

and

$$\Delta=\Delta_0\left[1+\beta\frac{(v-v_0)}{v_0}\right], \tag{22}$$

where α and β are some constants expected to be of the order of unity. Then substituting these equations into Eq. (18) and (20) one gets

$$\frac{\rho_n}{\rho}=2.507vx^{5/2}T^{3/2}\exp[-\Delta_0 y/kT]\left[1+\frac{\theta}{t}\frac{1.446}{\sqrt{xT}}\right], \tag{23}$$

and

$$S=0.137\,vx^{3/2}T^{3/2}\exp[-\Delta_0 y/kT][5/2+\Delta_0 y/kT]$$
$$\times\left[1+\frac{\theta}{t}\frac{(2+\Delta_0 y/kT)}{(5/2+\Delta_0 y/kT)}\cdot\frac{1.446}{\sqrt{xT}}\right]\text{cal. gm}^{-1}\text{. deg}^{-1}. \tag{24}$$

where $x=\mu/\mu_0$ and $y=\Delta/\Delta_0$. We shall now adjust the values of x and y, and hence of α and β, from the above two equations by observing (i) that the maximum of C_v-T curve for a film about five atoms thick ($t=18$Å) occurs at the transition temperature $T=1.95$°K and (ii) that the specific entropy of this film at 1.4°K is 0.072 cal. gm.$^{-1}$ deg.$^{-1}$.

In order to calculate the contribution of Eq. (24) towards this entropy at 1.4°K we observe that the phonon contribution at any temperature for a film of thickness t is given by the relation

$$s_{ph}=3.25\times10^{9}\frac{vT^3}{c^3}\left[1+\frac{\theta}{t}\frac{c}{T}4.98\times10^{-12}\right]\text{cal. gm.}^{-1}\text{ deg.}^{-1}, \tag{25}$$

where c is the velocity of sound which for any thickness may be calculated by using Atkins' result[10]

$$\left[\frac{\partial c}{\partial v}\right]_{p=0}=-7.61\times10^{3}\text{ C.G.S. units.}$$

For $t=18\times10^{-8}$ cm and $T=1.4$°K one gets

s_{ph} [cal. gm^{-1}. deg^{-1}.]	θ
0.004	+1
0.003	0
0.001	−1

so that subtracting from the total entropy we get for the contribution from Eq. (24) at $t=18$Å and $T=1.4$°K

s_r [cal. gm^{-1}. deg^{-1}.]	θ
0.068	+1
0.069	0
0.071	−1.

(26)

Moreover it is known that the phonon contribution to ρ_n/ρ at the transition temperature is quite negligible as compared with the roton contribution at this temperature. Thus in order to get α and β we are to solve the two simultaneous equations: (i) Eq. (23) with $\rho_n/\rho=1$ and (ii) Eq. (24) with the left-hand side given by the values of the roton entropy as listed above. We get the following values of the constants α and β

α	β	θ
1.18	2.45	+1
1.18	2.65	0
1.26	2.88	−1.

It is seen that both the constants are positive and thus the effective mass as well as the energy gap decreases with increasing density. However, Z. Mikura has shown[3] that if one uses such an energy spectrum in order to understand the density dependent properties of liquid helium II in bulk, the experimental data are explicable only if the density dependence of the parameter μ is taken opposite to that found above. In the next section we take into account the contributions arising from the presence of surface tension waves on the liquid surface and find that these contributions become quite important for the film thickness under consideration.

§ 5. The surface waves

The phase velocity of surface waves present on a film of thickness t is given by the relation[11]

$$v_\omega^2=\left[\frac{3\gamma\lambda}{2\pi t^4}+\frac{2\pi\sigma}{\rho\lambda}\right]\tanh\left[\frac{2\pi t}{\lambda}\right], \tag{27}$$

where λ is the wavelength, σ the surface tension, ρ the density and γ a constant connected with the Van der Waals interaction between the film and the wall. It is readily seen that for films having thickness more than about 10Å, a valid approximation is

$$v_\omega^2=\frac{2\pi\sigma}{\rho\lambda}. \tag{28}$$

Using this relation and remembering that the number of surface modes with wave number between K and $K+dK$ is equal to $2\pi KdK$, one gets for the specific entropy of such waves at temperatures close to the absolute zero

$$s_\omega=\frac{7\pi}{3\rho t}\left[\frac{\rho}{2\pi\sigma}\right]^{2/3}k\left[\frac{kT}{h}\right]^{4/3}\Gamma(7/3)\zeta(7/3), \tag{29}$$

$$=\frac{0.216\times10^{-8}}{t\rho^{1/3}}\ \text{cal. gm}^{-1}\text{. deg}^{-1}. \tag{29a}$$

For a film with $t=18$Å and $T=1.4$°K; $s_\omega=0.022$ cal. gm.$^{-1}$ deg.$^{-1}$ Subtracting this and the phonon contribution from the total entropy, we get for the contribution from Eq. (24)

s_r [cal. gm^{-1}. deg^{-1}.]	θ
0.046	+1
0.047	0
0.049	−1.

The inertial density ρ_n for such waves is given by the relation

$$\frac{\rho_n}{\rho}=\frac{2\pi}{3}\frac{h}{\rho t}\left[\frac{\rho}{2\pi\sigma}\right]^{4/3}\left[\frac{kT}{h}\right]^{5/3}\Gamma(8/3)\zeta(5/3), \tag{30}$$

and its contribution at the transition temperature is found to be negligible as compared to that of rotons. Consequently, we make use of Eq. (23) with the left-hand side equated to unity and Eq. (24) with the left-hand given by the values listed in (30) and get the following values of α and β

α	β	θ
-1.70	0.91	$+1$
-1.65	1.11	0
-1.53	1.32	-1.

Now, the theoretical density dependence of the parameters Δ and μ comes out to be qualitatively the same as that used by Z. Mikura. With these values of α and β and making use of Eqs. (23), (24), (25), (29a) and (14), one may now calculate the values of the specific entropy and the transition temperature for various film thicknesses. These values with $\theta=+1$, $\theta=0$ and $\theta=-1$ are plotted in Figs. 1 and 2. The curves for three different values of θ coincide, showing thereby that the corrections arising from the better enumeration of the eigen-functions, in this case, is not of much importance. Moreover, keeping in view the inaccuracies of the experimental data, there is a good quantitative agreement between the experimental and the theoretical results.

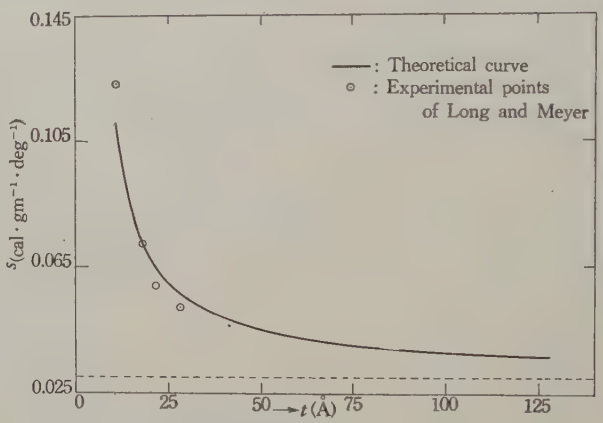

Fig. 1. The specific entropy s (in cal. gm^{-1}. deg^{-1}.) as a function of the film thickness t (in Å). The curves for the different values of θ coincide.

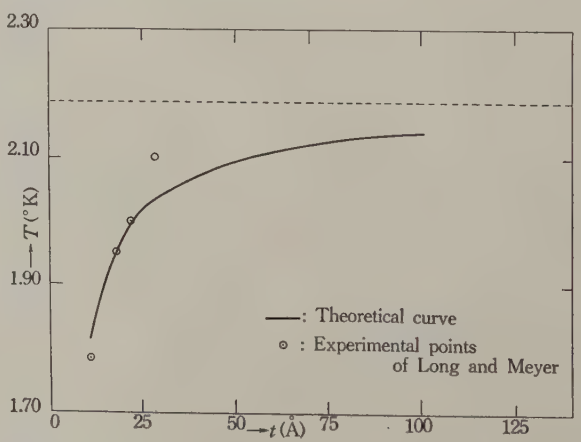

Fig. 2. The transition temperature T as a function of the film thickness t. The curves for the different values of θ coincide.

§ 6. Conclusion

From the above discussion it becomes clear that, when the variation of the specific volume of the film with its thickness is neglected, the ideal

gas approximation for liquid helium can explain the experimental results only if one invokes the boundary condition $\partial\psi/\partial n=0$ rather than $\psi=0$. This condition, however, seems difficult to understand, especially at the solid wall on which the film is formed.

Since, however, it is well known that even in the bulk liquid, the transition temperature is lowered and the entropy is raised when it is subject to a high pressure, one may expect that the observed phenomena in the case of films may be due to the increased film density. We find, however, that this effect in itself is insufficient to account for the experimental observations.

Next, when we consider the non-ideal gas approximation for the liquid, the increased density becomes more effective by influencing the values of the parameters Δ and μ. Moreover, in this case the contributions from the surface waves that are present on a liquid surface become quite important; if we neglect these contributions, then the theoretical variation of the parameter μ is in contradiction with what one expects from the pressure dependent behaviour of the bulk liquid. Further, the corrections arising from a better enumeration of the eigenfunctions do not play so important a part. It may also be noted that the properties of the film become appreciably different from those of the bulk only when its transverse dimension becomes less than about fifteen times the atomic distance.

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Appendix I

Here we derive the expressions for the free energy of an ideal (degenerate) Bose gas confined within a cuboidal assembly of length l, breadth b and thickness t, with the eigenfunctions obeying the general boundary conditions. In this case the total energy u is given by

$$u=\sum_{\varepsilon}\varepsilon[\exp(\varepsilon/kT)-1]^{-1}, \tag{1}$$

with the eigenvalues

$$\varepsilon=\frac{h^2}{8m}\left[\frac{(n_x+\eta)^2}{l^2}+\frac{(n_y+\eta)^2}{b^2}+\frac{(n_z+\eta)^2}{t^2}\right]. \tag{2}$$

Here, the quantum numbers n_x, n_y and n_z have the values $0, 1, 2 \cdots$ and η is a constant which depends upon the boundary conditions obeyed by the eigenfunctions. In particular, $\eta=1$ if the eigenfunction ψ vanishes at the boundary and $\eta=0$ if $\partial\psi/\partial n$ vanishes at the boundary; for intermediate boundary conditions η may assume values lying between these two limits.

Replacing the summation over n_x and n_y by integration, we get

$$u=\frac{\pi lbkT}{\lambda^2}\left[\frac{\lambda^2}{4t^2}\sum_{n_z=0}^{\infty}(n_z+\eta)^2\int_0^{\infty}\frac{dq}{\exp\{\lambda^2(n_z+\eta)^2/4t^2+q\}-1}\right.$$

$$\left.+\sum_{n_z=0}^{\infty}\int_0^{\infty}\frac{q\cdot dq}{\exp\{(\lambda^2/4t^2)(n_z+\eta)^2+q\}-1}\right], \tag{3}$$

where

$$q=\frac{\lambda^2}{4}\left[\frac{(n_x+\eta)^2}{l^2}+\frac{(n_y+\eta)^2}{b^2}\right], \tag{4}$$

and

$$\lambda=[h^2/2mkT]^{1/2} \text{ is the mean thermal wave length.}$$

In order to evaluate the summations coming in Eq. (3) we apply the method of Mellin Transforms.[12] For the first sum $\sum_1$ in (3) the Mellin Transform will be

$$F_1(s)=\int_0^{\infty}x^2\int_0^{\infty}\frac{dq}{\exp[b'x^2+q]-1}\cdot x^{s-1}dx=\frac{1}{2}b'^{-(s+2)/2}\Gamma\left(\frac{s+2}{2}\right)\zeta\left(\frac{s+4}{2}\right), \tag{5}$$

where

$$b'=\lambda^2/4t^2.$$

This gives for $\sum_1$

$$\sum_1=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}F_1(s)\zeta(s,\eta)\,ds. \tag{6}$$

The integrand has poles at $s=1$ and $s=-2$ with residua $\frac{1}{2}b'^{-3/2}\Gamma(3/2)\zeta(5/2)$ and $2\zeta'(-2,\eta)$, respectively. For the second sum $\sum_2$ in (3) the Mellin Transform $F_2(s)$ will be

$$F_2(s)=\int_0^{\infty}\int_0^{\infty}\frac{q\cdot dq}{\exp[b'x^2+q]-1}\cdot x^{s-1}dx=\frac{1}{2}b'^{-s/2}\Gamma(s/2)\zeta\left(\frac{s+4}{2}\right). \tag{7}$$

This gives for $\sum_2$

$$\sum_2=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}F_2(s)\zeta(s,\eta)\,ds. \tag{8}$$

The integrand has poles at $s=1$, $s=0$ and $s=-2$ with residua

$$\frac{1}{2}b'^{-1/2}\Gamma(1/2)\zeta(5/2)\,;\ \zeta(2)\zeta(0,\eta) \quad \text{and} \quad -2b'\zeta'(-2,\eta),$$

respectively. Combining these two results we get

$$u=\frac{\pi lbkT}{\lambda^2}[b'\sum_1+\sum_2]=\frac{\pi lbkT}{\lambda^2}\left[\frac{3}{4}b'^{-1/2}\sqrt{\pi}\,\zeta(5/2)+\zeta(2)\left(\frac{1}{2}-\eta\right)\right]$$

$$=v\cdot\frac{3}{2}\frac{\pi^{3/2}kT}{\lambda^3}\zeta(5/2)\left[1+(1-2\eta)\frac{\zeta(2)}{3\sqrt{\pi}\zeta(5/2)}\frac{\lambda}{t}\right], \tag{9}$$

or

$$u=v\cdot\frac{3}{2}\frac{\pi^{3/2}kT}{\lambda^3}\zeta(5/2)\left[1+\theta\frac{\zeta(2)}{3\sqrt{\pi}\zeta(5/2)}\frac{\lambda}{t}\right], \tag{10}$$

where $\theta=-1$ for $\eta=1$ and $\theta=+1$ for $\eta=0$. From Eq. (10) the free energy may be calculated by using the relation $F=-T\int u/T^2\cdot dT$. The free energy in the case of phonons may also be calculated in a similar way.

Appendix II

In this appendix we shall prove that the change in the effective density of the film caused by the thickness dependence of the excitation energy is almost negligible as compared with that caused by the dependence of the zero point energy on the film thickness.

Atkins has derived the following results for the dependence of the zero point energy of the Debye waves on the film thickness,

$$Z=Z_0\left[1-\frac{\lambda_c}{6t}\right], \tag{1}$$

$$=Z_0[1+\sigma_z], \tag{2}$$

where Z_0 is the value of Z for the bulk and

$$\sigma_z=-\frac{\lambda_c}{6t}=-\frac{0.765\times10^{-8}}{t}. \tag{3}$$

Making use of Eq. (1) of Section 2 it can be shown that the free energy per gramme of the phonons is given by

$$f_{ph}=f_{ph_0}\left[1+\frac{\theta}{t}\frac{0.14hc}{kT}\right], \tag{4}$$

$$=f_{ph_0}[1+\sigma_{ph}], \tag{5}$$

where again f_{ph_0} is the value of f_{ph} for the bulk liquid and $\sigma_{ph}=\theta/t\{0.14hc/kT\}$.

Similarly for the excitations of the form used in Section 3

$$f_{ex}=f_{ex_0}\left[1+\frac{\theta}{t}0.28\left(\frac{h^2}{2\mu kT}\right)^{1/2}\right], \tag{6}$$

$$=f_{ex_0}[1+\sigma_{ex}], \tag{7}$$

where

$$\sigma_{ex}=\theta\left(\frac{h^2}{2\mu kT}\right)^{1/2}\left(\frac{0.28}{t}\right). \tag{8}$$

Further, the free energy per gramme due to the presence of surface tension waves in the film surface is

$$f_w=-\left[\frac{1}{2\pi\sigma}\right]^{2/3}\cdot\frac{4\pi}{3}\,\frac{\Gamma(7/3)\zeta(7/3)}{t}\,kT\left(\frac{kT}{h}\right)^{4/3}v^{1/3}=-\sigma_w v^{1/3} \tag{9}$$

where σ_w stands for $[0.033T^{7/3}/t]$. In the vicinity of v_0 the difference is the energy of the film and the bulk can be approximated as

$$\Delta u=Z_0\sigma_z+f_{ph_0}\sigma_{ph}+f_{ex_0}\sigma_{ex}-\sigma_w v^{1/3}+\frac{1}{2K_T}\frac{[v-v_0]^2}{v_0}, \tag{10}$$

where K_T is the isothermal compressibility of liquid helium.

The specific volume changes in such a way as to make Δu a minimum, i.e., $\partial/\partial v[\Delta u]=0$. This gives

$$\frac{v-v_0}{v_0}=-K_T\left[\sigma_z\frac{\partial Z_0}{\partial v}+\sigma_{ph}\frac{\partial f_{ph_0}}{\partial v}+\sigma_{ex}\frac{\partial f_{ex_0}}{\partial v}-\frac{1}{3}\sigma_w v^{-2/3}\right], \tag{11}$$

whence

$$\frac{v-v_0}{v_0}=-\left[\frac{+0.765\times10^{-8}}{t}-\frac{\theta}{t}\{0.102\times10^{-8}T^2\exp(-\Delta_0/kT)\}\right.$$
$$\left.-\frac{\theta}{t}\{5\times10^{-12}T^3\}-\frac{3.7\times10^{-11}T^{7/3}}{t}\right]. \tag{12}$$

For the film thicknesses under investigation, we find that the second, third and fourth terms are negligible as compared with the first one.

References

1) E. Long and L. Meyer, Advances in Phys. **2** (1953), 1.
2) A. D. Singh and R. K. Pathria, Nature **183** (1959), 668.
3) Z. Mikura, Prog. Theor. Phys. **11** (1954), 207.
4) D. S. Kothari and F. C. Auluck, Nature **159** (1947), 204.
5) S. Franchetti, Nuovo Cimento **5** (1957), 183.
6) F. C. Auluck and D. S. Kothari, Proc. Roy. Irish Academy, Session 1946–47, p. 13.
7) S. Franchetti, Nuovo Cimento **4** (1956), 1504.
8) K. R. Atkins, Cand. Jour. Phys. **32** (1954), 347.
9) F. London, *Superfluids*, Vol. II (John Wiley and Co. Inc., New York, 1954).
10) K. R. Atkins and R. A. Stasior, Cand. Jour. Phys. **31** (1953), 1156.
11) K. R. Atkins, Phys. Rev. **113** (1959), 962.
12) G. G. Macfarlane, Phil. Mag. **40** (1949), 188.

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High Energy Scattering of Nucleon from Nuclei
——*A Proposed Model and Its Experimental Verification*——

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In the present paper a new model of the high energy scattering of nucleon from complex nuclei is presented. Under the assumption that the short-range correlation between nucleons and the Pauli principle in the target nucleus in its ground state plays an essential role in the high energy elastic scattering, one can predict the angular distribution of the high energy elastic scattering from heavy nuclei to be characterized by the following important results:

(i) the positions of maxima and minima coincide respectively with those given by the impulse approximation,

(ii) the absolute value of the angular distribution is multiplied by a constant factor $|\lambda|^2$ to that given by the impulse approximation,

(iii) this factor is a constant independent of mass number and nearly independent of the incident energy in heavy nuclei, and finally,

(iv) the magnitude of $\sqrt{|\lambda|^2}$ is of order 0.4~0.5. This factor results from the Pauli principle in the target nucleus.

In fact, analysis of the experimental data shows that the above expectation is valid. Here we take the scattering of 95 Mev proton. The factor $\sqrt{|\lambda|^2}$ is equal to 0.55 for Cu and heavier nuclei to Th, in the case of meson theoretical phase shift. ($\sqrt{|\lambda|^2}=1$ for C, 0.75 for Al.)

§ 1. Introduction

The recent remarkable development in analysis of the nucleon-nucleon scattering led immediately to the analysis of the high energy scattering of nucleon from nuclei based on the idea of the direct collision. Indeed, the idea of the direct collision goes back to 1937 when Heisenberg[1] proposed it originally. Later, Serber[2] put forward the same idea. However, it is only recent that one can analyse the high energy nuclear scattering using the detailed knowledge of the nuclear force.

In the previous papers,[3)~5)] we analysed the high energy n-d and n-He^4 scattering and showed that we can reproduce the experimental results by the superposition of the nucleon-nucleon scattering amplitudes. Others[6)~10)] reached the same conclusion in analysing nuclei lighter than carbon. However, there is no similar analysis for nuclei heavier than carbon. Indeed, as to the high energy elastic scattering from heavy nuclei, it has been established phenomenologically that the optical model[11] is very useful in reproducing the experimental data. As a method of clarifying the mechanism of the scattering, however, the optical model is only of limited use. In the present paper, a new model of the high energy scattering

of nucleon from complex nuclei is presented. This is the detailed and revised description of a preliminary report.[12]

In the ground state of the target nucleus, it seems probable that all the particles are free at a time in some potential well, except a pair.[13] Then, since two particles strongly correlate to each other at any point of the target nucleus, we can take this correlating pair as if it were one particle as a whole at a given instance. Really, the partner may change from time to time. But in high energy scattering the collision seems to occur impulsively, hence the struck particle has no chance to change the partner during the collision time. Accordingly, when an incident nucleon hits a target nucleon, the incident nucleon always finds an attendant following the struck particle. Then what will be the shape of and the magnitude of the angular distribution of the elastically scattered particle? Our concern will be concentrated on these points.

As to the mathematical description, see the Appendix. The original impulse approximation[14),15] is described in Appendix I, where the main attention is paid on its physical meaning. In Appendix II, the present model, as it may be called by the name of the " modified impulse approximation ", where the pair correlation in the target nucleus is taken as essential, is described. Accordingly, the discussion of the next section is mathematically based on the contents in Appendix II. As the main feature of the process is, however, easily understood without somewhat complicated mathematical manipulation, the next section will be devoted to the physical presentation of the scattering mechanism.

§ 2. A proposed model and resulting predictions

In the usual impulse approximation, the cross section is proportional to the absolute square of the dynamical factor times the sticking factor:*

$$\text{(the cross section)} = \text{(the kinematical factor)} \times \left| \sum \text{(the dynamical factor)} \times \text{(the sticking factor)} \right|^2, \tag{1}$$

where the sum runs over all the target particles. The dynamical factor means the two-body scattering amplitude between the incident particle and target particles:

$$\text{(the dynamical factor)} = (\phi_f | V_{op} | \psi_i(op)). \tag{2}$$

Here we name the incident particle o, and a specific particle in the target p. ϕ_f is the plane wave state, $\psi_i(op)$ the scattering state of the particle o with the particle p. The sticking factor is a Fourier transform of the each particle density $\rho(\boldsymbol{r})$:

$$\text{(the sticking factor)} = \int e^{-i\Delta \boldsymbol{k} \boldsymbol{r}} \cdot \rho(\boldsymbol{r}) d\boldsymbol{r}, \tag{3}$$

* In other texts, this is called by the name of *the square root* of the sticking factor.

where

$$\Delta\boldsymbol{k}=\boldsymbol{k}_f-\boldsymbol{k}_i=2\frac{A}{A+1}k_{\text{lab}}\cdot\sin\theta/2.$$

In deriving the above formula, one assumes that there is no direct correlations between the target nucleons, though each nucleon may move in a some common Hartree potential. However, under the situation that one always finds a partner near the struck particle, the above description is not suitable. In this case, the cross section can be expressed as

$$\begin{aligned}&(\text{the cross section})=(\text{the kinematical factor})\\&\qquad\times|\sum_{\text{pair}}(\text{the dynamical factor})\\&\times(\text{the sticking factor to a specific pair in the target})\\&\times(\text{the sticking factor corresponding to the center of mass}\\&\quad\text{motion of the specific pair})|^2.\end{aligned}\tag{4}$$

See (A·11) in Appendix II. Corresponding to (1), the sum runs over all the pairs. This is discussed in more detail later. The wave function describing the center of mass motion of the pair depends on the relative coordinate in the pair as well as the center of mass of the pair.

Let the pair correlation function between a typical chosen particle p and its partner q be $X_\alpha(\boldsymbol{r}_\alpha)$, $\boldsymbol{r}_\alpha$ being the relative coordinate of the partner q to the particle p: $\boldsymbol{r}_\alpha=\boldsymbol{r}_q-\boldsymbol{r}_p$. Let the wave function describing the center of mass motion of the pair be $Y_\beta(\boldsymbol{r}_\alpha, \boldsymbol{R}_\alpha)$, $\boldsymbol{R}_\alpha$ meaning the relative coordinate of the center of mass of the pair p and q to the center of mass of the target. (We assume here $1\gg 1/A$). Then, as will be seen in Appendix II, the last two factors in the right-hand side of the expression (4) is as follows.

$$\begin{aligned}&(\text{the sticking factor to a specific pair in the target})\\&\times(\text{the sticking factor corresponding to the center of mass motion}\\&\quad\text{of the specific pair})\\&=\iint d\boldsymbol{r}_\alpha d\boldsymbol{R}_\alpha\exp[i\Delta\boldsymbol{k}\boldsymbol{r}_\alpha/2]|X_\alpha(\boldsymbol{r}_\alpha)|^2\cdot\exp[i\Delta\boldsymbol{k}\boldsymbol{R}_\alpha]|Y_\beta(\boldsymbol{r}_\alpha, \boldsymbol{R}_\alpha)|^2.\end{aligned}\tag{5}$$

At present, we are not sure of the detailed property of the correlation function $X_\alpha(\boldsymbol{r}_\alpha)$. However, we know some of the gross properties of this function. According to the investigation by Walecka, Gomes and Weisskopf,[13] the pair correlation function contains many Fourier components in a relatively small region, $|\boldsymbol{r}_\alpha|<D$, where D is of the order of the inter nucleon distance, say, 2×10^{-13} cm (or say, $1.6\sim2.2\times10^{-13}$ cm). This value is estimated by $D=(4\pi/3)^{1/3}r_0$. In other region, this function contains only one Fourier component. This is due to the exclusion principle, hence the integration over $\boldsymbol{r}_\alpha$ in (5) vanishes for $|\boldsymbol{r}_\alpha|>D$. Under such a circumstance, $Y_\beta(\boldsymbol{r}_\alpha, \boldsymbol{R}_\alpha)$ is practically independent of $\boldsymbol{r}_\alpha$ and

$|Y_\beta(\boldsymbol{R}_a)|^2$ spreads over the whole nucleus. Then the right-hand side of Eq. (5) is factorized.

$$\text{Eq. (5)} \simeq \int_{|\boldsymbol{r}_a|<D} d\boldsymbol{r}_a \exp[i\Delta\boldsymbol{k}\boldsymbol{r}_a/2]|X_\alpha(\boldsymbol{r}_a)|^2 \cdot \int d\boldsymbol{R}_a \exp[i\Delta\boldsymbol{k}\boldsymbol{R}_a]\cdot|Y_\beta(\boldsymbol{R}_a)|^2. \quad (6)$$

This expression embodies the model that when an incident particle hits a target nucleon, the incident nucleon always finds a partner following the struck particle.*

Before going into the discussion of the dynamical factor appearing in (4), which is of course different from the expression given in (2) at least at the starting point of our discussion, we shall discuss the nature of the factors in (6).

As D is very much smaller than the dimension of the whole nucleus, $|Y_\beta(\boldsymbol{R}_a)|^2$ spreads over the whole nucleus as described before. Meanwhile we assume that $|Y_\beta(\boldsymbol{R}_a)|^2$ is spherically symmetric and spreads uniformly over the whole nucleus. Then

$$|Y_\beta(\boldsymbol{R}_a)|^2 = C \quad \text{for} \quad |\boldsymbol{R}_a| < R.$$
$$= 0 \quad \text{for} \quad |\boldsymbol{R}_a| > R.$$

R is the nuclear radius. C is any constant. Under this assumption

$$\int d\boldsymbol{R}_a \exp[i\Delta\boldsymbol{k}\boldsymbol{R}_a]|Y_\beta(\boldsymbol{R}_a)|^2 d\boldsymbol{R}_a = \frac{4\pi C}{(\Delta k)^3}[\sin(\Delta kR) - (\Delta kR)\cos(\Delta kR)], \quad (7)$$

hence this factor gives the main feature of the angular distribution. In fact this factor gives the similar angular distribution as the impulse approximation, in which the effect of the pair correlation is not taken into account. In next section, we assume the form of $|Y_\beta(\boldsymbol{R}_a)|^2$ to be the Saxon-type. Thereby, this feature of this factor is by no means changed. As to the absolute value of this factor, it is just the same as that given by the impulse approximation (3), provided that both $\rho(\boldsymbol{r})$ and $|Y_\beta(\boldsymbol{R}_a)|^2$ are normalized to give the value 1 when each of these is integrated over the whole space.

Next we take up the first factor in (6). We shall discuss the angular distribution, again under the assumption that $|X_\alpha(\boldsymbol{r}_a)|^2$ is spherically symmetric and spreads uniformly over the region $|\boldsymbol{r}_a| < D$:

$$|X_\alpha(\boldsymbol{r}_a)|^2 = K \quad \text{for} \quad |\boldsymbol{r}_a| < D.$$

K is any constant. This leads to the formula

$$\int_{|\boldsymbol{r}_a|<D} d\boldsymbol{r}_a \exp[i\Delta\boldsymbol{k}\boldsymbol{r}_a/2]|X_\beta(\boldsymbol{r}_a)|^2 d\boldsymbol{r}_a$$
$$= \frac{4\pi K}{(\Delta k/2)^3}[\sin(\Delta k\, D/2) - (\Delta k\, D/2)\cos(\Delta k\, D/2)]. \quad (8)$$

---

* $X_\alpha(\boldsymbol{r}_a)$ is not the residual correlation. In the notation by Gomes et al.[13] $X_\alpha(\boldsymbol{r}_a)Y_\beta(\boldsymbol{R}_a) = \varphi_s(\boldsymbol{p})\varphi_t(\boldsymbol{q}) + \sum_{\gamma\gamma'>F} \varphi_\gamma(\boldsymbol{p})\varphi_{\gamma'}(\boldsymbol{q})$, where $s, t < F$.

As far as Δk is not large, we can regard the right-hand side of (8) as a constant compared with that of (7), hence this factor does not concern the angular distribution. As ΔkD is small, (8) is practically independent of Δk, i.e. the incident energy. As for the effect of this factor on the absolute value of the cross section, however, this factor gives a conspicuous results. As $|X_\alpha(\boldsymbol{r}_\alpha)|^2$ itself extends to the region $|\boldsymbol{r}_\alpha| \gtrsim D$, $|X_\alpha(\boldsymbol{r}_\alpha)|^2$ is not normalized such as to give the value 1 when it is integrated inside the region $|\boldsymbol{r}_\alpha| < D$. For the discussion of this factor, recognition of the following two situations is essential.

One is the situation that in such a small region that $|\boldsymbol{r}_\alpha| < D$, the density corresponding to the relative coordinate $\boldsymbol{r}_\alpha$ is nearly the same as the corresponding density of a free deuteron system. The other situation is that we are dealing with the high energy scattering such that $\lambda\!\!\!^{-} = 1/0.22\sqrt{E}(\text{Mev}) < 1$. That is $\lambda\!\!\!^{-} \ll D$. Under such a situation, the scattering of the incident particle from the correlated pair is described as the superposition of the scatterings from each particle, as in the *n-d* scattering, provided that $\lambda\!\!\!^{-} \ll R_d$. ($R_d$: the deuteron radius). Then, the first two factors in the absolute square of (4) is of the form,

$$t_{01}S(12)+t_{02}S(21)+t_{03}S(34)+t_{04}S(43)+\cdots, \tag{9}$$

where

$$t_{0p}=(\phi_f|V_{0p}|\psi_i(op)) \quad \text{and} \quad S(ij)=\int_{|\boldsymbol{r}_{ij}|<D} d\boldsymbol{r}_{ij}\exp[i\Delta\boldsymbol{k}\boldsymbol{r}_{ij}/2]|X_\alpha(\boldsymbol{r}_{ij})|^2.$$

If we singled out the correlated pair from the rest of target particles and calculated the cross section, we would get the following result, remembering that the *n-d* scattering is well-reproduced by the impulse approximation.[4]

$$\sigma_{o-pn}=\sigma_{nd}\times(\pi D^2/\pi R_d^2)=\sigma_{\text{impulse}}\times(\pi D^2/\pi R_d^2), \tag{10}$$

where R_d is of the order 4.31×10^{-13} cm.[16] By σ_{impulse}, we mean its sticking factor is calculated neglecting the effect of the Pauli principle, but its dynamical factor is calculated as the impulse approximation. In $\sigma_{o\text{-}pn}$, the sticking factor is given by the first factor in (6) and the dynamical factor is calculated as the impulse approximation, hence the second factor in (10) results from the Pauli principle in the target nucleus. D/R_d is of order $0.4\sim0.5$, independent of the incident energy. In what follows, we use the notation $\sqrt{|\bar{\lambda}|^2}$ for D/R_d. We shall call $\sqrt{|\bar{\lambda}|^2}$ the reducing factor.

Combining (4), (5), (6), (10) and the discussions developed between Eqs. (8) and (10), we obtain the cross section of the elastic scattering of high energy nucleon from heavy nuclei as

$$\text{(the cross section)} = \text{(the kinematical factor)}$$
$$\cdot|\lambda|^2\cdot\Big|\sum_p{}'(\phi_f|V_{op}|\psi_i(op))\cdot\int d\boldsymbol{R}_\alpha\exp[i\Delta\boldsymbol{k}\boldsymbol{R}_\alpha]\cdot|Y_\beta(\boldsymbol{R}_\alpha)|^2. \tag{11}$$

Comparing Eq. (11) with Eq. (1) substituted by Eqs. (2) and (3) we get a simple result,

$$\text{(the cross section)} = |\lambda|^2 \cdot \text{(the cross section given by the impulse approximation)}. \tag{12}$$

According to the above discussions we get the following predictions. (i) The shape of the angular distribution is just the same as that given by the impulse approximation, (ii) the absolute value of the angular distribution is multiplied by a constant factor $|\lambda|^2$ to that given by the impulse approximation, as expressed in Eq. (12). (iii) $|\lambda|^2$ is independent of mass number, and nearly independent of the incident energy and the scattering angle. (iv) $\nu|\lambda|^2$ is of order $0.4 \sim 0.5$. This factor results from the Pauli principle in the target nucleus.

In the following section, we shall examine these predictions by direct comparison with the experimental data. On the first point (i), the author reported in a previous note.[17]

§ 3. Analysis of the experimental data

In this section, we examine the experimental data directly and try to see whether the predictions given in the previous section holds.

In § 3b, we write the result of the analysis of the experimental data. In this analysis $\nu|\lambda|^2$ is considered to be an adjustable parameter, and we examine its mass number dependence. Thereby, the pion theoretical phase shifts[18] are used in the two-body scattering amplitude. The set of the pion theoretical phase shifts gave the best fit among others in our previous analysis of light nuclei.[4),5)]

§ 3a. Formula used in the experimental analysis

The differential cross section of the elastic scattering is expressed, in general,

$$d\sigma(\theta) = \frac{(AM/(A+1))^2}{4\pi^2\hbar^4} \frac{1}{2} \sum_{i=1}^{2} |H_{fi}|^2.$$

Here M is the nucleon mass and H_{fi} is the transition matrix element. The sum $\sum_{i=1}^{2}$ means the two spin states (up and down) of the incident nucleon. Using the coefficients of the spin matrix, the differential cross section (12) is expressed as

$$d\sigma(\theta) = 4\left(\frac{A}{A+1}\right)^2 |\lambda|^2 \left\{\frac{1}{16}|G+B+N|^2 + |C|^2\right\} \times S(k), \tag{13}$$

in the case that the target nucleus is even-even nucleus.* Here G, B, N and C are the coefficients of the scattering spin-matrix between particles with spin $1/2$, related to the phase shifts of the two-body scatterings by Wright.[19] When we

* In reference 5), the differential cross section is erroneously written as factor 2 larger.

mean the neutron and the proton number with n and z, respectively, G, etc., are given by

$G=nG_{nn}+zG_{pn}$, etc., for neutron-nucleus scattering, and

$G=zG_{pp}+nG_{pn}$, etc., for proton-nucleus scattering.

The angle of the two-body scattering and that of the $(A+1)$-body scattering is related by

$$(A/A+1)\,(\sin\theta/2)_{(A+1)\text{-body}}=(1/2)\,(\sin\theta/2)_{2\text{-body}}.$$

In the actual calculation, the following must be noted. After giving the explicit expression of the coefficients of the spin matrix in terms of the phase shifts for like particles, Wright describes the method for getting the expression for neutron proton scattering: "The formulas apply to neutron proton scattering if $\alpha=e^2/\hbar v$ is put equal to zero, the sums are extended over all l save for the absent $^3S_0$ state." However, as it will be evident after a moment's consideration, if we calculate the coefficient in the case of neutron proton scattering following this method, we get the value twice as large as the true value. Therefore, we must divide the value which we get after the method by 2.

In even-odd or odd-odd nuclei, the expression of the differential cross section is somewhat more complicated. However, when we use the expression (13) for these nuclei also, the error introduced is of order $1/A$. We can show this fact by the Born approximation, though we do not reproduce the result explicitly here.

$S(k)$ is square of the sticking factor given by Eq. (3). As the density distribution, we adopt the Saxon-type (Fermi-type):

$$\rho(r)=\rho_0\left[1+\exp\left(\frac{r-R}{a}\right)\right]^{-1}. \tag{14}$$

ρ_0 is normalized so that $4\pi\int_0^\infty\rho(r)r^2dr=1$.

For actual calculation, it is convenient to use the following formula. After some calculations, we can show that

$$S(k)=\frac{4\pi\rho_0}{k}\left\{\int_0^R f(r)dr+\frac{\pi^2}{6}a^2\left(\frac{df}{dr}\right)_{r=R}+\frac{7\pi^4}{360}a^4\left(\frac{d^3f}{dr^3}\right)_{r=R}\right.$$
$$\left.+\frac{31\pi^6}{15120}a^6\left(\frac{d^5f}{dr^5}\right)_{r=R}+\cdots\cdots\right\},$$

where

$$f(r)=r\sin kr.$$

The first two factors in the curly brackets is sometimes called the Sommerfeld-formula.[20] Performing the differentiation, we get

$$S(k)=\frac{4\pi\rho_0}{k}\left[\left(\frac{1}{k^2}+\frac{\pi^2}{6}a^2-\frac{7\pi^4a^4}{120}k^2+\frac{31}{3024}\pi^6a^6k^4+\cdots\right)\sin kR\right.$$
$$\left.-(kR)\left(\frac{1}{k^2}-\frac{\pi^2a^2}{6}+\frac{7\pi^4a^4}{360}k^2-\frac{31}{15120}\pi^6a^6k^4+\cdots\right)\cos kR\right]. \qquad (15)$$

If we put $a=0$, this formula corresponds to the Fourier transform of the uniform distribution of the nuclear density used in § 2. Errors introduced for the uniform density distribution were estimated in the previous note.[17] In the present paper, we adopt the value of

$$a=0.49\times10^{-13}\text{ cm},$$
$$R=1.32\,A^{1/3}\times10^{-13}\text{ cm}.$$

As the convergence of the coefficients of $\sin kR$ is worse than that of $\cos kR$, we can estimate the utility of this formula by examining the convergence of the coefficients of $\sin kR$. As it is expected that the denominator of the higher order terms than terms written explicitly in (15) becomes larger and larger, we estimate the convergence with those terms which are written explicitly in the expression (15). Denoting the i-th coefficient of $\sin kR$ by α_i, we get the relation,

for
$$\alpha_1/\alpha_2>1, \quad k<\sqrt{6}/(\pi a),$$
$$\alpha_2/\alpha_3>1, \quad k<\sqrt{(20/7)}/(\pi a),$$
$$\alpha_3/\alpha_4>1, \quad k<\sqrt{7\times126/155}/(\pi a).$$

Then the upper limit of k is estimated by the requirement $\alpha_2/\alpha_3>1$. Taking

$$k\simeq0.44\sqrt{E_{\text{lab}}(\text{Mev})}\sin\theta/2,$$

we get the applicable region of the formula (16), that

for $E_{\text{lab}}\simeq50$ Mev, $\theta<40°$,
$50\text{ Mev}<E_{\text{lab}}<100\text{ Mev}$, $\theta<30°$,
$100\text{ Mev}<E_{\text{lab}}<200\text{ Mev}$, $\theta<20°$,
$200\text{ Mev}<E_{\text{lab}}<350\text{ Mev}$, $\theta<15°$.

§ 3b. Analysis of the experimental data

In this section, we analyse the experimental result[21] of the scattering of 95 Mev protons from various nuclei. The effect of the Coulomb force is neglected. See Fig. 1. In Fig. 1, the small circle

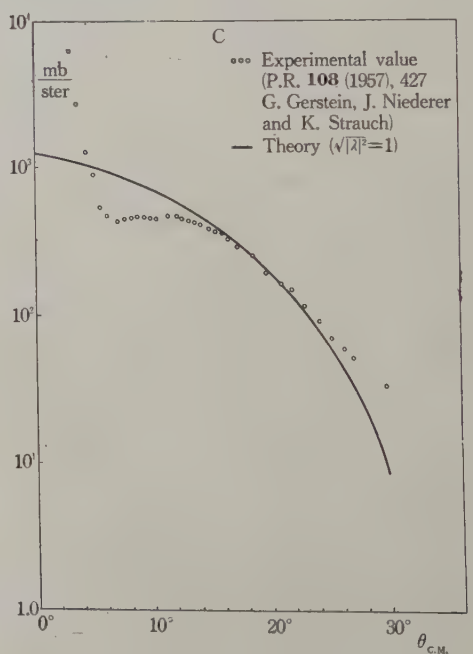

Fig. 1 (1)

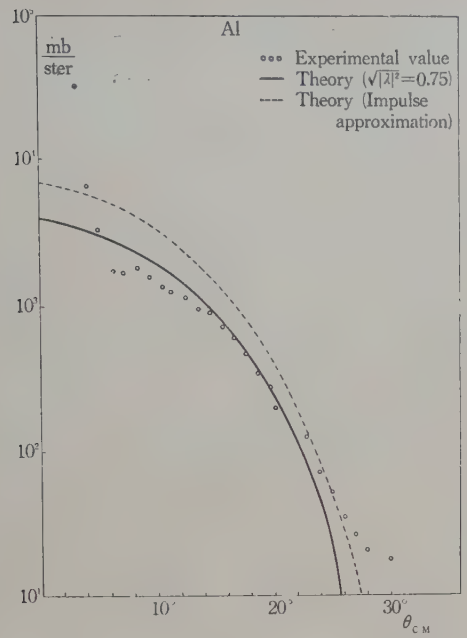

Fig. 1 (2)

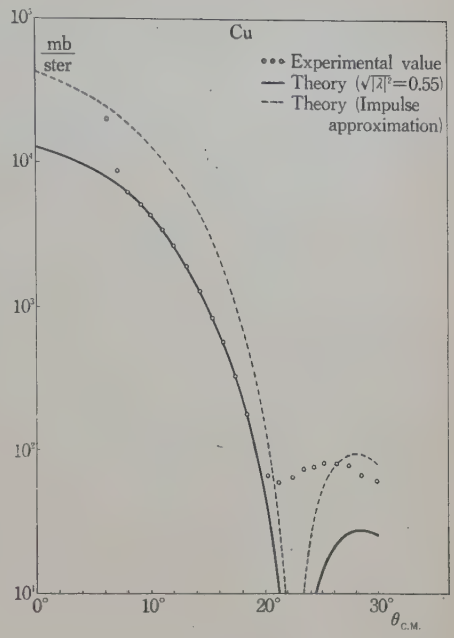

Fig. 1 (3)

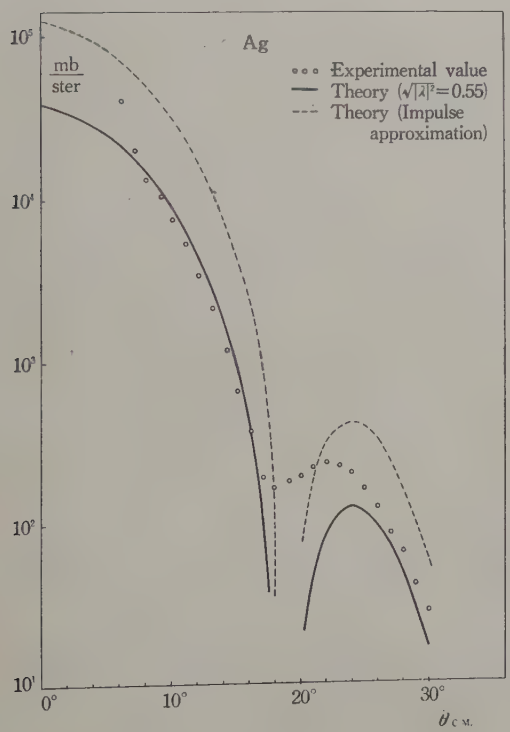

Fig. 1 (4)

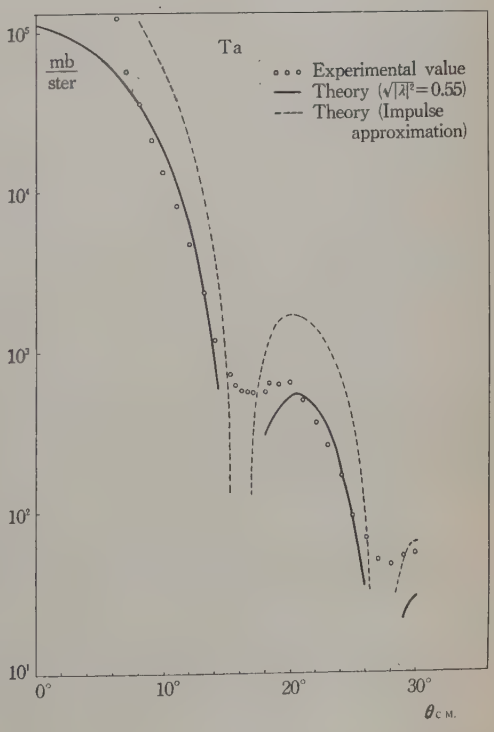

Fig. 1 (5)

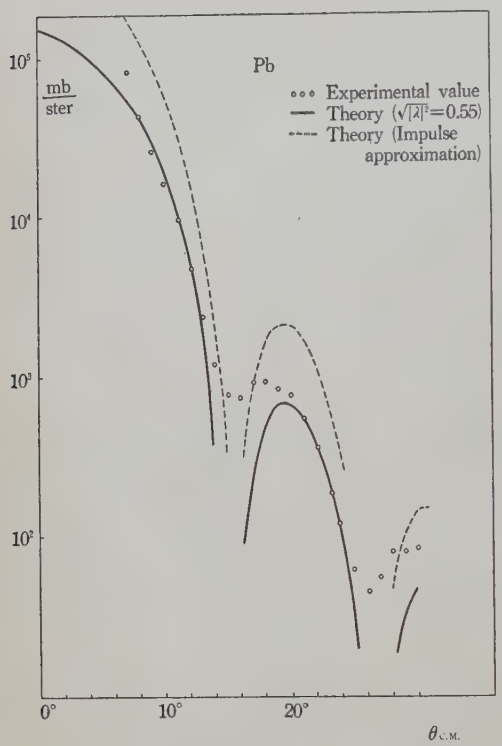

Fig. 1 (6)

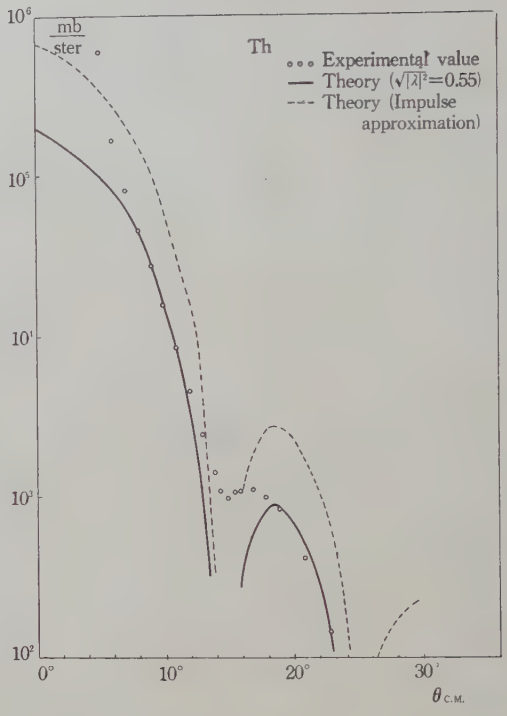

Fig. 1 (7)

is the experimental value. The dotted line is got by putting $\sqrt{|\lambda|^2}=1$ in the formula (13). The solid line is got by putting $\sqrt{|\lambda|^2}=1$ for C. $\sqrt{|\lambda|^2}=0.75$ for Al and $\sqrt{|\lambda|^2}=0.55$ for Cu and heavier nuclei to Th. Thus, the result of the analysis proves that the theoretical predictions developed in § 2 is true. The patterns of the angular distribution is the same as that given by the impulse approximation. The value of $\sqrt{|\lambda|^2}$ is a constant for heavy nuclei and it is very close to the predicted value.

As for the energy dependence of $\sqrt{|\lambda|^2}$, the author is analysing the experimental data with Sakamoto and Takemiya. The result seems positive.*

§ 4. Discussions

As seen in § 3, the analysis of the experimental results seems well realizing the predictions in § 2. Here we add comments on a few points upon which we did not touch in § 2.

(a) Since the mean free path estimated by the formula $\Lambda=(\rho\sigma)^{-1}$, where ρ is the nucleon density in the nucleus and σ is the total collision cross section, is of

* See note added in proof: (1)

order 2×10^{-13} cm at about 100 Mev, the effect of the multiple scattering may be important. At present, the author has not estimated this effect. But so far as we limit ourselves to forward scatterings, all the predictions are well satisfied in the experimental data. Therefore, the multiple scattering may be important only for large angle scattering.*

(b) One may wonder if the incident nucleon may hit a lone particle without any partner. According to the independent pair model,[22] such a single particle state in the ground state is considered to form the Hartree potential. Therefore, if the independent pair model to the ground state of the target nucleus is valid, there is always a partner near the struck particle in the sense as discussed in § 2.

(c) In our discussion developed in § 2, the two conditions, the high energy ($\lambda\ll D$) and heavy nucleus ($1\gg1/A$), were essential. In light nuclei, all the particle are situated at the surface. And the situation in light nuclei is very much different from heavy nuclei. Then the usual impulse approximation is more applicable as shown in previous papers[3)~10)] and in § 3 for C and Al. At low energy, the simple model developed in the present paper will not be applicable.

§ 5. Conclusion

Under the assumption that the short-range correlation between nucleons and the Pauli principle in the target nucleus in its ground state plays an essential role in the high energy elastic scattering, one can predict the angular distribution of the high energy elastic scattering from heavy nuclei to be characterized by the conspicuous results that *the cross section is given by Eq.* (12) *or* (13). The meaning of this equation is:

(i) The shape of the angular distribution is very similar to that given by the impulse approximation,

(ii) the absolute value of the angular distribution is multiplied by a constant factor $|\lambda|^2$ to that given by the impulse approximation,

(iii) this factor is a constant independent of angle, and nearly independent of mass number and the incident energy and

(iv) the magnitude of $\sqrt{|\lambda|^2}$ is of order $0.4\sim0.5$. This factor represents the effect of the Pauli principle in the target nucleus.

The analysis of the experimental data at 95 Mev shows that above predictions are indeed satisfied. As for higher energies than 95 Mev, the calculation is now being carried out by the present author in collaboration with Y. Sakamoto and T. Takemiya. The result seems positive. (See note added in proof: (1))

Acknowledgement

The author would like to thank Professor M. Kobayasi for his encouragement.

* See note added in proof: (2)

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Appendix I

The physical meaning of the impulse approximation

The impulse approximation was presented by Chew and Wick.[14] In the high energy scattering, the process described by this approximation plays an essential role, especially in light nuclei.[3)~10)] Here we shall discuss the physical meaning of the impulse approximation.

Usually, one is apt to think that in the impulse approximation one treats the scattering problem as if it were a superposition of two body scatterings. It is true. But, to understand the physical meaning of the impulse approximation, it seems more suitable to say that in this approximation one treats the scattering problem as if it were a three-body problem, in the sense that one uses three coordinates to describe the approximation.

Let the incident particle be labeled by o, a specific particle in the target by p, and the residual particles as a whole in the target by w. We treat the problem in the center of mass system. To avoid unessential complications, we do not take account of the spin and the isotopic spin. These were taken into account in our previous paper on n-d scattering.[4] We restrict ourselves to the elastic scattering.

The transition matrix element corresponding to the interaction of the particle p with the incident particle o is given in the impulse approximation as

$$H_{fi}(p)=\sum_{mn}(\Psi_f,\ \phi_m)(\phi_m,\ V_{op}\psi_n(p))(\phi_n,\ \Psi_i). \qquad (A\cdot 1)$$

Of course, to get the transition matrix element for the interaction between the incident particle and the target nucleus, we should superpose $H_{fi}(p)$ for all the particles in the target.

Here

$$\left.\begin{aligned}
&\phi_m=(1/2\pi)^{3/2\times 3}\exp[i(\boldsymbol{k}_o\boldsymbol{r}_o+\boldsymbol{k}_p\boldsymbol{r}_p+\boldsymbol{k}_w\boldsymbol{r}_w)],\\
&\phi_n=(1/2\pi)^{3/2\times 3}\exp[i(\boldsymbol{k}_o{}^0\boldsymbol{r}_o+\boldsymbol{k}_p{}^0\boldsymbol{r}_p+\boldsymbol{k}_w{}^0\boldsymbol{r}_w)],\\
&\psi_n(p)=(1/2\pi)^{3/2\times 2}\exp[i(\boldsymbol{k}_w{}^0\boldsymbol{r}_w)]\exp\left[i(\boldsymbol{k}_o{}^0+\boldsymbol{k}_p{}^0)\frac{\boldsymbol{r}_o+\boldsymbol{r}_p}{2}\right]\\
&\qquad\qquad\cdot\chi_{(1/2)(k_o{}^0-k_p{}^0)}(\boldsymbol{r}_0-\boldsymbol{r}_p).
\end{aligned}\right\} \qquad (A\cdot 2)$$

$$\Psi_{i,f}=(1/2\pi)^{3/2\times 2}\exp\left[i\boldsymbol{k}_{i,f}\left(\boldsymbol{r}_o-\frac{\boldsymbol{r}_p+(A-1)\boldsymbol{r}_w}{A}\right)\right]\varphi_0(\boldsymbol{r}_p-\boldsymbol{r}_w). \qquad (A\cdot 3)$$

Then m means $\boldsymbol{k}_o$ $\boldsymbol{k}_p$ and $\boldsymbol{k}_w$ as a whole and n means $\boldsymbol{k}_o^0$ $\boldsymbol{k}_p^0$ and $\boldsymbol{k}_w^0$. Here $\chi_{(1/2)(k_o{}^0-k_p{}^0)}(\boldsymbol{r}_o-\boldsymbol{r}_p)$ describes the solution of the scattering problem between the particles o and p. The initial and the final state of the target particle p is described by $\varphi_0(\boldsymbol{r}_p-\boldsymbol{r}_w)$. Properly speaking, we should write the total wave function describing the ground state of the target in place of $\varphi_0(\boldsymbol{r}_p-\boldsymbol{r}_w)$. But because

here we assume the single particle model for the ground state of the target nucleus and all the particles except p is not excited during the interaction $(\phi_m V_{op} \psi_n(p))$, we do not write other wave functions corresponding to other degrees of freedom. This is also the reason why in ϕ_m, ϕ_n and $\psi_n(p)$ in (A·2) only the coordinates $\boldsymbol{r}_o$, $\boldsymbol{r}_p$ and $\boldsymbol{r}_w$ appear. When $1 \gg 1/A$, $\boldsymbol{r}_p - \boldsymbol{r}_w$ is practically identical with the coordinate of p relative to the center of mass of the target nucleus. In this case, $\Psi_{i,f}$ can be expressed as

$$\Psi_{i,f} = (1/2\pi)^{3/2\times 2} \exp[i\boldsymbol{k}_{i,f}(\boldsymbol{r}_o - \boldsymbol{r}_w)]\varphi_0(\boldsymbol{r}_p - \boldsymbol{r}_w). \qquad (A\cdot 3')$$

But as the use of (A·3) is more handy than the use of (A·3′), we shall proceed with (A·3) and later introduce the assumption $1 \gg 1/A$. At first sight, the introduction at the final step of the reduction of the assumption $1/A$ may seem unnecessary. However, we singled out $\varphi_0(\boldsymbol{r}_p - \boldsymbol{r}_w)$ from the total wave function, which is not explicitly given here, of the ground state of the target nucleus, i.e. we adopted the single particle model. The single particle model seems applicable only under the assumption of $1 \gg 1/A$. The following is the reason for it. Due to the conservation of the center of mass of the target, the energy of the one particle should depend on the position of other particles. But if $1 \gg 1/A$, the other particles as a whole is much heavier than the one particle under consideration and the energy of the one particle does not depend on the position of other particles, hence the condition $1 \gg 1/A$ seems necessary for the validity of the single particle model.

As to the wave function, the following seems worth noting. The wave function $\varphi_0(\boldsymbol{r}_p - \boldsymbol{r}_w)$ should not be taken as a plane wave. If the particle p were free from the residual particles, the impulse approximation would not be applicable to the *elastic* scattering. But in actual situation the particle p is not free from the residual particles. At least it is bounded by the Hartree potential. Under this situation, as we shall see in the following, the impulse approximation is applicable even for the elastic scattering, provided that the multiple scattering can be neglected.

As to the physical meaning of Eq. (A·1), one may say as " we analyse the initial wave function to the Fourier-components. Each component undergoes the two-body collision. Each component gets free again." However, we shall add more realistic description of Eq. (A·1) in the following.

Writing $\boldsymbol{r} = \boldsymbol{r}_p - \boldsymbol{r}_w$, and using (A·2), (A·3), we can express Eq. (A·1) as

$$\begin{aligned} H_{fi}(p) = & \int d\boldsymbol{k}_o\, d\boldsymbol{k}_p\, d\boldsymbol{k}_w\, d\boldsymbol{k}_o^0\, d\boldsymbol{k}_p^0\, d\boldsymbol{k}_w^0 \\ & \cdot (1/2\pi)^{3/2\times 3} \delta(\boldsymbol{k}_o^0 - \boldsymbol{k}_i)\delta(\boldsymbol{k}_p^0 + \boldsymbol{k}_w^0 + \boldsymbol{k}_i)\delta(\boldsymbol{k}_w - \boldsymbol{k}_w^0) \\ & \cdot \delta(\boldsymbol{k}_o + \boldsymbol{k}_p - \boldsymbol{k}_o^0 - \boldsymbol{k}_p^0)\delta(\boldsymbol{k}_o - \boldsymbol{k}_f)\delta(\boldsymbol{k}_p + \boldsymbol{k}_w + \boldsymbol{k}_f) \\ & \cdot \int \exp\left[-i\left(\frac{(A-1)\boldsymbol{k}_p^0 - \boldsymbol{k}_o^0}{A}\right)\boldsymbol{r}\right]\varphi_0(\boldsymbol{r})\, d\boldsymbol{r} \end{aligned}$$

$$\cdot \int \exp\left[-i\frac{\boldsymbol{k}_o-\boldsymbol{k}_p}{2}\boldsymbol{r}_{op}\right] V_{op}\chi_{(1/2)(k_0{}^0-k_p{}^0)}(\boldsymbol{r}_{op})\,d\boldsymbol{r}_{op}$$

$$\cdot \int \exp\left[i\left(\frac{(A-1)\boldsymbol{k}_p-\boldsymbol{k}_w}{A}\right)\boldsymbol{r}'\right]\varphi_o(\boldsymbol{r}')\,d\boldsymbol{r}'. \qquad (A\cdot 4)$$

Naturally, the total momentum conserves at all steps. The momentum change $\boldsymbol{k}_o-\boldsymbol{k}_o{}^0$ of the incident particle is equal to $\boldsymbol{k}_f-\boldsymbol{k}_i$, and it is also equal to the momentum change $\boldsymbol{k}_p+\boldsymbol{k}_w-(\boldsymbol{k}_p{}^0+\boldsymbol{k}_w{}^0)$ of the target nucleus. Hereby, the relative motion described by $\varphi_0(\boldsymbol{r})$ in the target nucleus remains unchanged. Then in the impulse approximation in the elastic scattering, the following two facts are important and not to be forgotten.

(1) The motion of the center of mass of the target is not neglected.

(2) A specific particle, here labeled by p, is bounded tightly to the target. Consequently, after the collision, the center of mass of the target carries the recoil. At this time, the residual "particle" w moves as before, as a delta function of Eq. (A·4) shows. Then we may express the physical meaning of the impulse approximation as follows. The target nucleus is considered as if it were an asymmetric dumb-bell, one end being a typical particle p and another the residual "particle" w. The impulse approximation corresponds to hit the end labeled by p. Then motion of the rod of the dumb-bell is changed. At this time, however, the other end moves as before.

Eliminating the delta-functions from Eq. (A·4), we get

$$H_{fi}(p)=(1/2\pi)^{3/2\times 3}\int d\boldsymbol{k}_w\int \exp\left[-i\left(\frac{(A-1)(-\boldsymbol{k}_i-\boldsymbol{k}_w)-\boldsymbol{k}_w}{A}\right)\boldsymbol{r}\right]\varphi_0(\boldsymbol{r})\,d\boldsymbol{r}$$

$$\cdot \int \exp\left[-i\frac{\boldsymbol{k}_f+(\boldsymbol{k}_w+\boldsymbol{k}_f)}{2}\boldsymbol{r}_{op}\right] V_{op}\chi_{(1/2)(k_i+k_i+k_w)}(\boldsymbol{r}_{op})\,d\boldsymbol{r}_{op}$$

$$\cdot \int \exp\left[-i\left(\frac{-(A-1)(\boldsymbol{k}_w+\boldsymbol{k}_f)-\boldsymbol{k}_w}{A}\right)\boldsymbol{r}'\right]\varphi_0{}^*(\boldsymbol{r}')\,d\boldsymbol{r}'. \qquad (A\cdot 5)$$

For further reduction, we put

$$\int \exp[-i(\boldsymbol{k}_f+\boldsymbol{k}_w/2)\boldsymbol{r}_{op}]V_{op}\chi_{(k_i+k_w/2)}(\boldsymbol{r}_{op})\,d\boldsymbol{r}_{op}$$

$$\simeq \int \exp[-i\boldsymbol{k}_f\boldsymbol{r}_{op}]V_{op}\chi_{k_i}(\boldsymbol{r}_{op})\,d\boldsymbol{r}_{op}. \qquad (A\cdot 6)$$

Eq. (A·6) holds strictly in the Born approximation. We introduce the right-hand side of Eq. (A·6) into the right-hand side of Eq. (A·5) under the situation that the $\boldsymbol{k}_w$-dependence of the left-hand side of Eq. (A·6) is slower than the $\boldsymbol{k}_w$-dependence of the other factors in Eq. (A·5). This situation holds especially in the high energy scattering. The right-hand side of Eq. (A·6), which we shall call the dynamical factor, is the two-body scattering amplitude, because now we are treating the elastic scattering and $|\boldsymbol{k}_f|=|\boldsymbol{k}_i|$.

Introducing the approximation given by Eq. (A·6) into the right-hand side of Eq. (A·5), we get

$$H_{fi}(p)=(1/2\pi)^{3/2}\int\exp[-i(\boldsymbol{k}_f-\boldsymbol{k}_i)\boldsymbol{r}]\rho(\boldsymbol{r})d\boldsymbol{r}$$
$$\cdot\int\exp[-i\boldsymbol{k}_f\boldsymbol{r}_{op}]V_{op}\chi_{k_i}(\boldsymbol{r}_{op})d\boldsymbol{r}_{op}. \qquad (A\cdot7)$$

Here we used the $1/A$-approximation. $\rho(\boldsymbol{r})$ is given by

$$\rho(\boldsymbol{r})=|\varphi_0(\boldsymbol{r})|^2.$$

This gives the density distribution of the particle p. Eq. (A·7) is the transition matrix element for the impulse approximation.

In the impulse approximation, the incident particle is assumed to interact only with one particle in the target nucleus. (However, of course, we superpose $H_{fi}(p)$ for verious particles.) But the target particle correlates strongly with other target particles. Consequently, we must take into account the effect of such a correlation. In Appendix II, we describe the impulse approximation taking such a correlation in the target nucleus into account.

Appendix II

The modified impulse approximation

The physical meaning of the "modified impulse approximation," where the pair correlation in the target nucleus is taken as essential, which consequences are described in § 2 in the text. So in Appendix II, we wish to derive only Eq. (4) in § 2, which forms the basis of the discussions throughout § 2.

Again we start from Eq. (A·1),

$$H_{fi}(p)=\sum_{mn}(\Psi_f,\ \phi_m)(\phi_m,\ V_{op}\psi_n(p))(\phi_n,\ \Psi_i).$$

In the present case, we take into account a pair correlation of the target particles in Ψ_i and Ψ_f. Let us designate the struck particle as p, its partner q, and the residual particles as a whole s. Then, by analogy with Appendix I, ϕ_m, ϕ_n and $\psi_n(p)$ in Eq. (A·1) are given by

$$\phi_m=(1/2\pi)^{3/2\times4}\exp[i(\boldsymbol{k}_o\boldsymbol{r}_o+\boldsymbol{k}_p\boldsymbol{r}_p+\boldsymbol{k}_q\boldsymbol{r}_q+\boldsymbol{k}_s\boldsymbol{r}_s)]$$
$$\phi_n=(1/2\pi)^{3/2\times4}\exp[i(\boldsymbol{k}_o^0\boldsymbol{r}_o+\boldsymbol{k}_p^0\boldsymbol{r}_p+\boldsymbol{k}_q^0\boldsymbol{r}_q+\boldsymbol{k}_s^0\boldsymbol{r}_s)]$$
$$\psi_n=(1/2\pi)^{3/2\times3}\exp[i(\boldsymbol{k}_q^0\boldsymbol{r}_q+\boldsymbol{k}_s^0\boldsymbol{r}_s)]\exp\left[i(\boldsymbol{k}_o^0+\boldsymbol{k}_p^0)\frac{(\boldsymbol{r}_o+\boldsymbol{r}_p)}{2}\right]$$
$$\cdot\chi_{(1/2)(k_o^0-k_p^0)}(\boldsymbol{r}_o-\boldsymbol{r}_p). \qquad (A\cdot8)$$

In this case, m stands for $\boldsymbol{k}_o$, $\boldsymbol{k}_p$, $\boldsymbol{k}_q$ and $\boldsymbol{k}_s$ as a whole and n stands for $\boldsymbol{k}_o^0$, $\boldsymbol{k}_p^0$, $\boldsymbol{k}_q^0$ and $\boldsymbol{k}_s^0$ as a whole. The meaning of $\chi_{(1/2)(k_o^0-k_p^0)}(\boldsymbol{r}_o-\boldsymbol{r}_p)$ is the same as in Appendix I.

Now we assume that the wave function of the target nucleus be $X_\alpha(\boldsymbol{r}_\alpha)Y_\beta(\boldsymbol{r}_\alpha\boldsymbol{R}_\alpha)$, where $\boldsymbol{r}_\alpha=\boldsymbol{r}_q-\boldsymbol{r}_p$ and $\boldsymbol{R}_\alpha=\boldsymbol{r}_s-\frac{1}{2}(\boldsymbol{r}_p+\boldsymbol{r}_q)$. Since the other internal coordinates in the target nucleus does not concern in the transition matrix element (A·1), we omit to write explicitly the wave functions corresponding to these coordinates. Then

$$\Psi_{i,f}=(1/2\pi)^{3/2\times 2}\exp\left[i\boldsymbol{k}_{i,f}\left(\boldsymbol{r}_o-\frac{\boldsymbol{r}_p+\boldsymbol{r}_q+(A-2)\boldsymbol{r}_s}{A}\right)\right]\cdot X_\alpha(\boldsymbol{r}_\alpha)\,Y_\beta(\boldsymbol{r}_\alpha\boldsymbol{R}_\alpha). \tag{A·9}$$

Using (A·8) and (A·9), we get from (A·1)

$$\begin{aligned}H_{fi}(p)=&(1/2\pi)^{3/2\times 5}\int d\boldsymbol{k}_o\,d\boldsymbol{k}_p\,d\boldsymbol{k}_q\,d\boldsymbol{k}_s\,d\boldsymbol{k}_o^{\,0}\,d\boldsymbol{k}_p^{\,0}\,d\boldsymbol{k}_q^{\,0}\,d\boldsymbol{k}_s^{\,0}\\&\cdot\delta(\boldsymbol{k}_i-\boldsymbol{k}_o)\,\delta(\boldsymbol{k}_i+\boldsymbol{k}_p^{\,0}+\boldsymbol{k}_q^{\,0}+\boldsymbol{k}_s^{\,0})\\&\cdot\delta(\boldsymbol{k}_f-\boldsymbol{k}_o)\,\delta(\boldsymbol{k}_f+\boldsymbol{k}_p+\boldsymbol{k}_q+\boldsymbol{k}_s)\\&\cdot\delta(\boldsymbol{k}_q-\boldsymbol{k}_q^{\,0})\,\delta(\boldsymbol{k}_s-\boldsymbol{k}_s^{\,0})\,\delta(\boldsymbol{k}_o+\boldsymbol{k}_p-\boldsymbol{k}_o^{\,0}-\boldsymbol{k}_p^{\,0})\\&\cdot\iint d\boldsymbol{r}_\alpha'\,d\boldsymbol{R}_\alpha'\exp\left[i\frac{\boldsymbol{r}_\alpha'}{2}(\boldsymbol{k}_q-\boldsymbol{k}_p)\right]X_\alpha(\boldsymbol{r}_\alpha')\\&\cdot\exp\left[i\frac{\boldsymbol{R}_\alpha'}{A}(-(A-2)(\boldsymbol{k}_p+\boldsymbol{k}_q)+2\boldsymbol{k}_s)\right]Y_\beta(\boldsymbol{r}_\alpha'\boldsymbol{R}_\alpha')\\&\cdot\int\exp\left[-i\frac{1}{2}(\boldsymbol{k}_0-\boldsymbol{k}_p)\boldsymbol{r}_{op}\right]V(|\boldsymbol{r}_{op}|)\chi_{(1/2)(\boldsymbol{k}_0{}^0-\boldsymbol{k}_p{}^0)}(\boldsymbol{r}_{op})\,d\boldsymbol{r}_{op}\\&\cdot\iint d\boldsymbol{r}_\alpha\,d\boldsymbol{R}_\alpha\exp[-i(\boldsymbol{r}_\alpha/2)(\boldsymbol{k}_q^{\,0}-\boldsymbol{k}_p^{\,0})]X_\alpha(\boldsymbol{r}_\alpha)\\&\cdot\exp[-i(\boldsymbol{R}_\alpha/A)(-(A-2)(\boldsymbol{k}_p^{\,0}+\boldsymbol{k}_q^{\,0})+2\boldsymbol{k}_s^{\,0})]Y_\beta(\boldsymbol{r}_\alpha\boldsymbol{R}_\alpha).\end{aligned} \tag{A·10}$$

Eliminating the delta-functions from the above equation, and adopting the same approximation as (A·6), we get

$$\begin{aligned}\text{Eq. (A·10)}=&(1/2\pi)^{3/2}\int\exp[-i\boldsymbol{k}_f\boldsymbol{r}_{op}]V(|\boldsymbol{r}_{op}|)\chi_{k_i}(\boldsymbol{r}_{op})\,d\boldsymbol{r}_{op}\\&\cdot\iint d\boldsymbol{r}_\alpha\,d\boldsymbol{R}_\alpha\exp\left[i\Delta\boldsymbol{k}\,\frac{\boldsymbol{r}_\alpha}{2}\right]|X_\alpha(\boldsymbol{r}_\alpha)|^2\cdot\exp\left[i\Delta\boldsymbol{k}\cdot\frac{A-2}{A}\cdot\boldsymbol{R}_\alpha\right]|Y_\beta(\boldsymbol{r}_\alpha\boldsymbol{R}_\alpha)|^2,\end{aligned}$$

where $\Delta\boldsymbol{k}=\boldsymbol{k}_f-\boldsymbol{k}_i$.

Finally, under the condition $1\gg 2/A$,

$$\begin{aligned}\text{Eq. (A·10)}=&(1/2\pi)^{3/2}\int\exp[-i\boldsymbol{k}_f\boldsymbol{r}_{op}]V(|\boldsymbol{r}_{op}|)\chi_{k_i}(\boldsymbol{r}_{op})\,d\boldsymbol{r}_{op}\\&\cdot\iint d\boldsymbol{r}_\alpha\,d\boldsymbol{R}_\alpha\exp\left[i\Delta\boldsymbol{k}\,\frac{\boldsymbol{r}_\alpha}{2}\right]|X_\alpha(\boldsymbol{r}_\alpha)|^2\cdot\exp[i\Delta\boldsymbol{k}\,\boldsymbol{R}_\alpha]|Y_\beta(\boldsymbol{r}_\alpha\boldsymbol{R}_\alpha)|^2.\end{aligned} \tag{A·11}$$

This must be superposed over all p. This is the explicit form of (A·1), expressed as Eq. (4) in the text.

In this Appendix, the wave function of the target was not antisymmetrized. However, we get just the same final expression (A·11) in the text, if the target wave function is antisymmetrized from the start, provided that in this case we use

$$\exp\left[i\boldsymbol{k}_{i,f}\left(\boldsymbol{r}_o-\frac{(\boldsymbol{r}_1+\cdots+\boldsymbol{r}_p+\boldsymbol{r}_q+\cdots+\boldsymbol{r}_A)}{A}\right)\right] \text{ in the place of}$$

$$\exp\left[i\boldsymbol{k}_{i,f}\left(\boldsymbol{r}_o-\frac{(\boldsymbol{r}_p+\boldsymbol{r}_q+(A-2)\boldsymbol{r}_s)}{A}\right)\right] \text{ in (A·3). In what follows,}$$

we shall acount for this situation.

If we use the antisymmetrized wave function for the target, we get a sum over terms similar to (A·11). In some of these terms, $|X_\alpha(\boldsymbol{r}_\alpha)|^2|Y_\beta(\boldsymbol{r}_\alpha\boldsymbol{R}_\alpha)|^2$ may result from the uncorrelated wave functions for p and q. But these uncorrelated wave functions contain a very limited range of the Fourier component (only one component if we adopt the Fermi gas model). And then the density made up from these wave functions contains naturally the very limited range of the Fourier component. Under such a circumstance the integration

$$\iint d\boldsymbol{r}_\alpha d\boldsymbol{R}_\alpha \exp\left[i\Delta\boldsymbol{k}\frac{\boldsymbol{r}_\alpha}{2}\right]|X_\alpha(\boldsymbol{r}_\alpha)|^2\exp[i\Delta\boldsymbol{k}\boldsymbol{R}_\alpha]|Y_\beta(\boldsymbol{r}_\alpha\boldsymbol{R}_\alpha)|^2$$

practically vanishes. This integral have non-vanishing value only if the density have the Fourier component corresponding to $\Delta\boldsymbol{k}/2$ for $\boldsymbol{r}_\alpha$ and to $\Delta\boldsymbol{k}$ for $\boldsymbol{R}_\alpha$. Note $\Delta\boldsymbol{k}$ is given by the equation below (3) in the text. $\Delta\boldsymbol{k}$ takes any value equal to and greater than zero. Accordingly, we can expect a non-vanishing value of the above integral from the correlated density but not from the uncorrelated one.

In the remaining part of these terms, p and $s(\neq q)$ may be correlated and p and q uncorrelated. In this case also the integral practically vanishes; so even if we use the antisymmetrized wave function from the start, we get the same result (A·11).

One may wonder if we are *not* dealing with the elastic scattering, because the Fourier component of the target changes from $-\boldsymbol{k}_i$ to $-\boldsymbol{k}_f$. Answer to this question is as follows. Since $|\boldsymbol{k}_i|=|\boldsymbol{k}_f|$, the energy of the target is conserved by this process. And as our derivation shows, the momentum of the total system conserves. Accordingly, by the process, only the direction of the target changes from $-\boldsymbol{k}_i$ to $-\boldsymbol{k}_f$, due to the recoil, without changing its internal energy. Then we are *actually* dealing with the elastic scattering.

Note added in proof:

(1) Using the set of phase shifts by Gammel and Thaler, we find that the value of $\nu'|\lambda|^2$ is about 0.4~0.5 nearly independent of energy (95 Mev~310 Mev) and mass number ($A \geq 50$). This result completely matches the predictions given in the present paper.

(2) In fact, we can show this. The detailed foundation on this point will appear in a future paper now in preparation.

References

1) W. Heisenberg, Leipziger Ber. **89** (1937), 369.
2) R. Serber, Phys. Rev. **72** (1947), 1114.
3) Y. Sakamoto and T. Sasakawa, Prog. Theor. Phys. **19** (1958), 745.
4) Y. Sakamoto and T. Sasakawa, Prog. Theor. Phys. **21** (1958), 879.
5) Y. Sakamoto and T. Sasakawa, Prog. Theor. Phys. **22** (1959), 299.
6) H. A. Bethe, Ann. Phys. **3** (1958), 190.
7) S. Ohnuma, Phys. Rev. **111** (1958), 1173.
8) J. Sawicki and S. Watanabe, Nuclear Phys. **10** (1959), 151.
9) L. Castillejo and L. S. Singh, Nuovo Cimento **XI** (1959), 131.
10) A. H. Cromer, Phys. Rev. **113** (1959), 1607.
11) For example, A. E. Glassgold and P. J. Kellog, Phys. Rev. **109** (1958), 1291.
12) T. Sasakawa, Prog. Theor. Phys. **23** (1960), 177.
13) Gomes, Walecka and Weisskopf, Ann. Phys. **5** (1958), 2821.
14) G. F. Chew and G. C. Wick, Phys. Rev. **85** (1952), 636.
15) G. F. Chew and M. L. Goldberger, Phys. Rev. **87** (1952), 778.
16) J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, New York, 1952), P. 52.
17) T. Sasakawa, Prog. Theor. Phys. **22** (1959), 738.
18) S. Otsuki, Prog. Theor. Phys. **20** (1958), 171.
W. Watari, Prog. Theor. Phys. **20** (1958), 181
19) S. C. Wright, Phys. Rev. **99** (1955), 996.
20) See C. Van Lier and G. E. Uhlenbeck, Physica, IV (1937), 531.
21) Gerstein, Niederer and Strauch, Phys. Rev. **108** (1957), 427.
The author is grateful for their sending experimental data.
22) A. de Shalit and V. F. Weisskopf, Ann. Phys. **5** (1958), 451.

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Symmetries of the Interactions among Strongly Interacting Particles and Non-Leptonic Decays of Hyperons

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A symmetry law which can hold true in both strong and weak interactions among the strongly interacting particles is investigated and an example is given. The requirement of invariance under the proposed symmetry transformation and some assumptions lead to the interactions which give consistent result with experiments of $\Sigma^{\pm}$ and Λ decays.

Implications of such a symmetry law are discussed and the problems concerning the K-meson decay into two pions are mentioned.

§ 1. Introduction

Many works concerning the symmetry properties of the strong interaction have been carried out for the purpose of reducing the independent number of the coupling constants of the strong interaction,[1] or, with the view of getting the parity-conserving strong interaction under the assumptions of CP invariance and some symmetry laws in the internal space (e.g. charge independence).[2] In these works, the strong interaction Hamiltonian is required to possess more internal symmetry than is implied by the conventional charge independence, and some of the conditions thus required lead to invariance of the strong interaction under various formal discrete substitution transformations. These attempts seem to be very interesting in the sense that they are made to investigate a possible connection between symmetry properties in the Lorentz space and those in the internal space. Most of these conditions or symmetry laws, however, concern only the strong interaction and have nothing to do with the weak interaction; it seems difficult to prove the validity of these conditions experimentally, and at the utmost we can only say that the conditions with an extremely high degree of internal symmetry, such as the global symmetry with the same pion-baryon coupling constants, are inconsistent with experimental facts;[3] too many conditions are required in order to get the parity-conserving strong interaction.

In this note, an attempt is made to find symmetry laws which can reasonably hold true in the strong interaction as well as in the weak interaction among the strongly interacting particles (abbreviated as SIP's). We may say that CP invariance is one of such symmetry laws. Symmetry laws which we now examine have relation not only to the Lorentz space but also to the internal space. If we

can find such a law that governs both the strong and the weak interactions, we shall have possible methods to establish the validity of the symmetry law by using the weak interaction.

In § 2, a new symmetry law is given as one of the possible examples. In § 3, implications of the symmetry law are discussed. The problems concerning the K-meson decay into two pions are also mentioned.

§ 2. A possible symmetry law

We define, according to Pais,[3)4)] charge doublets of baryons as follows:

$$N_1 \equiv \begin{bmatrix} p \\ n \end{bmatrix}, \quad N_2 \equiv \begin{bmatrix} \Sigma^+ \\ Y^0 \end{bmatrix}, \quad N_3 \equiv \begin{bmatrix} Z^0 \\ \Sigma^- \end{bmatrix}, \quad N_4 \equiv \begin{bmatrix} \Xi^0 \\ \Xi^- \end{bmatrix}, \tag{1}$$

$$Y^0 \equiv (\Lambda - \Sigma^0)/\sqrt{2}, \quad Z^0 \equiv (\Lambda + \Sigma^0)/\sqrt{2}. \tag{2}$$

We assume, at the stage of switching off the K meson-baryon interaction, the doublet structure for all of the baryons, and assume that the masses of (N_1, N_4) and (N_2, N_3) are degenerate respectively. It is also possible to assume for

$$N_2' \equiv \begin{bmatrix} \Sigma^+ \\ -Z^0 \end{bmatrix} \text{ and } \quad N_3' \equiv \begin{bmatrix} -Y^0 \\ \Sigma^- \end{bmatrix}$$

the same as for N_2 and N_3. Here the doublet structure for N_2 and N_3 is assumed tentatively, and this is not essential for the later discussion. We introduce "4-component spinors" as follows:

$$\Psi \equiv \begin{bmatrix} N_1 \\ N_4 \end{bmatrix}, \qquad \Phi \equiv \begin{bmatrix} N_2 \\ N_3 \end{bmatrix}, \tag{3}$$

$$\Psi' \equiv \begin{bmatrix} p \\ \Xi^0 \\ \rho n \\ \rho \Xi^- \end{bmatrix} = \begin{bmatrix} N_1' \\ N_4' \end{bmatrix}, \qquad \Phi' \equiv \begin{bmatrix} \Sigma^+ \\ -Z^0 \\ -Y^0 \\ \Sigma^- \end{bmatrix} = \begin{bmatrix} N_2' \\ N_3' \end{bmatrix}, \tag{4}$$

where ρ is a phase factor which is to be determined experimentally.

Now, *we will write by using these quantities all of the pion-baryon interactions of Yukawa type which are consistent with the strangeness selection rule* (*Assumption I*). Taking the viewpoint of H. Umezawa[5)] or D. Itô, et al.,[6)] we assume that *the strong and the weak Yukawa type interactions are of direct and derivative coupling respectively* (*Assumption II*). Further, it is assumed that the interaction Hamiltonians satisfy the following requirement: *the strong pion-baryon interaction is, in the sense of the doublet structure, of rotational invariance in the charge space, i.e. of charge independence, and the weak interaction includes not only charge independent terms but also "apparently" charge independent terms in the sense that* N_1', N_2', N_3' *and* N_4' *are doublets* (*Assumption III*). The meaning of the last assumption is stated as follows. As Salam and Polkinghorne[12)] have already discussed, each multiplet of SIP's can

correspond to the irreducible representation, $D(I^+, I^-)$, of the rotational group in the 4-Euclidian space. In order to give a meaning to the "charge independent" interaction forms, we should assume that *each doublet and the pion-field correspond to the irreducible representations written in Tables I and II*, and that, *in all interactions, terms that conserve* $\mathbf{I}^+$ *and* $\mathbf{I}^-$ *not only in the sense of Table I but also in the sense of Table II must be contained* (*Assumption III*). (Note that we have assumed the strangeness selection rule to hold true independently.) Some comments concerning Assumption III will be given in the last section.

Table I.

	I_3^+	I_3^-
N_1	$\pm 1/2$	$+1/2$
N_4	$\pm 1/2$	$-1/2$
N_2	$\pm 1/2$	$+1/2$
N_3	$\pm 1/2$	$-1/2$
π	$+1, 0, -1$	0

Table II.

	I_3^+	I_3^-
N_1'	$+1/2$	$\pm 1/2$
N_4'	$-1/2$	$\pm 1/2$
N_2'	$+1/2$	$\pm 1/2$
N_3'	$-1/2$	$\pm 1/2$
π	0	$+1, 0, -1$

Now we define the "*generalized space inversion*" $\mathcal{P}$, which means the simultaneous transformation of the space reflection in the Lorentz space (P), and the charge symmetry transformations in both I^+- and I^--spaces. (The latter transformations are denoted as S^+ and S^-, respectively.) Giving the meaning described above to the Assumption III, we can easily see that all of the possible interactions among pions and baryons, which include parity-conserving and parity-nonconserving terms, satisfy the requirement of invariance under $\mathcal{P}=PS^+S^-$ in the sense of Table I as well as in the sense of Table II. Under $\mathcal{P}$,

$$\Psi(x)\to\Psi_P(-\boldsymbol{x}, x_0)=\eta\gamma_4\cdot\begin{bmatrix} 0 & i\tau_2 \\ -i\tau_2 & 0 \end{bmatrix}\cdot\begin{bmatrix} N_1(x) \\ N_4(x) \end{bmatrix}$$

$$\equiv -\eta\cdot\gamma_4\Gamma_2\cdot\Psi(x), \tag{5}$$

$$\Phi(x)\to\Phi_P(-\boldsymbol{x}, x_0)=\xi\gamma_4\cdot\begin{bmatrix} 0 & i\tau_2 \\ -i\tau_2 & 0 \end{bmatrix}\cdot\begin{bmatrix} N_2(x) \\ N_3(x) \end{bmatrix}$$

$$\equiv -\xi\cdot\gamma_4\Gamma_2\cdot\Phi(x), \tag{6}$$

and

$$(\pi_1(x), \pi_2(x), \pi_3(x))\to(\pi_{1P}(-\boldsymbol{x}, x_0), \pi_{2P}(-\boldsymbol{x}, x_0), \pi_{3P}(-\boldsymbol{x}, x_0))$$

$$=(\pi_1(x), -\pi_2(x), \pi_3(x)), \tag{7}$$

where we have put

$$\Gamma_2\equiv\begin{bmatrix} 0 & -i\tau_2 \\ i\tau_2 & 0 \end{bmatrix}=\rho_2\times\tau_2, \tag{8}$$

and, η and ξ are phase factors. Under $\mathcal{P}$, Ψ' and Φ' then transform as

$$\Psi' \to \eta \cdot \gamma_4 \begin{bmatrix} 0 & i\rho^* \tau_2 \\ -i\rho\tau_2 & 0 \end{bmatrix} \cdot \begin{bmatrix} N_1' \\ \rho N_4' \end{bmatrix}, \tag{9}$$

$$\Phi' \to -\xi \cdot \gamma_4 \Gamma_2 \cdot \Phi'. \tag{10}$$

We take $\rho = -1$ in order that the parity-conserving and parity-nonconserving terms coexist in the weak interaction Hamiltonian. Then, (9) reduces to the transformation

$$\Psi' \to \eta\, \gamma_4\, \Gamma_2 \cdot \Psi'. \tag{11}$$

It is evident that the strong interaction is parity-conserving, the explicit form of which is

$$\begin{aligned} & g_1 \bar{\Psi} i\gamma_5 (\boldsymbol{\sigma}\cdot\boldsymbol{\pi}) \Psi + g_2 \bar{\Phi} i\gamma_5 (\boldsymbol{\sigma}\cdot\boldsymbol{\pi}) \Phi + g_2' \bar{\Phi}' i\gamma_5 (\boldsymbol{\sigma}\cdot\boldsymbol{\pi}) \Phi' \\ & = g_1' [\bar{N}_1 i\gamma_5 (\boldsymbol{\tau}\cdot\boldsymbol{\pi}) N_1 + \bar{N}_4 i\gamma_5 (\boldsymbol{\tau}\cdot\boldsymbol{\pi}) N_4] \\ & \quad + G_2 [(\bar{\boldsymbol{\Sigma}}\cdot\boldsymbol{\pi}) i\gamma_5 \Lambda + \bar{\Lambda} i\gamma_5 [\boldsymbol{\Sigma}\cdot\boldsymbol{\pi})] - iG_2' ([\bar{\boldsymbol{\Sigma}} i\gamma_5 \times \boldsymbol{\Sigma}]\cdot\boldsymbol{\pi}), \end{aligned} \tag{12}$$

where $\bar{\Psi} = \Psi^+ \gamma_4$, $\boldsymbol{\sigma} \equiv \begin{bmatrix} \boldsymbol{\tau} & 0 \\ 0 & \boldsymbol{\tau} \end{bmatrix}$, $G_2 \equiv g_2 - g_2'$ and $G_2' \equiv g_2 + g_2'$.

The possible weak interaction is written down as follows:

$$\begin{aligned} & f \bar{\Phi} \gamma_\mu (\boldsymbol{\sigma}\cdot\partial_\mu \boldsymbol{\pi}) \Psi + f' \cdot \bar{\Phi}' \gamma_5 \gamma_\mu (\boldsymbol{\sigma}\cdot\partial_\mu \boldsymbol{\pi}) \Psi' + H.C. \\ & = f \Big[(\bar{\Sigma}^+, \bar{Y}^0) \gamma_\mu (\boldsymbol{\tau}\cdot\partial_\mu \boldsymbol{\pi}) \begin{bmatrix} p \\ n \end{bmatrix} + (\bar{Z}^0, \bar{\Sigma}^-) \gamma_\mu (\boldsymbol{\tau}\cdot\partial_\mu \boldsymbol{\pi}) \begin{bmatrix} \Xi^0 \\ \Xi^- \end{bmatrix} \Big] \\ & + f' \Big[(\bar{\Sigma}^+, -\bar{Z}^0) \gamma_5 \gamma_\mu (\boldsymbol{\tau}\cdot\partial_\mu \boldsymbol{\pi}) \begin{bmatrix} p \\ \Xi^0 \end{bmatrix} \\ & \qquad - (-\bar{Y}^0, \bar{\Sigma}^+) \gamma_5 \gamma_\mu (\boldsymbol{\tau}\cdot\partial_\mu \boldsymbol{\pi}) \begin{bmatrix} n \\ \Xi^- \end{bmatrix} \Big] + H.C. \end{aligned} \tag{13}$$

Here we have taken $\eta \cdot \xi^* = -1$. If we take $\eta \cdot \xi^* = +1$, γ_μ and $\gamma_5 \gamma_\mu$ are substituted for $\gamma_5 \gamma_\mu$ and γ_μ, respectively, in (13). As far as $f^2 \approx f'^2$, it is not able to distinguish experimentally between these two choises of the phase. Putting $f' = \lambda f$, we get the following interaction Hamiltonians corresponding to non-leptonic decays of $\Sigma^\pm$ and Λ particles.

$$\begin{aligned} H_{\text{int}}(\Sigma^\pm \to N\pi) = f \Big[& \sqrt{2} \bar{\Sigma}^+ \gamma_\mu n \cdot \partial_\mu \pi^+ + \bar{\Sigma}^+ (1 + \lambda\gamma_5) \gamma_\mu p \cdot \partial_\mu \pi^0 \\ & - \lambda \sqrt{2}\, \bar{\Sigma}^- \gamma_5 \gamma_\mu n \cdot \partial_\mu \pi^- \Big] + H.C., \end{aligned} \tag{14}$$

$$H_{\text{int}}(\Lambda \to N\pi) = f \left[\bar{\Lambda} (1 - \lambda\gamma_5) \gamma_\mu p \cdot \partial_\mu \pi^- - \frac{1}{\sqrt{2}} \bar{\Lambda} (1 - \lambda\gamma_5) \gamma_\mu n \cdot \partial_\mu \pi^0 \right] + H.C. \tag{15}$$

It is easily seen that the form (13) is very much similar but not equivalent to the interaction given by d'Espagnat and Prentki.[7] However, (14) and (15)

are equivalent to their $|\Delta I|=1/2$ global symmetric interaction. In other words, both interaction forms can equally well explain the experimental facts concerning non-leptonic decays of $\Sigma^{\pm}$ and Λ.† It should be noted that in (13) the asymmetry coefficients of $\Xi^- \to \Lambda+\pi^-$ and $\Xi^0 \to \Lambda+\pi^0$ are predicted to have the same sign, which is different from that of $\Lambda \to N+\pi$.†† This is a point in which (13) differs from the $|\Delta I|=1/2$ global symmetric interaction.[7] (See Appendix I.)

§ 3. Discussions

3–1. *Implications of the transformation* $\mathcal{P}$

In the preceding section, a possible and simple example of symmetry law, which can hold true in both the strong and the weak interactions among SIP's, was given, and there may be some possibilities other than the example given above. As a typical example of them, using

$$\Psi_1 \equiv \begin{bmatrix} N_1 \\ -N_4 \end{bmatrix}, \qquad \Phi_1 \equiv \begin{bmatrix} N_2 \\ -N_3 \end{bmatrix},$$

$$\Psi_1' \equiv \begin{bmatrix} p \\ -\Xi^0 \\ -n \\ \Xi^- \end{bmatrix} \equiv \begin{bmatrix} N_1'' \\ -N_4'' \end{bmatrix}, \qquad \Phi_1' \equiv \begin{bmatrix} N_2' \\ -N_3' \end{bmatrix},$$

we can obtain the following $\mathcal{P}$-invariant weak interaction.†††

$$\begin{aligned} & f\cdot\bar{\Phi}_1\gamma_\mu(\boldsymbol{\sigma}\cdot\partial_\mu\boldsymbol{\pi})\Psi_1 + f'\cdot\bar{\Phi}_1'\gamma_5\gamma_\mu(\boldsymbol{\sigma}\cdot\partial_\mu\boldsymbol{\pi})\Psi_1' + H.C. \\ & = f\Big[(\bar{\Sigma}^+, \bar{Y}^0)\gamma_\mu(\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\pi})\begin{bmatrix} p \\ n \end{bmatrix} + (\bar{Z}^0, \bar{\Sigma}^-)\gamma_\mu(\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\pi})\times \\ & \times\begin{bmatrix} \Xi^0 \\ \Xi^- \end{bmatrix}\Big] + f'\Big[-(\bar{Z}^0, \bar{\Sigma}^+)\gamma_5\gamma_\mu(\boldsymbol{\tau}^T\cdot\partial_\mu\boldsymbol{\pi})\begin{bmatrix} \Xi^0 \\ p \end{bmatrix} \\ & \quad + (\bar{\Sigma}^-, \bar{Y}^0)\gamma_5\gamma_\mu(\boldsymbol{\tau}^T\cdot\partial_\mu\boldsymbol{\pi})\begin{bmatrix} \Xi^- \\ n \end{bmatrix}\Big] + H.C. \end{aligned} \tag{17}$$

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† On the basis of the $V\mp A$ interaction and the global symmetry, Takeda and Katô[8] proposed a model which explains the observed asymmetry coefficients of $\Sigma^{\pm}$ and Λ decays as the effects due to the mesonic structure of hyperons. Their primary weak interaction contains no term corresponding to the decay mode, $\Sigma^- \to n+\pi^-$. Contrary to the view of Takeda and Katô, our view is, as already mentioned in §1, that the weak interaction is used to establisn the validity of the symmetry law *directly*. As for comparison of the values predicted by (13) or (17) with the experimental results, see Hara's paper.[9]

†† Because of Assumption III, we have omitted the following interaction being consistent with the strangeness selection rule.

$$F\cdot\bar{\Phi}'\gamma_\mu\ (\sigma\cdot\partial_\mu\pi)\ \Psi + F'\cdot\bar{\Phi}\gamma_5\gamma_\mu\ (\sigma\cdot\partial_\mu\pi)\ \Psi' + H.C. \tag{16}$$

If we take $f'=F=0$ and $F'=\lambda f$, we get an interaction Hamiltonian with $-\lambda$ for λ in (15), which gives the same sign to the asymmetry coefficients of Λ and $\Sigma^{\pm}$ decays.

††† Note that $\bar{\Psi}_1\boldsymbol{\sigma}\Psi_1 = \bar{\Psi}\boldsymbol{\sigma}\Psi$, $\bar{\Phi}_1\boldsymbol{\sigma}\Phi_1 = \Phi\boldsymbol{\sigma}\Phi$.

Here we have used the transformation formulae of Ψ_1, Ψ_1', Φ_1 and Φ_1' under $\not{\mathcal{P}}$: $\Psi_1 \to \eta \cdot \gamma_4 \Gamma_2 \cdot \Psi_1$, $\Psi_1' \to -\eta \cdot \gamma_4 \Gamma_2 \cdot \Psi_1'$, $\Phi_1 \to \xi \cdot \gamma_4 \Gamma_2 \cdot \Phi_1$, $\Phi_1' \to \xi \cdot \gamma_4 \Gamma_2 \cdot \Phi_1'$, and put $\eta \cdot \xi^* = -1$. Then, instead of Eqs. (14) and (15), the following expressions are obtained.

$$H_{\text{int}}(\Sigma^{\pm} \to N\pi) = f\Big[\bar{\Sigma}^{+}(1+\lambda\gamma_5)\gamma_\mu p \cdot \partial_\mu \pi^0 + \sqrt{2}\,\bar{\Sigma}^{+}\gamma_\mu n \cdot \partial_\mu \pi^{+} + \sqrt{2}\,\lambda\,\bar{\Sigma}^{-}\gamma_5\gamma_\mu n \cdot \partial_\mu \pi^{-}\Big] + H.C., \quad (18)$$

$$H_{\text{int}}(\Lambda \to N\pi) = f\Big[\bar{\Lambda}(1-\lambda\gamma_5)\gamma_\mu p \cdot \partial_\mu \pi^{-} - \frac{1}{\sqrt{2}}\bar{\Lambda}(1+\lambda\gamma_5)\gamma_\mu n \cdot \partial_\mu \pi^0\Big] + H.C. \quad (19)$$

Many possibilities due to the choice of phases in constructing the " 4-component spinors " give interaction forms, each of which can explain the asymmetry coefficients of $\Sigma^{\pm}$ and Λ decays equivalently to either (13) or (17). The interaction (19) gives the asymmetry coefficients for Λ decays with the same magnitude but different sign from each other. At present, (13), (17) and the $|\Delta I| = 1/2$ global symmetric interaction (A·1) can explain the experimental facts equally well. (See Appendices.)

Next, we make some comments concerning the transformation $\not{\mathcal{P}}$. This transformation seems at first sight to be a simultaneous transformation of the space inversion in the Lorentz space (P), the space inversion in the 4-dimensional charge space (P') and the charge symmetry transformation in the ordinary charge space (S). However, some difficulty arises in such an interpretation. We have implicitly assumed that the pion field is a self dual tensor in order to avoid introducing a scalar $\pi^{0\prime}$, hence the 4-dimensional charge space must be Euclidian.[10]* When we consider only pion-baryon interactions, the nucleon and Ξ (or Λ and Σ) should be attributed to the spinor representation of the rotational group in the 4-Euclidian space, $D(1/2, 0)$, or the vector representation, $D(1/2, 1/2)$. Consequently, we cannot interpret the transformation $N \leftrightarrow \Xi$ as the reflection, analogously to the reflection in the Lorentz space.

Next, according to Assumption III, we must introduce two kinds of pions, each of which corresponds to the different irreducible representation of the rotation group in the 4-Euclidian space. This corresponds to the note given by d'Espagnat and Prentki about Eqs. (15) and (16) in ref. 7). (See (A·1) in Appendix. See also Sakurai's paper.[17])

If we assume that the 4-dimensional internal space is of the Lorentz metric, it can be said that the transformation $\not{\mathcal{P}}$ means simultaneous transformations of

* If we take the 4-dimensional charge space to be of the Lorentz metric, there arise many difficulties other than that mentioned above,[10][11]

P, P' and S. The interactions are written, for instance, as follows:
The strong interaction:

$$g_1\Big[(\bar{N}_1 i\gamma_5 \boldsymbol{\tau} N_1+\bar{N}_4 i\gamma_5 \boldsymbol{\tau} N_4)\boldsymbol{\pi}+(-\bar{N}_1 N_1+\bar{N}_4 N_4)\pi^{0\prime}\Big]$$

$$+g_2\Big[(\bar{N}_2 i\gamma_5 \boldsymbol{\tau} N_2+\bar{N}_3 i\gamma_5 \boldsymbol{\tau} N_3)\boldsymbol{\pi}+(-\bar{N}_2 N_2+\bar{N}_3 N_3)\pi^{0\prime}\Big], \qquad (20)$$

and the weak interaction:

$$f\Big[(\bar{N}_2 \gamma_\mu \boldsymbol{\tau} N_1+\bar{N}_3 \gamma_\mu \boldsymbol{\tau} N_4)\partial_\mu\boldsymbol{\pi}+(-\bar{N}_2 \gamma_5\gamma_\mu N_1+\bar{N}_3 \gamma_5\gamma_\mu N_4)\partial_\mu\pi^{0\prime}\Big]$$

$$+f'\Big[(\bar{N}_2' \gamma_5\gamma_\mu \boldsymbol{\tau} N_1'+\bar{N}_3'\gamma_5\gamma_\mu \boldsymbol{\tau} N_4')\partial_\mu\boldsymbol{\pi}+(-\bar{N}_2'\gamma_\mu N_1'+\bar{N}_3'\gamma_\mu N_4')\partial_\mu\pi^{0\prime}\Big]+H.C. \qquad (21)$$

Here we have taken $\eta\cdot\hat{\xi}^*=-1$ and assumed that π_μ is a 4-pseudovector in the internal space, in other words, under PP', $(\boldsymbol{\pi}, \pi^{0\prime})$ is transformed into $(-\boldsymbol{\pi}, -\pi^{0\prime})$. (See the appendix of ref. 10).)

Although there are many difficulties for the case of the Lorentz metric of the internal space[10)11)] and it may not be able to give a geometrical interpretation to the "charge independent" forms of the interaction,† the Lorentz metric seems more favourable than the Euclidian metric from a view that we search for a possible connection between the Lorentz space and the internal space.

3–2. *K meson-baryon interactions*

Untill now, we have discussed symmetry laws assuming that the K-meson interaction is absent. If we assume the transformation law of the K-meson field under $\mathcal{P}$ to be

$$\begin{bmatrix}K\\ \dot{K}\end{bmatrix}\to\zeta\cdot\Gamma_2\begin{bmatrix}K\\ \dot{K}\end{bmatrix}, \qquad K\equiv\begin{bmatrix}K^+\\ K^0\end{bmatrix}, \qquad \dot{K}\equiv i\tau_2 K^*, \qquad (22)$$

then we can find, for instance, a K meson-baryon interaction which is invariant under $\mathcal{P}$:

$$g_3\,\overline{\Psi}_1 i\gamma_5 \mathcal{K}\Phi_1+g_3'\,\overline{\Psi}_1 i\gamma_5 \mathcal{K}\Phi_1'+H.C. \qquad (23)$$

$$(=g_3\,\overline{\Psi} i\gamma_5 \mathcal{K}'\Phi+g_3'\overline{\Psi} i\gamma_5 \mathcal{K}'\Phi'+H.C.) \qquad (24)$$

† As for the interpretation of the "charge independent" forms of the interactions, for the present we cannot help being satisfied with regarding these forms as a kind of symmetries which reflect the internal structure of elementary particles at the stage of switching off the K-interaction.

$$=\frac{i}{\sqrt{2}}(g_3-g'_3)\left[(\bar{N}_1\ K)\gamma_5\Lambda+\bar{N}_1\cdot\gamma_5(\boldsymbol{\tau}\cdot\boldsymbol{\Sigma})\,K\right]+H.C.$$

$$+\frac{i}{\sqrt{2}}(g_3+g'_3)\left[(\bar{N}_4\,\dot{K})\gamma_5\Lambda+\bar{N}_4\gamma_5(\boldsymbol{\tau}\cdot\boldsymbol{\Sigma})\dot{K}\right]+H.C., \qquad (25)$$

where we have put $\mathcal{K}\equiv\begin{bmatrix} K^0 & -K^+ \\ K^- & K^0 \end{bmatrix}$, $\mathcal{K}'\equiv\begin{bmatrix} K^0 & K^+ \\ -K^- & K^0 \end{bmatrix}$, and $\gamma^*\cdot\xi\cdot\zeta=-1$ and $+1$ for (23) and (24) respectively. We cannot give to (23) or (24) a meaning such as described in 3–1, and we have no guiding principles how to construct the K meson-baryon interaction. (In this connection, we have to pay attention to the paper by Feinberg and Gürsey.[2]) If we put $g_3=0$ or $g_3'=0$, the strong interaction thus obtained has a too narrow symmetry, and one cannot account for the N-Ξ mass difference.[13]* For the present, we assume that $g_3\cdot g_3'=$ is not equal to zero.

Next, as already mentioned, the weak interactions obtained above are similar but not equivalent to the $|\Delta I|=1/2$ global symmetric interaction given by d'Espagnat and Prentki,[7] but they can give predictions equally consistent with the present experimental facts about non-leptonic decays of $\Sigma^\pm$ and Λ. One might be able to distinguish these interactions by invesigating the K-meson decays into two pions. In the case of (13), it is not possible to get a reasonable explanation of the difference between K^+- and K^0-lifetimes, because (13) is consistent with the $|\Delta I|=1/2$ rule. In the case of (17), although a quantitative argument is difficult, it seems that we can obtain consistent values of the difference of lifetime and the branching ratio of the K-meson decays into two pions. (See Appendix II.) Anyhow, it seems interesting to investigate possible symmetry laws which, like the $|\Delta I|=1/2$ rule, are able to explain the experimental results of $\Sigma^\pm$ and Λ decays equally well, and, different from the $|\Delta I|=1/2$ rule, can give an explanation for the K-meson decay.

3–3. *Final remarks*

From the view that it is important to explore phenomenologically a possible connection between the degree of freedom belonging to the Lorentz space and that belonging to the internal space, we have investigated a symmetry law which can be applied to both the strong and the weak interactions of SIP's. As an example of such a smmetry law, the transformation $\mathcal{P}$, which comes into operation combined with the ordinary space reflection and some kind of transformations in the internal space, was given and the invariance of all the interactions of SIP's under this transformation was required. Although we can give a good explanation of the

* If $g_3=0$ or $g_3'=0$, we cannot require $\mathcal{P}$ invariance of the K-baryon interaction. We are, according to d'Espagnat and Prentki,[14] led to a view that, by introducing the K-field, one must obtain the 3-dimensional and not 4-dimensional representation of the baryon fields, i.e. $D(1)$ for Σ, $D(0)$ for Λ and $D(1/2)$ for N, Ξ and K.

experimental facts about $\Sigma^{\pm}$ and Λ decays, it is desirable to modify this transformation suitably in order to give it a clear physical meaning and obtain a unique form of the interactions. It also seems desirable to find a connection between the symmetry law proposed in this paper, basing on the local-field theory and the theory of multipole model of pion-baryon interactions[6] recently developed by D. Itô and the author.[15]

In spite of the similarity between the weak coupling constants of SIP's and those of weakly interacting particles, we take the view that some symmetry law that holds true commonly both in strong and weak interactions should be looked for. The reason why we explore such a symmetry law is as follows. The degrees of freedom belonging to the internal space, which would describe the internal structure of elementary particles, may be important not only in the strong interaction but also in the weak interaction, because in decay processes, the internal structure of elementary particles seems to play an essential role, and in fact, the $|\Delta I|=1/2$ rule is able to give many predictions consistent with the experimental results.

From the substantialistic viewpoint, it seems important for us to make clear what fundamental structure of elementary particles the symmetry properties proposed in this paper reflect.

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Appendix I

Details of (13) (or (17)) are given as follows.

$$f\Big[\sqrt{2}\,\bar{\Sigma}^{+}\gamma_\mu n\cdot\partial_\mu\pi^{+}+\bar{\Sigma}^{+}(1+\lambda\gamma_5)\gamma_\mu p\cdot\partial_\mu\pi^{0}-_{(+)}\lambda\sqrt{2}\bar{\Sigma}^{-}\gamma_5\gamma_\mu n\cdot\partial_\mu\pi^{-}$$

$$-\bar{\Sigma}^{0}(1+\lambda\gamma_5)\gamma_\mu p\cdot\partial_\mu\pi^{-}+\frac{1}{\sqrt{2}}\bar{\Sigma}^{0}(1-_{(+)}\lambda\gamma_5)\gamma_\mu n\cdot\partial_\mu\pi^{0}$$

$$+\bar{\Lambda}(1-\lambda\gamma_5)\gamma_\mu p\cdot\partial_\mu\pi^{-}-\frac{1}{\sqrt{2}}\bar{\Lambda}(1-_{(+)}\lambda\gamma_5)\gamma_\mu n\cdot\partial_\mu\pi^{0}\Big]+H.C.$$

$$+f\Big[\sqrt{2}\,\bar{\Xi}^{-}\gamma_\mu\Xi^{0}\cdot\partial_\mu\pi^{-}-\bar{\Sigma}^{-}(1-\lambda\gamma_5)\gamma_\mu\Xi^{-}\cdot\partial_\mu\pi^{0}+_{(-)}\lambda\sqrt{2}\,\bar{\Sigma}^{+}\gamma_5\gamma_\mu\times$$

$$\times\Xi^{0}\cdot\partial_\mu\pi^{+}+\bar{\Sigma}^{0}(1-\lambda\gamma_5)\gamma_\mu\Xi^{-}\cdot\partial_\mu\pi^{+}+\frac{1}{\sqrt{2}}\bar{\Sigma}^{0}(1+_{(-)}\lambda\gamma_5)\gamma_\mu\times$$

$$\times\Xi^{0}\cdot\partial_\mu\pi^{0}+\bar{\Lambda}(1+\lambda\gamma_5)\gamma_\mu\Xi^{-}\cdot\partial_\mu\pi^{+}+\frac{1}{\sqrt{2}}\bar{\Lambda}(1+_{(-)}\lambda\gamma_5)\gamma_\mu\Xi^{0}\cdot\partial_\mu\pi^{0}\Big]+H.C.$$

If we substitute $-\lambda$ for λ into the first square brackets in the above expression of (13), we get the expression equivalent to that given by d'Espagnat and Prentki :[7]

$$f\left[\bar{\Sigma}_k^{\dot{n}}\gamma_\mu\partial_\mu\pi_k^m B_m^{\dot{n}}+\lambda\bar{\Sigma}_k^{\dot{n}}\gamma_5\gamma_\mu B_k^{\dot{m}}\cdot\partial_\mu\pi_{\dot{m}}^{\dot{n}}\right]+H.C.$$

$$=f\left[\bar{\Sigma}\gamma_\mu(\boldsymbol{\tau}\times\dot{1})B\cdot\partial_\mu\boldsymbol{\pi}+\lambda\bar{\Sigma}\gamma_5\gamma_\mu(1\times\ddot{\boldsymbol{\tau}}^T)B\cdot\partial_\mu\boldsymbol{\pi}\right]+H.C., \qquad (A\cdot1)$$

where we have put

$$\Sigma=\left[\Sigma_k^{\dot{n}}\right]=\begin{bmatrix}\Sigma^0 & \Sigma^+\\ \Sigma^- & Y^0\end{bmatrix},\quad B=\left[B_k^{\dot{n}}\right]=\begin{bmatrix}\Xi^0 & p\\ \Xi^- & n\end{bmatrix},\ \text{and}$$

$$(\pi_k^m)=(\pi_{\dot{k}}^{\dot{m}})=\begin{bmatrix}\pi^0/\sqrt{2} & \pi^+\\ \pi^- & -\pi^0/\sqrt{2}\end{bmatrix}.$$

Appendix II

Writing all Feynman diagrams corresponding to the lowest order perturbation and using (17), we can get the following K-meson decay amplitudes :

In the case of pseudoscalar K meson,

$$A(K^+\to\pi^+\pi^0)\sim 2(a\cdot A_1+A_2)g_{NN\pi}\,g_{\Sigma NK}+(b\cdot A_4+A_5)g_{\Sigma\Sigma\pi}\,g_{\Sigma NK}\Big]$$

$$+2\Big[(a'\cdot B_1+B_2)g_{\Xi\Xi\pi}\,g_{\Sigma\Xi K}+(b\cdot B_4+B_5)g_{\Sigma\Sigma\pi}\,g_{\Sigma\Xi K}\Big],$$

$$A(K_1^0\to 2\pi^0)\sim\frac{1}{2}\Big[(a\cdot A_1+3A_2)g_{NN\pi}\,g_{\Sigma NK}+\qquad(ab\cdot A_3+b\cdot A_4$$

$$+2A_5)g_{\Sigma\Sigma\pi}\,g_{\Sigma NK}\Big]$$

$$+\frac{1}{2}\Big[(a'\cdot B_1+3B_2)g_{\Xi\Xi\pi}\,g_{\Sigma\Xi\pi}+(a'b\cdot B_3+b\cdot B_4+2B_5)g_{\Sigma\Sigma\pi}\,g_{\Sigma\Xi K}\Big],$$

$$A(K_1^0\to\pi^+\pi^-)\sim\Big[(-a\cdot A_1+A_2)g_{NN\pi}\,g_{\Sigma NK}+(-b\cdot A_4+A_5)g_{\Sigma\Sigma\pi}\,g_{\Sigma NK}\Big]$$

$$+\Big[(-a'\cdot B_1+B_2)g_{\Xi\Xi\pi}\,g_{\Sigma\Xi K}+(-b\cdot B_4+B_5)g_{\Sigma\Sigma\pi}\,g_{\Sigma\Xi K}\Big];$$

and *in the case of scalar K meson*, $A(K^+\to\pi^+\pi^0)=0$, (if we neglect mass differences between the particles belonging to the same multiplet),

$$A(K_1^0\to 2\pi^0)\sim\frac{1}{2}\Big[(a\cdot A_1+3A_2)g_{NN\pi}\,g_{\Sigma NK}+(ab\cdot A_3+b\cdot A_4+2A_5)g_{\Sigma\Sigma\pi}\,g_{\Sigma NK}$$

$$+\frac{1}{2}\Big[(a'\cdot B_1+3B_2)g_{\Xi\Xi\pi}\,g_{\Sigma\Xi K}+(a'b\cdot B_3+b\cdot B_4+2B_5)g_{\Sigma\Sigma\pi}\,g_{\Sigma\Xi K}\Big],$$

$$A(K_1^0 \to \pi^+ \pi^-) \sim \Big[(a \cdot A_1 + 3A_2) g_{NN\pi}\, g_{\Sigma NK} + (ab \cdot A_3 + b \cdot A_4 + 2A_5) g_{\Xi\Xi\pi}\, g_{\Sigma NK} \Big]$$
$$- \Big[(a' \cdot B_1 + 3B_2) g_{\Xi\Xi\pi}\, g_{\Sigma\Xi K} + (a'b \cdot B_3 + b \cdot B_4 + 2B_5) g_{\Sigma\Sigma\pi}\, g_{\Sigma\Xi K} \Big],$$

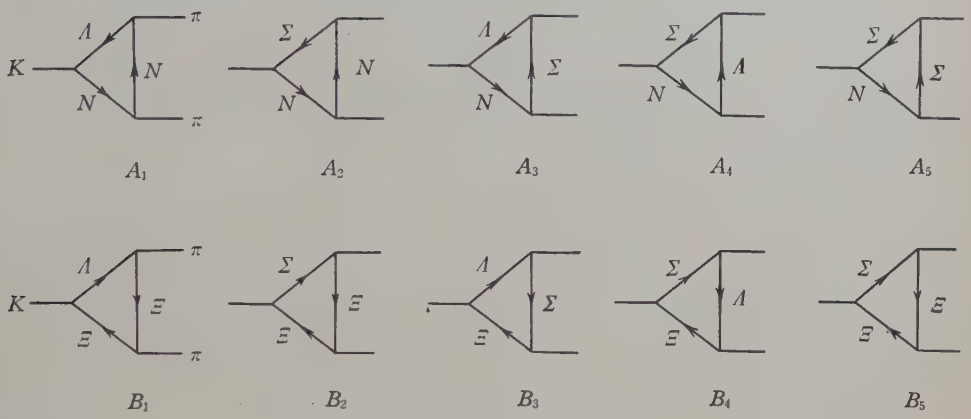

Fig. 1.

where we have put $a \equiv g_{\Lambda NK}/g_{\Sigma NK}$, $a' \equiv g_{\Lambda\Xi K}/g_{\Sigma\Xi K}$, and $b \equiv g_{\Sigma\Lambda\pi}/g_{\Sigma\Sigma\pi}$, the latter being equal to -1 for (12). (See also Fig. 1.)

For the case of pseudoscalar-charged K meson, if we put $a \approx -1$ and $a' \approx -1$, it may be possible to obtain $P(K^+ \to \pi^+\pi^0) \ll P(K_1^0 \to 2\pi^0$ and $\pi^+\pi^-)$ and $R = P(K_1^0 \to 2\pi^0)/P(K_1^0 \to \pi^+\pi^-) \sim 1/4$, the latter being different from the value predicted by the $|\Delta I| = 1/2$ rule, i.e. $1/2$, but in good agreement with the present experimental result.[16] It should be noted that the predicted value of R may be independent of the parity of K^0 meson.

References

1) B. d'Espagnat, J. Prentki and A. Salam, Nuclear Physics **3** (1957), 446.
J. C. Polkinghorne, Nuovo Cimento **6** (1957), 864.
B. d'Espagnat and J. Prentki, Progress in Elementary Particle and Cosmic Ray Physics, Vol. IV (1958).
2) V. G. Soloviev, Zhur. Exptl. i. Teo. Fiz. **33** (1957), 537, 796 (translation: Soviet Physics JETP, **6** (1958), 419, 613); Nuclear Physics **6** (1958), 618.
C. Iso, Prog. Theor. Phys. **20** (1958), 410.
J. J. Sakurai, Phys. Rev. **113** (1959), 1679.
G. Feinberg and F. Gürsey, Phys. Rev. **114** (1959), 1153.
3) A. Pais, Phys. Rev. **110** (1958), 574; **112** (1958), 624.
K. Kawarabayashi, Prog. Theor. Phys. **22** (1959), 451.
T. Sakuma and S. Furui, Prog. Theor. Phys. **23** (1960), 522; **24** (1960), 18.
M. H. Ross and G. L. Shaw, Phys. Rev. **115** (1959), 1773.
4) M. Gell-Mann, Phys. Rev. **106** (1957), 1296.
5) H. Umezawa, Suppl. Prog. Theor. Phys. **7** (1959), 67.
6) D. Itô, S. Minami and H. Tanaka, Prog. Theor. Phys. **22** (1959), 159.

7) B. d'Espagnat and J. Prentki, Phys. Rev. **114** (1959), 1366.
8) G. Takeda and M. Katô, Prog. Theor. Phys. **21** (1959), 441.
9) O. Hara, Nuovo Cimento **14** (1959), 114.
10) B. d'Espagnat and J. Prentki, Progress in Elementary Particles and Cosmic Ray Physics. Vol. IV (1958), p. 37.
11) P. W. Higgs, Nuclear Physics **4** (1957), 221.
12) A. Salam and J. C. Polkinghorne, Nuovo Cimento **2** (1955), 685. See also ref. 10), p. 41.
13) For example, ref. 10), p. 46.
14) Ref. 10), 3. 7. Remarks.
15) K. Fujii and D. Itô, Prog. Theor. Phys. **23** (1960), 815.
16) 1958 Annual International Conference on Hige Energy Physics at *CERN*, Session 9, Table VII.
17) J. J. Sakurai, Phys. Rev. **115** (1959), 1304.

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On a Model of Nucleon Structure and Meson-Nucleon Interaction. I

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The problems of π-nucleon collision and nucleon-antinucleon annihilation are treated in this paper (I) using a phenomenological model in which we assume the existence of the repulsive core for nucleon-nucleon potential and the attractive core for nucleon-antinucleon potential.

The results give rather good agreement with experimental evidence not only with respect to π-nucleon collision but also to nucleon-antinucleon annihilation.

We shall treat the problems of nucleon structure by using the same model in the following paper (II).

§ 1. Introduction

In the field theory, the problem of nucleon structure is the most interesting one of late years and many authors proposed various theories from various standpoints. For example, analyses based on the dispersion relation method were achieved by Chew[1] and Federbush et al.[2] on the electromagnetic structure. They were not able to explain simultaneously the value of the magnetic moment and the radii of the charge and momentum distributions. Recently, however, Frazer et al.[3] has suggested that the difficulty may be removed by introducing strong π-π interaction proposed by Takeda[4] and Dyson.[5]

On the other hand, Hiida and Nakanishi[6] calculated also the Feynman diagrams which contribute to the electromagnetic structure of nucleon on the basis of perturbation theory. By their enormous calculation, they showed that meson theory is not necessarily powerless for the inner domain of the nucleon structure.

It seems to us that to study the nucleon structure we must analyse not only the electromagnetic structure but also the phenomena which are closely connected with mesonic cloud such as high energy π-nucleon collision or antinucleon annihilation from some unifying standpoint. Therefore, it might be meaningful to use a more drastic phenomenological model for this analysis. In this paper, we analyse these phenomena not from the direct application of meson theory to the inner part of the nucleon structure but using an entirely phenomenological method. Namely, we substitute a phenomenological potential instead of the effects of some meson lines in the Feynman diagrams.

In order to construct our model, we write up the phenomena which have close

correlation with the nucleon structure and of which we have exact experimental data. Namely,

a) The scattering lengths of S-wave in π-nucleon collision are $+0.16\mu^{-1}$ and $-0.1\mu^{-1}$ for iso-spin $1/2$ and $3/2$ states respectively;[7] (μ=meson mass).

b) In π-nucleon scattering, there exist the second and third maximums for iso-spin $1/2$ state and their resonance energies are 615 ± 40 Mev and 775 ± 40 Mev (c.m.s.).[8] On the other hand, in γ-π production process, there also exist the resonances at the same energies.[9] The second resonance is considered as $D_{3/2}$ state, though there is left some possibility of it being $P_{3/2}$ state.

Moreover, π-$\boldsymbol{K}$ production has also a sharp resonance near the threshold which corresponds to the third resonance of π-nucleon collision.[10]

c) In nucleon antinucleon annihilation, the multiplicity of produced π is 4.82 ± 0.14.[11),12)] It is difficult to explain this multiplicity by Fermi's statistical theory without using very large interaction volume; ($R=2.5\mu^{-1}$). Moreover, the production of $\boldsymbol{K}$-particles is very few compared with statistical theory.[13]

d) With respect to the charge distribution of the nucleon, the r.m.s. charge radius of neutron is nearly zero, viz:

$$\langle r^2\rangle_1^S=\langle r^2\rangle_1^V.$$[14]

e) The iso-scalar part of anomalous magnetic moment of nucleon is very small in the absolute value; ($\mu_s=-0.06$).

It seems to us that there is some key point in experiment b) which predicts the coincidence of resonances of π-nucleon scattering, γ-π production and π-$\boldsymbol{K}$ production. This remarkable fact might be understood as that these processes are the various decay channels of some mediate state which has discrete eigenvalue energetically just as Bohr's compound nucleus model. Therefore, we should put an assumption for the substance of this mediate state. For example, we may consider this as the isobar state of nucleon which is based on strong coupling theory, just as Miyazawa and Fujimoto's model. But, in this paper, we regard this as the strong bound state of $n+1$ nucleons and n antinucleons. In order to get such a strong bound state, there must be very strong attractive potential between two nucleons or between a nucleon and an antinucleon. As it is well-known from experimental facts that the potential between two nucleons has the hard core about $(3\mu)^{-1}$ range, the potential between a nucleon and an antinucleon must have strong attractive core just as Fermi and Yang's model.[15]

Thus, we reach the following fundamental assumption.

Assumption

The potential between a nucleon and an antinucleon is given by

$$V(r)=V^0(r)+\boldsymbol{\tau}^{(1)}\boldsymbol{\tau}^{(2)}V^\tau(r)+\boldsymbol{\sigma}^{(1)}\boldsymbol{\sigma}^{(2)}V^\sigma(r)+\boldsymbol{\tau}^{(1)}\boldsymbol{\tau}^{(2)}\boldsymbol{\sigma}^{(1)}\boldsymbol{\sigma}^{(2)}V^{\tau\sigma}(r)+S^{12}V^T(r),$$

where $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$ represent τ-spin matrix and spin matrix respectively, and indices 1 and 2 represent the nucleon and the antinucleon. V^0, V^τ, V^σ, $V^{\tau\sigma}$, V^T are central potentials and S^{ij} represent the tensor force between i and j particles.

On the other hand, the potential between two nucleons is given by

$$U(r)=U^0(r)+\boldsymbol{\tau}^{(1)}\boldsymbol{\tau}^{(3)}U^\tau(r)+\boldsymbol{\sigma}^{(1)}\boldsymbol{\sigma}^{(3)}U^\sigma(r)+\boldsymbol{\tau}^{(1)}\boldsymbol{\tau}^{(3)}\boldsymbol{\sigma}^{(1)}\boldsymbol{\sigma}^{(3)}U^{\tau\sigma}(r)+S^{13}U^T(r),$$

where index 3 represents another nucleon.

Among these potentials, V^0 and U^0 have a strong attractive and repulsive core of the radius $(3\mu)^{-1}$ respectively. Namely, V^0 and U^0 are very strong ane the other potentials are comparatively weak.

Therefore, our model may be regarded as a model of the nucleon core.

In Section 2, we will take $PS(pv)$ coupling as π-nucleon interaction. But the results of our model do not sensitively depend on the coupling type.

Moreover, we use the following method of calculation:

i) We consider that the potentials are of the square-well.

ii) We use non-relativistic approximation for the Hamiltonian of nucleon and antinucleon.*

iii) We neglect the pair creation by the potentials V, U.

By this method, we shall discuss the general feature of the S-wave π-nucleon scattering in Section 2 and we get some information about the core. In Section 3, we shall also aquire considerable information concerning these potentials from the analysis of these bound state. In Section 4, we show some application of our model. Section 5 is devoted to the determination of the radius and the depth of V^T-U^T in which we use the width of the resonance of π-nucleon collision. At last, we treat the problems of nucleon-antinucleon annihilation, using the "*nucleoneum*" which is the strong bound state between a nucleon and an antinucleon.

§ 2. *S*-wave scattering in π-nucleon collision

In the following, we denote a nucleon and an antinucleon by N and $\bar{N}$ respectively, and write the core radius as r_0; $(r_0=(3\mu)^{-1})$. Moreover, we name the virtual pair of N and $\bar{N}$ the "pair".

In our model, the potential between the pair and N turns into the potential between π and N through the process

$$\pi\rightarrow\text{pair}, \quad \text{(Fig. 1)}. \tag{2·1}$$

In Fig. 1, we denote N and the pair by 1 and 2 $(\bar{N})$, 3(N) respectively. Then the matrix elements of the S-wave scattering through these processes are given by

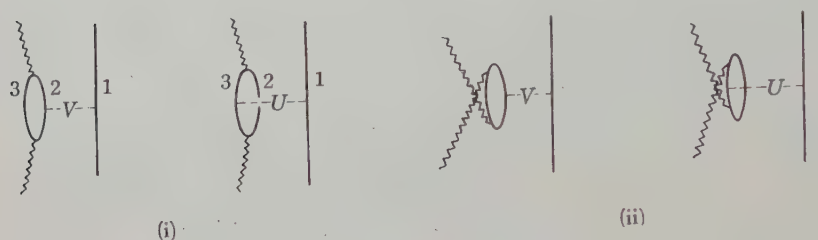
* But we shall sometimes use the relativistic relation between the energy and the momentum on the way of the calculation.

$$\left\langle f|H_a^{23}\frac{1}{E-2M-H^{23}}(V^{12}+U^{13})\frac{1}{E-2M-H^{123}}H_c^{23}|i\right\rangle$$

(for Fig. 1 (i)),

$$\left\langle f|H_a^{23}\frac{1}{-E-2M-H^{23}}(V^{12}+U^{13})\frac{1}{-E-2M-H^{123}}H_c^{23}|i\right\rangle$$

(for Fig. 1 (ii)), (2·2)



(i) (ii)

Fig. 1.

in the first approximation, where $|i\rangle$, $|f\rangle$ represent the initial and the final state respectively. H^{23} and H^{123} are the Hamiltonians of 2, 3 particles and of 1, 2, 3 particles. V^{12} and U^{13} represent the potentials between 1, 2 particles and between 1, 3 particles. H_c^{23} and H_a^{23} are the pair creation part and the pair annihilation part of the π-N interaction with respect to 2, 3 particles, and E is the total energy of incident π.

On the other hand, Fig. 2 shows the other diagrams which are effective to the S-wave scattering. The matrix elements are given by

$$\left\langle f|H_a^{12}\frac{1}{E-2M-H^{123}}H_c^{23}|i\right\rangle$$

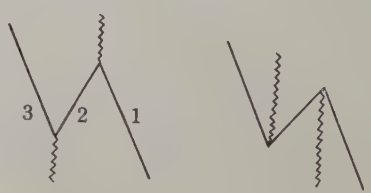
(for Fig. 2 (i)),

$$\left\langle f|H_a^{12}\frac{1}{-E-2M-H^{123}}H_c^{23}|i\right\rangle$$

(for Fig. 2 (ii)). (2·3)

But the calculation of these matrix elements is very difficult. Therefore, we shall discuss only the general feature of S-wave scattering in this paper.

First, we remark that the matrix elements (2·3) are larger than that of the ordinary PS (pv) theory. This is due to the expansion of the pair by the strong attractive potential V. To illustrate this, we regard the depth of V as $2M$



(i) (ii)

Fig. 2.

tentatively, because this becomes actually $2M$ in the highest as will be shown in Section 3. Considering that

$$H^{23}=p^2/M+V^{23}(s),$$

we get

$$\frac{1}{-2M-H^{23}}H_c^{23}|i>=\begin{cases}-\sqrt{\frac{2}{\mu}}\frac{fM}{4\pi s}\left(1-\frac{\alpha s}{r_0}\right)|> & (s\leq r_0),\\ -\sqrt{\frac{2}{\mu}}\frac{fM}{4\pi s}\frac{e^{-\sqrt{2}M(r-r_0)}}{1+\sqrt{2}Mr_0}|> & (s>r_0),\end{cases}$$

$$\alpha=\frac{\sqrt{2}Mr_0}{1+\sqrt{2}Mr_0}, \qquad (2\cdot4)$$

where s, p are the relative coordinate and the momentum between 2, 3 particles, and $|>$ represents the τ- and σ-spin state.

Therefore, if we neglect E, V^{12} and U^{13} in formula (2·3), the corresponding scattering lengths are given by

$$-0.04\mu^{-1}. \qquad (2\cdot5)$$

On the other hand, the $PS(pv)$ theory gives a small value, $-0.01\mu^{-1}$. Thus, our assertion is actually valid.

Second, we remark that the potential between N and the pair may not have the core. To illustrate this, we neglect V^{12} and U^{13} in the denominator of formula (2·2). Then the corresponding potentials between N and π are given by

$$\begin{aligned}&A(r)+B(r) \quad (\tau=3/2),\\ &A(r)-2B(r) \quad (\tau=1/2),\\ &A(r)=(I+I')(V^0(r)+U^0(r)),\\ &B(r)=(I-I')(V^\tau(r)+U^\tau(r)),\\ &I=|\langle\text{pair}|\frac{1}{E-2M-H^{23}}H_c^{23}|\pi\rangle|^2,\\ &I'=|\langle\text{pair}+\pi|\frac{1}{-E-2M-H^{23}}H_c^{23}|\text{vacuum}\rangle|^2.\end{aligned} \qquad (2\cdot6)$$

If we takes the value (2·5), the scattering lengths which correspond to A and B are given by $0.027\mu^{-1}$ and $-0.087\mu^{-1}$ respectively. Clearly, these are very small compared with core radius r_0. This strongly suggests that $A(r)$ is very small and has not a core. Since we shall assume in the following paper that I and I′ are not small, this means also that V^0+U^0 has not a core. Therefore, the cores of V^0 and U^0 must cancel each other strongly, and the potential between N and the pair does not have a core.

By the analysis of formula (2·6), we are able to acquire some information

concerning $V^\tau + U^\tau$. But the τ-dependent part of the phase shift is derived also from the dispersion relation theory. So, we should discuss this problem more carefully.

§ 3. Resonances of π-N scattering

As we mentioned in the Introduction, the resonances of π-N scattering correspond to the bound states of $n+1$ N and n $\bar{N}$. In this section, we shall analyse the structure of these bound states.

In our model, the two-body system of an N and an $\bar{N}$ composes strong S-wave bound states, as V has a very strong attractive core. We shall call these two-body bound states "***nucleoneum***".*

We concluded in the previous section that the potential between the pair and N may be weak and not have a core. So, we assume that the potential between N and nucleoneum or between nucleoneums is also comparatively weak. By this assumption, we may consider the above stated bound states as weakly bounded compound systems of one N and nucleoneums and these bound states are all in S-state (though there may be another possibility of the existence of P-wave bound states). We write the n-body bound states as

$$^nS_j,$$

in which j represents the total angular momentum. In this paper, we shall consider only the case $n=3$, 5. Then, taking the parity into consideration, the correspondence of the partial waves and the bound states are as follows,

$$\begin{aligned} S_{1/2} &\quad {}^3S_{1/2}, \\ P_{1/2} &\quad {}^5S_{1/2}, \\ P_{3/2} &\quad {}^5S_{3/2}, \\ D_{3/2} &\quad {}^3S_{3/2}, \\ F_{5/2} &\quad {}^5S_{5/2}. \end{aligned}$$

But these bound states may not necessarily exist.

Now, let us adapt our model to the experimental data. It is inferred that the angular momentum of the second resonance of π-N collision is 3/2 from the experiments. If we assume that this resonance is the $P_{3/2}$ state, there must be another resonance in the lower energy domain, because one of the ${}^5S_{1/2}$ and ${}^5S_{5/2}$ must have a lower energy level than the level of ${}^5S_{3/2}$ as far as we take two-body square-well potential. This contradicts the data. Therefore, we must consider that the second resonance is $D_{3/2}$ state and the data[9] of γ-π production seem to support this viewpoint.

* There may be also the P-wave bound states, but their rest masses are generally fairly larger than that of S-wave bound states. Therefore, we shall neglect their effects in the following discussion.

The third resonance must be $F_{5/2}$ state, because the angular momentum is considered to be 5/2 at least in the experiments. The experimental data also show that the second resonance is almost forward scattering and the third resonance accompanies a large backward scattering on the contrary.[8] This sharp change of the angular distribution strongly suggests that these two resonances have opposite parity to each other. Thus, our description seems to be quite reasonable.

Let us go into further details of the levels of the above assignment. The data show that the other resonances do not exist in the energy range up to 1.5 Bev. This indicates that ${}^3S_{1/2}$, ${}^5S_{1/2}$ and ${}^5S_{3/2}$ are unfavorable bound states. So, if there are such bound states, their levels must be higher enough than ${}^3S_{3/2}$ and ${}^5S_{5/2}$ levels or coincide with these. If there are no such coincidences, the energy difference between ${}^3S_{3/2}$ and ${}^3S_{1/2}$ levels must be 3μ at least and the difference between ${}^5S_{5/2}$ and ${}^5S_{3/2}$, ${}^5S_{1/2}$ levels must be 2μ at least, as 1.5 Bev is equivalent to 7.5μ(c.m.s.).

Since the iso-spins of these resonances are 1/2, we must consider also that the levels with iso-spin 3/2 are higher than that of 1/2. It is well known that the former ($\tau=3/2$) has a large diffused peak at about 7μ. Consequently, if there are $\tau=3/2$ bound states, it is natural to consider that these correspond to the diffused peak, viz. the energy difference between $\tau=1/2$ and 3/2 levels is 2.8μ for ${}^3S_{3/2}$ and 1.7μ for ${}^5S_{5/2}$.

These splittings of the levels are due to the τ- and σ-dependent terms of V and U. Therefore, we are able to aquire some information concerning these potentials by the analysis of these levels; (Appendix I). Of course, we are also able to estimate the rest masses of nucleoneums. In Appendix I, we shall show that these may be

$$5\mu\sim 7\mu.$$

In this case, the depth of V is estimated as

$$12\mu\sim 14\mu.$$

§ 4. π-production

In this section, we show an application of our model. We remark that our model is a kind of isobar model. Therefore, our model fits to the description proposed by Lindenbaum and Steinheimer[16] in the reaction

$$\pi+N\rightarrow\pi+\pi+N.$$

But in our model the first resonance is based on the theory of Chew and Low[17] contrary to their model, and the second and third resonances correspond to the true isobar states ${}^3S_{3/2}$ and ${}^5S_{5/2}$. So, the description is not the same.

In the case of 1.4 Bev incident π,[18] the produced π has a diffused first peak in the momentum 0.1~0.3 Bev/c and a relatively clear second peak at 0.5~0.6 Bev/c. Lindenbaum explained these first and second peaks by the "*decay* π" of (3/2, 3/2)

isobar state and the "*extra* π" respectively. But we can also explain these data by our model. Namely, using ${}^3S_{3/2}$ and ${}^5S_{5/2}$ as the isobar states, the extra π has the peaks at

$$0.30 \text{ Bev}/c \quad (\text{for } {}^3S_{3/2}),$$

$$0.17 \text{ Bev}/c \quad (\text{for } {}^5S_{5/2}),$$

and the decay π has the peaks at

$$0.48 \text{ Bev}/c \quad (\text{for } {}^3S_{3/2}),$$

$$0.59 \text{ Bev}/c \quad (\text{for } {}^5S_{5/2}).$$

These fit to the first and second peaks fairly well.

But this description is not applicable to the N-N scattering, because such a decay π is not observed in the experiment of 2.75 Bev incident N.[19]

Thus, our model is powerful only in the high energy π-N scattering.

§ 5. Tensor force

Although we considered that the resonance $D_{3/2}$ corresponds to ${}^3S_{3/2}$ in Section 3, there must be some spin-orbit coupling in order to change $D_{3/2}$ wave into ${}^3S_{3/2}$ state. We consider that this coupling is tensor-force.

Since the strength of the tensor force must closely correlate to the width of the resonance, we can presume the strength from the width. We denote the momentum and the energy of the incident π by k and ω respectively and put

$$T=S^{12}V^T+S^{13}U^T.$$

Then the width Γ is given by

$$\Gamma=4N'^{-2}\left|\left\langle \begin{matrix}3\\B\end{matrix}\right| T \frac{1}{E-2M-H^{23}} H_c^{23}|D_{3/2}\rangle|^2 \frac{\sqrt{M^2+k^2}}{\sqrt{M^2+k^2}+\omega}\omega k,$$

in the first approximation, where $|D_{3/2}\rangle$ and $\left|\begin{matrix}3\\B\end{matrix}\right\rangle$ represent the initial state and ${}^3S_{3/2}$ respectively, and particles 2, 3 are the members of the pair. N' is the normalization factor of the spherical Bessel function. Since the nucleoneum changes only the σ-spin by the potential T, we may write

$$\Gamma=4N'^{-2}\left|\left\langle \begin{matrix}3\\B\end{matrix}\right| T \left|\begin{matrix}2\\B\end{matrix}\right\rangle\right|^2 \left|\left\langle \begin{matrix}2\\B\end{matrix}\right| \frac{1}{E-2M-H^{23}} H_c^{23}|D_{3/2}\rangle\right|^2 \frac{\sqrt{M^2+k^2}}{\sqrt{M^2+k^2}+\omega}\omega k,$$

where $\left|\begin{matrix}2\\B\end{matrix}\right\rangle$ represents the nucleoneum state which is substituted for incident π. Since the probability I_R of the process

$$\pi \rightarrow \text{pair} \rightarrow \text{nucleoneum}$$

is given by

$$\left|\left\langle \begin{matrix}2\\B\end{matrix}\right| \frac{1}{E-2M-H^{23}} H_c^{23}|D_{3/2}\rangle\right|^2,$$

we get

$$\Gamma=4N'^{-2}I_B\left|\left\langle \begin{matrix}3\\B\end{matrix}\right| T \left|\begin{matrix}2\\B\end{matrix}\right\rangle\right|^2 \frac{\sqrt{M^2+k^2}}{\sqrt{M^2+k^2}+\omega}\omega k.$$

In this formula, we are able to put

$$\left|\begin{matrix}3\\B\end{matrix}\right\rangle=\left|\begin{matrix}32\\0\end{matrix}\right\rangle\frac{\phi(r)}{r}+\left|\begin{matrix}32\\1\end{matrix}\right\rangle\frac{\psi(r)}{r},$$

where $\boldsymbol{r}$ is the relative coordinate between N and nucleoneum and $\left|\begin{matrix}32\\j\end{matrix}\right\rangle$ represents the nucleoneum state in which the τ-spin is j. Then, we get

$$\left\langle \begin{matrix}3\\B\end{matrix}\right| T \left|\begin{matrix}2\\B\end{matrix}\right\rangle^2=12\pi N'^2\left|\int_0^\infty (V^T(r)-U^T(r))\psi(r)j_2(kr)\,r\,dr\right|^2,$$

where j_2 is the spherical Bessel function.

In order to evaluate this formula, we tentatively put

$$\phi(r),\ \psi(r)=\begin{cases}\alpha r,\ \beta r & (r<\bar{r}),\\ 0 & (r\geq\bar{r}).\end{cases}$$

Then, we get the following relation,

$$\bar{I}|t|=\begin{cases}13.0\mu & (r^*=r_0),\\ 3.1\mu & (r^*=(2\mu)^{-1}),\\ 1.4\mu & (r^*=2r_0),\end{cases}$$

$$(\Gamma=0.4\mu),$$

$$\bar{I}^2=I_B\int_{r\leq r^*}|\psi(r)|^2\,\boldsymbol{dr},$$

$$r^*=\min(\bar{r},\ r^T),$$

where t and r^T represent the depth and the radius of the potential V^T-U^T. From the definition, $\bar{I}$ must satisfy inequality $\bar{I}\leq 1$.

This $\bar{I}|t|$ gives a very close value to one pion exchange tensor force at $r=r^*$. Therefore, we may consider that our phenomenological tensor force is derived from meson theory, if the $\bar{I}$ is not very small. But we remark that our tensor force is rather strong as compared with the conventional phenomenological tensor force which is used in the problems of N-N collision and this suggests that the V^T is not weak.

In the above calculation, we neglected the Pauli-principle with respect to the 1 and 3 particles. This means the omission of the diagrams which correspond to

Fig. 2. But we can actually neglect this effect, as the radius is r_0 at most and smaller than $\bar{r}$.

§ 6. Nucleon-antinucleon annihilation

There are two problems in the annihilation process of N and $\bar{N}$

$$N+\bar{N}\to\pi+\pi+\cdots, \tag{6·1}$$

viz.,

i) the range where the annihilation occurs,

ii) the multiplicity of pions.

In this section, we shall discuss these problems using our model.

In our model, there are nucleoneum formation processes in addition to the direct process (6·1), e.g.

$$N+\bar{N}\to\text{nucleoneum}+\pi, \tag{6·2}$$

$$N+\bar{N}\to 2 \text{ nucleoneums} \quad \text{(Fig. 3)}. \tag{6·3}$$

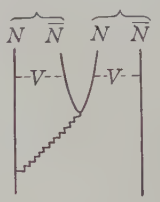


Fig. 3.

The strengths of these reactions are not weak in general. For example, with respect to the S-wave, the imaginary phase shift of the process (6·3) is given by

$$\delta=i\,AHf^2(2\varepsilon/\mu)^{3/2}(k/\mu),$$

$$A=12f^2(\mu M_n)^{5/2}MI^2,$$

$$H=\frac{1}{9}\sum_f|\langle f|\frac{\tau_i^{(2)}\boldsymbol{\sigma}^{(2)}+\tau_i^{(3)}\boldsymbol{\sigma}^{(3)}}{2}|i\rangle|^2,$$

$$I=\frac{1}{4\pi}\left(\int\varphi(r)\,\boldsymbol{dr}\right)\left(\int\varphi(r)\exp(-\mu r)\left(1+\frac{1}{\mu r}\right)\boldsymbol{dr}\right),$$

where ε and k are the available energy per one nucleoneum and the momentum of incident $\bar{N}$, and M_n is the mass of nucleoneum. $|i\rangle$ and $|f\rangle$ represent the initial and final τ- and σ-spin states respectively, and the τ-spin matrix τ_i represents also the creation of the pair. $\varphi(r)$ is the wave function of the nucleoneum.

If we take the minimum value $M_n=5.0\mu$ (Appendix I), we get

$$\delta=2.8f^2(2\varepsilon/\mu)^{3/2}(k/\mu)Hi.$$

Therefore, it seems that this is small in the case of the $PS(p\tau)$ theory; $(f^2=0.08)$. But if we calculate the correction with respect to the virtual pair (Fig. 4) using the formula (2·4), we get a fairly large value,

$$f^2=1.0,$$

in virtue of the strong attractive potential V. So, the reaction (6·3) is not weak except in the neighborhood of the threshold. If we take the $PS(ps)$ theory, this becomes very large; $(f^2=15)$. Of course, the reactions (6·1), (6·2) are, also, very strong.

It is natural to consider that the branching ratio of such strong reactions depends on the ranges in which the reactions occur. Therefore, we shall discuss the ranges in the following.

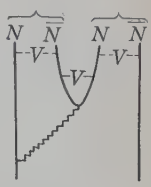


Fig. 4.

The absorptive reactions (6·2), (6·3), ⋯, are a kind of nonlocal interaction and the form factor is the wave function of nucleoneum. The radius of the wave function is about core radius r_0. Consequently, there is a black hole of radius r_0 in these processes just as in Ball and Chew's theory.[20]

On the other hand, it is well known in perturbation theory that the range of the process (6·1) is very small. Therefore, direct process (6·1) may be negligible.

In order to evaluate the ranges where the processes (6·2), (6·3) occur more correctly, we use mean square distance $\sqrt{\langle t^2 \rangle_{\text{Av}}}$, where t represents the distance between N and $\bar{N}$ and the weight function represents the strength of the reaction in the distance. The results are as follows,

$$0.19\mu^{-1} \quad \text{(case 6.2)},$$
$$0.30\mu^{-1} \quad \text{(case 6.3)};$$
$$\text{(Appendix II)}.$$

The relatively small values are due to the short tail of the wave function of nucleoneum.

This results show that the range in which the annihilation occurs extends, if the number of emitted nucleoneums inceases. This indicates that the emission of nucleoneums has strong tendency to multiple production. But the emission of three nucleoneums is energetically not allowable, because the rest mass of nucleoneum is larger than 5μ; (Appendix I). Therefore, the most probable case may be of the two nucleoneums production, if it is allowable.

We now turn to the second problem ii). The nucleoneums which are produced through these processes immediately decay into pions. Therefore, the process (6·1) is divided into two successive processes. Namely,

$$N+\bar{N} \to \text{nucleoneums}+\pi+\cdots,$$
$$\text{nucleoneums} \to \pi+\pi+\cdots.$$

Since the decay modes of nucleoneum are determined by the τ- and σ-spin, we write the nucleoneum as (i, j), where i and j represent the τ- and σ-spin respectively. The π-nucleoneum interactions are given by

$$f\boldsymbol{k}\boldsymbol{\sigma}^{(1)}\tau_i^{(1)}-f\boldsymbol{k}\boldsymbol{\sigma}^{(2)}\tau_i^{(2)}, \quad H_a^{12}, \; H_c^{12}.$$

Therefore, the allowable processes are given by

$$(1,\ 0) \to 3\pi,$$
$$(1,\ 1) \to 2\pi,$$

$$(0,\ 1) \rightarrow 3\pi .$$

In the first process, we considered energy conservation law. If the $(1, 0)$ is not free, the process

$$(1,\ 0) \rightarrow \pi$$

is allowable.

On the contrary, the $(0, 0)$ cannot decay into pions by these interactions but it is shown that this decays into 4π by the mediation of the virtual pair.

The mediate states are the various combinations of these nucleoneums and pions. But some of these mediate states should vanish in the case of two nucleoneums production by the effect of the interference of the two nucleoneums in the rearrangement of $2N$ and $2\bar{N}$. We show the allowable mediate states which are produced through processes $(6\cdot 2)$, $(6\cdot 3)$ and the number of the pions in the final state in Table I (l=even) and Table II (l=odd), where l represents the orbital angular momentum of the incident $\bar{N}$.

As we mentioned above, the most probable case is the two nucleoneums production, if it is energetically allowable. Therefore, the multiplicity of pions correlates sharply to the rest masses of nucleoneums.

If we neglect the many-body force, we can roughly estimate the masses. But we are not able to use such a rough value in this problem. Fortunately, they are correlated also to the electromagnetic structure of the nucleon, which we will discuss in the following paper. By this reason, we shall discuss the general feature of our model in this paper, using two special cases. Namely,

i) The example which is shown in Appendix I.

$$M(0, 1) = 5.0\mu, \quad M(1, 1) = 6.9\mu,$$
$$M(0, 0) = 7.0\mu, \quad M(1, 0) = 8.9\mu.$$

Table I. (l=even)

initial state	mediate state	No. of emitted pions
$\tau=0,\ \sigma=0.$	$(0, 1)+(0, 1)$	6
	$(1, 1)+(1, 1)$	4
$\tau=0,\ \sigma=1.$	$(1, 1)+\pi$ $(l=0)$	3
$\tau=1,\ \sigma=0.$	$(1, 0)+(1, 1)$	5
	$(0, 1)+(1, 1)$	5
	$(1, 1)+\pi$ $(l=0)$	3
$\tau=1,\ \sigma=1.$	$(1, 1)+(0, 0)$	6
	$(1, 0)+(0, 1)$	6
	$(1, 0)+(1, 0)$	6
	$(1, 1)+(1, 1)$	4
	$(0, 1)+\pi$ $(l=0)$	4
	$(1, 0)+\pi$	4

Table II. (l=odd)

j represents the total angular momentum.

initial state	mediate state	No. of emitted pions
$\tau=0,\ \sigma=0.$	$(0,\ 0)+(0,\ 1)$	7
	$(1,\ 0)+(1,\ 1)$	5
	$(1,\ 1)+\pi$ $(l=1)$	3
$\tau=0,\ \sigma=1.$	$(0,\ 0)+(0,\ 0)$ $(j=\text{even})$	8
	$(1,\ 0)+(1,\ 0)$ $(j=\text{even})$	6
	$(0,\ 1)+(0,\ 1)$ $(j=\text{even})$	6
	$(1,\ 1)+(1,\ 1)$ $(j=\text{even})$	4
	$(1,\ 0)+\pi$ $(j=0)$	4
$\tau=1,\ \sigma=0.$	$(1,\ 1)+(0,\ 0)$	6
	$(1,\ 0)+(0,\ 1)$	6
	$(1,\ 1)+(1,\ 1)$	4
	$(0,\ 1)+\pi$ $(l=1)$	4
$\tau=1,\ \sigma=1.$	$(0,\ 1)+(1,\ 1)$	5
	$(0,\ 0)+(1,\ 0)$ $(j=\text{even})$	7
	$(1,\ 1)+\pi$ $(j=1)$	3
	$(0,\ 0)+\pi$ $(j=0)$	5

ii) $M(0,1)=M(1,1)=M(0,0)=M(1,0)$,

$M(i,j)<M$,

where $M(i,j)$ is the mass of (i,j).

If we neglect the kinetic energy, the allowable mediate states of the process (6·3) are limited to

$$(0,1)+(0,1),$$

$$(0,1)+(0,0),$$

$$(0,1)+(1,1),$$

in the first case. So, using Tables I and II, we get the following multiplicity

$$4.0\ \ (l=0),$$

$$5.0\ \ (l=1).$$

Of course, we neglect process (6·2), if process (6·3) is simultaneously allowable.

On the other hand, it becomes

$$5.0\ \ (l=0),$$

$$5.3\ \ (l=1),$$

in the second case.

We show the experimental data in Table III.

Table III.

	Rome	Berkeley+Rome[11]	Bubble chambers[12]
in flight	5.10±0.24	5.12±0.22	5.10±0.12
at rest	4.60±0.15	4.77±0.14	4.72±0.11
combined	4.82±0.14	4.92±0.13	4.89±0.12

If we consider that the annihilation at rest occurs from S-state and the annihilation in flight occurs from P-state mainly, the increase of the multiplicity in flight may be naturally explained in both cases.

But the value in the first case ($l=0$) seems to be slightly small. This is due to the prohibition of the processes

$$(1, 1)+(0, 0),$$

$$(1, 0)+(0, 1),$$

in the $\tau=\sigma=1$ initial state. Therefore, at least one of them may be allowable.

In this calculation, we neglected the processes

$$N+\bar{N}\rightarrow(i, j)+\pi+\pi,$$

$$N+\bar{N}\rightarrow(i, j)+(m, n)+\pi,$$

$$N+\bar{N}\rightarrow(i, j)+(1, 0) \quad \text{(Fig. 5)}.$$

The first process increases the multiplicity fairly in the case i), because the $\tau=0$, $\sigma=1$, $l=0$ initial state decays into 5π in this process. Of course, the multiplicity of the second and third processes is very large.

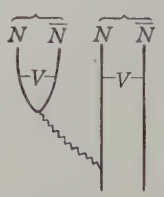


Fig. 5.

Thus, it is not difficult to explain the experimental data with our model.

At last, we discuss the emission of $\boldsymbol{K}$-meson. As the rest mass of the emitted nucleoneum is $5\sim7\mu$, the process

$$\text{nucleoneum}\rightarrow\boldsymbol{K}+\bar{\boldsymbol{K}}$$

is energetically hardly allowable. Therefore, the production of $\boldsymbol{K}$-meson should be considerably checked and this does not contradict the experiments.

§ 7. Discussion

Our main purpose in this paper (I) was to establish a clear explanation for the second and third resonances in π-nucleon collision.

Indeed, there is not yet any positive theory with respect to these resonances. The most important character of our model in this problem is the point that the second resonance may be $D_{3/2}$ state. Therefore, it may be a support to our model,

if it is clearly proved by experiment. Moreover, in our model, the third resonance must be $F_{5/2}$ state and we showed in Section 3 that this is rather favorable to the explanation of the angular distribution.

With respect to the π-production in high energy π-nucleon collisions, we took just the opposite description to Lindenbaum's one.[16] For nucleon-nucleon collisions, Lindenbaum's model is more powerful than our model. But their model involves some questions, because $(\frac{3}{2}, \frac{3}{2})$ resonance does not correspond to isobar state according to Chew and Low's theory.

We divided nucleon-antinucleon annihilation process into two successive processes,

i) production of nucleoneums,

ii) decay of nucleoneums,

because the radius of the range where the process i) occurs is core radius r_0 at least and may be larger than that of direct production of pions.

We have pointed out that the reaction

$$N+\bar{N}\rightarrow 2 \text{ nucleoneums}$$

is very strong in the case of $PS(ps)$ coupling in virtue of the large coupling constant and this may be strong even in the case of $PS(pv)$ coupling too. Moreover, we have shown that the range of this process is more extensive than that of one nucleoneum production process. By this reason, we have concluded that this is the main process, if it is energetically allowable.

It is remarkable that the ranges in which these processes occur correspond to the black hole in Ball and Chew's description.[20]

On the other hand, the second process ii) is quite different from Fermi's statistical model,[21] Takeda-Koba's model[22] and the models based on strong π-π interaction.[23),24),25] In these theories, the pions were produced directly but in our model the emitted pions are mainly the secondaries of nucleoneums. By this reason, the multiplicity of pions correlates strongly to the decay modes of nucleoneum and the number of the produced nucleoneums.

We have shown that the results depend on the rest masses of the nucleoneums using two special cases. Nevertheless, the results have given rather good agreement with experimental data.

It is our point to give the mechanism of Ball and Chew's description and also to give the reason of large multiplicity of pions.

It is also a remarkable fact that the depths of our potentials may be smaller than 1μ except the cores of V^0 and U^0 (Appendix I), because this indicates that our potentials are of the same order with the one derived from meson theory. But we concluded in Section 5 that the tensor force between a nucleon and an antinucleon is not weak. This fact will be used in the following paper.

Our model resembles Sakata's model in various points.[26] But the idea is quite different. Namely, their main purpose is to simplify the concept of

strangeness and iso-spin by a composite model. But our purpose is to unify the various features of π-nucleon interaction and nucleon structure, and π, $\boldsymbol{K}$, etc., are still elementary particles. But we hope there may be some connection with each other in future development.

Appendix I

Our main purpose in this Appendix is to estimate the rest mass of nucleoneum from the resonances in π-nucleon collision.

For simplicity, we neglect the many-body force, the tensor force and $V^{\tau\sigma}$, $U^{\tau\sigma}$ terms in this calculation. Therefore, the results in this Appendix are not general at all.

As we mentioned in Section 3, we must note the following points.

i) If there is bound state ${}^3S_{1/2}$, the energy difference between ${}^3S_{3/2}$ and ${}^3S_{1/2}$ levels must be 3μ at least.
If there is bound state ${}^5S_{3/2}$ or ${}^5S_{1/2}$, the energy difference between ${}^5S_{5/2}$ and ${}^5S_{3/2}$, ${}^5S_{1/2}$ levels must be 2μ at least.

ii) If there is $\tau=3/2$ bound state, the energy difference between $\tau=1/2$ and $3/2$ levels should be 2.8μ or above 2.8μ for ${}^3S_{3/2}$ and 1.7μ or above 1.7μ for ${}^5S_{5/2}$.

iii) There must be the bound states ${}^3S_{3/2}$ and ${}^5S_{5/2}$.

As n_1N and $n_2\bar{N}$ are confined in the interior of the potentials, the spin dependent potential energy is given by

$$\varepsilon^s\boldsymbol{\sigma}^{(N)}\boldsymbol{\sigma}^{(\bar{N})}+\eta^s\boldsymbol{\sigma}^{(N)}\boldsymbol{\sigma}^{(N)}+\eta^s\boldsymbol{\sigma}^{(\bar{N})}\boldsymbol{\sigma}^{(\bar{N})},$$

where $\boldsymbol{\sigma}^{(N)}$ and $\boldsymbol{\sigma}^{(\bar{N})}$ represent the spin matrices of N and $\bar{N}$. ε^s and η^s are the depths of V^σ and U^σ respectively. So, the eigenvalue is gived by

$$2J(J+1)\varepsilon^s+2[I_1(I_1+1)+I_2(I_2+1)]\lambda^s-\frac{3}{2}(n_1+n_2)\lambda^s-\frac{3}{2}(n_1+n_2)\varepsilon^s,$$

$$\lambda^s=\eta^s-\varepsilon^s,$$

where J, I_1, I_2 represent the spin of total system, N-system, and $\bar{N}$-system.

In the case of $J=1/2$ and $3/2$, there are three energy levels with respect to the spin states,

$$\begin{aligned}
&3\varepsilon^s+\lambda^s+C \qquad (J=3/2),\\
&-3\varepsilon^s+\lambda^s+C \qquad (J=1/2,\ I_1=1),\\
&-3\varepsilon^s-3\lambda^s+C \qquad (J=1/2,\ I_1=0).
\end{aligned}$$

Considering the condition i), ε^s must be negative, and

$$\begin{aligned}
6\varepsilon^s+4\lambda^s\leq-3\mu \quad (\lambda^s>0),\\
6\varepsilon^s\leq-3\mu \quad (\lambda^s\leq 0).
\end{aligned}$$

If there is not ${}^3S_{1/2}$ state, we should use the condition

$$10\varepsilon^s+8\lambda^s\leq-2\mu \quad (\lambda^s>0),$$
$$10\varepsilon^s\leq-2\mu \quad (\lambda^s\leq 0),$$

instead of the above condition. From these, we get

$$\varepsilon^s\leq-0.5\mu \quad (\text{or } -0.2\mu). \tag{1}$$

By the same calculation, we get from condition ii)

$$6\varepsilon^\tau+4\lambda^\tau\geq 2.8\mu \quad (\lambda^\tau>0),$$
$$6\varepsilon^\tau\geq 2.8\mu \quad (\lambda^\tau\leq 0),$$
$$\lambda^\tau=\eta^\tau-\varepsilon^\tau, \tag{2}$$

where ε^τ and η^τ are the depths of V^τ and U^τ. If there is not ${}^3S_{3/2}$ ($\tau=3/2$) state, we should take the value 1.7μ instead of 2.8μ in this formula.
Moreover, if there are two $\tau=1/2$ bound states, we must consider that the energy difference between the two states is nearly 0 or above 3μ (or 2μ) by the same discussion with respect to the spin state, viz.,

$$4\lambda^\tau=0,$$
$$\text{or} \quad 4|\lambda^\tau|\geq 3\mu \quad (\text{or } 2\mu). \tag{3}$$

As to the condition iii), we regard the system as the compound system of one N and n-nucleoneums as we mentioned in Section 3. Considering that the spin is the highest and $\tau=1/2$ in this case, we get the total potential energy

$$nv+n^2(d+\eta^s)+n(n+1)\varepsilon^s-3n\varepsilon^\tau-3n\lambda^\tau \quad (\lambda^\tau>0),$$
$$nv+n^2(d+\eta^s)+n(n+1)\varepsilon^s-3n\varepsilon^\tau+n\lambda^\tau \quad (\lambda^\tau\leq 0),$$
$$d=v+u, \tag{4}$$

where v, u are the depths of V^0 and U^0.

On the other hand, the potential energy where the N and nucleoneums are separated from each other is given by

$$nv+n\varepsilon^s-3n\varepsilon^\tau \quad (\varepsilon^\tau>0),$$
$$nv+n\varepsilon^s+n\varepsilon^\tau \quad (\varepsilon^\tau\leq 0). \tag{5}$$

Since the potential energy between N and nucleoneums is the difference between (4) and (5), the negative ε^τ contributes as repulsive force. So, we assume ε^τ is positive in the following. Then the effective potential energy becomes

$$n^2(d+\eta^s+\varepsilon^s)-3n\lambda^\tau \quad (\lambda^\tau>0),$$
$$n^2(d+\eta^s+\varepsilon^s)+n\lambda^\tau \quad (\lambda^\tau\leq 0), \tag{6}$$

and this must be larger than $nK(r')$, where $K(r')$ is the necessary energy per one nucleoneum to compose the bound state which relates mainly to the range r' of the potentials V^σ, U^σ, V^τ, U^τ and V^0+U^0.

Considering that the second resonance corresponds to the $n=1$ case and the energy level is 4.35μ, we are able to estimate the mass of nucleoneum $M(i, j)$, where i and j represent the τ- and σ-spin respectively, namely,

$$\begin{aligned} M(0,1) &= 4.35\mu - (d+\eta^s+\varepsilon^s) + 3\lambda^\tau - K \quad (\lambda^\tau > 0), \\ &\quad 4.35\mu - (d+\eta^s+\varepsilon^s) - \lambda^\tau - K \quad (\lambda^\tau \leq 0), \\ M(1,1) &= M(0,1) + 4\varepsilon^\tau, \\ M(0,0) &= M(0,1) - 4\varepsilon^s, \\ M(1,0) &= M(0,1) + 4\varepsilon^\tau - 4\varepsilon^s. \end{aligned} \tag{7}$$

Since the third resonance corresponds to the $n=2$ case and the energy level is 5.5μ, we must put

$$d+\eta^s+\varepsilon^s = -1.6\mu, \tag{8}$$

in formula (6). The $-\eta^s$ obtained from this formula is 1μ at most, for the d is small as we mentioned in Section 2.

As we considered ε^τ is positive, formula (7) indicates that $M(0, 1)$ is the smallest. Moreover, as $M(0, 1)$ becomes the minimum at $\lambda=0$, we get

$$M(i, j) \geq 4.35\mu - (d+\eta^s+\varepsilon^s) - K.$$

This result depends on K, but $M(i, j)$ must be larger than 4.35μ, because the mass of free nucleoneum must be larger than the mass of bounded one.

In this Appendix, we take tentative value $r'=2r_0$ as we suggested in Section 3 and put

$$K = \frac{1}{M}\left(\frac{\pi}{2}\right)^2 r'^{-2} = 0.9\mu.$$

Of course, this value does not contradict condition iii). Then we get

$$M(i, j) \geq 5.0\mu.$$

Since λ^τ, ε^τ and ε^s are unknown, we put $\lambda^\tau=0$ and $6\varepsilon^\tau=2.8\mu$, $\varepsilon^s=-0.5\mu$ tentatively taking into account condition 1), 2), 3).

Then we get

$$\begin{aligned} M(0,1) &= 5.0\mu, \quad M(1,1) = 6.9\mu, \\ M(0,0) &= 7.0\mu, \quad M(1,0) = 8.9\mu. \end{aligned}$$

Of course, these results are no more than one example that satisfies conditions 1), 2), 3).

Appendix II

In this Appendix, we shall estimate the range of the process shown in Fig. 3 as an example.

We write the wave function of nucleoneum and the wave function between the nucleoneums as $\varphi(r)$ and $\varphi^0(\boldsymbol{r})$ respectively. Moreover, we write the incident $\bar{N}$-wave as $\varphi^i(\boldsymbol{r})$. Then the initial and final states are given by

$$\varphi^i(\boldsymbol{t})F(\boldsymbol{r}),$$

$$\varphi(|\boldsymbol{r}-\boldsymbol{s}/2|)\varphi(\boldsymbol{t}-\boldsymbol{r}-\boldsymbol{s}/2|)\varphi^0\left(\frac{\boldsymbol{t}+\boldsymbol{s}}{2}\right),$$

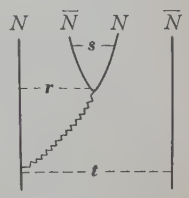


Fig. 6.

where the meaning of the variables $\boldsymbol{s}$, $\boldsymbol{t}$, $\boldsymbol{r}$ is illustrated in Fig. 6. $F(\boldsymbol{r})$ represents the state of the emitted π.

The matrix element $\boldsymbol{M}$ is given by

$$\boldsymbol{M}=\langle\text{two nucleoneums}|H_c|N+\bar{N}+\pi\rangle.$$

Therefore, we get

$$\boldsymbol{M}=C_1\int\varphi(r)\varphi(|\boldsymbol{t}-\boldsymbol{r}|)\varphi^0(\boldsymbol{t}/2)F(\boldsymbol{r})\varphi^i(\boldsymbol{t})\,d\boldsymbol{r}\,d\boldsymbol{t}.$$

For simplicity, we assume φ^i is S-wave. Then φ^0 must be P-wave. We write the momentum of φ^0 as k. In this calculation we assume k is small and put

$$\varphi^0(\boldsymbol{r})=\frac{1}{3}\sqrt{\frac{3}{4\pi}}N'k\boldsymbol{r},$$

$$\varphi^i(\boldsymbol{r})=\frac{N'}{\sqrt{4\pi}}G(r),$$

$$F(\boldsymbol{r})=C_2\frac{\boldsymbol{r}}{r^2}f(r).$$

Then, we get

$$\boldsymbol{M}=C_3\int\varphi(|\boldsymbol{t}-\boldsymbol{r}|)\varphi(r)f(r)G(t)\boldsymbol{t}\cdot\boldsymbol{r}/2r^2\,d\boldsymbol{r}\,d\boldsymbol{t}.$$

Since t is the distance between N and $\bar{N}$, the function of t,

$$\int\varphi(r)\varphi(|\boldsymbol{t}-\boldsymbol{r}|)f(r)\boldsymbol{t}\cdot\boldsymbol{r}/2r^2\,d\boldsymbol{r},$$

gives the strength of the reaction at the distance. As this function is always positive, we may consider that the mean square distance $\langle t^2\rangle$ by this function gives the range of the reaction.

By the substitution

$$\boldsymbol{t}-\boldsymbol{r}\to\boldsymbol{t},$$

this becomes

$$\frac{5}{3}\left(\frac{\int\varphi(t)t^2\,d\boldsymbol{t}}{\int\varphi(t)\,d\boldsymbol{t}}+\frac{\int f(r)\varphi(r)r^2\,d\boldsymbol{r}}{\int f(r)\varphi(r)\,d\boldsymbol{r}}\right).$$

If we put

$$f(r)=\exp(-\mu r),$$

we get

$$\sqrt{\frac{\langle t^2\rangle}{6}}=0.30\,\mu^{-1}.$$

References

1) G. F. Chew et al., Phys. Rev. **110** (1958), 265.
2) Federbush, Goldberger and Treiman, Phys. Rev. **112** (1958), 642.
3) W. R. Frazer and J. R. Fulco, Phys. Rev. Letters **2** (1959), 365.
4) G. Takeda, Phys. Rev. **100** (1955), 155.
5) F. J. Dyson, Phys. Rev. **99** (1955), 1037.
6) K. Hiida, N. Nakanishi et al., Prog Theor. Phys. **22** (1959), 247, 351, 863.
7) *CERN Conference* (1958), 39.
8) H. C. Burrowes et al., Phys. Rev. Letters **2** (1959), 119.
9) *Kiev Conference* (1959),1.
10) *CERN Conference* (1958), 147.
11) E. Amaldi et al., Nuovo Cimento **14** (1959), 977.
O. Chamberlain et al., Phys. Rev. **113** (1959), 1615.
12) L. Agnew, thesis, UCRL (1959) 8785.
N. Horowitz et al., UCRL 8591, in press in Phys. Rev.
13) S. C. Frautschi, Prog. Theor. Phys. **22** (1959), 15.
14) R. Hofstadter et al., Rev. Mod. Phys. **30** (1958), 482.
15) E. Fermi and C. N. Yang, Phys. Rev. **76** (1949), 1739.
16) S. J. Lindenbaum and R. B. Steinheimer, Phys. Rev. **105** (1957), 1874; **109** (1958), 1723.
17) G. F. Chew and F. E. Low, Phys. Rev. **101** (1956), 1570.
18) L. M. Eisberg et al., Phys. Rev. **97** (1955), 797.
19) M. M. Block et al., Phys. Rev. **103** (1956), 1484.
20) J. S. Ball and G. F. Chew, Phys. Rev. **109** (1958), 1385.
21) E. Fermi, Prog. Theor. Phys. **5** (1950), 570.
22) Z. Koba and G. Takeda, Prog. Theor. Phys. **19** (1958), 192.
23) E. Ebirle, Nuovo Cimento **8** (1958), 610.
T. Goto, Nuovo Cimento **8** (1958), 625.
24) N. Yajima and K. Kobayakawa, Prog. Theor. Phys. **19** (1958), 192.
25) I. Pomeranchuk, Doklady Akad. Nauk SSSR **78** (1958), 889.
26) S. Sakata, Prog. Theor. Phys. **16** (1956), 686.

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Lagrangian Formalism for a Classical Relativistic Particle Endowed with Internal Structure

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§ 1. General conservation relations

The problem of the classical motion of a relativistic particle endowed with proper angular momentum was approached many years ago by several authors.[1] Although they reached the same system of equations from very different starting points, none of their treatments seems us to be entirely satisfactory. The first one, Frenkel,[2] starts from the non-relativistic motion of a spinning dipole and uses a variational procedure. But the Lagrange variation he puts forward is not a total differential, and thus he is not able to express the Lagrangian completely. Mathisson and his followers[3] in a series of beautiful works use the frame of the general relativity and Einstein's field equations. But in the case of an electromagnetic field—which is just the case we are here dealing with—they include the Maxwell field into the metric, which does not seem to be correct. Afterwards Weyssenhoff[4] established and integrated the equations in a very smart way, but his starting point is purely axiomatic and fairly arbitrary. Bohm and Vigier[5] in a very important paper, which opens the way to their present theory of elementary particles, obtained the same equations as describing the global motion of an extended droplet of classical relativistic fluid. But their treatment is very narrowly bound to the consideration of a definite model, and the extension to the case in which an external field is acting, tentatively treated by the author,[6] appears to be fairly clumsy and arbitrary. Finally the treatment of Bouchiat and Michel,[7] performed from the standpoint of the bare and abstract group theory, appears paradoxically to be restricted to a particular case (linear momentum colinear to the velocity).

However, this purely classical problem has today acquired a great practical interest, because in the present giant accelerators the quantum effects can be neglected, while both the relativity effects and spin effects must be taken into account. The aim of the present paper is to approach again this problem with no other aid than the principles of classical mechanics considered in variational form. We consider a single particle travelling in an external field A_μ and moving along a definite curved line $\mathcal{L}$ in space-time. We associate to it a Lagrangian L taking into account the field by suitable interaction terms, and we submit this Lagrangian to a suitable variational principle concerning its integral along the line $\mathcal{L}$. The

feature of this treatment is that the state of the particle—unlike the Newtonian point-particle—is not assumed to be entirely defined by its position and velocity, but requires some supplementary quantities, the so-called internal variables, which describe its internal structure and are needed in order to express its proper rotation. The physical idea underlying this theory is that it is meaningless to speak about a proper rotation for a point-like particle. A particle whose proper rotation may be taken into account must necessarily admit of some distinct parts able to rotate with respect to one another. Thus the rotating particle is of necessity extended into space.

Now we can summarize in many ways the internal degrees of freedom of such a particle by global quantities derived by integration of some local densities over the whole space covered by it, and attached to some definite point looked at as its "center". The more accurate the description of the internal structure will be, the more numerous we must take these internal variables. Thus we can describe up to a definite degree of approximation a structure extended in space by various global quantities localized at a same single point, in the same way by which the MacLaurin development of a function describes the latter in a finite domain with the aid of several successive derivatives computed for a single definite value into this domain. The representation of a rotating fluid mass by Møller,[8] and further by the above quoted paper of Bohm and Vigier,[5] gives a pictorial instance of such a procedure. And it is obvious that all the other treatments of the rotating particle problem introduce this conception more or less explicitly.

But here these ideas will play no explicit role. We assume only that our particle can be suitably described by its reduction to a point-like center endowed with a set of internal variables which we admit to be vectors $q_\mu^{(r)}$, and which stand in the Lagrangian, including the interaction terms. Let $L(x_\mu, \dot{x}_\mu, q_\mu^{(r)}, \dot{q}_\mu^{(r)})$ be the Lagrangian of the particle at a point $M(x_\mu)$ on the line $\mathcal{L}$. The dot stands for the derivative with respect to the proper time τ, that is the derivative along the line $\mathcal{L}$, and we assume L to depend as usually on the variables and their first derivatives in τ. The dependence of L on the coordinates x_μ is assumed to be due exclusively to the interaction with the external field A_μ and thus to manifest itself roundabout by the customary dependence on $A_\mu(x)$ and its first derivatives $\partial_\nu A_\mu(x)$ (∂_ν denotes the operator $\partial/\partial x_\nu$). We shall use indifferently for the partial derivatives of L the equivalent forms $\partial_\mu L$ or $\partial L/\partial A_\nu \cdot \partial_\mu A_\nu + \partial L/\partial(\partial_\lambda A_\nu)\,\partial_\mu \partial_\lambda A_\nu$.

We first submit L to Hamilton's variational principle: the line-integral $\int_{M_1}^{M_2} L d\tau$ between two fixed points M_1 and M_2 is stationary along the actual line $\mathcal{L}$ with respect to all varied motions, the total interval of proper time $\int_{M_1}^{M_2} d\tau$ being kept constant for these varied motions. The usual calculation yields as an immediate outcome the ordinary Lagrange equations:

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{x}_\mu} = \frac{\partial L}{\partial x_\mu}, \qquad \frac{d}{d\tau}\frac{\partial L}{\partial \dot{q}_\mu^{(r)}} = \frac{\partial L}{\partial q_\mu^{(r)}}.$$

Now we wish to put into evidence the so-called linear and angular momenta by means of convenient " conservation relations " appearing as consequences of the invariance properties of the Lagrangian. Such a procedure was used successfully by J. P. Vigier, T. Takabayashi and ourselves[6),9),10)] in order to express the linear and angular momentum densities of a relativistic fluid endowed with intrinsic rotation. We used the well-known formalism of Belinfante[11)] and Rosenfeld[12)] for the classical field theory, the most convenient way to derive the latter formalism starting from a principle stated of old by E. Noether[13)] and the importance of which was recently emphasized by J. Winogradski[14)] : In the hydrodynamical treatment, Noether's principle applies to the integral of a Lagrangian density taken over a four-dimensional domain and considers a variation under a change of frame, so that the variations do not vanish on the limiting surface. After integration by parts, and having taken into account Lagrange's equations, the integrated part, that is a surface integral, remains alone.

In order to transpose that procedure to our line integral, we consider an infinitesimal change of frame, and express the invariance of the integral $\int_{M_1}^{M_2} L d\tau$: $\delta \int_{M_1}^{M_2} L d\tau = 0$. We have $\delta L = L'(x', \dot{x}', q', \dot{q}') - L(x, \dot{x}, q, \dot{q})$, $x', \dot{x}', q', \dot{q}'$ being the transformed values of the variables, and L' being the transformed form of the Lagrange function after the change of frame. The total variation can be split into two parts :

$$\delta L = L'(x', \dot{x}', q', \dot{q}') - L'(x, \dot{x}, q, \dot{q}) + L'(x, \dot{x}, q, \dot{q}) - L(x, \dot{x}, q, \dot{q}) \equiv \delta_1 L + \delta_2 L.$$

The first difference is the ordinary variation of a definite function L' under an infinitesimal change of frame acting on the vector variables, that is,

$$\delta_1 L = \frac{\partial L'}{\partial x_\mu} \delta x_\mu + \frac{\partial L'}{\partial \dot{x}_\mu} \delta \dot{x}_\mu + \frac{\partial L'}{\partial q_\mu^{(\nu)}} \delta q_\mu^{(\nu)} + \frac{\partial L'}{\partial \dot{q}_\mu^{(\nu)}} \delta \dot{q}_\mu^{(\nu)},$$

or, neglecting some second order variations,

$$\delta_1 L = \frac{\partial L}{\partial x_\mu} \delta x_\mu + \frac{\partial L}{\partial \dot{x}_\mu} \delta \dot{x}_\mu + \frac{\partial L}{\partial q_\mu^{(\nu)}} \delta q_\mu^{(\nu)} + \frac{\partial L}{\partial \dot{q}_\mu^{(\nu)}} \delta \dot{q}_\mu^{(\nu)}.$$

Using Noether's procedure, we integrate by parts without vanishing of the integrated parts ; and, taking into account Lagrange's equations, we are left with these integrated parts, that is,

$$\delta_1 L = \frac{d}{d\tau} \left(\frac{\partial L}{\partial \dot{x}_\mu} \delta x_\mu + \frac{\partial L}{\partial \dot{q}_\mu^{(\nu)}} \delta q_\mu^{(\nu)} \right).$$

The second difference, $\delta_2 L = L'(x, \dot{x}, q, \dot{q}) - L(x, \dot{x}, q, \dot{q})$, appears to be more difficult to work out because of the hybrid character of its first term, which belongs at a time to the former frame Σ and to the new frame Σ'. We shall consider that this first term is entirely computed in the frame Σ', the coordinates x_μ being those of a point M' different from M and having as coordinates *relative to* Σ'

the same coordinates x_μ as those of M *relative to* Σ. In a way of picture, let us consider a simple rotation of the axes x_1, x_2 in their own plane. The same rotation applied to the point M around the origin leads us to a point M^+ which has the required coordinates in Σ'. Repeating this operation with each point of the line $\mathcal{L}$, we get a hinged line $\mathcal{L}^+$. We can also take the set of vectors $q_\mu^{(r)}$ attached to each point M and carry them to corresponding point M^+, each of them hinging from the same angle, so that the components of the hinged vectors with respect to Σ' are the same as those of the former vectors with respect to Σ. The proper time τ on $\mathcal{L}^+$ having obviously the same value at each M^+ as at the corresponding M of $\mathcal{L}$, the vectors $\dot{x}_\mu$, $\dot{q}_\mu^{(r)}$ at M^+ have also the same components in the new system as the corresponding vectors at M in the old one.

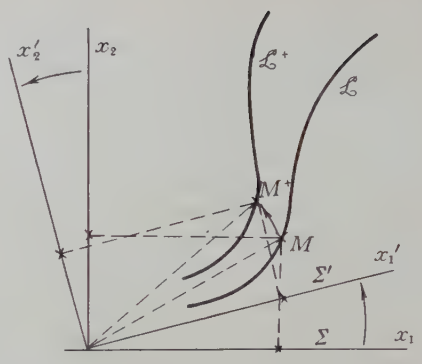

Fig. 1.

Thus between the expression of the Lagrangian L' on the line $\mathcal{L}^+$ in Σ' and that of L on the line $\mathcal{L}$ in Σ, the only difference arises from the external field, because M^+ is not at the same place of this field as M, and the vector A_μ and tensor $\partial_\nu A_\mu$ at M^+ are the project on different axes as those considered at M. This remark immediately yields our difference $\delta_2 L$. Let us notice that, if no external field is acting, this difference is zero. For a free particle, the function L has the same form in all frames. This is the very simplified case we have treated with J. P. Vigier in two recent papers.[20]

Now if δx_μ stands for the variation of the coordinates of a given point brought about by an infinitesimal change of frame, the displacement MM^+ that has to cancel that variation is $-\delta x_\mu$. At the point M^+ the field A_μ and its derivatives $\partial_\nu A_\mu$ consequently become,

$$A_\mu{}^+ = A_\mu + \partial_\nu A_\mu(-\delta x_\nu), \quad \text{and} \quad (\partial_\nu A_\mu)^{+\prime} = \partial_\nu A_\mu + \partial_\lambda \partial_\nu A_\mu(-\delta x_\lambda),$$

if they are referred to the old frame Σ. But because in the expression L', they are to be referred to the new frame Σ', they become

$$A_\mu^{+\prime} = A_\mu{}^+ + \delta A_\mu{}^+, \quad (\partial_\nu A_\mu)^{+\prime} = (\partial_\nu A_\mu)^+ + \delta(\partial_\nu A_\mu)^+,$$

and, if we neglect second order differences, we finally get,

$$A_\mu^{+\prime} = A_\mu - \partial_\nu A_\mu \delta x_\nu + \delta A_\mu, \quad (\partial_\nu A_\mu)^{+\prime} = \partial_\nu A_\mu - \partial_\lambda \partial_\nu A_\mu \delta x_\lambda + \delta(\partial_\nu A_\mu).$$

Let us recall that the symbol δ stands for the variations undergone by the coordinates x_μ, the vector A_μ and tensor $\partial_\nu A_\mu$ under a given infinitesimal transformation of frame. Thus we easily obtain

$$\delta_2 L=\frac{\partial L}{\partial A_\mu}(\delta A_\mu-\partial_\nu A_\mu \delta x_\nu)+\frac{\partial L}{\partial(\partial_\nu A_\mu)}[\delta(\partial_\nu A_\mu)-\partial_\lambda \partial_\nu A_\mu \delta x_\lambda].$$

Now we can express the invariant character of the integral $\int L d\tau$ taken on an arbitrary arc of the line of motion $\mathcal{L}$:

$$\int \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}_\mu}\delta x_\mu+\frac{\partial L}{\partial \dot{q}_\mu^{(r)}}\delta q_\mu^{(r)}\right)d\tau+\int \frac{\partial L}{\partial A_\mu}(\delta A_\mu-\partial_\nu A_\mu \delta x_\nu)\,d\tau$$

$$+\int \frac{\partial L}{\partial(\partial_\nu A_\mu)}[\delta(\partial_\nu A_\mu)-\partial_\lambda \partial_\nu A_\mu \delta x_\lambda]d\tau=0.$$

Because of the arbitrariness of the arc M_1M_2 (which can be chosen infinitesimal) it is required that the integrand vanishes, so that

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}_\mu}\delta x_\mu+\frac{\partial L}{\partial \dot{q}_\mu^{(r)}}\delta q_\mu^{(r)}\right)$$

$$=\frac{\partial L}{\partial A_\mu}(\partial_\nu A_\mu \delta x_\nu-\delta A_\mu)+\frac{\partial L}{\partial(\partial_\nu A_\mu)}[\partial_\lambda \partial_\nu A_\mu \delta x_\lambda-\delta(\partial_\nu A_\mu)]$$

Now we have to specify the kind of the infinitesimal transformation, in order to get the expected conservation relations.

Let us first take a translation, in which $\delta x_\mu=\varepsilon_\mu$, a constant, all the other variations being zero. Thus we get

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}_\mu}\varepsilon_\mu\right)=\frac{\partial L}{\partial A_\mu}\partial_\nu A_\mu \varepsilon_\nu+\frac{\partial L}{\partial(\partial_\nu A_\mu)}\partial_\lambda \partial_\nu A_\mu \varepsilon_\lambda,$$

or, dropping ε_μ as a constant factor,

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}_\mu}\right)=\frac{\partial L}{\partial A_\nu}\partial_\mu A_\nu+\frac{\partial L}{\partial(\partial_\lambda A_\nu)}\partial_\mu \partial_\lambda A_\nu.$$

This is just one of the Lagrange equations written down above. In order to fit the hydrodynamical formalism, we set

$$\frac{\partial L}{\partial \dot{x}_\mu}=G_\mu. \tag{1}$$

G_μ can be called the *canonical linear momentum* as in the Hamiltonian formalism, and we have then the first conservation relation :

$$\dot{G}_\mu=\frac{\partial L}{\partial A_\nu}\partial_\mu A_\nu+\frac{\partial L}{\partial(\partial_\lambda A_\nu)}\partial_\mu \partial_\lambda A_\nu. \tag{2}$$

Then we take for the variation δ an infinitesimal Lorentz transformation $\varepsilon_{[\mu\nu]}$. For a vector V_μ and a tensor $T_{\mu\nu}$, we have the well-known variations :

$$\delta V_\mu=\delta_{\mu\nu}^{\alpha\beta}\varepsilon_{\alpha\beta}V_\nu,\quad \delta T_{\mu\nu}=(\delta_{\mu\lambda}^{\alpha\beta}T_{\lambda\nu}+\delta_{\nu\lambda}^{\alpha\beta}T_{\mu\lambda})\varepsilon_{\alpha\beta},$$

where $\delta^{\alpha\beta}_{\mu\nu}=\frac{1}{2}(\delta_{\alpha\mu}\delta_{\beta\nu}-\delta_{\alpha\nu}\delta_{\beta\mu})$, and thus our general equation becomes, apart from the arbitrary constant factor $\varepsilon_{\alpha\beta}$,

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}_\alpha}x_\beta-\frac{\partial L}{\partial \dot{x}_\beta}x_\alpha+\frac{\partial L}{\partial \dot{q}^{(r)}_\alpha}q^{(r)}_\beta-\frac{\partial L}{\partial \dot{q}^{(r)}_\beta}q^{(r)}_\alpha\right)=-\frac{\partial L}{\partial A_\mu}(\partial_\alpha A_\mu x_\beta-\partial_\beta A_\mu x_\alpha)$$
$$-\frac{\partial L}{\partial A_\alpha}A_\beta+\frac{\partial L}{\partial A_\beta}A_\alpha+\frac{\partial L}{\partial(\partial_\nu A_\mu)}(\partial_\alpha\partial_\nu A_\mu x_\beta-\partial_\beta\partial_\nu A_\mu x_\alpha)$$
$$-\frac{\partial L}{\partial(\partial_\nu A_\alpha)}\partial_\nu A_\beta+\frac{\partial L}{\partial(\partial_\nu A_\beta)}\partial_\nu A_\alpha-\frac{\partial L}{\partial(\partial_\alpha A_\nu)}\partial_\beta A_\nu+\frac{\partial L}{\partial(\partial_\beta A_\nu)}\partial_\alpha A_\nu.$$

We introduce the so-called *canonical angular momentum*:

$$S_{\alpha\beta}=q^{(r)}_\alpha\frac{\partial L}{\partial \dot{q}^{(r)}_\beta}-q^{(r)}_\beta\frac{\partial L}{\partial \dot{q}^{(r)}_\alpha}. \tag{3}$$

We take into account the expression (1) of G_μ and Eq. (2):

$$G_\alpha\dot{x}_\beta-G_\beta\dot{x}_\alpha-\dot{S}_{\alpha\beta}=A_\alpha\frac{\partial L}{\partial A_\beta}-A_\beta\frac{\partial L}{\partial A_\alpha}-\frac{\partial L}{\partial(\partial_\nu A_\alpha)}\partial_\nu A_\beta$$
$$+\frac{\partial L}{\partial(\partial_\nu A_\beta)}\partial_\nu A_\alpha-\frac{\partial L}{\partial(\partial_\alpha A_\nu)}\partial_\beta A_\nu+\frac{\partial L}{\partial(\partial_\beta A_\nu)}\partial_\alpha A_\nu, \tag{4}$$

which is the second conservation relation. Equations (2) and (4) generalize the well-known Frenkel-Weyssenhoff equations.

§ 2. Lagrangian formalism for the spinning electric particle

Our purpose is now to apply the preceding general conservation equations to the case of the relativistic spinning particle moving classically in an electromagnetic field. We must first put forward a Lagrangian for the free spinning particle admitting of suitable internal variables. This will be done on account of the corresponding formulation which was recently brought about for a continuous spinning fluid[6),15) 16),17),18),19)] with the aid of the so-called Einstein-Kramers variables. We express the internal state of motion of the particle by a set of four unitary vectors $b^{(\xi)}_\mu$ orthogonal to one another (ξ is not a tensor suffix, but an index labelling the vectors, which runs from 1 to 4). Thus we have the orthonormality relations:

$$b^{(\xi)}_\mu b^{(\eta)}_\mu=\delta^{\xi\eta},\quad b^{(\xi)}_\mu b^{(\xi)}_\nu=\delta_{\mu\nu}. \tag{5}$$

The fourth of these vectors is supposed to be colinear to the velocity of the particle, so that $\dot{x}_\mu=icb^{(4)}_\mu$. The three remaining vectors express the proper rotation of the particle in the following way: Its generalized angular velocity (kinematical description) has the expression

$$w_{\mu\nu}=\frac{1}{2}(\dot{b}^{(\xi)}_\mu b^{(\xi)}_\nu-\dot{b}^{(\xi)}_\nu b^{(\xi)}_\mu)=\dot{b}^{(\xi)}_\mu b^{(\xi)}_\nu,\ \text{because of (5)}.$$

Further its proper angular momentum (dynamical description) is

$$M_{\mu\nu}=h_0(b_\mu^{(1)}b_\nu^{(2)}-b_\nu^{(1)}b_\mu^{(2)}),$$

(h_0 being the magnitude of the spin), so that the energy of proper rotation $T=\frac{1}{2}M_{\mu\nu}\omega_{\mu\nu}$ takes the form

$$T=\frac{1}{2}h_0(b_\mu^{(1)}b_\nu^{(2)}-b_\nu^{(1)}b_\mu^{(2)})\dot{b}_\mu^{(\xi)}b_\nu^{(\xi)},$$

or, taking the relations (5) into account,

$$T=h_0 b_\mu^{(1)}\dot{b}_\mu^{(2)}.$$

This is not the most general expression using the $b_\mu^{(\xi)}$ variables[19], but it is the simplest one and covers probably the most significant cases. Let us remark that we thus note the "physical" angular momentum $M_{\mu\nu}$ derived from dynamical considerations, by another symbol than the above "canonical" angular momentum $S_{\mu\nu}$ which arises from the preceding formal theory. It is the eventual agreement of their two expressions after calculation which will secure the correctness of our Lagrangian.

The Lagrangian must moreover include a classical proper energy term $\frac{1}{2}m_0\dot{x}_\mu\dot{x}_\mu$ and a Lagrange term $\lambda^{\xi\eta}(b_\mu^{(\xi)}b_\mu^{(\eta)}-\delta^{\xi\eta})$ which assures the orthonormality relations (the $\lambda^{\xi\eta}$ are Lagrange multipliers which are evidently symmetrical in ξ and η). However, we see that the third internal vector $b_\mu^{(3)}$ does not appear in the mechanical terms, so that it may be left out, the value $\xi, \eta=3$ being excluded. On the other hand, the product $\dot{x}_\mu\dot{x}_\mu$ appears twice, because

$$\lambda^{44}b_\mu^{(4)}b_\mu^{(4)}=-\frac{1}{c^2}\lambda^{44}\dot{x}_\mu\dot{x}_\mu.$$

Thus we may bring the two terms together in one and write

$$\frac{1}{2}\mathfrak{m}_0(\dot{x}_\mu\dot{x}_\mu+c^2).$$

But now $\mathfrak{m}_0$ is a variable Lagrange multiplier, as the other $\lambda^{\xi\eta}$'s and it does not longer stand for the constant proper mass m_0 of the particle. Finally, after replacement of $b_\mu^{(4)}$ by $1/ic\cdot\dot{x}_\mu$, we get the following Lagrangian for a free particle:

$$L^*=-h_0 b_\mu^{(1)}\dot{b}_\mu^{(2)}+\frac{1}{2}\mathfrak{m}_0(\dot{x}_\mu\dot{x}_\mu+c^2)+\lambda^{11}(b_\mu^{(1)}b_\mu^{(1)}-1)+\lambda^{22}(b_\mu^{(2)}b_\mu^{(2)}-1)$$
$$+2\lambda^{12}b_\mu^{(1)}b_\mu^{(2)}+2\frac{\lambda^{14}}{ic}b_\mu^{(1)}\dot{x}_\mu+2\frac{\lambda^{24}}{ic}b_\mu^{(2)}\dot{x}_\mu.$$

Now we assume the particle to be an electric dipole having a total electric charge e, and a relativistic magnetoelectric moment connected with its proper angular momentum by the well-known classical relation:

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* This expression was at first established and kindly communicated to the author by Prof. T. Takabayashi.

$$L_{\alpha\beta}=\frac{e}{m_0 c}M_{\alpha\beta}=\frac{eh_0}{m_0 c}(b_\alpha^{(1)}b_\beta^{(2)}-b_\alpha^{(2)}b_\beta^{(1)}).$$

These are the initial assumptions made by Frenkel.[2] In an electromagnetic field defined by the electromagnetic potential A_μ and by the Maxwell tensor $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$, the interaction energy of our dipole is, as is well known,

$$\frac{e}{c}A_\mu\dot{x}_\mu+\frac{1}{2}\mu_{\alpha\beta}F_{\alpha\beta}=\frac{e}{c}A_\mu\dot{x}_\mu+\frac{eh_0}{m_0 c}b_\mu^{(1)}b_\nu^{(2)}(\partial_\mu A_\nu-\partial_\nu A_\mu).$$

These are the two terms we have to add to the preceding Lagrangian so as to take the action of the field into account. Thus we finally get,

$$L=\frac{1}{2}m_0(\dot{x}_\mu\dot{x}_\mu-1)-h_0 b_\mu^{(1)}\dot{b}_\mu^{(2)}+\frac{e}{c}A_\mu\dot{x}_\mu+\frac{eh_0}{m_0 c}b_\mu^{(1)}b_\nu^{(2)}(\partial_\mu A_\nu-\partial_\nu A_\mu)$$
$$+\lambda^{11}(b_\mu^{(1)}b_\mu^{(1)}-1)+\lambda^{22}(b_\mu^{(2)}b_\mu^{(2)}-1)+2\lambda^{12}b_\mu^{(1)}b_\mu^{(2)}+2\frac{\lambda^{14}}{ic}b_\mu^{(1)}\dot{x}_\mu+2\frac{\lambda^{24}}{ic}b_\mu^{(2)}\dot{x}_\mu.$$

In order to express the conservation equations, let us first derive the expressions of the two canonical momenta,

$$G_\mu=\frac{\partial L}{\partial\dot{x}_\mu}=m_0\dot{x}_\mu+\frac{e}{c}A_\mu+2\frac{\lambda^{14}}{ic}b_\mu^{(1)}+2\frac{\lambda^{24}}{ic}b_\mu^{(2)},$$

$$S_{\mu\nu}=b_\mu^{(2)}\frac{\partial L}{\partial\dot{b}_\nu^{(2)}}-b_\nu^{(2)}\frac{\partial L}{\partial\dot{b}_\mu^{(2)}}=h_0(b_\mu^{(1)}b_\nu^{(2)}-b_\nu^{(1)}b_\mu^{(2)}).$$

As we hoped, the canonical angular momentum appears to have the same expression as the physical one. We can blot out the provisional notation $M_{\mu\nu}$. We must still eliminate the subsidiary variables λ^{14}, λ^{24}. For this aim, we write down the two Lagrange equations relative to $b_\mu^{(1)}$ and $b_\mu^{(2)}$:

$$-h_0\dot{b}_\mu^{(2)}+\frac{eh_0}{m_0 c}F_{\mu\nu}b_\nu^{(2)}+2\lambda^{11}b_\mu^{(1)}+2\lambda^{12}b_\mu^{(2)}+2\frac{\lambda^{14}}{ic}\dot{x}_\mu=0,$$

$$\dot{h}_0 b_\mu^{(1)}+h_0\dot{b}_\mu^{(1)}-\frac{eh_0}{m_0 c}F_{\mu\nu}b_\nu^{(1)}+2\lambda^{22}b_\mu^{(2)}+2\lambda^{12}b_\mu^{(1)}+2\frac{\lambda^{24}}{ic}\dot{x}_\mu=0.$$

If we contract these equations by $\dot{x}_\mu$ and take the relations (5) into account, we are left with

$$-h_0\dot{b}_\mu^{(2)}\dot{x}_\mu+\frac{eh_0}{m_0 c}F_{\mu\nu}\dot{x}_\mu b_\nu^{(2)}+2ic\lambda^{14}=0,$$

$$h_0\dot{b}_\mu^{(1)}\dot{x}_\mu-\frac{eh_0}{m_0 c}F_{\mu\nu}\dot{x}_\mu b_\nu^{(1)}+2ic\lambda^{24}=0,$$

which yields the expected expressions for λ^{14} and λ^{24},

$$2\frac{\lambda^{14}}{ic}=-\frac{h_0}{c^2}\dot{b}_\mu^{(2)}\dot{x}_\mu+\frac{1}{c_2}\frac{eh_0}{m_0 c}F_{\mu\nu}\dot{x}_\mu b_\nu^{(1)},$$

$$2\frac{\lambda^{24}}{ic}=\frac{h_0}{c^2}\dot{b}_\mu^{(1)}\dot{x}_\mu-\frac{1}{c^2}\frac{eh_0}{m_0c}F_{\mu\nu}\dot{x}_\mu b_\nu^{(1)}.$$

Or further, because $\dot{x}_\mu b_\mu^{(1)}=\dot{x}_\mu b_\mu^{(2)}=0$,

$$2\frac{\lambda^{14}}{ic}=\frac{h_0}{c^2}\left(\ddot{x}_\mu-\frac{e}{m_0c}F_{\mu\nu}\dot{x}_\nu\right)b_\mu^{(2)},$$

$$2\frac{\lambda^{24}}{ic}=\frac{h_0}{c^2}\left(-\ddot{x}_\mu+\frac{e}{m_0c}F_{\mu\nu}\dot{x}_\nu\right)b_\mu^{(1)}.$$

Thus, carrying these expressions into G_μ, we get

$$G_\mu=\mathfrak{m}_0\dot{x}_\mu+\frac{e}{c}A_\mu+\frac{h_0}{c^2}\left(\ddot{x}_\nu-\frac{e}{m_0c}F_{\nu\lambda}\dot{x}_\lambda\right)(b_\nu^{(2)}b_\mu^{(1)}-b_\nu^{(1)}b_\mu^{(2)}).$$

That is, finally,

$$G_\mu=\mathfrak{m}_0\dot{x}_\mu+\frac{e}{c}A_\mu+\frac{1}{c^2}S_{\mu\nu}\left(\ddot{x}_\nu-\frac{e}{m_0c}F_{\nu\lambda}\dot{x}_\lambda\right). \tag{6}$$

Now we can write down the conservation relations. We have the derivatives $\partial L/\partial A_\nu=e/c\cdot x_\nu$, $\partial L/\partial(\partial_\lambda A_\nu)=(eh_0/m_0c)(b_\lambda^{(1)}b_\nu^{(2)}-b_\lambda^{(2)}b_\nu^{(1)})=e/m_0c\cdot S_{\lambda\nu}$. Thus the first conservation relation takes the form

$$\dot{G}_\mu=\frac{e}{c}\dot{x}_\nu\partial_\mu A_\nu+\frac{e}{m_0c}S_{\lambda\nu}\partial_\mu\partial_\lambda A_\nu.$$

In order to fit in the formalism of Weyssenhoff, it is convenient to take away from G_μ its electromagnetic part, and thus to introduce a "purely mechanical linear momentum" differing from the canonical one:

$$K_\mu=G_\mu-\frac{e}{c}A_\mu=\mathfrak{m}_0\dot{x}_\mu+\frac{1}{c^2}S_{\mu\nu}\left(\ddot{x}_\nu-\frac{e}{m_0c}F_{\nu\lambda}\dot{x}_\lambda\right). \tag{7}$$

Because we can write $\dot{A}_\mu=\dot{x}_\nu\partial_\nu A_\mu$, our first conservation relation becomes

$$\dot{K}_\mu=-\frac{e}{c}\dot{x}_\nu\partial_\nu A_\mu+\frac{e}{c}\dot{x}_\nu\partial_\mu A_\nu+\frac{e}{m_0c}S_{\lambda\nu}\partial_\mu\partial_\lambda A_\nu,$$

that is,

$$\dot{K}_\mu=\frac{e}{c}F_{\mu\nu}\dot{x}_\nu+\frac{1}{2}\frac{e}{m_0c}S_{\lambda\nu}\partial_\mu F_{\lambda\nu} \tag{8}$$

which is exactly the equation postulated by Weyssenhoff. We recognize in the right-hand terms the Lorentz force $f_\mu=e/c\cdot F_{\mu\nu}\dot{x}_\nu$ exerted on an elementary electric charge, and the Stern-Gerlach force undergone by an elementary dipole. On the other hand, the second conservation relation appears easily to be

$$G_\mu \dot{x}_\nu - G_\nu \dot{x}_\mu - \dot{S}_{\mu\nu} = \frac{e}{c}(A_\mu \dot{x}_\nu - A_\nu \dot{x}_\mu)$$

$$+\frac{e}{m_0 c}(-S_{\alpha\mu}\partial_\alpha A_\nu + S_{\alpha\nu}\partial_\alpha A_\mu - S_{\mu\alpha}\partial_\nu A_\alpha + S_{\nu\alpha}\partial_\mu A_\alpha)$$

$$= \frac{e}{c}(A_\mu \dot{x}_\nu - A_\nu \dot{x}_\mu) + \frac{e}{m_0 c}(S_{\alpha\mu}F_{\nu\alpha} - S_{\alpha\nu}F_{\mu\alpha}).$$

Introducing still the vector K_μ, we get

$$\dot{S}_{\mu\nu} = K_\mu \dot{x}_\nu - K_\nu \dot{x}_\mu + \frac{e}{m_0 c}(S_{\mu\alpha}F_{\nu\alpha} - S_{\nu\alpha}F_{\mu\alpha}). \tag{9}$$

This is also an equation postulated by Weyssenhoff, where we recognize in the right-hand side, besides the orbital term, the ordinary expression for the torque acted on an elementary dipole by the Maxwell field. Finally, we note that the contracted expression $S_{\mu\nu}\dot{x}_\nu$ is zero, as a consequence of the expression of $S_{\mu\nu}$ and the orthogonality conditions, that is as a result of our particular choice for the internal variables. Thus, Eqs. (8) and (9) together with the relation

$$S_{\mu\nu}\dot{x}_\nu = 0, \tag{10}$$

yield the complete set of Weyssenhoff's equations, with all the outcomes which can be derived from them, and the most important ones we shall now briefly recall.

Let us denote the unitary velocity $\dot{x}_\mu$ by u_μ ($u_\mu u_\mu = -c^2$) and the Lorentz force $e/c \cdot F_{\mu\nu}\ u_\nu$ by $f_\mu (f_\mu u_\mu = 0)$. Further, Eq. (7) shows it is suitable to introduce the subsidiary vector

$$a_\mu = \dot{u}_\mu - \frac{1}{m_0} f_\mu.$$

In the case of a classical point-particle without internal rotation, we should have $m_0 \dot{u}_\mu = f_\mu$, that is, $a_\mu = 0$, and thus a_μ appears to be in the general case any supplementary acceleration due to the proper rotation. We have of course, $a_\mu u_\mu = 0$.

With the help of the latter notation, Eq. (7) becomes

$$K_\mu = \mathfrak{m}_0 u_\mu + \frac{1}{c^2} S_{\mu\nu} a_\nu. \tag{11}$$

Further, the expression $K_\mu u_\nu - K_\nu u_\mu$ becomes

$$\frac{1}{c_2}(S_{\mu\lambda}a_\lambda u_\nu - S_{\nu\lambda}a_\lambda u_\mu) = \frac{1}{c^2}[S_{\mu\lambda}(a_\lambda u_\nu - a_\nu u_\lambda) - S_{\nu\lambda}(a_\lambda u_\mu - a_\mu u_\lambda)],$$

under addition of two vanishing terms. Thus, Eq. (9) becomes

$$\dot{S}_{\mu\nu} = \frac{e}{m_0 c}\left\{S_{\mu\lambda}\left[F_{\nu\lambda} + \frac{1}{c^2}\frac{m_0 c}{e}(u_\nu a_\lambda - a_\nu u_\lambda)\right]\right.$$

$$\left. - S_{\nu\lambda}\left[F_{\mu\lambda} + \frac{1}{c^2}\frac{m_0 c}{e}(u_\mu a_\lambda - a_\mu u_\lambda)\right]\right\}.$$

Now it is suitable to define once more a subsidiary expression, namely the skew-symmetrical tensor,[2]

$$F'_{\mu\nu}=F_{\mu\nu}+\frac{1}{c^2}\frac{m_0 c}{e}(u_\mu a_\nu-a_\mu u_\nu).$$

Thus, Eq. (9) eventually becomes

$$\dot{S}_{\mu\nu}=\frac{e}{m_0 c}(S_{\mu\lambda}F'_{\nu\lambda}-S_{\nu\lambda}F'_{\mu\lambda}). \tag{12}$$

It is useful to point out that the above definition of $F'_{\mu\nu}$ draws identically for the acceleration the expression

$$\dot{u}_\mu=\frac{e}{m_0 c}F'_{\mu\nu}u_\nu,$$

which may be compared to the classical one relative to the spinless point-particle: $\dot{u}_\mu=e/m_0 c\cdot F_{\mu\nu}u_\nu$.

Now, starting from the above transformed equations, we can first establish the constancy of the magnitude of the proper angular momentum, putting $S_0^2=\frac{1}{2}\cdot S_{\mu\nu}S_{\mu\nu}$. We have $\dot{S}_0 S_0=\frac{1}{2}\cdot\dot{S}_{\mu\nu}S_{\mu\nu}$, or, using (12), $\dot{S}_0 S_0=\frac{1}{2}(S_{\mu\lambda}F'_{\nu\lambda}-S_{\nu\lambda}F'_{\mu\lambda})S_{\mu\nu}$ which vanishes because of the antisymmetry of $F'_{\mu\nu}$. Thus, S_0 is constant.

Further, we can study the proper mass $\mathfrak{m}_0$, which can be expressed by contraction of (11) with u_μ: $-\mathfrak{m}_0 c^2=K_\mu u_\mu$. Derivative with respect to the proper time provides the time variation of the proper mass: $-\dot{\mathfrak{m}}_0 c^2=\dot{K}_\mu u_\mu+K_\mu\dot{u}_\mu$ or, using (8) and (11): $-\dot{\mathfrak{m}}_0 c^2=\frac{1}{2}\cdot e/m_0 c\cdot S_{\lambda\nu}u_\mu\partial_\mu F_{\lambda\nu}+1/c^2\cdot S_{\mu\nu}a_\nu\dot{u}_\mu$. We recall that the operator $u_\mu\partial_\mu$ denotes the derivative with respect of the proper time, and thus we get after integration by parts

$$-\dot{\mathfrak{m}}_0 c^2=\frac{d}{d\tau}\left(\frac{1}{2}\frac{e}{m_0 c}S_{\lambda\nu}F_{\lambda\nu}\right)-\frac{1}{2}\frac{e}{m_0 c}F_{\lambda\nu}\dot{S}_{\lambda\nu}-\frac{1}{c^2}\dot{S}_{\mu\nu}a_\nu u_\mu,$$

which becomes, after a short calculation,

$$-\dot{\mathfrak{m}}_0 c^2=\frac{d}{d\tau}\left(\frac{1}{2}\frac{e}{m_0 c}S_{\lambda\nu}F_{\lambda\nu}\right)-\frac{1}{2}\frac{e}{m_0 c}\dot{S}_{\mu\nu}\left[F_{\mu\nu}+\frac{1}{c^2}\frac{e}{m_0 c}(u_\mu a_\nu-a_\mu u_\nu)\right].$$

In the second term, we recognize the tensor $F'_{\mu\nu}$ and see that this term becomes, according to (9), $-\frac{1}{2}\cdot(e/m_0 c)^2\cdot(S_{\mu\lambda}F'_{\nu\lambda}-S_{\nu\lambda}F'_{\mu\lambda})F'_{\mu\nu}$ which vanishes because of the antisymmetry of $S_{\mu\nu}$. Thus, we are left with $\dot{\mathfrak{m}}_0=-d/d\tau(\frac{1}{2}\cdot 1/c^2\cdot e/m_0 c\cdot F_{\mu\nu}S_{\mu\nu})$. As the proper mass $\mathfrak{m}_0$ evidently reduces to the constant mass m_0 when the angular momentum vanishes, we can write

$$\mathfrak{m}_0=m_0-\frac{1}{2}\frac{1}{c^2}\frac{e}{m_0 c}F_{\mu\nu}S_{\mu\nu}, \tag{13}$$

the second term expressing the mass of the potential energy of the dipole in the electromagnetic field.

We shall now express the relativistic spin vector which is defined as usually[6] by

$$s_\alpha = \frac{i}{2c}\varepsilon_{\beta\mu\nu\alpha}u_\beta S_{\mu\nu}, \quad (s_\mu u_\mu = 0,\ \varepsilon_{\beta\mu\nu\alpha} \text{ is the Levi-Civita symbol}).$$

Since $S_{\mu\nu}u_\nu = 0$, the angular momentum can be conversely written as

$$S_{\mu\nu} = \frac{i}{c}\varepsilon_{\mu\nu\alpha\beta}u_\alpha s_\beta, \quad (\text{so that } S_{\mu\nu}s_\nu = 0). \tag{14}$$

These expressions show that the vector s_μ lies along the third vector $b_\mu^{(3)}$ of our basic tetrad. As we have established, this vector remains constant in Weyssenhoff's case (no external field). Here we can derive its time variation,

$$\dot{s}_\alpha = \frac{i}{2c}\varepsilon_{\beta\mu\nu\alpha}(u_\beta \dot{S}_{\mu\nu} + \dot{u}_\beta S_{\mu\nu})$$

or, using (11) and (12),

$$\dot{s}_\alpha = \frac{i}{2c}\varepsilon_{\beta\mu\nu\alpha}\left(\frac{2e}{m_0 c}u_\beta S_{\mu\lambda}F'_{\nu\lambda} + \frac{e}{m_0 c}F'_{\beta\lambda}u_\lambda S_{\mu\nu}\right).$$

Putting again for $S_{\mu\lambda}$ and $S_{\mu\nu}$ expression (14), we easily get after a short calculation

$$\dot{s}_\alpha = \frac{e}{m_0 c}F'_{\alpha\beta}s_\beta \tag{15}$$

(Contraction by s_α shows that the spin has a constant magnitude s_0). Such a shortened expression must become of a meaning by carrying into it the definition of the subsidiary tensor $F'_{\alpha\beta}$ in terms of the physical quantities. The calculation provides

$$\dot{s}_\alpha = \frac{e}{m_0 c}\left(F_{\alpha\beta}s_\beta - \frac{1}{c^2}u_\alpha F_{\beta\lambda}u_\lambda s_\beta\right) + \frac{1}{c^2}u_\alpha \dot{u}_\beta s_\beta,$$

which can be put in the form

$$\left(\dot{s}_\beta - \frac{e}{m_0 c}F_{\beta\lambda}s_\lambda\right)\left(\delta_{\beta\alpha} + \frac{1}{c^2}u_\beta u_\alpha\right) = 0,$$

by replacing $\dot{u}_\beta s_\beta$ by $-u_\beta \dot{s}_\beta$. This is a relation projected on the "proper space" hyperplane orthogonal to u_μ. Nevertheless no clear interpretation can be given to such an equation, because the proper frame is not a Galilean one, its motion being accelerated, so that the projection of $\dot{s}_\mu$ is not the derivative of the projection of s_μ. Moreover, the study of the free particle case shows that the point x_μ follows a helicoidal travel which describes in reality a complicated internal motion inside the particle and obviously escapes from the experiment.

Consequently, it is of interest to pass to another frame by considering the simpler motion of another point. We have established by studying the free par-

ticle case that Weyssenhoff's formalism describes in reality a " bilocal " edifice composed of the above considered point x_μ and of another point y_μ playing the role of a center of gravity. Indeed, if we put

$$M_0^2 c^2 = -K_\mu K_\mu, \quad R_\mu = \frac{1}{M_0^2 c^2} S_{\mu\nu} K_\nu, \quad y_\mu = x_\mu - R_\mu,$$

we easily show that the point y_μ travels parallel to K_μ in a uniform motion. This is the " motion of the circle " put into evidence by Weyssenhoff. In the general case, according to Eq. (8), K_μ may be considered as defining an inertial frame more suitable to be taken as reference for the description of the proper rotation. It is doubtless such an inertial frame which is taken into account by the experimentalist as he refers to the motion of the particle considered as a whole. Further, we must then choose as the four-vector describing the proper rotation a vector orthogonal to K_μ just as the above considered s_μ was orthogonal to u_μ. It appears immediately that s_μ has just the required property: if we contract Eq. (11) by s_μ, we see that $K_\mu s_\mu = 0$. Thus we are led to study the motion of the spin s_μ (with constant magnitude s_0) with respect to some axes bound to the linear momentum K_μ, or in other words, with respect to an inertial frame carried by the center of gravity. Our starting point will once more be Eq. (15).

§ 3. Precession of the spin in an electromagnetic field

Our purpose is now to use the procedure initiated by Bargmann, Michel and Telegdi[21] (abbreviated henceforward as B.M.T.) which is particularly suitable in order to connect up the theoretical treatment with the practical experiments. Let us briefly recall that these authors start from a set of equations which can be derived from the above equations (8) and (9) by considering the particular case where K_μ is colinear to u_μ. This assumption entails a considerable simplification, but on the other hand, the equations seem to be no longer derivable from a variational principle such as the above considered one, and further, it appeared to be necessary to introduce an anomalous gyromagnetic ratio in order to account for the experimental results. If we first do away with the latter assumption, the equations become merely

$$\dot{u}_\mu = \frac{e}{m_0 c} F_{\mu\nu} u_\nu, \tag{16}$$

$$\dot{s}_\mu = \frac{e}{m_0 c} F_{\mu\nu} s_\nu. \tag{17}$$

Now B.M.T. use a particular frame bound both to the inertial frame (proper frame) and to the laboratory frame. Let v_k be the space-velocity in the laboratory frame. We consider an orthonormal tetrad $n_\mu^{(\xi)}$: $n_\mu^{(4)} = (\gamma \cdot V_k/ic, \gamma)$ lies along the four-velocity u_μ. Then $n_\mu^{(1)} = (n_k^{(1)}, 0)$ and $n_\mu^{(2)} = (n_k^{(2)}, 0)$ lie at the intersection of the space-hyperplanes of the laboratory and of the proper frame. Finally $n_\mu^{(3)} =$

$(\gamma V_k/V,\ \gamma \cdot iV/c)$ $[\gamma=(1-V^2/c^2)^{-1/2}]$. The so-called longitudinal and transversal parts of the spin are considered to be the projections of the four-vector s_μ respectively on the four-vector $n_\mu^{(3)}$ and orthogonal to it. Thus, as s_μ is always orthogonal to $n_\mu^{(4)}$, we can write: $s_\mu=s_0(n_\mu^{(3)}\cos\varphi+N_\mu\sin\varphi)$, N_μ lying in the plane $n_\mu^{(1)}$, $n_\mu^{(2)}$, that is, in particular in the plane orthogonal to the velocity v_k in the laboratory space. This formalism, together with the covariant equations (16), (17), enables us to compute $\dot{\varphi}$, or rather $\Omega=d\varphi/dt$, the rate of change of the spin from longitudinal to transverse, that is, the precession velocity. The result got by B.M.T. in the normal case is in vector form

$$\Omega=-\frac{e}{m_0\gamma^2}\boldsymbol{E}\cdot\frac{\boldsymbol{N}}{V},$$

if we assume that $F_{\mu\nu}$ splits into two three-vectors on the laboratory space: electric field $E_k=1/c\cdot icF_{k4}$ and magnetic field: $H_k=1/c\cdot\varepsilon_{ijk}F_{ij}$.

Now the case here treated is more complicated, the starting equations being different:

$$\dot{K}_\mu=\frac{e}{c}F_{\mu\nu}u_\nu, \tag{18}$$

$$\dot{s}_\mu=\frac{e}{m_0c}F'_{\mu\nu}s_\nu$$

(we have neglected the last term of Eq. (8) which is connected with the slope of $F_{\mu\nu}$, as B.M.T. do explicitly).

As we said above, we shall first introduce a frame $n_\mu^{(\xi)}$ connected with an inertial unitary velocity colinear to the linear momentum K_μ, namely K_μ/M_0 $(K_\mu K_\mu=-M_0^2c^2)$. We put $V_k=K_k/M_0\gamma$, $\gamma=K_4/M_0ic=(1-V^2/c^2)^{-1/2}$. Thus we have $n_\mu^{(4)}=K_\mu/icM_0$, $n_\mu^{(3)}=(K_k/M_0V,\ K_4/M_0V\cdot V^2/c^2)$. Putting $s_\mu=s_0(n_\mu^{(3)}\cos\varphi+N_\mu\sin\varphi)$ with $N_\mu=(N_k,\ 0)$, $N_kn_k^{(3)}=0$, we have

$$\dot{s}_\mu=s_0(\dot{n}_\mu^{(3)}\cos\varphi+\dot{N}_\mu\sin\varphi-n_\mu^{(3)}\sin\varphi\cdot\dot{\varphi}+N_\mu\cos\varphi\cdot\dot{\varphi}). \tag{19}$$

On the other hand, Eq. (15) becomes

$$\dot{s}_\mu=\frac{e}{m_0c}F'_{\mu\nu}s_0(n_\nu^{(3)}\cos\varphi+N_\nu\sin\varphi). \tag{20}$$

Contracting both (19) and (20) by $n_\mu^{(3)}$, we get, $s_0(\dot{N}_\mu n_\mu^{(3)}\sin\varphi-\dot{\varphi}\sin\varphi)=e/m_0c\cdot F'_{\mu\nu}s_0n_\mu^{(3)}N_\nu\sin\varphi$. That is; $\dot{\varphi}=(e/m_0c\cdot F'_{\mu\nu}n_\nu^{(3)}-\dot{n}_\mu^{(3)})N_\mu$, or, since $N_4=0$,

$$\dot{\varphi}=\left(\frac{e}{m_0c}F'_{k\nu}n_\nu^{(3)}-\dot{n}_k^{(3)}\right)N_k. \tag{21}$$

Now, from the definition of $n_k^{(3)}$, we get $\dot{n}_k^{(3)}=\dot{K}_k/M_0V-K_k(\dot{M}_0V+M_0\dot{V})/M_0^2V^2$. That is, since $N_kK_k=0$, $\dot{n}_k^{(3)}N_k=\dot{K}_kN_k/M_0V=F_{k\nu}u_\nu N_k\,e/M_0cV$. Finally, replacing u_ν by $(K_\nu-1/c^2\cdot S_{\nu\lambda}a_\lambda)/m_0$, we have

$$\dot{n}_k^{(3)} N_k = \frac{e}{c} \frac{1}{\mathfrak{m}_0 M_0 V} \left(F_{k\nu} K_\nu - \frac{1}{c^2} F_{k\nu} S_{\nu\lambda} a_\lambda \right) N_k .$$

On the other hand, replacing $n_\nu^{(3)}$ and $n_4^{(3)}$ by their definitions, we have

$$F'_{k\nu} n_\nu^{(3)} = F'_{k3} \frac{K_3}{M_0 V} + F'_{k4} \frac{K_4}{M_0 V} \frac{V^2}{c^2} = F'_{k\nu} \frac{K_\nu}{M_0 V} - \frac{1}{\gamma^2} F'_{k4} \frac{K_4}{M_0 V} .$$

Then, replacing the subsidiary tensor $F'_{\mu\nu}$ by its expression, we have

$$\frac{e}{m_0 c} F'_{k\nu} n_\nu^{(3)} = \frac{1}{M_0 V} \left[\frac{e}{m_0 c} F_{k\nu} K_\nu - \frac{e}{m_0 c} \frac{1}{\gamma^2} F_{k4} K_4 + \mathfrak{m}_0 a_k - \frac{1}{c^2 \gamma^2} (u_k a_4 - u_4 a_k) K_4 \right].$$

If we once more replace u_μ by $(K_\mu - 1/c^2 \cdot S_{\mu\lambda} a_\lambda)/\mathfrak{m}_0$ and contract by N_k, $(N_k K_k = 0)$,

$$\frac{e}{m_0 c} F'_{k\nu} n_\nu^{(3)} N_k = \frac{1}{M_0 V} \left[\frac{e}{m_0 c} F_{k\nu} K_\nu - \frac{e}{m_0 c} \frac{1}{\gamma^2} F_{k4} K_4 + \mathfrak{m}_0 a_k - \frac{1}{\mathfrak{m}_0 c^2 \gamma^2} \left(-\frac{1}{c^2} S_{k\lambda} a_\lambda a_4 + \frac{1}{c^2} S_{4\lambda} a_\lambda a_k - K_4 a_k \right) \right] N_k .$$

According to $K_4 K_4 = -M_0^2 c^2 \gamma^2$, two terms in a_k in brackets reduce to $(\mathfrak{m}_0^2 - M_0^2) a_k / \mathfrak{m}_0$ and we are left with

$$\frac{e}{m_0 c} F'_{k\nu} n_\nu^{(3)} N_k = \frac{1}{M_0 V} \left[\frac{e}{m_0 c} F_{k\nu} K_\nu - \frac{e}{m_0 c \gamma^2} F_{k4} K_4 + \frac{\mathfrak{m}_0^2 - M_0^2}{\mathfrak{m}_0} a_k + \frac{a_\lambda K_4}{\mathfrak{m}_0 c^4 \gamma^2} (S_{k\lambda} a_4 - S_{4\lambda} a_k) \right] N_k .$$

Finally, carrying these results into Eq. (21), we get

$$\dot{\varphi} = \frac{1}{M_0 V} \left\{ -\frac{M_0}{m_0 \gamma} \frac{e}{c} i c F_{k4} + \frac{\mathfrak{m}_0 - m_0}{\mathfrak{m}_0 m_0} \frac{e}{c} F_{k\nu} K_\nu + \frac{\mathfrak{m}_0^2 - M_0^2}{\mathfrak{m}_0} a_k + \frac{1}{\mathfrak{m}_0 c^2} \left[\frac{e}{c} F_{k\nu} S_{\nu\lambda} + \frac{i c M_0}{\gamma c^2} (S_{k\lambda} a_4 - S_{4\lambda} a_k) \right] a_\lambda \right\} N_k . \qquad (22)$$

Now we think that this complicated result can be reduced approximately to its first two terms. Indeed we must keep in mind that in the cases met in the experiments the magnitude of the proper angular momentum $S_{\mu\nu}$, which differentiates the here studied particle from the point-particle one (by the non-colinearity of velocity and linear momentum), is a very small quantity, of the size of $\hbar$, so that the motion under consideration differs very little from the classical motion obeying the law: $m_0 \dot{u}_\mu = e/c \cdot F_{\mu\nu} u_\nu$, which is just the law used by B.M.T. The conclusion, in particular, is that the supplementary acceleration a_μ is little with respect to the classical one $\dot{u}_\mu$, or with respect to $e/m_0 c \cdot F_{\mu\nu} u_\nu$, and also with respect to $e/m_0^2 c \cdot F_{\mu\nu} K_\nu$. The reasonable assumption that the magnitude of a_μ is negligible

with respect to that of $e/m_0^2 c \cdot F_{\mu\nu} K_\nu$ will entail the above announced result, as can be checked for each term.

For instance, if we compare the two terms in $F_{k\nu}$, we get, irrespective of their common coefficients: $F_{k\nu}[(\mathfrak{m}_0 - m_0)/m_0 \cdot K_\nu + 1/c^2 \cdot S_{\nu\lambda} a_\lambda]$. That is, taking Eq. (13) into account:

$$\frac{1}{c^2} F_{k\nu} \left(-\frac{1}{2} \frac{e}{m_0 c} F_{\alpha\beta} S_{\alpha\beta} \frac{1}{m_0} K_\nu + S_{\nu\lambda} a_\lambda \right).$$

Now in order to compare the size of the two vectors roughly, we may compare their magnitudes, or rather the product of the magnitudes of their coefficients. By this procedure, the magnitudes into the bracket become

$$|S_{\alpha\beta}| \cdot \left(-\frac{1}{2} \frac{e}{m_0^2 c} |F_{\alpha\beta}| \cdot |K_\nu| + |a_\lambda| \right).$$

Because $|a_\lambda|$ is assumed to be negligible beside $e/m_0^2 c \cdot |F_{\alpha\beta} K_\beta|$, it is *a fortiori* negligible beside $e/m_0^2 c |F_{\alpha\beta}| \cdot |K_\nu|$, so that the second term inside the bracket can be dropped. It is easy to check in the same way that all the next terms in the right-hand side of Eq. (22) may be dropped by comparison to $(\mathfrak{m}_0 - m_0)/\mathfrak{m}_0 m_0 \cdot e/c |F_{k\nu} K_\nu|$ so that we may restrict ourselves up to a reasonable degree of approximation to the first two terms, namely

$$\dot{\varphi} = \frac{1}{M_0 V} \left(-\frac{M_0}{m_0 \gamma} \frac{e}{c} icF_{k4} + \frac{\mathfrak{m}_0 - m_0}{\mathfrak{m}_0 m_0} \frac{e}{c} F_{k\nu} K_\nu \right) N_k.$$

It is now easy to see:

I. That the first term is much larger than the second one, since the coefficient $(\mathfrak{m}_0 - m_0)$ contains the angular momentum, according to (13), so that the first approximation for the precession is given by the "normal" result of B.M.T.:

$$\Omega = \frac{d\varphi}{dt} = -\frac{1}{m_0 \gamma^2 V} \frac{e}{c} icF_{k4} N_k = -\frac{e}{m_0 \gamma^2} \boldsymbol{E} \cdot \frac{\boldsymbol{N}}{V}.$$

II. Further, the second approximation, neglecting the last terms of (22), yields

$$\Omega = -\frac{e}{m_0 \gamma^2} \boldsymbol{E} \cdot \frac{\boldsymbol{N}}{V} + \frac{e}{m_0} \frac{\mathfrak{m}_0 - m_0}{m_0} \left[(\boldsymbol{V} \times \boldsymbol{H}) \cdot \frac{\boldsymbol{N}}{V} \right].$$

This result agrees qualitatively with that obtained by the latter authors as they make the supplementary assumption of an anomalous gyromagnetic ratio, the role of their "anomaly" $(g/2 - 1)$ being here played by our coefficient $(\mathfrak{m}_0 - m_0)/m_0$ which amounts to $-\frac{1}{2} \cdot 1/m_0^2 c^2 \cdot e/c \cdot F_{\mu\nu} S_{\mu\nu}$. Of course, unlike the assumed "anomaly", our coefficient is in general not constant.

Nevertheless this result derived from throughout classical assumptions seems us noteworthy. We may put forward the opinion that the described bilocal theory can be considered as providing a concrete interpretation for the gyromagnetic anomaly

in a completely classical frame. I wish to express my deepest thanks to Professors T. Takabayasi and L. Michel for many helpful suggestions and improvements.

References

1) All the here quoted works are related with some details in F. Halbwachs, Théorie des fluides à spin (Paris, Gauthier Villars 1959).
2) J. Frenkel, Zs. f. Phys. **37** (1926), 243.
3) M. Mathisson, Zs. f. Phys. **67** (1931), 270, 826.
M. Mathisson, Acta Phys. Pol. **6** (1937), 163, 218.
Lubanski, Acta Phys. Pol. **6** (1937), 356.
4) J. Weyssenhoff, Acta Phys. Pol. **9** (1947), 8.
5) D. Bohm and J. P. Vigier, Phys. Rev. **109** (1958), 1882.
6) F, Halbwachs, Théorie des fluides à spin (Paris, Gauthier Villars, 1959).
7) C. Bouchiat and L. Michel, Phys. Rev. **106** (1955), 170.
7′) L. Michel, Séminaire de physique théorique et de physique nucléaire, Ecole Polytechnique, (Paris, 24–2, 1959).
8) C. Møller, Ann. Inst. H. Poincaré, **10** (1949) 5.
9) F. Halbwachs, Lochak and J. P. Vigier, C. R. Acad. Sc. **241** (1955), 692-744.
10) T. Takabayasi, Séminaire L. de Broglie, Institut Henri Poincaré, Paris, No. **10** (1957).
11) F. J. Belinfante, Physica **7** (1939), 887.
12) L. Rosenfeld, Acad. Roy. Belgique **18** (1940), 6.
13) E. Noether, Götting. Nachr. (1918), 235.
14) J. Winogradski, Cahiers de Phys. **16** (1959).
15) F. Gürsey, Nuovo Cimento **5** (1957), 784.
16) A. Aymard, C. R. Acad. Sc. **243** (1956), 885, 1100.
17) P. Unal and J.P. Vigier, C. R. Acad. Sc. **245** (1957), 1787, 1891.
18) T. Takabayasi, Nuovo Cimento **7** (1958), 118.
19) F. Halbwachs, P. Hillion and J. P. Vigier, Nuovo Cimento **10** (1958), 817.
20) F. Halbwachs and J. P. Vigier, C. R. Acad. Sc. **248** (1959), 490, 1124.
21) Bargmann, Michel and Telegdi, Phys. Rev. Letters **2** (1959), 435.

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Universality of Coupling Constants and Individual Γ_5-Invariance

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The characteristics of the interaction of elementary particles are summarized as "universality" and a sort of invariance which will be called "individual Γ_5-invariance."

That these are the requisites of forthcoming theories is pointed out. We also refer to the limit of applicability of $(V\text{-}A)$ type-weak interaction in the high-energy region.

§ 1. Introduction

Attempts to construct the model of elementary particles and their interaction are now in progress. It seems necessary, at this time, to select and confirm the well-grounded materials obtained in the last few years. The present work has an important bearing on this point.

It is to be emphasized that the forthcoming theory of elementary particles must pass through two *filters*, the "universality"[1] and the "individual Γ_5-invariance"[2].

These two requisites, particularly the latter, have hitherto been expressed in various ways, for instance, two-component neutrino theory, one-to-one law, etc. But most of them have proved to be incomplete and ambiguous.

We have clarified differences among these expressions including our own, and in the course of study we found that the "$I\Gamma_5$-law" has a decided advantage over other theories for its accurate reproduction of experimental evidences.

We have developed our arguments as follows.

Universality of coupling constants (§ 2): The magnitude of interaction coupling constant is shown to be universal only when we use "Yukawa unit."[1]

Individual Γ_5-invariance (§ 3): It is proved (1) that the two-component neutrino theory is not equivalent to the $I\Gamma_5$-law and involves ambiguity; (2) that is, $I\Gamma_5$-law is as useful as Fleming's law in electromagnetism.

What is the use of these *filters* in developing future theories? It seems to us that the "universality" and the "$I\Gamma_5$-law" play a role similar to that of Balmer formula, precursor of Rydberg-Ritz's principle, in the course of constructing quantum mechanics.

Finally, we also refer to the limit of applicability of $(V\text{-}A)$ type-weak interaction in high-energy region.

§ 2. Universality of coupling constants

(The magnitude of interaction coupling constant is universal only when "Yukawa unit" is used.)

Recently there appeared some arguments that the coupling constants are universal when they are measured in the unit of length. But this is not true, as can be seen from the following Table.

Coupling constant comes out to be universal only when we use "Yukawa unit", which is the unit system where $\hbar$, c and $r_0(\sim 10^{-13}\,\mathrm{cm})$ are all taken equal to unity.

Table 1. Coupling constants of *WI* are universal only when we use "Yukawa unit". There is no universality in the magnitude of constant measured in length.

Interaction	cgs unit	Yukawa unit ($\hbar=1$, $c=1$, $r_0=1$)	measured in length ($\hbar=1$, $c=1$)
$f_F\bar{\psi}\psi\bar{\psi}\psi$ $(n\to p+e+\nu)$	$10^{-49}\ erg\ cm^3$	$10^{-7}(\hbar c r_0^2)$	$10^{-16}\ cm$
$f_{BF}\bar{\psi}\psi\varphi$	$10^{-15}\ erg^{1/2}cm^{1/2}$	$10^{-7}\ (\sqrt{\hbar c})$	
$g_{BF}\bar{\psi}\psi\partial\varphi$ $(\pi\to\mu+\nu)$	$10^{-28}\ erg^{1/2}cm^{3/2}$	$10^{-7}\ (\sqrt{\hbar c}\cdot r_0)$	$10^{-19}\ cm$
$f_{BB}\phi\phi\varphi$	$10^{+14}\ erg^{-1/2}cm^{-3/2}$	$10^{-7}(1/\sqrt{\hbar c r_0^2})$	$10^{-6}\ cm$

This universality is not trivial, since we can predict transition probability of each decay process in so far as we take account of "Yukawa unit" and the universal constants, $\sim 10^0$, $1/\sqrt{137}$, and $\sim 10^{-7}$ for *SI*, *EMI*, and *WI**, respectively.

[Example 1.]

Suppose that the value of coupling constant $f_F(\sim 10^{-49}\ erg\ cm^3)$ in β-decay, $n\to p+e+\nu$, has escaped from our memory. In such a case, we can readily recall it by taking the following steps.

First, we seek for the dimension of this coupling constant. Hamiltonian density is given by

$$f_F\bar{\psi}\,\psi\,\bar{\psi}\,\psi,$$

the dimension of f_F can be determined from the dimentional equation

$$[f_F\bar{\psi}\,\psi\,\bar{\psi}\,\psi\,\Omega]=[\mathrm{erg}]$$

where Ω is normalization volume.

We have, on the other hand,

$$[\psi]=\left[\frac{1}{\sqrt{\Omega}}\right]$$

* *SI*: strong interaction, *EMI*: electromagnetic interaction, *WI*: weak interaction.

so that

$$\left[f_F \frac{1}{\sqrt{\Omega}} \frac{1}{\sqrt{\Omega}} \frac{1}{\sqrt{\Omega}} \frac{1}{\sqrt{\Omega}} \Omega\right]=[\text{erg}],$$

or

$$[f_F]=[\text{erg cm}^3].$$

Next, we combine $\hbar$, c and r_0 so as to form the dimension $[erg\ cm^3]$. The result is $\hbar c r_0^2$. The coupling constant of WI measured in Yukawa unit is $\sim 10^{-7}$; we therefore obtain

$$f_F \simeq 10^{-7}\ \hbar c r_0^2,$$

or, inserting the values of $\hbar$, c and r_0 in *cgs* unit,

$$f_F \simeq 10^{-7} \cdot 10^{-27} \cdot 3 \times 10^{10} \cdot (10^{-13})^2 \text{ erg cm}^3$$

$$\sim 10^{-49} \text{ erg cm}^3.$$

This is nothing but the coupling constant of Fermi interaction expressed in *cgs* unit.

[Example 2.]

The question of how to get the value of g_{BF} of the so-called weak Boson-Fermion interaction,[3] where

$$H'=g_{BF}(\bar{\psi}_B \gamma_\mu (1+\gamma_5)\psi_A)\partial_\mu \varphi_c,$$

can be worked out also by simple application of Yukawa unit. In a way similar to Example 1, the dimension of g_{BF} is determined by

$$[g_{BF}(\bar{\psi}_B \gamma_\mu (1+\gamma_5)\psi_A)\partial_\mu \varphi_c \cdot \Omega]=[\text{erg}].$$

Noticing that

$$[\psi]=\left[\frac{1}{\sqrt{\Omega}}\right],$$

$$[\varphi]=\left[\frac{\hbar c}{\sqrt{2\Omega E}}\right],$$

$$[\partial]=\left[\frac{1}{r_0}\right],$$

we get

$$\left[g_{BF} \frac{1}{\sqrt{\Omega}} \frac{1}{\sqrt{\Omega}} \frac{1}{r_0} \frac{\hbar c}{\sqrt{2\Omega E}} \Omega\right]=[\text{erg}],$$

that is,

$$[g_{BF}]=[\text{erg}^{1/2}\ \text{cm}^{3/2}].$$

As $[g_{BF}]$ has the same dimension as $\sqrt{\hbar c}\, r_0$, we obtain, by applying the principle of Yukawa unit,

$$g_{BF} \simeq 10^{-7}\sqrt{\hbar c}\cdot r_0$$
$$\sim 10^{-7}\sqrt{10^{-27}\cdot 3\times 10^{10}}\cdot 10^{-13}\mathrm{erg}^{1/2}\mathrm{cm}^{3/2}$$
$$\sim 10^{-28}\ \mathrm{erg}^{1/2}\ \mathrm{cm}^{3/2}.$$

g_{BF}, as expressed in unit of length, turns out to be

$$\frac{10^{-28}\ \mathrm{erg}^{1/2}\ \mathrm{cm}^{3/2}}{\sqrt{\hbar c}} \sim 10^{-19}\ \mathrm{cm}$$

which means the length where the 2nd kind interaction is enhanced, as was pointed out by H. Umezawa et al.,[4] and D. Ito et al.[5] As seen from Table I, however, we can find no universal character for such a unit system.

The conclusion we arrived at may be summarized once more as follows: There are seemingly many kinds of interactions having various values and dimensions. Nevertheless, if we adopt Yukawa unit, the values of coupling constants are all reduced to a universal constant $\sim 10^{-7}$.

The most important point of the Yukawa unit system is to adopt $r_0(\sim 10^{-13}\ \mathrm{cm})$ as a new unit. But the question still remains to be solved why such a beautiful universality appears only when we take $\sim 10^{-13}$ cm as r_0.

The discovery of such universality suggests the existence of enclosed logic within the frame of WI. The universality combined with parity violation in WI and the Sakata model,[6] *has led us to the $I\Gamma_5$-law,*[2]* which is the main subject of the next section.

The basic idea which led to the $I\Gamma_5$-law was as follows. In 1957, characteristic difference between *SI* and *WI* was established to be conspicuous not only in magnitude of coupling constant but also in strangeness and parity, and what is curious, strangeness bears a parallel to parity, in that if strangeness is conserved parity is likewise conserved (*SI, EMI*), and if strangeness is not conserved parity is violated (*WI*).

This correlation seems to be natural, provided that the Sakata model and $I\Gamma_5$-invariance are admitted, because the change in strangeness means, in terms of the Sakata model, that different species of spinors are contained in a bispinor constructing the interaction Hamiltonian, while it also means, as required by $I\Gamma_5$-invariance, that the bispinor must contain $\gamma_\mu(1+\gamma_5)$. Thus strangeness change should necessarily be accompanied by parity violation.

§ 3. Individual Γ_5-Invariance

1. *$I\Gamma_5$-law exactly reproduces the well-established experimental facts.*

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* In the course of this work, the author received a preprint of S. Hori & A. Wakasa's paper,[7] for which he is much obliged.

The parity violation law has hitherto been expressed equivocally in various ways, for instance, two-component neutrino theory,[8] one-to-one law,[9] etc. But it must be emphasized that as will be seen below, the $I\Gamma_5$-law has an advantage over others for accurate reproduction of experimental evidences. We define the $I\Gamma_5$-law as follows.

> $I\Gamma_5$-law* :—Under individual Γ_5 transformation, $\psi_a \to \gamma_5\psi_a$, the interaction Hamiltonian must remain invariant and lepton number must also be conserved. Here a is taken for p, n, Λ, ν, e^- and μ^-. If Fermi interaction between them is of the form $f(\bar{a}O_i a)(\bar{b}O_i b)$ ($\supset a \neq b$, $a=b$), it is SI and the coupling constant f is $\sim 10^{-42} erg\ cm^3$, while otherwise the coupling constant f is $\sim 10^{-49}\ erg\ cm^3$. Lepton has no SI.

The experimental facts that can be precisely reproduced by this law are:

a) The $e^-(e^+)$ emitted from a polarized nucleus is almost completely anti-polarized (polarized) along the direction of its motion.[11]

b) The emission of e^+ in $\pi^+ - \mu^+ - e^+$ chain predominates in backward direction relative to the motion of meson.[12] The e^+ emitted in this chain is almost completely polarized along the direction of its motion, according to the experimental result obtained by the California group.[13]

c) The state of polarization of μ^+ emitted from π^+, which is experimentally yet unknown.

d) The situation similar in every respect to the case of $\pi^+ - \mu^+ - e^+$ chain is also realized in $K^+ - \mu^+ - e^+$ chain.[14]

In the following, we shall show how to derive these evidences from the $I\Gamma_5$-law. We shall treat this problem in the order of

[$\mu^+ \to e^+$ decay],

[π-decay],

[β-decay],

[K-decay],

[μ-capture].

[$\mu^+ \to e^+$ decay] :—The allowable interaction according to the $I\Gamma_5$-law is

$$f_F[\bar{e}^-\gamma_\mu(1+\gamma_5)\mu^-][\bar{\nu}\gamma_\mu(a+b\gamma_5)\nu].$$

From this, we obtain[2]

$$\begin{cases} (\textit{angular distribution of } \boldsymbol{p}_{e^-} \textit{ relative to } \boldsymbol{\sigma}_{\mu^-}) \\ \qquad\qquad \propto (1-\tfrac{1}{3}\cos(\boldsymbol{\sigma}_{e^-}\cdot\boldsymbol{p}_{\mu^-})) \\ (\textit{spirality of emitted } e^-) : \textit{left}. \end{cases}$$

* The author is indebted to Drs. K. Fujii, K. Furui, T. Sakuma and T. Tsuchida for this complete form of $I\Gamma_5$ -law.

Therefore, the correlation between $\boldsymbol{\sigma}_{\mu^-}$, $\boldsymbol{\sigma}_{e^-}$ and $\boldsymbol{p}_{e^-}$ is such as illustrated in Fig. 1.

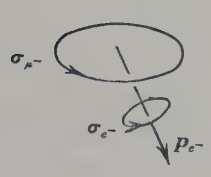

Fig. 1.

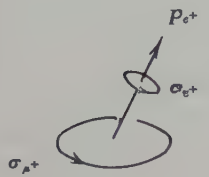

Fig. 2.

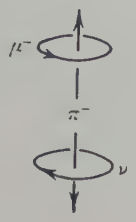

Fig. 3.

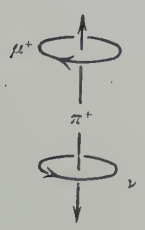

Fig. 4.

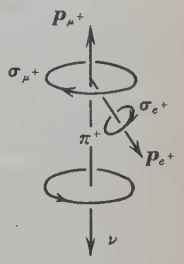

Fig. 5.

The correlation for $\mu^+ \to e^+$ decay process can be easily obtained from Fig. 1, by performing charge conjugation.* The result is given in Fig. 2.

[π-decay]:—The leptonic part of the interaction inducing $\pi^- \to \mu^- + \nu$ is of the form

$$[\bar{\mu}^- \gamma_\mu (1+\gamma_5)\nu],$$

as required by the $I\Gamma_5$-law. From this, one can see at once that neutrino is emitted with right spirality, so that, in the rest system of π^-, μ^- emitted in the direction opposite to has also right spirality as in Fig. 3.

Performance of charge conjugation as Fig. 1 $\to$Fig. 2 yields the result shown in Fig. 4.

Combining Fig. 2 and Fig. 4, we obtain the relative configuration of $\boldsymbol{\sigma}_{\mu^+}$, $\boldsymbol{p}_{\mu^+}$, $\boldsymbol{\sigma}_{e^+}$ and $\boldsymbol{p}_{e^+}$ in $\pi^+ - \mu^+ - e^+$ decay chain, as is shown in Fig. 5.

[β-decay]:—It follows from the $I\Gamma_5$-law that the leptonic part of Hamiltonian is

$$[\bar{e}^- \gamma_\mu (1+\gamma_5)\nu].$$

Therefore, the spirality of emitted e^- is left and that of ν is right.

[K-decay]:—The same as π-decay.

[μ-capture]:—Leptonic part must take the form

$$[\bar{\nu} \gamma_\mu (1+\gamma_5)\mu^-],$$

owing to $I\Gamma_5$-invariance. Emitted ν has left spirality.

It is thus evident that *the $I\Gamma_5$-law reproduces the well-established experimental facts.*

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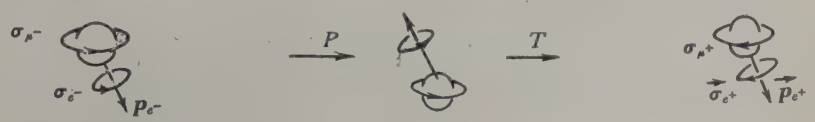
* Charge conjugation is equivalent to TP operation,[15] so that we have



2. *Two-component neutrino theory is not equivalent to $I\Gamma_5$-law and involves ambiguity.*

In an early work of Lee and Yang,[8] the two-component neutrino theory was defined as: "The neutrino field always appears in interactions in the combination $(1-\gamma_5)\psi_\nu$". But at present it has been proved by experiment that this is wrong and $(1-\gamma_5)\psi_\nu$ must be replaced by $(1+\gamma_5)\psi_\nu$.

The two-component neutrino theory cannot give such a strong correlation between $\boldsymbol{\sigma}_\mu$, $\boldsymbol{\sigma}_e$ and $\boldsymbol{p}_e$ for μ-e decay process as the $I\Gamma_5$-law does. It is because the interaction determined by the two-component neutrino theory is

$$[\bar{e}^- \gamma_\mu(a+b\gamma_5)\mu^-][\bar{\nu}\gamma_\mu(1+\gamma_5)\nu],$$

containing undetermined factors a and b in $(e\mu)$-bispinor. We obtain therefrom

(angular distribution of $\boldsymbol{p}_e$ relative to $\boldsymbol{\sigma}_\mu$)

$$\propto[(|a|^2+|b|^2)-\tfrac{2}{3}Re(ab^*)\cos(\boldsymbol{\sigma}_\mu\cdot\boldsymbol{p}_e)];$$

that is, undetermined factors a and b distruct the correlation between $\boldsymbol{\sigma}_\mu$, $\boldsymbol{\sigma}_e$ and $\boldsymbol{p}_e$. Why the $I\Gamma_5$-law imposes strong correlation upon them comes from the fact that $(e\mu)$-bispinor contains no undetermined factor. More strictly speaking, the origin of strong correlation is $(e\mu)$-bispinor but not $(\nu\nu)$. Even if $(\nu\nu)$-bispinor contains undetermined factors, we obtain strong correlation such as

$$1-\tfrac{1}{3}\cos(\boldsymbol{\sigma}_\mu\cdot\boldsymbol{p}_e),$$

so long as $(e\mu)$-bispinor contains no undetermined factor.

3. *The $I\Gamma_5$-law is as useful as Fleming's law in electromagnetism.*

The most remarkable property of the result following from the $I\Gamma_5$-law is the strong correlation between $\boldsymbol{\sigma}_\mu$, $\boldsymbol{\sigma}_e$ and $\boldsymbol{p}_e$ in the μ-e decay process. To become familiar with this correlation, we have only to remember Fig. 1 and the procedure of charge conjugation given in the footnote of § 3, 1.

One can readily notice that the direction of ring $\boldsymbol{\sigma}_\mu$ and $\boldsymbol{\sigma}_e$ in μ^+ and μ^- decay is always the same (see Fig. 6). This rule is quite worth remembering.

Fig. 6.

The most crucial test of $I\Gamma_5$-invariance is the as yet unfinished experiment (c) mentioned in 1, *where the $I\Gamma_5$-invariance predicts left spirality for μ^+ in the decay product of $\pi^+-\mu^+-\nu$ decay.*[2]

§ 4. Discussion

A brief comment may be added here on the relation between the *Nagoya model*[10] and our $I\Gamma_5$-law. In the *Nagoya model*, the *WI* is reduced to

$$H_W = f_F\, j_\mu\, j_\mu^\dagger$$

in which

$$j_\mu = (\bar{e}\,\gamma_\mu(1+\gamma_5)\nu) + (\bar{\mu}\,\gamma_\mu(1+\gamma_5)\nu).*$$

This form of Hamiltonian, the so-called $(V\text{-}A)$ type, is the same as that given by the $I\Gamma_5$-law.

We have now two projects in view which are based on the results gained for the last few years: One is to explain parity conservation in *SI* (Parity-conserving interaction is $I\Gamma_5$-invariant but the converse is not true.), the other is to investigate into the reason why parity violation suppresses the coupling constant factor $\varepsilon(\sim 10^{-7})$.**

In addition to these, it will become necessary to make investigation as to the limit of applicability of $(V\text{-}A)$ type-*WI* in the high-energy region.

In decay processes, the momentum transfer is at most of the order of 100 *Mev*. In this limited region, there has been, as a matter of fact, no fault to be found in applying the $(V\text{-}A)$ type-*WI*. But if we continue to describe the high-energy region by this type of interaction, copious production of lepton is to be expected. Since, however, in such low-energy region as decay processes, the effect of *WI* is no more than $(10^{-7})^2$ times as much as that of *SI*, it will become comparable to *SI* in the high-energy region,[19] the reason for this is that *WI* has strong momentum dependence according to the degree of dimension $[r_0]$ contained in the coupling constant.

From this standpoint, H. Nagai and D. Itô[20] analyzed the two competing processes,

$$\mathfrak{N} + (\mathfrak{N}) \to (\mathfrak{N}) + \mathfrak{N} + \pi \qquad (1)$$

$$\mathfrak{N} + (\mathfrak{N}) \to (\mathfrak{N}) + \mathfrak{N} + \mu + \nu, \qquad (2)$$

using $(V\text{-}A)$ type-*WI*, and found that the cross section in (2) must be comparable to that in (1) in the high-energy region, where the momentum transfer is

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* The necessity of γ_μ-term in Fermi interaction was pointed out ten years ago by Taketani et al.[16] The reason why γ_μ is necessary is that the frequency of $\pi \to e+\nu$ is rare in comparison with that of $\pi \to \mu+\nu$, so that one must adopt axial-vector coupling as Fermi interaction (See ref. 16), Table III).

This seemed to be in conflict with Steinberger's experiment,[17] which showed that the ratio $W(\pi \to e+\nu)/W(\pi \to \mu+\nu)$ can never attain the value 1.28×10^{-4} predicted by Taketani et al., that is, the decay $\pi \to e+\nu$ is practically forbidden. According to more recent experimental data, however, this ratio amounts to $(1.18\pm 0.08)\times 10^{-4}$,[18] which may well be considered to be in accordance with the theoretical value.

In 1955, the Nagoya group discovered "universality",[3] by applying the interaction containing γ_μ to all kinds of decay, and in 1957, Fujii and Iwata,[2] starting from this universality, in combination with the Sakata model and parity violation, succeeded in deriving the $I\Gamma_5$-law, which exactly gives the $V\text{-}A$ type coupling.

** The *Nagoya model* has not yet succeeded in explaining either of them.

more than 10^{13} *ev*.

Process (2) seems to have some connection with the bunch of high-energy μ-meson that appeared in recent air-shower experiments.[21]

These data suggest that the research on the limit of applicability of (V-A) type-WI in the high-energy region is not necessarily merely academic. The deviation from the prediction of (V-A) type-WI will offer powerful suggestion for forthcoming theories.

In conclusion, the author wishes to express his sincere gratitude to Profs. D. Ito, Y. Ono, H. Tanaka for their interest taken in this work. The author is also grateful to all members of theoretical physics of Hokkaido University for discussions.

References

1) S. Ogawa, B. Sakita and Y. Taguchi, Prog. Theor. Phys. **12** (1954), 691.
S. Ogawa, Soryushiron Kenkyu (mimeographed circular in Japanese) **9** (1955), 283.
S. Ogawa, *Lecture given at Summer School held at Kisaki in* 1956.
2) K. Fujii and K. Iwata, Prog. Theor. Phys. **18** (1957), 666; **19** (1958), 475 and 589; **20** (1958), 126.
3) K. Iwata, S. Ogawa, H. Okonogi, B. Sakita and S. Oneda, Prog. Theor. Phys. **13** (1955), 19.
4) H. Umezawa, Suppl. Prog. Theor. Phys. **7** (1959), 67.
5) D. Ito et al., Prog. Theor. Phys. **22** (1959), 159 and 168.
6) S. Sakata, Prog. Theor. Phys. **16** (1956), 686.
7) S. Hori and A. Wakasa, Nuovo Cimento **6** (1957), 304.
8) T. D. Lee and C. N. Yang, Phys. Rev. **105** (1957), 1671.
9) H. Umezawa, M. Konuma and K. Nakagawa, Nuclear Physics **7** (1958), 169.
10) Private communication from Dr. Ohnuki (to be published).
11) *Rehovoth Conference* (1957), report by C. S. Wu (see Table I), p. 346.
12) Garwin et al., Phys. Rev. **105** (1957), 1415.
13) P. C. Macq et al., Phys. Rev. **112** (1958), 2061.
14) Coombes et al., Phys. Rev. **108** (1957), 1348.
15) T. D. Lee, R. Oehme and C. N. Yang, Phys. Rev. **106** (1957), 340.
16) S. Nakamura, H. Fukuda, K. Ono, M. Sasaki, and M. Taketani, Prog. Theor. Phys. **5** (1950), 244.
17) S. Lokanathan and J. Steinberger, Phys. Rev. **98** (1955), 240 (A).
18) *Kiev Conference* (1959), report by A. I. Arikhanov.
19) D. I. Blokhintsev, Usp. Fiz. Nauk **62** (1957) 381.
20) H. Nagai and D. Ito, Prog. Theor. Phys. **23** (1960), 1210.
S. Vernov et al., JETF **37** (1959), 1193.
Private communication from Prof. I. Miura (Institute for Nuclear Study, Tokyo University, Tokyo).

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A New Approach to the Theory of Classical Fluids. II

——*Multicomponent Systems*——

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The results that have been derived in paper I of this series are generalized to the case of multicomponent systems. The argument is made in an almost parallel way with the one in I, though the derivation of the integral equation for the pair distribution function is given in a more intuitive way than in I. The present theory is also a generalization of Morita's theory in which the multicomponent systems were treated in the hyper-netted chain approximation.

§ 1. Introduction

In the previous paper,[1] which will be referred to as I, one-component fluid systems have been investigated by starting from the customary density expansion for the pair distribution function. An exact integral equation has been found for the pair distribution function. The integral equation is of different nature from those which are usually known in the theory of classical fluids, in the point that it involves an infinite series in itself. The Helmholtz free energy also has been expressed in a form of expansion which may be expected to converge considerably more rapidly than its customary density expansion. In the present paper the results that have been obtained in I will be generalized to the case of multicomponent systems.

In the case of multicomponent systems as well as one-component systems, a theory can be constructed by means of the integral equation method which is usually based on the Kirkwood superposition approximation.[2),3),4)]* However, the inconsistencies which are implied in this approximation reveal themselves more drastically for the multicomponent systems than for the one-component systems. In the case of binary mixtures, for example, *four* integral equations are derived to determine *three* kinds of the pair distribution functions. It is known[4),5)] that these integral equations are mutually inconsistent and also that further approximation must be introduced to convert them into consistent equations. It may be said that such a difficulty is one of the serious defects of the integral equation method.

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* In fact it is in the theory of multicomponent systems that Kirkwood used the superposition approximation for the first time.[2)]

The partial sum method for the multicomponent systems was adopted for the first time by Mayer in his theory of ionic solutions.[6] Mayer was able to obtain the Debye-Hückel limiting law by considering the terms corresponding to ring diagrams alone. After that, several authors have adopted this method mainly to investigate the ionic solutions.[4),7)] Recently Morita[8] has proposed the hyper-netted chain approximation to deal with the multicomponent systems in general. Morita's theory contains the previous works as its special case. Just as the hyper-netted chain approximation for the one-component systems appeared as the zeroth approximation to the theory presented in I, it will be shown that the above Morita's theory is contained in the present theory as a special case. In such a sense the present theory may be said to be a generalization of Morita's theory.

The content of the present paper is as follows. In § 2 the definition of the functions to be used in the following sections are given. This section corresponds to § 2 of I. In § 3 an exact integral equation for the pair distribution function is derived by starting from the density expansions of the functions which are defined in § 2. Though this section corresponds to § 3 of I, the derivation of the integral equation is given in a more intuitive way than in § 3 of I. In § 4 the expression for the Helmholtz free energy is derived and it is shown that the integral equation derived in § 3 can also be obtained from this expression for the free energy by means of a variational principle. The argument in § 4 is made in quite a parallel way with the one in § 4 and § 5 of I. In § 5 a summary of the results that are obtained in § 3 and § 4 is given and some remarks are also given.

§ 2. Notation and definition

We consider a σ-component fluid in a volume Ω and at temperature T, composed of N_1 particles of species 1, N_2 of species 2, $\cdots$, and N_σ of species σ. The interaction potential of our system is assumed to be the sum of pair interaction potentials:

$$\frac{1}{2}\sum_{\substack{i=1\\ i\neq j}}^{N}\sum_{j=1}^{N}\phi_{\nu_i\nu_j}(r_{ij}), \tag{2·1}$$

where N denotes the total number of the particles and $\nu_i (i=1, 2, \cdots, N)$ denotes the species of the particle at $\boldsymbol{r}_i$. The quantity $b_{\nu_i\nu_j}(r_{ij})$ is defined in terms of the pair interaction potential by

$$b_{\nu_i\nu_j}(r_{ij}) = e^{-\phi_{\nu_i\nu_j}(r_{ij})/kT} - 1. \tag{2·2}$$

The number density of species ν is denoted by $\rho_\nu (=N_\nu/\Omega)$.

The pair distribution function, $g_{\alpha\beta}(r_{12})$, of two particles of species α and β is given by the equation[9]

$$g_{\alpha\beta}(r_{12}) = \exp\left[-\frac{\phi_{\alpha\beta}(r_{12})}{kT} + w_{\alpha\beta}(r_{12})\right], \tag{2·3}$$

where

$$w_{\alpha\beta}(r_{12}) = \sum_{m_1}\sum_{m_2}\cdots\sum_{m_\sigma} \frac{\rho_1^{m_1}\rho_2^{m_2}\cdots\rho_\sigma^{m_\sigma}}{m_1!\,m_2!\cdots m_\sigma!} \iint\cdots\int d\{m_1\}\,d\{m_2\}\cdots d\{m_\sigma\}$$

$$\times \sum_{m_1+m_2+\cdots+m_\sigma+2\geq i>j\geq 1}^{(W)} \prod b_{\nu_i\nu_j}(r_{ij})$$

$$= \sum_{\mathbf{m}} \frac{\rho^{\mathbf{m}}}{\mathbf{m}!} \int d\{\mathbf{m}\} \sum_{\mathbf{m}+2\geq i>j\geq 1}^{(W)} \prod b_{\nu_i\nu_j}(r_{ij}). \qquad (2\cdot4)$$

In the first line of (2·4) the volume element of m_ν particles of species ν is denoted by $d\{m_\nu\}$ ($\nu=1, 2, \cdots, \sigma$). In the second line Mayer's notation for the multicomponent systems is adopted. While no explanation is given here on this notation, one will understand it if one compares the second line of (2·4) with the first line. As in I, the meaning of $\Sigma^{(W)}$ will be explained by means of the graphical representation of the product $\prod b_{\nu_i\nu_j}(r_{ij})$ in the following.

Particles 1 and 2 are represented by a white circle respectively. The set of $\mathbf{m}$ particles, over which the integration is to be performed, is represented by the set of numbered points (or numbered black circles). A factor $b_{\nu_i\nu_j}(r_{ij})$ is represented by a line (b-bond) which is drawn between two points i and j. Then a product $\prod_{\mathbf{m}+2\geq i>j\geq 1} b_{\nu_i\nu_j}(r_{ij})$ is represented by a bond diagram. Two particles i and j are said to be *directly* connected if $b_{\nu_i\nu_j}(r_{ij})$ appears in the product. An s-point is defined in the same way as in I. It is a point through which all possible paths going from particle 1 to particle 2 must pass. In other words, it is a point by which the bond diagram can be separated into two independent parts containing particles 1 and 2 respectively.

We shall introduce the following four kinds of restrictions which are to be imposed on the summation of diagrams:

(I) *Each particle of the set* $\mathbf{m}$ *is independently connected to particles 1 and 2.*
(II) *The particles of the set* $\mathbf{m}$ *are connected among themselves independently of particles 1 and 2.*
(III) *Particles 1 and 2 are not directly connected.*
(IV) *The diagram has no s-point.* (2·5)

Then $\Sigma^{(W)}$ in (2·4) denotes the summation over all diagrams being possible under restrictions (I), (II) and (III) (see Fig. 1 in I).[9]

Following the lines of I, we define the functions $x_{\alpha\beta}(r_{12})$, $z_{\alpha\beta}(r_{12})$ and $v_{\alpha\beta}(r_{12})$ as follows.*

$$x_{\alpha\beta}(r_{12}) = \sum_{\mathbf{m}} \frac{\rho^{\mathbf{m}}}{\mathbf{m}!} \int d\{\mathbf{m}\} \sum_{\mathbf{m}+2\geq i>j\geq 1}^{(X)} \prod b_{\nu_i\nu_j}(r_{ij}), \qquad (2\cdot6)$$

* The function $z_{S,\alpha\beta}(r_{12})$, corresponding to $z_S(r_{12})$ in I, may be defined in a similar way, though we shall have no need to use it in the following deduction.

where $\Sigma^{(X)}$ denotes the summation over all diagrams being possible under restrictions (I), (II), (III) and (IV).

$$z_{\alpha\beta}(r_{12}) = \sum_{\mathbf{m}} \frac{\rho^{\mathbf{m}}}{\mathbf{m}!} \int d\{\mathbf{m}\} \sum_{\mathbf{m}+2\geq i>j\geq 1}^{(Z)} \prod b_{\nu_i\nu_j}(r_{ij}), \tag{2·7}$$

where $\Sigma^{(Z)}$ denotes the summation over all diagrams being possible under restrictions (I) and (IV).

$$v_{\alpha\beta}(r_{12}) = \sum_{\mathbf{m}} \frac{\rho^{\mathbf{m}}}{\mathbf{m}!} \int d\{\mathbf{m}\} \sum_{\mathbf{m}+2\geq i>j\geq 1}^{(V)} \prod b_{\nu_i\nu_j}(r_{ij}), \tag{2·8}$$

where $\Sigma^{(V)}$ denotes the summation over all diagrams being possible under restriction (I) alone.

Some diagrams appearing in $\Sigma^{(X)}$ and $\Sigma^{(Z)}$ are shown for a one-component system in Figs. 2 and 4 of I.

The functions defined above are not independent of each other. The next section will be devoted to the study of the relations among them.

In concluding the present section, we introduce the Fourier transforms of $w_{\alpha\beta}(r)$, $x_{\alpha\beta}(r)$, $z_{\alpha\beta}(r)$ and $v_{\alpha\beta}(r)$ which will be denoted as $W_{\alpha\beta}(k)$, $X_{\alpha\beta}(k)$, $Z_{\alpha\beta}(k)$ and $V_{\alpha\beta}(k)$, respectively. For example, we have the relations

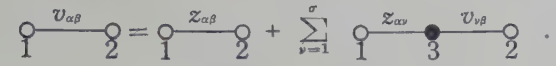
and

$$\left.\begin{aligned} z_{\alpha\beta}(r) &= \frac{1}{\Omega}\sum_{k} Z_{\alpha\beta}(k)\, e^{i\boldsymbol{k}\boldsymbol{r}} \\ Z_{\alpha\beta}(k) &= \int d\boldsymbol{r}\, z_{\alpha\beta}(r)\, e^{-i\boldsymbol{k}\boldsymbol{r}}. \end{aligned}\right\} \tag{2·9}$$

§ 3. Integral equation for the pair distribution function

We shall derive several relations among the functions which have been defined in the preceding section. Though the derivations may be made in quite a similar way as in the case of one-component systems of I, we shall here give more intuitive derivations. The relations will be seen to give an integral equation for the pair distribution function.

The diagrams appearing in $\Sigma^{(V)}$ can be divided into two groups. One group consists of the diagrams having no s-point, which are just those appearing in $\Sigma^{(Z)}$ (see below (2·7)). The other group consists of the diagrams which have at least one s-point. Let us consider a diagram belonging to the latter group. Let particle 3 of species ν be at the s-point nearest to particle 1. Then the diagrams which are to appear between particles 1 and 3 are restricted by restrictions (I) and (IV) of (2·5) where particles 1 and 2 must be replaced by particles 1 and 3. The diagrams which are to appear between particles 3 and 2 are restricted by restriction (I) where particles 1 and 2 must be replaced by particles 3 and 2. Therefore the total of the diagrams appearing in $v_{\alpha\beta}(r_{12})$ can be written in a symbolical way as



(3·1f)

This symbolical expression means (see below (3·4f) in I)

$$v_{\alpha\beta}(r_{12}) = z_{\alpha\beta}(r_{12}) + \sum_{\nu=1}^{\sigma} \rho_\nu \int d\boldsymbol{r}_3 z_{\alpha\nu}(r_{13}) v_{\nu\beta}(r_{32}), \tag{3·1}$$

which is rewritten, with the aid of Fourier transforms, as

$$V_{\alpha\beta}(k) = Z_{\alpha\beta}(k) + \sum_{\nu=1}^{\sigma} \rho_\nu Z_{\alpha\nu}(k) V_{\nu\beta}(k). \tag{3·1'}$$

If, in (3·1'), β is fixed and α is varied from 1 to σ, one obtains a set of linear equations which determines $V_{1\beta}$, $V_{2\beta}, \cdots$, and $V_{\sigma\beta}$ in terms of $Z_{\alpha\nu}$'s ($\alpha, \nu = 1, 2 \cdots, \sigma$). This set of linear equations can be solved in an elementary way by the use of a determinant. The result is as follows.

$$\rho_\alpha \rho_\beta V_{\alpha\beta}(k) = -\frac{1+\delta_{\alpha\beta}}{2} \frac{\partial \log D(k)}{\partial Z_{\alpha\beta}} - \rho_\alpha \delta_{\alpha\beta}, \tag{3·2}$$

where

$$D(k) = \begin{vmatrix} 1-\rho_1 Z_{11}(k) & -\rho_1 Z_{12}(k) & \cdots\cdots & -\rho_1 Z_{1\sigma}(k) \\ -\rho_2 Z_{21}(k) & 1-\rho_2 Z_{22}(k) & \cdots\cdots & -\rho_2 Z_{2\sigma}(k) \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ -\rho_\sigma Z_{\sigma 1}(k) & -\rho_\sigma Z_{\sigma 2}(k) & \cdots\cdots & 1-\rho_\sigma Z_{\sigma\sigma}(k) \end{vmatrix}, \tag{3·3}$$

which may be rewritten in a symmetrical form

$$D(k) = \begin{vmatrix} 1-\rho_1 Z_{11}(k) & -\sqrt{\rho_1\rho_2} Z_{12}(k) & \cdots\cdots & -\sqrt{\rho_1\rho_\sigma} Z_{1\sigma}(k) \\ -\sqrt{\rho_2\rho_1} Z_{21}(k) & 1-\rho_2 Z_{22}(k) & \cdots\cdots & -\sqrt{\rho_2\rho_\sigma} Z_{2\sigma}(k) \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ -\sqrt{\rho_\sigma\rho_1} Z_{\sigma 1}(k) & -\sqrt{\rho_\sigma\rho_2} Z_{\sigma 2}(k) & \cdots\cdots & 1-\rho_\sigma Z_{\sigma\sigma}(k) \end{vmatrix}. \tag{3·3'}$$

In the right-hand side of (3·2), $\delta_{\alpha\beta}=1$ for $\alpha=\beta$ and $\delta_{\alpha\beta}=0$ for $\alpha \neq \beta$. The differentiation with respect to $Z_{\alpha\beta}(k)$ is to be made by taking account of the fact that $Z_{\alpha\beta}(k) = Z_{\beta\alpha}(k)$.

If, in (3·1'), α is fixed and β is varied from 1 to σ, one obtains a set of linear equations which determines $Z_{\alpha 1}, Z_{\alpha 2}, \cdots$, and $Z_{\alpha\sigma}$ in terms of $V_{\nu\beta}$'s ($\nu, \beta = 1, 2, \cdots, \sigma$). This set of equations can also be solved to give the following result.

$$\rho_\alpha \rho_\beta Z_{\alpha\beta}(k) = -\frac{1+\delta_{\alpha\beta}}{2} \frac{\partial \log \bar{D}(k)}{\partial V_{\alpha\beta}} + \rho_\alpha \delta_{\alpha\beta}, \tag{3·4}$$

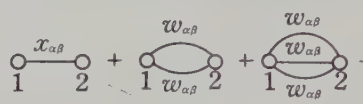
where

$$\bar{D}(k) = \begin{vmatrix} 1+\rho_1 V_{11}(k) & \rho_1 V_{12}(k) & \cdots\cdots & \rho_1 V_{1\sigma}(k) \\ \rho_2 V_{21}(k) & 1+\rho_2 V_{22}(k) & \cdots\cdots & \rho_2 V_{2\sigma}(k) \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \rho_\sigma V_{\sigma 1}(k) & \rho_\sigma V_{\sigma 2}(k) & \cdots\cdots & 1+\rho_\sigma V_{\sigma\sigma}(k) \end{vmatrix}. \tag{3·5}$$

The argument leading to (3·1) or (3·1′) is valid also in the case of the diagrams appearing in $w_{\alpha\beta}(r_{12})$. We get, corresponding to (3·1′),

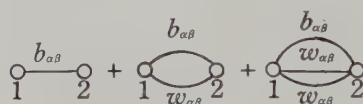
$$W_{\alpha\beta}(k)=X_{\alpha\beta}(k)+\sum_{\nu=1}^{\sigma}\rho_\nu Z_{\alpha\nu}(k)\,V_{\nu\beta}(k). \tag{3·6}$$

Next we shall consider the diagrams appearing in $z_{\alpha\beta}(r_{12})$ (see (2·7)). These diagrams can be grouped together by the number of parts into which the diagrams are divided when particles 1 and 2 are removed. It will be supposed first that particles 1 and 2 are not directly connected. Then the diagrams which are to appear in each part are restricted by restrictions (I), (II) and (III) of (2·5); hence the diagrams belong to $w_{\alpha\beta}(r_{12})$. Therefore the contribution, to $z_{\alpha\beta}(r_{12})$, of diagrams not containing $b_{\alpha\beta}(r_{12})$ is expressed symbolically as



(3·7f)

The reason why the first term is not $w_{\alpha\beta}$ but $x_{\alpha\beta}$ is that the diagrams in $z_{\alpha\beta}(r_{12})$ must not have s-points. In a similar way the contribution, to $z_{\alpha\beta}(r_{12})$, of diagrams containing $b_{\alpha\beta}(r_{12})$ turns out to be expressed symbolically as



(3·7′f)

The function $z_{\alpha\beta}(r_{12})$ is the sum of contributions (3·7f) and (3·7′f) so that $z_{\alpha\beta}(r_{12})$ is written in the form (see below (3·4f) in I):

$$\begin{aligned} z_{\alpha\beta}(r_{12}) &= x_{\alpha\beta}(r_{12})+\frac{1}{2!}w^2_{\alpha\beta}(r_{12})+\frac{1}{3!}w^3_{\alpha\beta}(r_{12})+\cdots\cdots \\ &\quad +b_{\alpha\beta}(r_{12})+\frac{1}{1!}b_{\alpha\beta}(r_{12})w_{\alpha\beta}(r_{12})+\frac{1}{2!}b_{\alpha\beta}(r_{12})w^2_{\alpha\beta}(r_{12})+\cdots\cdots \\ &= [b_{\alpha\beta}(r_{12})+1]e^{w_{\alpha\beta}(r_{12})}-1+x_{\alpha\beta}(r_{12})-w_{\alpha\beta}(r_{12}). \end{aligned} \tag{3·7}$$

As in I, it will be shown that the sum $\Sigma^{(X)}$ appearing in (2·6) can be reduced to a more restricted sum $\Sigma^{(X')}$ if one uses v-bonds in place of b-bonds. Let a diagram in $\Sigma^{(X)}$ have a pair of points (Let the particles on the points be particles 3 and 4.) by which the diagram is divided into two parts in the following way. The one is the part that includes particles 1 and 2 and cannot be divided any more by that pair of points. The other part consists of the rest. Diagrams to be included in the latter part are restricted only by restriction (I) of (2·5) where particles 1 and 2 must be replaced by particles 3 and 4. Therefore the latter part can be represented by $v_{\mu\nu}(r_{34})$ where μ and ν are the species of particles 3 and 4, respectively. By replacing the latter part by the $v_{\mu\nu}(r_{34})$-

bond, the original diagram is reduced to a simpler form. Such a reduction can be continued until such a pair of points appears no more. After all, (2·6) is reduced to

$$x_{\alpha\beta}(r_{12}) = \sum_{\mathbf{m}} \frac{\rho^{\mathbf{m}}}{\mathbf{m}!} \int d\{\mathbf{m}\} \sum_{\mathbf{m}+2\geq i>j\geq 1}^{(X')} \prod v_{\nu_i\nu_j}(r_{ij}), \tag{3·8}$$

where $\Sigma^{(X')}$ denotes the summation over all diagrams, composed of v-bonds, which are possible under the following restriction as well as (I), (II), (III) and (IV) of (2·5). The additional restriction is that there exists no part which is connected to the rest of the diagrams only by means of two points. As will be seen in the next section, it may be said that the diagrams appearing in (3·8) are those which are constructed by removing one v-bond from the diagrams appearing in the expression for the Helmholtz free energy (see below (4·9)). Some diagrams in $\Sigma^{(X')}$ are shown for a one-component system in Fig. 7 of I.

The set of Eqs. (3·1′), (3·2), (3·6) and (3·7) can be arranged as follows:

$$v_{\alpha\beta}(r) = [b_{\alpha\beta}(r)+1]e^{w_{\alpha\beta}(r)} - 1 \tag{3·9}$$

$$\rho_\alpha \rho_\beta V_{\alpha\beta}(k) = -\frac{1+\delta_{\alpha\beta}}{2} \frac{\partial \log D(k)}{\partial Z_{\alpha\beta}} - \rho_\alpha \delta_{\alpha\beta} \tag{3·10}$$

$$w_{\alpha\beta}(r) = x_{\alpha\beta}(r) + v_{\alpha\beta}(r) - z_{\alpha\beta}(r). \tag{3·11}$$

It is to be remembered that $Z_{\alpha\beta}(k)$ and $V_{\alpha\beta}(k)$ are the Fourier transforms of $z_{\alpha\beta}(r)$ and $v_{\alpha\beta}(r)$ respectively and that $D(k)$ is the determinant which has been defined by (3·3) or (3·3′). The function $v_{\alpha\beta}(r)$ is related to the pair distribution function $g_{\alpha\beta}(r)$ by

$$v_{\alpha\beta}(r) = g_{\alpha\beta}(r) - 1, \tag{3·12}$$

which is readily confirmed with the aid of (2·2), (2·3) and (3·9). The set of Eqs. (3·8)—(3·11) determines in principle the functions $v_{\alpha\beta}(r)$, $w_{\alpha\beta}(r)$, $z_{\alpha\beta}(r)$ and $x_{\alpha\beta}(r)$, and so the pair distribution function $g_{\alpha\beta}(r)$. Therefore we may say that this set of equations is an integral equation for the pair distribution function. It is to be noted that this integral equation is an exact one as far as the original density expansions (2·4), (2·6), (2·7) and (2·8) or their analytical continuations, if exist, are valid.

The above set of equations is reduced to the one which has been derived in I, if the system under consideration is a one-component system. As a matter of fact, the determinant $D(k)$ is reduced, in such a case, to $1-\rho Z(k)$ so that Eq. (3·10) is reduced to

$$\rho V(k) = \frac{1}{1-\rho Z(k)} - 1,$$

which is equivalent to (3·7) in I.

As in I, the zeroth approximation to solve the above integral equation will

be given by putting $x_{\alpha\beta}(r)$ to be zero. Then the set of equations is reduced to

$$\left.\begin{aligned} z_{\alpha\beta}(r) &= [b_{\alpha\beta}(r)+1]e^{w_{\alpha\beta}(r)}-1-w_{\alpha\beta}(r) \\ \text{and} \qquad & \\ \rho_\alpha\rho_\beta W_{\alpha\beta}(k) &= -\rho_\alpha\rho_\beta Z_{\alpha\beta}(k)-\frac{1+\delta_{\alpha\beta}}{2}\frac{\partial \log D(k)}{\partial Z_{\alpha\beta}}-\rho_\alpha\delta_{\alpha\beta}, \end{aligned}\right\} \tag{3·13}$$

which are to be called the equations in the hyper-netted chain approximation,[8] though they have not been written in an explicit form in reference 8). The next approximation will be to approximate $x_{\alpha\beta}(r)$ by the contributions arising from the diagrams $(\mu, \nu=1, 2, \cdots, \sigma)$ where the bold lines represent the v-bonds (see Fig. 7 in I).

§ 4. Expression for the Helmholtz free energy

It is known[10] that the interaction part of the Helmholtz free energy is in general expressed as

$$A_1=\frac{\Omega}{2}\int_0^1 d\xi\int d\boldsymbol{r}\sum_{\alpha=1}^{\sigma}\sum_{\beta=1}^{\sigma}\rho_\alpha\rho_\beta\phi_{\alpha\beta}(r)g_{\alpha\beta}(r\,;\xi), \tag{4·1}$$

where $g_{\alpha\beta}(r\,;\xi)$ is the pair distribution function for the case that all the pair interaction potentials are $\xi\phi_{\mu\nu}(r)$ $(\mu, \nu=1, 2, \cdots, \sigma)$. In such a case the functions $b_{\alpha\beta}(r)$, $w_{\alpha\beta}(r)$, and so on, are also dependent on ξ. In particular, $b_{\alpha\beta}(r)$ is of the form

$$b_{\alpha\beta}(r\,;\xi)=e^{-\xi\phi_{\alpha\beta}(r)/kT}-1. \tag{4·2}$$

By taking notice of (2·3) and (4·2), expression (4·1) can be rewritten as

$$\frac{A_1}{\Omega kT}=-\frac{1}{2}\int_0^1 d\xi\int d\boldsymbol{r}\sum_{\alpha=1}^{\sigma}\sum_{\beta=1}^{\sigma}\rho_\alpha\rho_\beta\frac{\partial b_{\alpha\beta}(r\,;\xi)}{\partial\xi}e^{w_{\alpha\beta}(r\,;\xi)}. \tag{4·1′}$$

In the following, the ξ-dependence of $b_{\alpha\beta}(r)$, $w_{\alpha\beta}(r)$, and so on, will not be explicitly written, for no confusion will arise.

Differentiating (3·9) with respect to ξ, we get

$$\frac{\partial b_{\alpha\beta}(r)}{\partial\xi}e^{w_{\alpha\beta}(r)}=\frac{\partial v_{\alpha\beta}(r)}{\partial\xi}-\frac{\partial w_{\alpha\beta}(r)}{\partial\xi}-v_{\alpha\beta}(r)\frac{\partial w_{\alpha\beta}(r)}{\partial\xi},$$

which is transformed, by the use of (3·11), into

$$\begin{aligned} \frac{\partial b_{\alpha\beta}(r)}{\partial\xi}e^{w_{\alpha\beta}(r)}=\frac{\partial}{\partial\xi}\Big[&v_{\alpha\beta}(r)-w_{\alpha\beta}(r)-w_{\alpha\beta}(r)v_{\alpha\beta}(r)+\frac{1}{2}v_{\alpha\beta}^2(r) \\ &-v_{\alpha\beta}(r)z_{\alpha\beta}(r)\Big]+v_{\alpha\beta}(r)\frac{\partial z_{\alpha\beta}(r)}{\partial\xi}+x_{\alpha\beta}(r)\frac{\partial v_{\alpha\beta}(r)}{\partial\xi}. \end{aligned} \tag{4·3}$$

When this expression is introduced into (4·1′) and use is made of the property of Fourier transforms, the interaction part of the free energy is written in the form

$$\frac{A_1}{\Omega kT}=-\frac{1}{2}\int d\boldsymbol{r}\sum_{\alpha=1}^{\sigma}\sum_{\beta=1}^{\sigma}\rho_\alpha\rho_\beta\int_0^1 d\xi\,\frac{\partial}{\partial\xi}\Big[v_{\alpha\beta}(r)-w_{\alpha\beta}(r)-w_{\alpha\beta}(r)v_{\alpha\beta}(r)$$

$$+\frac{1}{2}v_{\alpha\beta}^2(r)-v_{\alpha\beta}(r)z_{\alpha\beta}(r)\Big]-\frac{1}{2}\frac{1}{\Omega}\sum_{k}\int_0^1 d\xi\sum_{\alpha=1}^{\sigma}\sum_{\beta=1}^{\sigma}\rho_\alpha\rho_\beta V_{\alpha\beta}(k)$$

$$\times\frac{\partial Z_{\alpha\beta}(k)}{\partial\xi}+\frac{A_1'}{\Omega kT}, \tag{4·4}$$

where

$$\frac{A_1'}{\Omega kT}=-\frac{1}{2}\int_0^1 d\xi\int d\boldsymbol{r}\sum_{\alpha=1}^{\sigma}\sum_{\beta=1}^{\sigma}\rho_\alpha\rho_\beta x_{\alpha\beta}(r)\frac{\partial v_{\alpha\beta}(r)}{\partial\xi}. \tag{4·5}$$

The integration with respect to ξ is readily performed in the first term on the right-hand side of (4·4). By the use of the fact that $v_{\alpha\beta}$, $w_{\alpha\beta}$ and $z_{\alpha\beta}$ vanish at $\xi=0$, it turns out that the first term is written as

$$-\frac{1}{2}\int d\boldsymbol{r}\sum_{\alpha=1}^{\sigma}\sum_{\beta=1}^{\sigma}\rho_\alpha\rho_\beta\Big[v_{\alpha\beta}(r)-w_{\alpha\beta}(r)-w_{\alpha\beta}(r)v_{\alpha\beta}(r)$$

$$+\frac{1}{2}v_{\alpha\beta}^2(r)-v_{\alpha\beta}(r)z_{\alpha\beta}(r)\Big], \tag{4·6}$$

where the values of $v_{\alpha\beta}$, $w_{\alpha\beta}$ and $z_{\alpha\beta}$ are those at $\xi=1$. The integration over ξ is carried out also in the second term as follows:

$$-\frac{1}{2}\frac{1}{\Omega}\sum_{k}\int_0^1 d\xi\sum_{\alpha=1}^{\sigma}\sum_{\beta=1}^{\sigma}\rho_\alpha\rho_\beta V_{\alpha\beta}(k)\frac{\partial Z_{\alpha\beta}(k)}{\partial\xi}$$

$$=+\frac{1}{2}\frac{1}{\Omega}\sum_{k}\int_0^1 d\xi\sum_{\alpha=1}^{\sigma}\sum_{\beta=1}^{\sigma}\Big[\frac{1+\delta_{\alpha\beta}}{2}\frac{\partial\log D(k)}{\partial Z_{\alpha\beta}}+\rho_\alpha\delta_{\alpha\beta}\Big]\frac{\partial Z_{\alpha\beta}(k)}{\partial\xi}$$

$$=+\frac{1}{2}\frac{1}{\Omega}\sum_{k}\int_0^1 d\xi\frac{\partial}{\partial\xi}\Big[\log D(k)+\sum_{\alpha=1}^{\sigma}\rho_\alpha Z_{\alpha\alpha}(k)\Big]$$

$$=+\frac{1}{2}\frac{1}{\Omega}\sum_{k}\Big[\log D(k)+\sum_{\alpha=1}^{\sigma}\rho_\alpha Z_{\alpha\alpha}(k)\Big]. \tag{4·7}$$

In the last expression of (4·7), the values of $D(k)$ and $Z_{\alpha\alpha}(k)$ are those at $\xi=1$. In transforming the first expression of (4·7) into the second one, use has been made of (3·10). In transforming the second expression into the third one, the following relation has been used:

$$\frac{\partial \log D(k)}{\partial \xi} = \sum_{\alpha=1}^{\sigma} \sum_{\beta=1}^{\sigma} \frac{1+\delta_{\alpha\beta}}{2} \frac{\partial \log D(k)}{\partial Z_{\alpha\beta}} \frac{\partial Z_{\alpha\beta}(k)}{\partial \xi}, \tag{4·8}$$

which can be readily proved with the aid of the fact that the determinant $D(k)$ is dependent on ξ through $Z_{\alpha\beta}(k)$'s alone (see (3·3)). We can prove that the last term of (4·4), which is defined by (4·5), is expressed in a form

$$\frac{A_1'}{\Omega kT} = -\sum_{\mathbf{m}} \frac{\boldsymbol{\rho}^{\mathbf{m}}}{\mathbf{m}!} \frac{1}{\Omega} \int d\{\mathbf{m}\} \sum_{\mathbf{m}\geq i>j\geq 1}^{(A')} \prod v_{\nu_i \nu_j}(r_{ij}), \tag{4·9}$$

where $\sum^{(A')}$ denotes the summation over the diagrams which are obtained by introducing the $v_{\alpha\beta}(r_{12})$-bond into the diagrams appearing in $\sum^{(X')}$. If one takes notice of the restrictions which are imposed on the diagrams in $\sum^{(X')}$ (see below (3·8)), it can be verified that $\sum^{(A')}$ may be said to be the summation over all diagrams which are *more than doubly connected*, though the proof will be omitted here.

The proof of (4·9) is as follows. Let us consider the integral

$$\int d\{\mathbf{m}+2\} \frac{\partial v_{\alpha\beta}(r_{12})}{\partial \xi} \sum_{\mathbf{m}+2\geq i>j\geq 1}^{(X')} \prod v_{\nu_i \nu_j}(r_{ij}).$$

This integral can be transformed, with the aid of the definition of $\sum^{(A')}$, into the form

$$\frac{1+\delta_{\alpha\beta}}{(m_\alpha+1)(m_\beta+1+\delta_{\alpha\beta})} \frac{\partial}{\partial \xi} \int d\{\mathbf{m}+2\} \sum_{\mathbf{m}+2\geq i>j\geq 1}^{(A')} \prod v_{\nu_i \nu_j}(r_{ij}).$$

The factor $(1+\delta_{\alpha\beta})/(m_\alpha+1)(m_\beta+1+\delta_{\alpha\beta})$ enters because the total of diagrams appearing in $\sum^{(X')}$ is symmetrical in $\mathbf{m}$ particles while the total of diagrams appearing in $\sum^{(A')}$ is symmetrical in $(\mathbf{m}+2)$ particles which are $\mathbf{m}$ particles plus particles $1(\alpha$-species) and $2(\beta$-species). By the use of the above transformation, (4·5) is transformed in the following way (see (3·8)).

$$\begin{aligned}
\frac{A_1'}{\Omega kT} &= -\frac{1}{2} \int_0^1 d\xi \frac{1}{\Omega} \iint d\mathbf{r}_1 d\mathbf{r}_2 \sum_{\alpha=1}^{\sigma} \sum_{\beta=1}^{\sigma} \rho_\alpha \rho_\beta \frac{\partial v_{\alpha\beta}(r_{12})}{\partial \xi} x_{\alpha\beta}(r_{12}) \\
&= -\frac{1}{2} \int_0^1 d\xi \sum_{\alpha=1}^{\sigma} \sum_{\beta=1}^{\sigma} \rho_\alpha \rho_\beta \frac{1}{\Omega} \sum_{\mathbf{m}} \frac{\boldsymbol{\rho}^{\mathbf{m}}}{\mathbf{m}!} \int d\{\mathbf{m}+2\} \frac{\partial v_{\alpha\beta}(r_{12})}{\partial \xi} \sum_{\mathbf{m}+2\geq i>j\geq 1}^{(X')} \prod v_{\nu_i \nu_j}(r_{ij}) \\
&= -\frac{1}{2} \int_0^1 d\xi \sum_{\alpha=1}^{\sigma} \sum_{\beta=1}^{\sigma} \rho_\alpha \rho_\beta \frac{1}{\Omega} \sum_{\mathbf{m}} \frac{\boldsymbol{\rho}^{\mathbf{m}}}{\mathbf{m}!} \frac{1+\delta_{\alpha\beta}}{(m_\alpha+1)(m_\beta+1+\delta_{\alpha\beta})} \\
&\qquad \times \frac{\partial}{\partial \xi} \int d\{\mathbf{m}+2\} \sum_{\mathbf{m}+2\geq i>j\geq 1}^{(A')} \prod v_{\nu_i \nu_j}(r_{ij}) \\
&= -\frac{1}{2} \sum_{\alpha=1}^{\sigma} \sum_{\beta=1}^{\sigma} \rho_\alpha \rho_\beta \frac{1}{\Omega} \sum_{\mathbf{m}} \frac{\boldsymbol{\rho}^{\mathbf{m}}}{\mathbf{m}!} \frac{1+\delta_{\alpha\beta}}{(m_\alpha+1)(m_\beta+1+\delta_{\alpha\beta})} \\
&\qquad \times \int d\{\mathbf{m}+2\} \sum_{\mathbf{m}+2\geq i>j\geq 1}^{(A')} \prod v_{\nu_i \nu_j}(r_{ij})
\end{aligned}$$

$$= -\sum_{\mathbf{m}} \frac{\rho^{\mathbf{m}}}{\mathbf{m}!} \frac{1}{\Omega} \int d\{\mathbf{m}\} \sum_{\mathbf{m}\geq i>j\geq 1}^{(A')} \prod v_{\nu_i \nu_j}(r_{ij}).$$

The last expression is the one to be proved, and, consequently, the proof of (4·9) has been completed.

As a result of (4·6) and (4·7), the interaction part of the free energy is written in the form

$$\frac{A_1}{\Omega kT} = -\frac{1}{2} \int d\boldsymbol{r} \sum_{\alpha=1}^{\sigma} \sum_{\beta=1}^{\sigma} \rho_\alpha \rho_\beta \Big[v_{\alpha\beta}(r) - w_{\alpha\beta}(r) - w_{\alpha\beta}(r) v_{\alpha\beta}(r) + \frac{1}{2} v_{\alpha\beta}^2(r) - v_{\alpha\beta}(r) z_{\alpha\beta}(r) \Big]$$
$$+ \frac{1}{2} \frac{1}{\Omega} \sum_{k} \Big[\log D(k) + \sum_{\alpha=1}^{\sigma} \rho_\alpha Z_{\alpha\alpha}(k) \Big] + \frac{A_1'}{\Omega kT}, \qquad (4\cdot10)$$

where $A_1'/\Omega kT$ is given by (4·9). Expression (4·10) is rewritten, by the use of (3·9), as follows:

$$\frac{A_1}{\Omega kT} = -\frac{1}{2} \int d\boldsymbol{r} \sum_{\alpha=1}^{\sigma} \sum_{\beta=1}^{\sigma} \rho_\alpha \rho_\beta \Big[\{b_{\alpha\beta}(r)+1\} e^{w_{\alpha\beta}(r)} - 1 - w_{\alpha\beta}(r) - w_{\alpha\beta}(r) v_{\alpha\beta}(r) + \frac{1}{2} v_{\alpha\beta}^2(r) - v_{\alpha\beta}(r) z_{\alpha\beta}(r) \Big]$$
$$+ \frac{1}{2} \frac{1}{\Omega} \sum_{k} \Big[\log D(k) + \sum_{\alpha=1}^{\sigma} \rho_\alpha Z_{\alpha\alpha}(k) \Big] + \frac{A_1'}{\Omega kT}, \qquad (4\cdot11)$$

which will be more conveniently used than (4·10) in the following.

It can be easily seen that the above expression (4·11) is reduced to (4·11) in I if one considers a one-component system. It can also be seen that (4·10) is reduced to the expression in the hyper-netted chain approximation if one omits $A_1'/\Omega kT$ from (4·10), though in reference 8) $D(k)$ (which is identical to L in reference 8)) is not expressed explicitly in the form of determinant.

The expression for the free energy in a one-component system has been shown to have a stationary character.[1] It will be shown here that expression (4·11) also has the same character. The following relation is necessary to show it:

$$\frac{\delta}{\delta v_{\alpha\beta}(r)} \left(\frac{A_1'}{\Omega kT} \right) = -\frac{\rho_\alpha \rho_\beta}{1+\delta_{\alpha\beta}} x_{\alpha\beta}(r), \qquad (4\cdot12)$$

where the left-hand side denotes the functional derivative of $A_1'/\Omega kT$ with respect to $v_{\alpha\beta}(r)$.* The proof of (4·12) can be done by reversing the reasoning which

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* The functional derivative is in general defined as follows. Let P be a functional of $\alpha(r)$. When $\alpha(r)$ is varied by $\delta\alpha(r)$, the corresponding first order variation of P is denoted by δP. Then the functional derivative $\delta P/\delta\alpha(r)$ is defined by[1),11)]

$$\delta P = \int d\boldsymbol{r} \frac{\delta P}{\delta \alpha(r)} \delta\alpha(r).$$

was used in the derivation of (4·9) from (4·5), though the proof is omitted here. The first order variation of the free energy when in (4·11) the functions $w_{\alpha\beta}$, $v_{\alpha\beta}$ and $z_{\alpha\beta}$ are varied *independently* of each other, is shown to be, with the aid of (4·12) and (3·3),

$$\delta\left(\frac{A_1}{\Omega kT}\right)=-\frac{1}{2}\int d\boldsymbol{r}\sum_{\alpha=1}^{\sigma}\sum_{\beta=1}^{\sigma}\rho_\alpha\rho_\beta\Big[\{b_{\alpha\beta}(r)+1\}e^{w_{\alpha\beta}(r)}-1-v_{\alpha\beta}(r)\Big]\delta w_{\alpha\beta}(r)$$
$$-\frac{1}{2}\int d\boldsymbol{r}\sum_{\alpha=1}^{\sigma}\sum_{\beta=1}^{\sigma}\rho_\alpha\rho_\beta\Big[v_{\alpha\beta}(r)-w_{\alpha\beta}(r)-z_{\alpha\beta}(r)+x_{\alpha\beta}(r)\Big]\delta v_{\alpha\beta}(r)$$
$$+\frac{1}{2}\frac{1}{\Omega}\sum_{k}\sum_{\alpha=1}^{\sigma}\sum_{\beta=1}^{\sigma}\Big[\frac{1+\delta_{\alpha\beta}}{2}\frac{\partial\log D(k)}{\partial Z_{\alpha\beta}}+\rho_\alpha\rho_\beta V_{\alpha\beta}(k)+\rho_\alpha\delta_{\alpha\beta}\Big]\delta Z_{\alpha\beta}(k). \tag{4·13}$$

The coefficients of $\delta w_{\alpha\beta}(r)$, $\delta v_{\alpha\beta}(r)$ and $\delta Z_{\alpha\beta}(k)$ in the above expression are seen to vanish if there exist relations (3·9), (3·11) and (3·10) among $w_{\alpha\beta}$, $v_{\alpha\beta}$ and $z_{\alpha\beta}$. In other words, it has been proved that the free energy (4·11) is stationary with respect to the functions satisfying (3·9), (3·10) and (3·11). Conversely, it may be said that the set of Eqs. (3·9), (3·10) and (3·11) can be derived by means of such a variational principle that the free energy (4·11) is to be stationary with respect to the variations of $w_{\alpha\beta}$, $v_{\alpha\beta}$ and $z_{\alpha\beta}$.

The stationary character of (4·11) is useful in deriving the expressions for the pressure and the internal energy by means of thermodynamical relations, because in such cases the dependence of $w_{\alpha\beta}$, $v_{\alpha\beta}$ and $z_{\alpha\beta}$ on density and temperature can be neglected. It may be useful also in determining the approximate forms of $w_{\alpha\beta}$, $v_{\alpha\beta}$ and $z_{\alpha\beta}$, hence of the pair distribution function.

It is known that the pair distribution function is related to the free energy by[11),8)]

$$\rho_\alpha\rho_\beta g_{\alpha\beta}(r)=\frac{1+\delta_{\alpha\beta}}{\Omega}\frac{\delta A_1}{\delta\phi_{\alpha\beta}(r)}. \tag{4·14}$$

It can be readily verified, by the use of (4·11), that this relation is valid in our case as it should be.

§ 5. Summary and concluding remarks

The results which have been obtained can be summarized as follows. The interaction part of the Helmholtz free energy in a σ-component fluid system (the volume Ω, the temperature T and the number densities $\rho_1, \rho_2, \cdots, \rho_\sigma$) is written in the form

$$\frac{A_1}{\Omega kT}=-\frac{1}{2}\int d\boldsymbol{r}\sum_{\alpha=1}^{\sigma}\sum_{\beta=1}^{\sigma}\rho_\alpha\rho_\beta\Big[\{b_{\alpha\beta}(r)+1\}e^{w_{\alpha\beta}(r)}-1-w_{\alpha\beta}(r)$$
$$-w_{\alpha\beta}(r)v_{\alpha\beta}(r)+\frac{1}{2}v_{\alpha\beta}^2(r)-v_{\alpha\beta}(r)z_{\alpha\beta}(r)\Big]$$

$$+\frac{1}{2}\frac{1}{\Omega}\sum_{k}\left[\log D(k)+\sum_{\alpha=1}^{\sigma}\rho_\alpha Z_{\alpha\alpha}(k)\right]$$

$$-\sum_{\mathbf{m}}\frac{\boldsymbol{\rho}^{\mathbf{m}}}{\mathbf{m}!}\frac{1}{\Omega}\int d\{\mathbf{m}\}\sum_{\substack{\text{All products which are}\\ \text{more than doubly connected.}}}\prod_{\mathbf{m}\geq i>j\geq 1}v_{\nu_i\nu_j}(r_{ij}), \tag{5·1}$$

where $b_{\alpha\beta}(r)$ is defined in terms of the pair interaction potential $\phi_{\alpha\beta}(r)$ by

$$b_{\alpha\beta}(r)=e^{-\phi_{\alpha\beta}(r)/kT}-1 \tag{5·2}$$

and $D(k)$ is a determinant

$$D(k)=\begin{vmatrix} 1-\rho_1 Z_{11}(k) & -\rho_1 Z_{12}(k) & \cdots\cdots & -\rho_1 Z_{1\sigma}(k) \\ -\rho_2 Z_{21}(k) & 1-\rho_2 Z_{22}(k) & \cdots\cdots & -\rho_2 Z_{2\sigma}(k) \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ -\rho_\sigma Z_{\sigma 1}(k) & -\rho_\sigma Z_{\sigma 2}(k) & \cdots\cdots & 1-\rho_\sigma Z_{\sigma\sigma}(k) \end{vmatrix}. \tag{5·3}$$

Mayer's notation is adopted in the last term on the right-hand side of (5·1). The functions $w_{\alpha\beta}(r)$, $v_{\alpha\beta}(r)$ and $z_{\alpha\beta}(r)$ which are symmetric in α and β ($V_{\alpha\beta}(k)$ and $Z_{\alpha\beta}(k)$ denote the Fourier transforms of $v_{\alpha\beta}(r)$ and $z_{\alpha\beta}(r)$, respectively.) are to be determined in such a way that (5·1) is stationary with respect to the variations of these functions. Namely they must satisfy the set of equations:

$$v_{\alpha\beta}(r)=[b_{\alpha\beta}(r)+1]e^{w_{\alpha\beta}(r)}-1, \tag{5·4}$$

$$\rho_\alpha\rho_\beta V_{\alpha\beta}(k)=-\frac{1+\delta_{\alpha\beta}}{2}\frac{\partial\log D(k)}{\partial Z_{\alpha\beta}}-\rho_\alpha\delta_{\alpha\beta}, \tag{5·5}$$

and

$$w_{\alpha\beta}(r)=x_{\alpha\beta}(r)+v_{\alpha\beta}(r)-z_{\alpha\beta}(r). \tag{5·6}$$

The function $x_{\alpha\beta}(r)$ is defined by

$$x_{\alpha\beta}(r_{12})=\sum_{\mathbf{m}}\frac{\boldsymbol{\rho}^{\mathbf{m}}}{\mathbf{m}!}\int d\{\mathbf{m}\}\sum_{\mathbf{m}+2\geq i>j\geq 1}^{(X')}\prod v_{\nu_i\nu_j}(r_{ij}), \tag{5·7}$$

where $\Sigma^{(X')}$ denotes the summation over all the different diagrams which are to be constructed if one removes one bond $v_{\alpha\beta}(r_{12})$ from the diagrams appearing in the last term on the right-hand side of (5·1). The pair distribution function $g_{\alpha\beta}(r)$ is related to $v_{\alpha\beta}(r)$ by

$$g_{\alpha\beta}(r)=v_{\alpha\beta}(r)+1. \tag{5·8}$$

In some cases the above results may be exact ones as far as the expressions themselves are well determined, irrespective as to whether the original density expansions (2·4), (2·6), (2·7) and (2·8) from which the above results have been obtained are valid or not. For instance, in an ionic solution or a classical plasma, the interaction potentials are of long-range character so that the original

density expansions diverge term by term. There is good reason to expect that the above expressions have definite meanings even in such a case (see references 6), 7), 8) and 12)). An application of the present theory to such a system will be undertaken in a forthcoming paper.

If we restrict ourselves to a system without long-range interactions, the original density expansions converge at low enough densities so that the series appearing in the above expressions converge, perhaps more rapidly than the original expansions,* at such densities. The present author hopes that the series appearing in the above expressions may converge even at the densities which are larger than the maximum density for which the original expansions converge. It is also hoped that the maximum density for which the determinant $D(k)$ does not vanish for any k may be larger than the maximum density for which the original expansions converge.

At the present stage a physical meaning is not clear of the case where $D(k)$ vanishes for some k or where the series appearing in the above expressions diverge. The situation is the same as in I. Such a problem will be investigated in the near future.

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References

1) T. Morita and K. Hiroike, Prog. Theor. Phys. **23** (1960), 1003.
2) J. G. Kirkwood, J. Chem. Phys. **3** (1935), 300.
3) J. E. Mayer, J. Chem. Phys. **15** (1947), 187.
S. Ono, Prog. Theor. Phys. **6** (1951), 447.
4) G. S. Rushbrooke, Phil. Mag. **43** (1952), 1276.
5) G. S. Rushbrooke and H. I. Scoins, Phil. Mag. **42** (1951), 582.
6) J. E. Mayer, J. Chem. Phys. **18** (1950), 1426.
7) E. Haga, J. Phys. Soc. Japan **8** (1953), 714.
E. Meeron, J. Chem. Phys. **26** (1957), 804; **28** (1958), 630.
H. L. Friedman, Molecular Physics **2** (1959), 23, 190.
8) T. Morita, Prog. Theor. Phys. **21** (1959), 361.
9) E. Meeron, J. Chem. Phys. **27** (1957), 1238.
10) K. Hiroike, J. Phys. Soc. Japan **15** (1960), 771.
11) K. Hiroike, J. Phys. Soc. Japan **12** (1957), 864.
12) R. Abe, Prog. Theor. Phys. **22** (1959), 213.

* This may be inferred from the fact that the diagrams appearing in the above series are more restricted ones than the diagrams in the original expansions, though we cannot prove it in general.

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On the Collective Motion in Even-Even Spherical Nuclei*

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A method formally analogous to that developed by Bogoliubov, Tolmachev and Shirkov to investigate the collective excitations in superconductors is applied to even-even spherical nuclei in order to investigate the mechanism of the nuclear collective motion from the standpoint of particle excitations. Our theory leads, in principle, to the same results as those obtained by Belyaev. However, the method of description of the nuclear collective motion is quite different from that of Belyaev's paper in which the "cranking model" of Inglis is employed, and the various physical parameters used by Belyaev can be derived uniquely from the "first" principle. Thus, in so far as the vibrational motion is concerned, the physical implication underlying the nuclear collective model proposed by Bohr and Mottelson is made clear.

It is outside the scope of this paper to relate the effective inter-particle interactions used in this paper with the nuclear forces known from the two nucleon problems.

§ 1. Introduction

The main purpose of this paper is to investigate the mechanism of collective motion in even-even spherical nuclei from the standpoint of particle excitations. It is not our purpose to go into detailed quantitative calculations, but rather to develop the basic idea. The *j-j* coupling shell model is the starting point of our theory, and the pairing correlation between two nucleons is taken into account by means of the Bogoliubov transformation.[3]

Our theory leads, in principle, to the same results as those obtained by Belyaev.[4] However, the method of description of the nuclear collective motion is quite different from that of Belyaev's paper[4] in which the "cranking model" of Inglis[5] is employed. It is attempted to propose a method which is more fundamental than the "cranking model". The method is based on an extension of the theory of Sawada et al.[6] of the plasma oscillation, and is formally analogous to the method developed by Bogoliubov, Tolmachev and Shirkov[3] to investigate the collective excitations in superconductors.

To take into account the pairing correlation for which the "short range" part of the effective two-body interaction between particles plays an important role, the technique of the Bogoliubov transformation[3] used in the new theory of super-

* The basic idea which will be developed in this paper corresponds, in some sense, to the generalization of that proposed in our previous papers.[1),2)]

conductivity will be adopted, and the results of the application of the transformation to even-even spherical nuclei will be recapitulated in § 2, § 3 and § 4 in a form especially suited for our purpose.*

The fundamental idea and the physical basis of our method describing the nuclear collective vibrations will be developed in § 5, § 6 and § 7. In order to make clear that the essential idea of the theory of Sawada et al.[6] can be applied to the *finite size* nuclear system, the *explicit* relationship of this idea with that of the "time-dependent self-consistent field method" underlying the "cranking model"[5] will be established in § 5 and § 6, provided that the "long range" part of the effective two-body interaction between particles plays an important role for the collective vibration of even-even spherical nuclei. Thus it will be shown in § 7 that a method formally analogous to that developed by Bogoliubov, Tolmachev and Shirkov[3] to investigate the collective excitations in superconductors can describe the collective motion of even-even spherical nuclei.

By using this method it will be shown in § 8 that there are two kinds of quadrupole vibration-modes for even-even spherical nuclei. To display these collective modes, the "method of auxiliary variables"[7] will be employed in § 9. The relation between the collective variables thus obtained and the "deformation variables" used in the Bohr-Mottelson model[8] will be clarified in § 10. In § 11 and § 12, the normal vibrations of even-even spherical nuclei will be discussed, and the inertial parameter and the surface tension parameter for the quadrupole vibration will be uniquely determined. It will be shown that the results thus obtained are essentially equivalent to those obtained by Belyaev[4] and are in agreement with the observed trends.

It is hoped that our approach will be helpful to clarify the concept of the nuclear collective motion and to give a further insight into the various aspects of the unified model.

§ 2. The Bogoliubov transformation

An application of the Bogoliubov transformation[3] in the new theory of superconductivity to the nuclear system has been developed by Belyaev.[4] The close connection between the basic idea of this method and Racah's seniority concept in the *j-j* coupling shell model has also been discussed by many authors.[9] In this and next sections, we shall recapitulate the results in a form especially suited for our purpose.

Let us consider a system of nucleons which are moving in a spherically symmetrical self-consistent potential-well V and choose the wave functions of a nucleon in this well as the basic functions of the second quantization representation. If we adopt the *j-j* coupling shell model, such single-particle states are characterized

* The reader who is familiar with the application of the Bogoliubov transformation to the nuclear system need not read § 2 and § 3 in full but the final results (3·14) and (3·15).

by the quantum numbers (n, l, j, m). Hereafter we shall denote a set of the quantum numbers (n, l, j) by N. The Hamiltonian for the system of interacting nucleons is then

$$H=\sum_{\substack{Nm\\N'm'}}\langle Nm|T|N'm'\rangle a^{+}_{Nm}a_{N'm'}-\frac{1}{2}\sum_{(Nm)}\langle N_1m_1N_2m_2|G|N_2'm_2'N_1'm_1'\rangle$$
$$\times a^{+}_{N_1m_1}a^{+}_{N_2m_2}a_{N_2'm_2'}a_{N_1'm_1'}-\lambda\sum_{Nm}a^{+}_{Nm}a_{Nm}$$
$$=\sum_{\substack{Nm\\N'm'}}\{(\epsilon_N-\lambda)\delta_{NN'}\delta_{mm'}+\langle Nm|V|N'm'\rangle\}a^{+}_{Nm}a_{N'm'}$$
$$-\frac{1}{2}\sum_{(Nm)}\langle N_1m_1N_2m_2|G|N_2'm_2'N_1'm_1'\rangle a^{+}_{N_1m_1}a^{+}_{N_2m_2}a_{N_2'm_2'}a_{N_1'm_1'}, \qquad (2\cdot1)$$

where ϵ_N is the single-particle energy in N-state and λ is the well-known chemical potential. The sign of G and V is chosen to be positive for an attractive interaction and an attractive well respectively.

Now it is convenient for our purpose to divide the effective two-body interaction G approximately into three parts:

$$G=G^{(0)}+G^{(1)}+G^{(2)}, \qquad (2\cdot2)$$

where $G^{(0)}$ the part mainly contributing to the "pairing interaction", $G^{(1)}$ the part mainly contributing to the spherically symmetrical self-consistent field, and $G^{(2)}$ is the one that is responsible for the collective vibrations. The possibility of such an approximate division of G will be discussed in § 3. Corresponding to $(2\cdot2)$, the Hamiltonian $(2\cdot1)$ may be divided as follows:

$$H=H^{(0)}+H^{(\mathrm{res.})}, \qquad (2\cdot3a)$$
$$H^{(0)}=\sum_{\substack{Nm\\N'm'}}\{(\epsilon_N-\lambda)\delta_{NN'}\delta_{mm'}+\langle Nm|V|N'm'\rangle\}a^{+}_{Nm}a_{N'm'}$$
$$-\frac{1}{2}\sum_{(Nm)}\langle N_1m_1N_2m_2|G^{(0)}+G^{(1)}|N_2'm_2'N_1'm_1'\rangle a^{+}_{N_1m_1}a^{+}_{N_2m_2}a_{N_2'm_2'}a_{N_1'm_1'}, \qquad (2\cdot3b)$$
$$H^{(\mathrm{res.})}=-\frac{1}{2}\sum_{(Nm)}\langle N_1m_1N_2m_2|G^{(2)}|N_2'm_2'N_1'm_1'\rangle a^{+}_{N_1m_1}a^{+}_{N_2m_2}a_{N_2'm_2'}a_{N_1'm_1'}. \qquad (2\cdot3c)$$

Following Belyaev's,[4] let us introduce the new Fermi operators α_{Nm} by the Bogoliubov transformation:[3]

$$a_{Nm}=u_{Nm}\alpha_{Nm}+v_{Nm}\alpha^{+}_{N-m}, \qquad (2\cdot4)$$

where u_{Nm} and v_{Nm} are real numbers which obey the conditions

$$u_{Nm}=u_{N-m}=u_{N\nu}$$
$$v_{Nm}=-v_{N-m}=\begin{cases}+v_{N\nu} & m>0\\ -v_{N\nu} & m<0,\end{cases}\qquad \nu\equiv|m|, \qquad (2\cdot5)$$

$$u_{Nm}^2+v_{Nm}^2=1. \tag{2·6}$$

After the transformation (2·4), the Hamiltonian $H^{(0)}$ has the following structure:

$$H^{(0)}=U+H_{20}+H_{11}+H', \tag{2·7}$$

where

$$U=\sum_{N\nu}(\epsilon_N-\lambda)\cdot 2v_{N\nu}^2-\sum_{N\nu}\Delta_{N\nu}u_{N\nu}v_{N\nu}+\sum_{N\nu}\langle N\nu|V|N\nu\rangle v_{N\nu}^2, \tag{2·8}$$

$$H_{20}=\sum_{N\nu}\{(\epsilon_N-\lambda)\cdot 2u_{N\nu}v_{N\nu}-\Delta_{N\nu}(u_{N\nu}^2-v_{N\nu}^2)\}(\alpha_{N\nu}^{+}\alpha_{N-\nu}^{+}+\alpha_{N-\nu}\alpha_{N\nu}), \tag{2·9}$$

$$H_{11}=\sum_{N\nu}\{(\epsilon_N-\lambda)(u_{N\nu}^2-v_{N\nu}^2)+\Delta_{N\nu}\cdot 2u_{N\nu}v_{N\nu}\}(\alpha_{N\nu}^{+}\alpha_{N\nu}+\alpha_{N-\nu}^{+}\alpha_{N-\nu}). \tag{2·10}$$

H' is a small perturbation term which will be neglected hereafter. $\Delta_{N\nu}$ in (2·8)—(2·10) is defined by

$$\Delta_{N\nu}=\sum_{N_1\nu_1}\langle N\nu\ N\nu|G^{(0)}|N_1\nu_1\ N_1\nu_1\rangle u_{N_1\nu_1}v_{N_1\nu_1}, \tag{2·11}$$

where

$$\begin{aligned}\langle N\nu\ N\nu|G^{(0)}|N_1\nu_1\ N_1\nu_1\rangle\equiv&\langle N\ +\nu,\ N\ -\nu|G^{(0)}|N_1\ -\nu_1,\ N_1\ +\nu_1\rangle\\&-\langle N\ +\nu,\ N\ -\nu|G^{(0)}|N_1\ +\nu_1,\ N_1\ -\nu_1\rangle.\end{aligned} \tag{2·12}$$

The matrix elements $\langle N\nu|V|N'\nu'\rangle$ in (2·8) have the form

$$\langle N\nu|V|N'\nu'\rangle\equiv\langle N\ +\nu|V|N'\ +\nu'\rangle\delta_{\nu\nu'}=\langle N\ -\nu|V|N'\ -\nu'\rangle\delta_{\nu\nu'}. \tag{2·13}$$

In deriving (2·8)—(2·10), we have made an approximation

$$\begin{aligned}\langle N\nu\ N'\nu'|G^{(0)}|N_1\nu_1\ N_1\nu_1\rangle&=\langle N\nu\ N'\nu|G^{(0)}|N_1\nu_1\ N_1\nu_1\rangle\delta_{\nu\nu'}\\&\simeq 0\quad\text{for}\quad N\neq N',\end{aligned} \tag{2·14}$$

and used the relation

$$\begin{aligned}\langle N\nu|V|N'\nu'\rangle=\sum_{N_1\nu_1}\{&\langle N\ +\nu,\ N_1\ -\nu_1|G^{(1)}|N_1\ -\nu_1,\ N'\ +\nu'\rangle\\&-\langle N+\nu,\ N_1-\nu_1|G^{(1)}|N'\ +\nu',\ N_1\ -\nu_1\rangle+\langle N+\nu,\ N_1+\nu_1|G^{(1)}|N_1+\nu_1,\ N'+\nu'\rangle\\&-\langle N\ +\nu,\ N_1\ +\nu_1|G^{(1)}|N'\ +\nu',\ N_1\ +\nu_1\rangle\}v_{N_1\nu_1}^2,\end{aligned} \tag{2·15}$$

which means the definition of the self-consistent field V.

§ 3. Ground- and excited-states of $H^{(0)}$

Let us determine the transformation coefficients u_{Nm} and v_{Nm} in (2·4) by the condition

$$\langle c_0|\alpha_{N-\nu}\alpha_{N\nu}H^{(0)}|c_0\rangle=0, \tag{3·1}$$

where $|c_0\rangle$ is the vacuum of the quasi-particles α_{Nm}, i.e.

$$\alpha_{Nm}|c_0\rangle=0,\quad\langle c_0|\alpha_{Nm}^{+}=0. \tag{3·2}$$

The vacuum state $|c_0\rangle$ thus determined becomes the ground state of $H^{(0)}$ and cor-

responds to the "lowest seniority state" of even-even spherical nuclei.

The condition (3·1) means that $H_{20}=0$, i.e.

$$(\epsilon_N-\lambda)\cdot 2u_{N\nu}v_{N\nu}-\Delta_{N\nu}(u_{N\nu}^2-v_{N\nu}^2)=0. \tag{3·3}$$

Using (3·3), (2·6) and (2·11), we find the following equation for $\Delta_{N\nu}$:

$$\Delta_{N\nu}=\frac{1}{2}\sum_{N_1\nu_1}\frac{\langle N\nu\, N\nu|G^{(0)}|N_1\nu_1 N_1\nu_1\rangle}{\sqrt{(\epsilon_{N_1}-\lambda)^2+\Delta_{N_1\nu_1}^2}}\Delta_{N_1\nu_1}. \tag{3·4}$$

A. *Solutions of the equation for* $\Delta_{N\nu}$

Eq. (3·4) has a trivial solution

$$\Delta_{N\nu}=0 \quad \text{or} \quad u_{N\nu}v_{N\nu}=0, \tag{3·5}$$

which corresponds to the sharp Fermi surface. If $G^{(0)}$ is sufficiently weak, (3·5) is the only solution of (3·4). However, if $G^{(0)}$ satisfies the inequality

$$\frac{1}{2}\sum_{N_1\nu_1}\frac{\langle N\nu\, N\nu|G^{(0)}|N_1\nu_1 N_1\nu_1\rangle}{|\epsilon_{N_1}-\lambda|}>1, \tag{3·6}$$

then there exists also a non-trivial solution.

Hereafter we shall make the following approximations:

i) Since the most essential contribution to the sum in (3·4) is given by the transitions between the states in the same shell,* we shall neglect the transitions between different shells, i.e. we shall assume

$$\langle N\nu\, N\nu|G^{(0)}|N_1\nu_1 N_1\nu_1\rangle\simeq\langle N\nu\, N\nu|G^{(0)}|N\nu_1 N\nu_1\rangle\delta_{NN_1}. \tag{3·7}$$

ii) To simplify the problem, we shall assume further that the inequality (3·6) under (3·7) may be fulfilled only for the outermost partially-filled shell (the N_0-shell),** and the trivial solution (3·5) is the only solution for the other shells.***

iii) In order that the non-trivial solution exists for the N_0-shell, the matrix elements $\langle N_0\nu_0 N_0\nu_0|G^{(0)}|N_0\nu_0' N_0\nu_0'\rangle$ ("pairing interaction") must have a same sign, because otherwise there will occur a cancellation. So we shall approximate the matrix element $\langle N_0\nu_0 N_0\nu_0|G^{(0)}|N_0\nu_0' N_0\nu_0'\rangle$ by an average over the m_0-states, i.e.

$$\langle N_0\nu_0\, N_0\nu_0|G^{(0)}|N_0\nu_0'\, N_0\nu_0'\rangle\simeq\langle N_0\bar{\nu}_0\, N_0\bar{\nu}_0|G^{(0)}|N_0\nu_0'\, N_0\nu_0'\rangle=\overline{G^{(0)}}>0. \tag{3·8}$$

---

* As discussed by Belyaev,[4] transitions between different shells lead simply to some renormalization of $G^{(0)}$.

** Hereafter we shall denote the quantum numbers of the single-particle states in this outermost partially-filled shell by adding the suffix 0.

*** This assumption corresponds to that we are considering the spherical nuclei for which the level distances $|\epsilon_N-\epsilon_{N_0}|$ satisfy the condition

$$\overline{G_N^{(0)}}\,\Omega_N-\overline{G^{(0)}}\,\Omega(1-n/\Omega)<2(\epsilon_N-\epsilon_{N_0}) \quad \text{for} \quad \epsilon_N>\epsilon_{N_0},$$
$$\overline{G_N^{(0)}}\,\Omega_N+\overline{G^{(0)}}\,\Omega(1-n/\Omega)<2(\epsilon_{N_0}-\epsilon_N) \quad \text{for} \quad \epsilon_N<\epsilon_{N_0}.$$

Here $\overline{G_N^{(0)}}$ is an average of $\langle N\nu\, N\nu|G^{(0)}|N\nu' N\nu'\rangle$ over the m-states and $\Omega_N=(2j+1)/2$. $G^{(0)}$ and Ω are defined by (3·8) and (3·11) respectively, and n is the number of particles in the N_0-shell. For the nuclei for which such a condition is not satisfied, we must take into account that the inequality (3·6) under (3·7) may also be fulfilled for the shells near the N_0-shell.

Under these approximations the non-trivial solution, which exists only for the N_0-shell, is obtained by solving the equation

$$\Delta_{N_0}=\frac{1}{2}\overline{G^{(0)}}\sum_{\nu_0}\Delta_{N_0}/\sqrt{(\epsilon_{N_0}-\lambda)^2+\Delta_{N_0}^2}, \tag{3·9}$$

and becomes

$$\Delta_{N_0}=\{(\overline{G^{(0)}}^2\Omega^2/4)-(\epsilon_{N_0}-\lambda)^2\}^{1/2} \tag{3·10}$$

where

$$\Omega=(2j_0+1)/2. \tag{3·11}$$

B. *Elimination of the chemical potential* λ

The chemical potential λ is determined through the following equation:

$$\langle c_0|\sum_{m_0}a^+_{N_0m_0}a_{N_0m_0}|c_0\rangle=\sum_{\nu_0}2v^2_{N_0\nu_0}=\sum_{\nu_0}\{1-(\epsilon_{N_0}-\lambda)/\sqrt{(\epsilon_{N_0}-\lambda)^2+\Delta^2_{N_0}}\}=n, \tag{3·12}$$

where n is the number of particles in the N_0-shell, and is given by

$$\lambda=\epsilon_{N_0}-(\overline{G^{(0)}}\Omega/2)(1-n/\Omega). \tag{3·13}$$

By the use of (3·13), λ can be eliminated from all final results. The results in which λ is eliminated are summarized as follows:

$$\Delta_{N\nu}=\Delta_N=\begin{cases}(\overline{G^{(0)}}\Omega/2)\{1-(1-n/\Omega)^2\}^{1/2} & \text{for the } N_0\text{-shell.}\\ 0 & \text{for the shells other than the } N_0\text{-shell.}\end{cases} \tag{3·14}$$

$$u_{Nm}=u_{N-m}=u_N$$

$$v_{Nm}=-v_{N-m}=\begin{cases}+v_N & m>0\\ -v_N & m<0,\end{cases}$$

$$u_N=\begin{cases}(1-n/2\Omega)^{1/2} & \text{for the } N_0\text{-shell.}\\ 0 & \text{for the closed occupied shells.}\\ 1 & \text{for the open unoccupied shells.}\end{cases} \tag{3·15}$$

$$v_N=\begin{cases}(n/2\Omega)^{1/2} & \text{for the } N_0\text{-shell.}\\ 1 & \text{for the closed occupied shells.}\\ 0 & \text{for the open unoccupied shells.}\end{cases}$$

C. *Expression of* $H^{(0)}$ *and excited states*

With the aid of (3·14) and (3·15), $H^{(0)}$ becomes

$$H^{(0)}=U+\sum_{Nm}E_N\alpha^+_{Nm}\alpha_{Nm}, \tag{3·16}$$

where U is given by (2·8) and means the energy of the ground state (the vacuum state $|c_0\rangle$) and

$$E_N=\sqrt{(\epsilon_N-\lambda)^2+\Delta_N^2}$$

$$=\sqrt{\{\epsilon_N-\epsilon_{N_0}+(\overline{G^{(0)}}\Omega/2)(1-n/\Omega)\}^2+\Delta_N^2}. \tag{3·17}$$

The excited states of $H^{(0)}$ are characterized by those with the quasi-particles. It is well known that the excited states of even-even nuclei contain an even number of quasi-particles, e.g., for the ground state the number of quasi-particles is zero (the vacuum state $|c_0\rangle$) and the first excited states are described in terms of two quasi-particles in the N_0-shell, etc. The number of quasi-particles in the N_0-shell is thus a generalized seniority number.

§ 4. Phenomenological analysis of the two-body interaction G

We now come to the stage to discuss the possibility of the division of G into (2·2). Recently discussions on this possibility were made by many authors.[9),10)] Here we shall follow Weisskopf's argument.[10)]

Let us expand the two-body interaction $G(x_1, x_2)$ in spherical harmonics:*

$$G(x_1, x_2)=\sum_L G_L(r_1 r_2)\, Y_{L0}(\theta_{12})=\sum_L \sqrt{\frac{4\pi}{2L+1}}\, G_L(r_1 r_2) \sum_{M=-L}^{+L} Y_{LM}(\theta_1, \varphi_1) \times Y^*_{LM}(\theta_2, \varphi_2). \tag{4·1}$$

There is a close connection between the above sum and the division in (2·2). The spherically symmetrical self-consistent field V is determined mainly by the $L=0$ term in (4·1). This means that $G^{(1)}$ corresponds to the $L=0$ term. On the other hand, the inequality (3·6) under the approximation (3·7) suggests that $G^{(0)}$ is mainly composed of the high harmonics in (4·1).[4)] We may thus conclude that the $L=0$ term corresponds to $G^{(1)}$ and the sum over $L=1$, 2 and 3 give rise to $G^{(2)}$ and the sum over $L=4, 5, \cdots$ give rise to $G^{(0)}$. We may call $G^{(1)}$ and $G^{(2)}$ the "long range" part of G and call $G^{(0)}$ the "short range" part of G, though it is not really accurate.

The $L=1$ term is not really important, because its main contribution is only a displacement of the center of mass. The interesting terms composing $G^{(2)}$, thus, are those with $L=2$ and $L=3$.

It is the main purpose of this paper to investigate how such terms bring about the quadrupole- and octupole-vibrations of spherical nuclei. In the following, our main attention will be paid especially on the $L=2$ term. Then, $H^{(\mathrm{res.})}$ in (2·3c) becomes

$$H^{(\mathrm{res.})}=-\frac{1}{2}\sum_{(Nm)}\langle N_1 m_1 N_2 m_2|G^{(2)}|N_2' m_2' N_1' m_1'\rangle a^+_{N_1 m_1} a^+_{N_2 m_2} a_{N_2' m_2'} a_{N_1' m_1'}$$

$$=-\frac{1}{2}\sum_{M=-2}^{+2}\sum_{(Nm)}\langle N_1 N_2|l_2|N_2' N_1'\rangle(-1)^{2j_1'+2j_2'-1}(-1)^M(-1)^{-m_1'}(j_1 j_1' m_1-m_1'|2M) \times(-1)^{-m_2'}(j_2 j_2' m_2-m_2'|2-M)\, a^+_{N_1 m_1} a^+_{N_2 m_2} a_{N_2' m_2'} a_{N_1' m_1'}, \tag{4·2}$$

where

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* For simplicity, we neglect the spin dependence of G.

$$\langle N_1 N_2|g_2|N_2' N_1'\rangle \equiv \frac{1}{2L+1}\sqrt{\frac{(2j_1+1)(2j_1'+1)(2j_2+1)(2j_2'+1)}{4\pi(2L+1)}}$$
$$\times\left(j_1 j_1'\frac{1}{2}-\frac{1}{2}|L0\right)\left(j_2 j_2'\frac{1}{2}-\frac{1}{2}|L0\right)\langle n_1 l_1 n_2 l_2|G_L|n_2' l_2' n_1' l_1'\rangle\Big|_{L=2}. \tag{4·3}$$

Here $\langle n_1 l_1 n_2 l_2|G_L|n_2' l_2' n_1' l_1'\rangle$ means the radial integral part of the L-th term in (4·1) and $(jj'm\,m'|LM)$ denotes the Clebsch-Gordan coefficient.

§ 5. Concept of nuclear collective motion and the time-dependent self-consistent field method

In order to make clear the physical meaning of our method which will be developed in the next section, we shall, in this section, start from the general consideration of the nuclear collective motion.

As is well known, the remarkable success of the nuclear shell model has suggested that the main part of the inter-nucleon interaction can be treated as a spherically symmetrical self-consistent field V. The essential point of the nuclear collective model is to assume that an additional self-consistent field may be extracted from the remaining part of the interaction. Bohr and Mottelson,[8] and Hill and Wheeler[11] have suggested that such an additional self-consistent field is non-spherical and time-dependent, and have grasped the time-variation of the additional self-consistent field as the nuclear collective motion.

Such a fundamental picture on the nuclear collective motion is directly formulated in the framework of the time-dependent self-consistent field method (the TDSCF-method). The discussions of the nuclear collective motion in the framework of the TDSCF-method were made by Nogami[12] and Ferrell[13] for the vibrational motion of closed shell nuclei and by Shono and Tanaka[14] for the rotational motion. In this section, we shall recapitulate the TDSCF-method in a form convenient for later discussions.

Let us consider the Hamiltonian (2·3a) in the case of $G^{(0)}=0$:

$$H=H^{(0)}+H^{(\mathrm{res.})}$$
$$=\sum_{Nm}\epsilon_N a_{Nm}^{+}a_{N'm'}-\frac{1}{2}\sum_{(Nm)}\langle N_1 m_1 N_2 m_2|G^{(2)}|N_2' m_2' N_1' m_1'\rangle a_{N_1 m_1}^{+}a_{N_2 m_2}^{+}a_{N_2' m_2'}a_{N_1' N_1'} \tag{5·1}$$

where the second term is given by (4·2). Here we omit the chemical potential λ. In the coordinate representation, (5·1) is

$$H=\int\psi^{+}(x)\left(\frac{1}{2m}p^2-V(x)\right)\psi(x)dx$$
$$-\frac{1}{2}\int\psi^{+}(x_1)\psi^{+}(x_2)\{G_2(r_1 r_2)Y_{20}(\theta_{12})\}\psi(x_2)\psi(x_1)dx_1 dx_2,^{*} \tag{5·1'}$$

* The spin variable is assumed to be included in the coordinate x.

where $\psi(x)=\sum_{Nm} a_{Nm}\psi_{Nm}(x)$ and $\psi_{Nm}(x)$ are the single-particle wave functions in the spherically symmetrical self-consistent potential V.

The time derivative of $\psi(x)$ is expressed by

$$i\hbar\dot{\psi}(x)\equiv[\psi(x),\ H]=\mathsf{H}(p,\ x)\psi(x), \tag{5·2}$$

where

$$\mathsf{H}(p_1,\ x_1)=\frac{1}{2m}p_1^2-V(x_1)-\int dx_2\rho(x_2,\ x_2)\{G_2(r_1r_2)\,Y_{20}(\theta_{12})\}$$

$$\equiv\frac{1}{2m}p_1^2-V(x_1)-\boldsymbol{U}(x_1). \tag{5·3}$$

Here the density matrix operator $\rho(x,\ x')$ is defined by

$$\rho(x,\ x')=\psi^+(x')\psi(x)=\sum_{\substack{Nm\\N'm'}} a^+_{Nm}a_{N'm'}\psi_{N'm'}(x)\psi^*_{Nm}(x') \tag{5·4}$$

and the time derivative of ρ is given by

$$i\hbar\dot{\rho}\equiv[\mathsf{H},\ \rho]. \tag{5·5}$$

The expectation value of ρ at time t is

$$\rho(t)=\langle\Psi(t)|\rho|\Psi(t)\rangle, \tag{5·6}$$

where $|\Psi(t)\rangle$ is a state vector of H at time t, (and is not that of $H^{(0)}$). Using (5·5) we have the equation of motion for $\rho(t)$:

$$i\hbar\frac{\partial\rho(t)}{\partial t}=\langle\Psi(t)|[\mathsf{H},\ \rho]|\Psi(t)\rangle. \tag{5·7}$$

If we make an approximation which is essential to the Hartree method :

$$\langle\Psi(t)|[\mathsf{H},\ \rho]|\Psi(t)\rangle\simeq[\langle\Psi(t)|\mathsf{H}|\Psi(t)\rangle,\ \rho(t)], \tag{5·8}$$

we obtain the fundamental equation in the TDSCF-method :

$$i\hbar\frac{\partial\rho(t)}{\partial t}=[H(t),\ \rho(t)], \tag{5·9}$$

where

$$H(t)\equiv\langle\Psi(t)|\mathsf{H}|\Psi(t)\rangle=\frac{1}{2m}p^2-V(x)-U(x,\ t). \tag{5·10}$$

In the TDSCF-method, Eq. (5·9) is simplified through the following procedures.

i) $\rho(t)$ is expanded in the form

$$\rho(t)=\rho^{(0)}+\rho^{(1)}(t). \tag{5·11}$$

Here $\rho^{(0)}$ is the unperturbed density matrix and is equivalent to

$$\rho^{(0)}=\langle c_0|\rho|c_0\rangle \tag{5·12}$$

where $|c_0\rangle$ is the ground state of $H^{(0)}$, and $\rho^{(1)}$ is the density matrix which should be related to the time dependent self-consistent field $U(x, t)$:

$$U(x_1, t)=\int dx_2 \rho^{(1)}(x_2, x_2; t) G_2(r_1 r_2) Y_{20}(\theta_{12}). \tag{5·13}$$

ii) The first order self-consistent perturbation theory is used, which means that the products of the type $U\rho^{(1)}$ are neglected. Under this approximation, Eq. (5·9) is linearized and becomes

$$i\hbar \frac{\partial \rho^{(1)}}{\partial t}=\left[\frac{1}{2m}p^2-V, \rho^{(1)}\right]-[U, \rho^{(0)}]. \tag{5·14}$$

The main problem of the TDSCF-method is, as is well known, to solve Eq. (5·14) and to find a characteristic frequency corresponding to the collective vibration.

§ 6. Description of the nuclear collective vibration in the framework of second quantization

In this section we shall develop a method describing the nuclear collective vibration in the framework of second quantization. The essential idea of this method is based on an extension of the theory of Sawada et al.[6] of the plasma oscillation, and has been suggested by many authors.[15),16),17),1),2)]

Keeping a direct comparison with the TDSCF-method, we shall develop this idea in this section. In order to avoid unnecessary complications and to emphasize the physical implication of the method, we shall, however, illustrate here the fundamental idea by taking closed shell nuclei. The generalization of the method will be discussed in § 7.

We shall, at first, begin with rewriting Eq. (5·14) in a form more suitable for our purpose. Noting that for closed shell nuclei

$$\left.\begin{aligned}&\langle Nm|\rho^{(0)}|N'm'\rangle=\delta_{NN'}\delta_{mm'}\theta_N,\\&\langle Nm|\rho^{(1)}|N'm'\rangle=(1-\theta_N)\theta_{N'}\langle Nm|\rho^{(1)}|N'm'\rangle+(1-\theta_{N'})\theta_N\langle Nm|\rho^{(1)}|N'm'\rangle,^*\end{aligned}\right\} \tag{6·1}$$

let us introduce the following quantities with a definite angular momentum $(L=2, M)$:

$$\left.\begin{aligned}\overline{C^{+}_{2M}(NN')}&=\sum_{mm'}(-1)^{+m'}(jj'm-m'|2M)(1-\theta_N)\theta_{N'}\langle N'm'|\rho^{(1)}|Nm\rangle,\\\overline{C_{2M}(NN')}&=\sum_{mm'}(-1)^{-m'}(jj'm-m'|2M)(1-\theta_N)\theta_{N'}\langle Nm|\rho^{(1)}|N'm'\rangle.\end{aligned}\right\} \tag{6·2}$$

Here $\langle Nm|\rho|N'm'\rangle$ is given by

* In the TDSCF-method such a presupposition on the matrix elements of $\rho^{(1)}$ is usually made, in order to solve Eq. (5·14) and to find a characteristic frequency corresponding to the collective vibration.

$$\langle Nm|\rho|N'm'\rangle=\int\psi^*_{Nm}(x)\rho(x,x')\psi_{N'm'}(x')dxdx'$$

and θ_N is defined as follows:

$$\theta_N=\begin{cases}1 & \text{for the closed occupied shells.}\\ 0 & \text{for the open unoccupied shells.}\end{cases}\qquad(6\cdot3)$$

Using (6·1), (6·2) and (5·13), we find, after some calculations, that Eq. (5·14) becomes

$$i\hbar\frac{\partial}{\partial t}\overline{C^+_{2M}(NN')}=(\epsilon_{N'}-\epsilon_N)\overline{C^+_{2M}(NN')}$$
$$+\sum_{N_1N_1'}\langle NN_1|g_2|N_1'N'\rangle(1-\theta_N)\theta_{N'}\{\overline{C^+_{2M}(N_1N_1')}-(-1)^M\overline{C_{2-M}(N_1N_1')}\},\qquad(6\cdot4)$$

where $\langle N_1N_2|g_2|N_2'N_1'\rangle$ is given by (4·3) and has the property

$$\langle N_1N_2|g_2|N_2'N_1'\rangle=\langle N_1'N_2|g_2|N_2'N_1\rangle=\langle N_1N_2'|g_2|N_2N_1'\rangle.\qquad(6\cdot5)$$

The equation equivalent to (6·4) is also obtained in the framework of second quantization in the following way.

i) We pick up the following particle-hole pair terms from $H^{(\mathrm{res.})}$ in (5·1):

$$H^{(1)}=-\frac{1}{2}\sum_{(Nm)}(N_1m_1N_2m_2|G^{(2)}|N_2'm_2'N_1'm_1')\{(1-\theta_{N_1})a^+_{N_1m_1}\cdot\theta_{N_1'}a_{N_1'm_1'}$$
$$+\theta_{N_1}a^+_{N_1m_1}\cdot(1-\theta_{N_1'})a_{N_1'm_1'}\}\{(1-\theta_{N_2})a^+_{N_2m_2}\cdot\theta_{N_2'}a_{N_2'm_2'}+\theta_{N_2}a^+_{N_2m_2}\cdot(1-\theta_{N_2'})a_{N_2'm_2'}\}$$
$$=-\frac{1}{2}\sum_{M=-2}^{+2}\sum_{(N)}\langle N_1N_2|g_2|N_2'N_1'\rangle\{C^+_{2M}(N_1N_1')-(-1)^MC_{2-M}(N_1N_1')\}$$
$$\times\{C_{2M}(N_2N_2')-(-1)^MC^+_{2-M}(N_2N_2')\},\qquad(6\cdot6)$$

where

$$C^+_{2M}(NN')=\sum_{mm'}(-1)^{+m'}(jj'm-m'|2M)(1-\theta_N)\theta_{N'}a^+_{Nm}a_{N'm'},$$
$$C_{2M}(NN')=\sum_{mm'}(-1)^{-m'}(jj'm-m'|2M)(1-\theta_N)\theta_{N'}a^+_{N'm'}a_{Nm}.\qquad(6\cdot7)$$

ii) Commutation relations between C^+_{2M} and $C_{2M'}$ are approximated by the Boson-like commutation relations

$$\left.\begin{aligned}[C_{2M}(N_1N_1'),\ C^+_{2M'}(N_2N_2')]&\simeq[C_{2M}(N_1N_1'),\ C^+_{2M'}(N_2N_2')]|c_0\rangle\\ &=\delta_{MM'}\delta_{N_1N_2}\delta_{N_1'N_2'}(1-\theta_{N_1})\theta_{N_1'},\\ [C_{2M}(N_1N_1'),\ C_{2M'}(N_2N_2')]&=[C^+_{2M}(N_1N_1'),\ C^+_{2M'}(N_2N_2')]=0.\end{aligned}\right\}\qquad(6\cdot8)$$

Using (6·8), we find that

$$i\hbar\dot{C}^+_{2M}(NN')\equiv[C^+_{2M}(NN'),\ H^{(0)}+H^{(1)}]=(\epsilon_{N'}-\epsilon_N)C^+_{2M}(NN')$$
$$+\sum_{N_1N_1'}\langle NN_1|g_2|N_1'N'\rangle(1-\theta_N)\theta_{N'}\{C^+_{2M}(N_1N_1')-(-1)^MC_{2-M}(N_1N_1')\}.\qquad(6\cdot9)$$

Now it is clear that Eq. (6·9) is equivalent to Eq. (6·4) in the TDSCF-method.* From this fact we may conclude that the approximations made in i) and ii) correspond to those of (5·8) and ii) of § 5 in the TDSCF-method.

So far we have considered the simplified system with the Hamiltonian (5·1) where $G^{(0)}=0$ is assumed. A particular advantage of the present formulation in the framework of second quantization is that the extension of the method to the case of $G^{(0)}\neq 0$ is very easy. In the following, we shall use this method and shall investigate the collective motion in the system with the Hamiltonian (2·3).

§ 7. Reduction of the Hamiltonian

Let us consider the Hamiltonian (2·3) and perform the Bogoliubov transformation (2·4). We then obtain the transformed Hamiltonian, $H=H^{(0)}+H^{(\mathrm{res.})}$, where $H^{(0)}$ is given by (3·16) while

$$\begin{aligned} H^{(\mathrm{res.})}=-\frac{1}{2}\sum_{(Nm)}\langle N_1 m_1 N_2 m_2|G^{(2)}|N_2' m_2' N_1' m_1'\rangle\{u_{N_1 m_1}\alpha^{+}_{N_1 m_1}+v_{N_1 m_1}\alpha_{N_1-m_1}\} \\ \times\{u_{N_2 m_2}\alpha^{+}_{N_2 m_2}+v_{N_2 m_2}\alpha_{N_2-m_2}\}\{u_{N_2' m_2'}\alpha_{N_2' m_2'} \\ +v_{N_2' m_2'}\alpha^{+}_{N_2'-m_2'}\}\{u_{N_1' m_1'}\alpha_{N_1' m_1'}+v_{N_1' m_1'}\alpha^{+}_{N_1'-m_1'}\}, \end{aligned} \tag{7·1}$$

u_{Nm} and v_{Nm} being given by (3·15).

In order to extend the method developed in § 6 to the present system, we now introduce the operators

$$\left.\begin{aligned} C^{+}_{2M}(NN')&=\sum_{mm'}(-1)^{+m'}(jj'm-m'|2M)u_{Nm}v_{N'm'}\alpha^{+}_{Nm}\alpha^{+}_{N'-m'} \\ C_{2M}(NN')&=\sum_{mm'}(-1)^{-m'}(jj'm-m'|2M)u_{Nm}v_{N'm'}\alpha_{N'-m'}\alpha_{Nm}, \end{aligned}\right\} \tag{7·2}$$

and pick up the following quasi-particle pair terms from (7·1):

$$\begin{aligned} H^{(\mathrm{I})}=-\frac{1}{2}\sum_{M=-2}^{+2}\sum_{(N)}\langle N_1 N_2|g_2|N_2' N_1'\rangle\{C^{+}_{2M}(N_1 N_1')-(-1)^{M}C_{2-M}(N_1 N_1')\} \\ \times\{C_{2M}(N_2 N_2')-(-1)^{M}C^{+}_{2-M}(N_2 N_2')\}. \end{aligned} \tag{7·3}$$

For closed shell nuclei in the special case of $G^{(0)}=0$, we get, with the aid of (2·4) and (3·15),

$$\left.\begin{aligned} u_{Nm}\alpha^{+}_{Nm}&=(1-\theta_N)a^{+}_{Nm},\quad v_{Nm}\alpha^{+}_{N-m}=\theta_N a_{Nm}, \\ u_{Nm}\alpha_{Nm}&=(1-\theta_N)a_{Nm},\quad v_{Nm}\alpha_{N-m}=\theta_N a^{+}_{Nm}, \end{aligned}\right\} \tag{7·4}$$

and the operators in (7·2) are reduced to those in (6·7). The operators in (7·2), therefore, corresponds to the generalization of those in (6·7), and the generalization of (6·6) corresponds to (7·3).

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* In the course of performing this work, it has come to the author's notice that Ehrenreich and Cohen[18] have made a similar proof for the case of the many-electron system.

Now we decompose C^{+}_{2M} and C_{2M} in the following way:

$$\left.\begin{aligned} &C^{+}_{2M}(NN') = C^{(\mathrm{d})+}_{2M}(NN') + C^{(\mathrm{nd})+}_{2M}(NN'), \\ &C^{(\mathrm{d})+}_{2M}(NN') \equiv C^{+}_{2M}(NN')\delta_{NN'},\ C^{(\mathrm{nd})+}_{2M}(NN') \equiv C^{+}_{2M}(NN')(1-\delta_{NN'}). \end{aligned}\right\} \tag{7·5}$$

It follows from (3·15) that $C^{(\mathrm{d})+}_{2M}$ and $C^{(\mathrm{d})}_{2M}$ exist only for the *partially filled** N_0-shell and vanish for the other shells. It is also clear from (3·5) that, in the absence of the pairing interaction ($G^{(0)}=0$), $C^{(\mathrm{d})+}_{2M}$ and $C^{(\mathrm{d})}_{2M}$ vanish. Thus we are able to rewrite $H^{(1)}$ in (7·3) as

$$\begin{aligned} H^{(1)} = &-\frac{1}{2}\sum_{M=-2}^{+2}\sum_{(N)}\langle N_1 N_2|g_2|N_2' N_1'\rangle\{C^{(\mathrm{nd})+}_{2M}(N_1N_1') - (-1)^M C^{(\mathrm{nd})}_{2-M}(N_1N_1')\} \\ &\times\{C^{(\mathrm{nd})}_{2M}(N_2N_2') - (-1)^M C^{(\mathrm{nd})+}_{2-M}(N_2N_2')\} \\ &-\frac{1}{2}\sum_{M=-2}^{+2}\langle N_0 N_0|g_2|N_0 N_0\rangle\{C^{(\mathrm{d})+}_{2M}(N_0N_0) - (-1)^M C^{(\mathrm{d})}_{2-M}(N_0N_0)\} \\ &\times\{C^{(\mathrm{d})}_{2M}(N_0N_0) - (-1)^M C^{(\mathrm{d})+}_{2-M}(N_0N_0)\} \\ &-\frac{1}{2}\sum_{M=-2}^{+2}\sum_{NN'}[\langle N_0 N|g_2|N' N_0\rangle\{C^{(\mathrm{d})+}_{2M}(N_0N_0) - (-1)^M C^{(\mathrm{d})}_{2-M}(N_0N_0)\} \\ &\times\{C^{(\mathrm{nd})}_{2M}(NN') - (-1)^M C^{(\mathrm{nd})+}_{2-M}(NN')\} + \text{herm. conj.}]. \end{aligned} \tag{7·6}$$

In order to investigate the collective motion in the system with the Hamiltonian (2·3), we shall hereafter use the reduced Hamiltonian

$$H_{\mathrm{R}} = H^{(0)} + H^{(1)}, \tag{7·7}$$

where $H^{(0)}$ is given by (3·16) and $H^{(1)}$ is given by (7·6). Furthermore we shall hereafter employ the following commutation relations corresponding to the generalization of (6·8):

$$\left.\begin{aligned} &[C^{(\mathrm{nd})}_{2M}(N_1N_1'),\ C^{(\mathrm{nd})+}_{2M'}(N_2N_2')] \simeq [C^{(\mathrm{nd})}_{2M}(N_1N_1'),\ C^{(\mathrm{nd})+}_{2M'}(N_2N_2')]|c_0\rangle \\ &\qquad = \delta_{MM'}\delta_{N_1N_2}\delta_{N_1'N_2'}(u^2_{N_1}v^2_{N_1'})(1-\delta_{N_1N_1'}), \\ &[C^{(\mathrm{nd})}_{2M}(N_1N_1'),\ C^{(\mathrm{nd})}_{2M'}(N_2N_2')] = [C^{(\mathrm{nd})+}_{2M}(N_1N_1'),\ C^{(\mathrm{nd})+}_{2M'}(N_2N_2')] = 0, \end{aligned}\right\} \tag{7·8a}$$

$$\left.\begin{aligned} &[C^{(\mathrm{d})}_{2M}(N_0N_0),\ C^{(\mathrm{d})+}_{2M'}(N_0N_0)] \simeq [C^{(\mathrm{d})}_{2M}(N_0N_0),\ C^{(\mathrm{d})+}_{2M'}(N_0N_0)]|c_0\rangle \\ &\qquad = 2\delta_{MM'}(u^2_{N_0}v^2_{N_0}), \\ &[C^{(\mathrm{d})}_{2M}(N_0N_0),\ C^{(\mathrm{d})}_{2M'}(N_0N_0)] = [C^{(\mathrm{d})+}_{2M}(N_0N_0),\ C^{(\mathrm{d})+}_{2M'}(N_0N_0)] = 0, \end{aligned}\right\} \tag{7·8b}$$

$$\left.\begin{aligned} &[C^{(\mathrm{d})}_{2M}(N_0N_0),\ C^{(\mathrm{nd})+}_{2M'}(NN')] = [C^{(\mathrm{d})}_{2M}(N_0N_0),\ C^{(\mathrm{nd})}_{2M'}(NN')] = 0, \\ &[C^{(\mathrm{d})+}_{2M}(N_0N_0),\ C^{(\mathrm{nd})}_{2M'}(NN')] = [C^{(\mathrm{d})+}_{2M}(N_0N_0),\ C^{(\mathrm{nd})+}_{2M'}(NN')] = 0. \end{aligned}\right\} \tag{7·8c}$$

§ 8. Two modes of collective motion

The commutation relations (7·8) suggest that there exist two modes of col-

* It is easily seen from (3·15) that $C_{2M}{}^{(\mathrm{d})+}$ and $C_{2M}{}^{(\mathrm{d})}$ vanish when the N_0-shell is closed.

lective motion: One mode is connected with $C^{(\mathrm{ind})+}_{2M}$ and $C^{(\mathrm{ind})+}_{2M}$, and another is connected with $C^{(\mathrm{d})}_{2M}$ and $C^{(\mathrm{d})+}_{2M}$. In this section we shall discuss these two modes separately. For this purpose we now rewrite the reduced Hamiltonian (7·7) as

$$H_{\mathrm{R}}=H^{(0)}_{\mathrm{R}}+H'_{\mathrm{R}} \tag{8·1}$$

where

$$H'_{\mathrm{R}}=-\frac{1}{2}\sum_{M=-2}^{+2}\sum_{NN'}[\langle N_0 N|g_2|N' N_0\rangle\{C^{(\mathrm{d})+}_{2M}(N_0 N_0)-(-1)^M C^{(\mathrm{d})}_{2-M}(N_0 N_0)\}$$
$$\times\{C^{(\mathrm{ind})}_{2M}(NN')-(-1)^M C^{(\mathrm{ind})+}_{2-M}(NN')\}+\text{herm. conj.}], \tag{8·2}$$

and consider the system with the Hamiltonian $H^{(0)}_{\mathrm{R}}$ for the moment. The term H'_{R} will be taken into account later on in § 9.

A. *Collective mode connected with* $C^{(\mathrm{ind})}_{2M}$ *and* $C^{(\mathrm{ind})+}_{2M}$

Using (7·8a) and (7·8c), we get the equation corresponding to Eq. (6·9):

$$[C^{(\mathrm{ind})+}_{2M}(NN'),\ H^{(0)}_{\mathrm{R}}]=-(E_{N'}+E_N)C^{(\mathrm{ind})+}_{2M}(NN')$$
$$+\sum_{N_1'N_1'}\langle NN_1|g_2|N_1' N'\rangle(u^2_N v^2_{N'})(1-\delta_{NN'})\{C^{(\mathrm{ind})+}_{2M}(N_1 N_1')$$
$$-(-1)^M C^{(\mathrm{ind})}_{2-M}(N_1 N_1')\}, \tag{8·3}$$

where E_N is given by (3·17).

Now let us introduce an operator

$$B^+_{2M}=\sum_{NN'}a(NN')C^{(\mathrm{ind})+}_{2M}(NN')-\sum_{NN'}b(NN')(-1)^M C^{(\mathrm{ind})}_{2-M}(NN'), \tag{8·4}$$

where $a(NN')$ and $b(NN')$ are determined by the following simultaneous equations:

$$(E-E_0)a(NN')=(E_N+E_{N'})a(NN')-\sum_{N_1N_1'}a(N_1N_1')\langle NN_1|g_2|N_1'N'\rangle$$
$$\times(u^2_{N_1}v^2_{N_1'})(1-\delta_{N_1N_1'})+\sum_{N_1N_1'}b(N_1N_1')\langle NN_1|g_2|N_1'N'\rangle(u^2_{N_1}v^2_{N_1'})(1-\delta_{N_1N_1'}),$$
$$(E-E_0)b(NN')=-(E_N+E_{N'})b(NN')-\sum_{N_1N_1'}a(N_1N_1')\langle NN_1|g_2|N_1'N'\rangle$$
$$\times(u^2_{N_1}v^2_{N_1'})(1-\delta_{N_1N_1'})+\sum_{N_1N_1'}b(N_1N_1')\langle NN_1|g_2|N_1'N'\rangle(u^2_{N_1}v^2_{N_1'})(1-\delta_{N_1N_1'}), \tag{8·5}$$

and the normalization condition:

$$[B_{2M},\ B^+_{2M}]=1=\sum_{NN'}a(NN')^2(u^2_N v^2_{N'})(1-\delta_{NN'})$$
$$-\sum_{NN'}b(NN')^2(u^2_N v^2_{N'})(1-\delta_{NN'}). \tag{8·6}$$

Then it is easily shown with the aid of (8·3) that B^+_{2M} satisfies

$$[H^{(0)}_{\mathrm{R}},\ B^+_{2M}]=(E-E_0)B^+_{2M}.$$

This means that B^+_{2M} is the operator which creates an eigenstate $|\Psi\rangle$ with energy E from the ground state $|\Psi_0\rangle$ of $H^{(0)}_{\mathrm{R}}$ with energy E_0:

$$|\Psi\rangle = B^{+}_{2M}|\Psi_0\rangle.$$

We do not discuss here Eq. (8·5) in detail, and note only the following. If $\langle NN_1|g_2|N_1'N'\rangle$ are separable, i.e. $\langle NN_1|g_2|N_1'N'\rangle = \langle N|g_2|N'\rangle\langle N_1|g_2|N_1'\rangle$,* we obtain the eigenvalue equation from (8·5) :[1)]

$$1 = \sum_{NN'}{}' \left\{ \frac{1}{(E-E_0)+(E_{N'}+E_N)} - \frac{1}{(E-E_0)-(E_{N'}+E_N)} \right\} (u_N^2 v_{N'}^2)(1-\delta_{NN'}) \times \langle NN|g_2|N'N'\rangle. \tag{8·7}$$

This equation has a root $(E-E_0)$ which is real and positive and is smaller than the smallest pair (quasi-particle) excitation energy $(E_{N'}+E_N)$ with $N \neq N'$, provided that the condition

$$1 > \sum_{NN'}{}' \frac{2\langle NN|g_2|N'N'\rangle (u_N^2 v_{N'}^2)(1-\delta_{NN'})}{E_{N'}+E_N} \tag{8·8}$$

is satisfied. The state with such a lowest excitation energy, $(E-E_0)_{\min.} \equiv \hbar\omega_\beta$, can be regarded as that of the collective motion connected with $C^{(1,d)}_{2M}$ and $C^{(1,d)+}_{2M}$. In this case, $a(NN')$ and $b(NN')$ are given by

$$a(NN') = \frac{K\langle N|g_2|N'\rangle}{\hbar\omega_\beta - (E_N+E_{N'})}, \quad b(NN') = \frac{K\langle N|g_2|N'\rangle}{\hbar\omega_\beta + (E_N+E_{N'})}, \tag{8·9}$$

where the normalization constant K is determined by (8·6), i.e.

$$K = 1\Big/\Bigg[\sum_{NN'}{}' \frac{\langle NN|g_2|N'N'\rangle u_N^2 v_{N'}^2 (1-\delta_{NN'})}{\{\hbar\omega_\beta - (E_N+E_{N'})\}^2} - \sum_{NN'}{}' \frac{\langle NN|g_2|N'N'\rangle u_N^2 v_{N'}^2 (1-\delta_{NN'})}{\{\hbar\omega_\beta + (E_N+E_{N'})\}^2} \Bigg]^{1/2}. \tag{8·10}$$

B. *Collective mode connected with* $C^{(d)}_{2M}$ *and* $C^{(d)+}_{2M}$

In this case, we obtain the following equation corresponding to Eq. (6·9) :

$$[C^{(d)+}_{2M}(N_0N_0),\ H^{(0)}_{R}] = -2E_{N_0}C^{(d)+}_{2M}(N_0N_0) + 2\langle N_0N_0|g_2|N_0N_0\rangle (u^2_{N_0}v^2_{N_0})\{C^{(d)+}_{2M}(N_0N_0) - (-1)^M C^{(d)}_{2-M}(N_0N_0)\}. \tag{8·11}$$

Now let us consider the matrix element, $\langle\Psi'|(E'-H^{(0)}_{R})C^{(d)+}_{2M}(N_0N_0)|\Psi_0\rangle$, where $|\Psi'\rangle$ is an eigenstate of $H^{(0)}_{R}$ with energy E' and $|\Psi_0\rangle$ is the ground state of $H^{(0)}_{R}$. This vanishes, but can also be written as

$$0 = (E'-E_0)\langle\Psi'|C^{(d)+}_{2M}(N_0N_0)|\Psi_0\rangle - \langle\Psi'|[H^{(0)}_{R},\ C^{(d)+}_{2M}(N_0N_0)]|\Psi_0\rangle. \tag{8·12a}$$

Similarly, we find

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* The "quadrupole-quadrupole interaction" adopted by Elliott[19)] and Moszkowski[20)] is an example of such a type.

$$0=(E'-E_0)\langle\Psi'|(-1)^M C^{(\nu)}_{2-M}(N_0N_0)|\Psi_0\rangle-\langle\Psi'|[H^{(0)}_{\rm R},\ (-1)^M C^{(\nu)}_{2-M}(N_0N_0)]|\Psi_0\rangle. \tag{8·12b}$$

Using (8·11) and combining (8·12a) and (8·12b), we are led to the eigenvalue equation

$$1=\left\{\frac{1}{\hbar\omega_\eta+2E_{N_0}}-\frac{1}{\hbar\omega_\eta-2E_{N_0}}\right\}2\langle N_0N_0|g_2|N_0N_0\rangle u^2_{N_0}v^2_{N_0}, \tag{8·13}$$

where $\hbar\omega_\eta\equiv E'-E_0$ and $E_{N_0}=\sqrt{(\epsilon_{N_0}-\lambda)^2+\Delta^2_{N_0}}=\frac{1}{2}G^{(0)}\Omega$.

Solving Eq. (8·13), we obtain

$$\hbar\omega_\eta=G^{(0)}\Omega\sqrt{1-\theta_n/\theta'_{n_0}}, \tag{8·14}$$

where

$$\theta_n=\{1-(1-n/\Omega)^2\}, \tag{8·15a}$$

$$\theta'_{n_0}=G^{(0)}\Omega/4\langle N_0N_0|g_2|N_0N_0\rangle. \tag{8·15b}$$

θ_n is the occupation factor used by Belyaev (see ref. 4), Eq. (67)) and θ'_{n_0} represents a ratio of the strength of $G^{(0)}$ to that of $G^{(2)}$.

For the collective mode under consideration, the creation operator $B^{(\eta)+}_{2M}$, which leads to

$$|\Psi'\rangle=B^{(\eta)+}_{2M}|\Psi_0\rangle, \tag{8·16}$$

is easily obtained in the same way as that in the subsection **A.** The result is

$$B^{(\eta)+}_{2M}=a(N_0N_0)C^{(\nu)+}_{2M}(N_0N_0)-b(N_0N_0)(-1)^M C^{(\nu)}_{2-M}(N_0N_0), \tag{8·17}$$

where

$$a(N_0N_0)=\frac{\hbar\omega_\eta+G^{(0)}\Omega}{\sqrt{2\theta_nG^{(0)}\Omega\hbar\omega_\eta}},\quad b(N_0N_0)=\frac{\hbar\omega_\eta-G^{(0)}\Omega}{\sqrt{2\theta_nG^{(0)}\Omega\hbar\omega_\eta}}. \tag{8·18}$$

Here it should be noted that the collective mode under consideration vanishes in the absence of the pairing interaction ($G^{(0)}=0$).

C. *Derivation of collective variables*

Now let us introduce the following two kinds of collective coordinates $(q^{(\beta)}_M, q^{(\eta)}_M)$ and their conjugate momenta $(p^{(\beta)}_M, p^{(\eta)}_M)$:

$$\left.\begin{aligned} q^{(\beta)}_M&=-i\sqrt{\frac{\hbar}{2I_\beta\omega_\beta}}(B^{(\beta)+}_{2M}-(-1)^M B^{(\beta)}_{2-M}),\ q^{(\beta)+}_M=(-1)^M q^{(\beta)}_{-M},\\ p^{(\beta)}_M&=\sqrt{\frac{\hbar I_\beta\omega_\beta}{2}}(B^{(\beta)}_{2M}+(-1)^M B^{(\beta)+}_{2-M}),\ p^{(\beta)+}_M=(-1)^M p^{(\beta)}_{-M},\end{aligned}\right\} \tag{8·19}$$

$$\left.\begin{aligned} q^{(\eta)}_M&=-i\sqrt{\frac{\hbar}{2I_\eta\omega_\eta}}(B^{(\eta)+}_{2M}-(-1)^M B^{(\eta)}_{2-M}),\ q^{(\eta)+}_M=(-1)^M q^{(\eta)}_{-M},\\ p^{(\eta)}_M&=\sqrt{\frac{\hbar I_\eta\omega_\eta}{2}}(B^{(\eta)}_{2M}+(-1)^M B^{(\eta)+}_{2-M}),\ p^{(\eta)+}_M=(-1)^M p^{(\eta)}_{-M},\end{aligned}\right\} \tag{8·20}$$

which satisfy the commutation relations

$$[q_M^{(\beta)},\ p_{M'}^{(\beta)}]=i\hbar\delta_{MM'},\ [q_M'^{(\eta)},\ p_{M'}^{(\eta)}]=i\hbar\delta_{MM'}. \tag{8·21}$$

$B_{2M}^{(\beta)+}$ is the creation operator of the collective state with the excitation energy $\hbar\omega_\beta$ and is given by (8·4) with the corresponding solution $(a(NN'),\ b(NN'))$ of Eq. (8·5). I_β and I_η are the inertial parameters of the two kinds of collective modes, which will be determined in § 10.

§ 9. Use of the "method of auxiliary variables"

In the preceding section, we have found the collective variables of the system with the Hamiltonian $H_{\mathrm{R}}^{(0)}$. In this section we shall take H_{R}' (8·2) into account, which contains the coupling term between two modes $(q_M^{(\beta)},\ q_M^{(\eta)})$. For this purpose, we shall employ the "method of auxiliary variables",[7] which makes it possible to display the dependence of the Hamiltonian on the collective variables.

Let us consider the Hamiltonian (8·1) and start with the Schrödinger equation

$$H_{\mathrm{R}}|\Phi\rangle\equiv(H_{\mathrm{R}}^{(0)}+H_{\mathrm{R}}')|\Phi\rangle=\tilde{E}|\Phi\rangle. \tag{9·1}$$

Here we introduce two kinds of auxiliary variables, $(\beta_M,\ \pi_M^{(\beta)})$ and $(\eta_M,\ \pi_M^{(\eta)})$, which satisfy the commutation relations

$$[\beta_M,\ \pi_{M'}^{(\beta)}]=i\hbar\delta_{MM'},\ [\eta_M,\ \pi_{M'}^{(\eta)}]=i\hbar\delta_{MM'}. \tag{9·2}$$

Corresponding to $(q_M^{(\beta)},\ p_M^{(\beta)})$ and $(q_M^{(\eta)},\ p_M^{(\eta)})$, they have the conjugate relations

$$\left.\begin{aligned}\beta_M^+&=(-1)^M\beta_{-M},\ \pi_M^{(\beta)+}=(-1)^M\pi_{-M}^{(\beta)},\\ \eta_M^+&=(-1)^M\eta_{-M},\ \pi_M^{(\eta)+}=(-1)^M\pi_{-M}^{(\eta)}.\end{aligned}\right\} \tag{9·3}$$

To compensate the introduction of the auxiliary variables, we impose on $|\Phi\rangle$ the supplementary conditions

$$\beta_M|\Phi\rangle=0,\ \eta_M|\Phi\rangle=0. \tag{9·4}$$

Now let us successively perform the following unitary transformations:

$$|\Phi^{(1)}\rangle=U_{(1)}^{-1}|\Phi\rangle,\ U_{(1)}=\exp\{i\sum_M(\pi_M^{(\beta)}q_M^{(\beta)}+\pi_M^{(\eta)}q_M^{(\eta)})/\hbar\}, \tag{9·5a}$$

$$|\Phi^{(2)}\rangle=U_{(2)}^{-1}|\Phi^{(1)}\rangle,\ U_{(2)}=\exp\{-i\sum_M(p_M^{(\beta)}\beta_M+p_M^{(\eta)}\eta_M)/\hbar\}, \tag{9·5b}$$

$$|\Phi_{\mathrm{c}}\rangle=U_{(1)}^{-1}|\Phi^{(2)}\rangle,\ U_{(1)}=\exp\{i\sum_M(\pi_M^{(\beta)}q_M^{(\beta)}+\pi_M^{(\eta)}q_M^{(\eta)})/\hbar\}. \tag{9·5c}$$

Then the transformed Hamiltonian and the supplementary conditions become

$$\mathcal{H}=U_{(1)}^{-1}U_{(2)}^{-1}U_{(1)}^{-1}H_{\mathrm{R}}U_{(1)}U_{(2)}U_{(1)}, \tag{9·6}$$

$$q_M^{(\beta)}|\Phi_{\mathrm{c}}\rangle=0,\quad q_M^{(\eta)}|\Phi_{\mathrm{c}}\rangle=0. \tag{9·7}$$

This representation is the so-called "collective representation", in which Bohr

and Mottelson[8] have developed the nuclear unified model.

It should be noted that the Hamiltonian (9·6) contains neither $(q_M^{(\eta)}, p_M^{(\eta)})$ nor $(q_M^{(\beta)}, p_M^{(\beta)})$, i.e. it satisfies

$$[\mathcal{H}, q_M^{(\beta)}]=[\mathcal{H}, p_M^{(\beta)}]=0, \quad [\mathcal{H}, q_M^{(\eta)}]=[\mathcal{H}, p_M^{(\eta)}]=0. \tag{9·8}$$

This implies that the collective variables $(q_M^{(\beta)}, p_M^{(\beta)}; q_M^{(\eta)}, p_M^{(\eta)})$ in the original Hamiltonian H_R are completely replaced by the auxiliary variables $(\beta_M, \pi_M^{(\beta)}; \eta_M \pi_M^{(\eta)})$. In this representation, therefore, the collective modes of our system can be visualized through the auxiliary variables.

Employing the operator identity

$$\exp(iS)O\exp(-iS)=O+i[S, O]-\frac{1}{2}[S, [S, O]]+\cdots,$$

and using the commutation relations (7·8) and (9·2), we obtain the explicit form of $\mathcal{H}$ of the following structure:

$$\mathcal{H}=\mathcal{H}_{\text{coll.}}+\mathcal{H}_{\text{intr.}}+\mathcal{H}_{\text{coupl}}\,, \tag{9·9}$$

$$\mathcal{H}_{\text{coll.}}=\sum_M{}' \left\{\frac{1}{2I_\beta}|\pi_M^{(\beta)}|^2+\frac{1}{2}C_\beta|\beta_M|^2\right\}+\sum_M{}' \left\{\frac{1}{2I_\eta}|\pi_M^{(\eta)}|^2+\frac{1}{2}C_\eta|\eta_M|^2\right\}$$
$$-\frac{1}{2}\kappa\sum_M{}' \{\beta_M\eta_M^{+}+\eta_M\beta_M^{+}\}\,, \tag{9·10}$$

$$\mathcal{H}_{\text{intr.}}=H_R-\sum_M{}' \left\{\frac{1}{2I_\beta}|p_M^{(\beta)}|^2+\frac{1}{2}C_\beta|q_M^{(\beta)}|^2\right\}-\sum_M{}' \left\{\frac{1}{2I_\eta}|p_M^{(\eta)}|^2+\frac{1}{2}C_\eta|q_M^{(\eta)}|^2\right\}$$
$$+i\frac{\sqrt{I_\eta}}{\hbar}\sqrt{G^{(0)}\mathcal{Q}\theta_n}\sum_M{}'\sum_{NN'}{}'\langle N_0 N|g_2|N' N_0\rangle\{C_{2M}^{(\text{ud})}(NN')$$
$$-(-1)^M C_{2-M}^{(\text{ud})+}(NN')\}q_M^{(\eta)}, \tag{9·11}$$

$$\mathcal{H}_{\text{coupl.}}=\sum_M{}'\Big[\kappa q_M^{(\beta)+}-i\frac{\sqrt{I_\eta}}{\hbar}\sqrt{G^{(0)}\mathcal{Q}\theta_n}\sum_{NN'}{}'\langle N_0 N|g_2|N' N_0\rangle$$
$$\times\{C_{2M}^{(\text{ud})}(NN')-(-1)^M C_{2-M}^{(\text{ud})+}(NN')\}\Big]\cdot\eta_M, \tag{9·12}$$

where

$$C_\beta\equiv I_\beta\omega_\beta^2, \quad C_\eta\equiv I_\eta\omega_\eta^2, \tag{9·13}$$

$$\kappa=\frac{\sqrt{I_\eta}}{\hbar}\sqrt{G^{(0)}\mathcal{Q}\theta_n}\,\frac{1}{\hbar}\sqrt{\frac{\hbar I_\beta\omega_\beta}{2}}\sum_{NN'}{}'\{a(NN')-b(NN')\}(2u_N^2v_{N'}^2)(1-\delta_{NN'})$$
$$\times\langle N_0 N|g_2|N' N_0\rangle. \tag{9·14}$$

Here $\mathcal{H}_{\text{coll.}}$ represents the energy of collective motion, and $\mathcal{H}_{\text{intr.}}$ can be interpreted as the energy of the intrinsic (particle) motion. $\mathcal{H}_{\text{coupl.}}$ represents the interaction between the collective and the intrinsic motions.

For the spherical nuclei under consideration, we can safely treat $\mathcal{H}_{\text{coupl.}}$ as a small perturbation term. Therefore, we shall neglect this term hereafter. In this case the Schrödinger equation in the "collective representation" becomes

$$(\mathcal{H}_{\text{coll.}}+\mathcal{H}_{\text{intr.}})|\Phi_c\rangle=\tilde{E}|\Phi_c\rangle \tag{9·15}$$

with the supplementary condition (9·7).

Let us write $|\Phi_c\rangle$ as

$$|\Phi_c\rangle=|\chi_{\text{intr.}}\rangle|\phi_{\text{coll.}}\rangle. \tag{9·16}$$

Then $|\chi_{\text{intr.}}\rangle$ satisfies the equation

$$\mathcal{H}_{\text{intr.}}|\chi_{\text{intr.}}\rangle=\tilde{E}_{\text{intr.}}|\chi_{\text{intr.}}\rangle \tag{9·17}$$

with the supplementary condition $q_M^{(\beta)}|\chi_{\text{intr.}}\rangle=q_M^{(\gamma)}|\chi_{\text{intr.}}\rangle=0$, and $|\phi_{\text{coll.}}\rangle$ satisfies

$$\mathcal{H}_{\text{coll.}}|\phi_{\text{coll.}}\rangle=\tilde{E}_{\text{coll.}}|\phi_{\text{coll.}}\rangle. \tag{9·18}$$

The total energy of our system thus becoms $\tilde{E}=\tilde{E}_{\text{intr.}}+\tilde{E}_{\text{coll.}}$

The Schödinger equation (9·18) describes the collective motion of our system. In the following sections, we shall investigate the properties of Eq. (9·18).

§ 10. Mass quadrupole moments and "surface deformation variables"

Prior to discussing Eq. (9·18), we shall, in this section, clarify the connection between our variables (β_M, γ_M) and the "surface deformation variables" α_M used in the Bohr-Mottelson model.[8] For this purpose, we consider the mass quadrupole moment operator.

In the Bohr-Mottelson model,[8] as is well known, the quadrupole moment operator is givan by

$$\mathcal{Q}_M=\mathcal{Q}_M^{(\text{intr.})}+\mathcal{Q}_M^{(\text{coll.})}, \tag{10·1}$$

the first part of which is associated with the intrinsic structure.* The second part is due to the surface deformation and is given by

$$\mathcal{Q}_M^{(\text{coll.})}=\frac{3}{\sqrt{5\pi}}AR_0^2\alpha_M\equiv\bar{Q}\alpha_M, \tag{10·2}$$

where A is the atomic number and R_0 is the nuclear radius.

Now let us consider the quadrupole moment operator of our system. In the original representation the quadrupole moment operator is

$$\begin{aligned}Q_M&=\sqrt{\frac{16\pi}{5}}\sum_{\substack{Nm\\N'm'}}\langle Nm|r^2Y_{2M}|N'm'\rangle a_{Nm}^{+}a_{N'm'}\\&=i\sum_{NN'}\langle N|q_2|N'\rangle\sum_{mm'}(-1)^{-m'}(jj'm-m'|2M)a_{Nm}^{+}a_{N'm'},\end{aligned} \tag{10·3}$$

* For the even-even nuclei under consideration, $\langle\chi_{\text{intr.}}^{(0)}|\mathcal{Q}_M^{\text{intr.}}|\chi_{\text{intr.}}^{(0)}\rangle=0$, where $\chi_{\text{intr.}}^{(0)}$ is the ground state of $\mathcal{H}_{\text{intr.}}$.

where

$$\langle N|q_2|N'\rangle=\sqrt{\frac{16\pi}{5}}\sqrt{\frac{(2j+1)(2j'+1)}{20\pi}}\left(jj'\ \ \frac{1}{2}\ -\frac{1}{2}\ \ |20\right)\langle nl|r^2|n'l'\rangle. \quad (10\cdot4)$$

Performing the Bogoliubov transformation (2·4) with (3·15) and making use of the approximations discussed in § 6 and § 7, we have

$$\begin{aligned}Q_M=&-i\sum_{NN'}{}'\langle N|q_2|N'\rangle\{C^{(\mathrm{ind})+}_{2M}(NN')-(-1)^M C^{(\mathrm{ind})}_{2-M}(NN')\}\\&-i\langle N_0|q_2|N_0\rangle\{C^{(\mathrm{ind})+}_{2M}(N_0N_0)-(-1)^M C^{(\mathrm{ind})}_{2-M}(N_0N_0)\}. \quad (10\cdot5)\end{aligned}$$

In the "collective representation", (10·5) changes into

$$\mathcal{Q}_M=U^{-1}_{(1)}U^{-1}_{(2)}U^{-1}_{(1)}Q_M U_{(1)}U_{(2)}U_{(1)}=\mathcal{Q}^{(\mathrm{intr.})}_M+\mathcal{Q}^{(\mathrm{coll})}_M, \quad (10\cdot6)$$

$$\mathcal{Q}^{(\mathrm{intr})}_M=Q_M-k_1q^{(\beta)}_M-k_2q^{(\eta)}_M,* \quad (10\cdot6\mathrm{a})$$

$$\mathcal{Q}^{(\mathrm{coll})}_M=k_1\beta_M+k_2\eta_M, \quad (10\cdot6\mathrm{b})$$

where

$$k_1=\frac{1}{\hbar}\sqrt{\frac{\hbar I_\beta\omega_\beta}{2}}\sum_{NN'}{}'\langle N|q_2|N'\rangle\{a(NN')-b(NN')\}(2u^2_Nv^2_{N'})(1-\delta_{NN'}),$$

$$k_2=\frac{1}{\hbar}\sqrt{I_\eta}\langle N_0|q_2|N_0\rangle\sqrt{G^{(0)}\Omega\theta_n}.$$

Comparing (10·6b) with (10·2), we can find the connection between (β_M, η_M) and α_M:

$$\alpha_M=(\bar{Q}^{-1}k_1)\beta_M+(\bar{Q}^{-1}k_2)\eta_M. \quad (10\cdot7)$$

If we determine the inertial parameters (I_β, I_η) introduced in (8·19) and (8·20) as

$$\begin{aligned}I_\beta&=\frac{2\hbar\bar{Q}^2}{\omega_\beta}\Big[\sum_{NN'}{}'\langle N|q_2|N'\rangle\{a(NN')-b(NN')\}(2u^2_Nv^2_{N'})(1-\delta_{NN'})\Big]^{-2},\\I_\eta&=\hbar^2\bar{Q}^2/\overline{G^{(0)}\Omega\theta_n}\langle N_0|q_2|N_0\rangle^2, \quad (10\cdot8)\end{aligned}$$

Eq. (10·7) becomes

$$\alpha_M=\beta_M+\eta_M. \quad (10\cdot9)$$

Now the physical meaning of the variables β_M and η_M is clear. η_M represents the deformation associated with the configuration of the nucleons in the N_0-shell, and vanishes when the N_0-shell is closed. Thus, β_M represents the deformation of the core mainly, and itself becomes α_M when the N_0-shell is closed.

With the aid of (10·9), the polarizability of the core by the outside nucleons

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* Using the supplementary condition $q^{(\beta)}_M|\chi^{(0)}_{\mathrm{intr.}}\rangle=q^{(\eta)}_M|\chi^{(0)}_{\mathrm{intr.}}\rangle=0$, and making use of the approximation $|\chi^{(0)}_{\mathrm{intr.}}\rangle\simeq|c_0\rangle$, we find $\langle\chi^{(0)}_{\mathrm{intr.}}|\mathcal{Q}^{(\mathrm{intr.})}_M|\chi^{(0)}_{\mathrm{intr.}}\rangle=0$, where $|\chi^{(0)}_{\mathrm{intr.}}\rangle$ is the ground state of $\mathcal{H}_{\mathrm{intr.}}$.

can be obtained in the following way. Let us pick up the "potential energy" term for the collective motion from $\mathcal{H}_{\text{coll.}}$:

$$2U \equiv \sum_M \{C_\beta|\beta_M|^2 + C_\eta|\eta_M|^2 - \kappa(\beta_M\eta_M^+ + \eta_M\beta_M^+)\}$$

$$= \sum_M \{C_\beta|\alpha_M|^2 - (C_\beta+\kappa)(\eta_M\alpha_M^+ + \alpha_M\eta_M^+) + (C_\beta+C_\eta+2\kappa)|\eta_M|^2\}. \quad (10\cdot10)$$

For a fixed value of η_M, the equilibrium deformation α_M^0 is determined from (10·10) and is given by

$$\alpha_M^0(\eta_M) = \{1/(1-\varepsilon)\}\cdot\eta_M, \quad (10\cdot11)$$

where

$$\varepsilon = \kappa/(C_\beta+\kappa). \quad (10\cdot12)$$

With the aid of (10·2), (10·11) is written as

$$\mathcal{Q}_{M=0}^{(\eta)} = (1-\varepsilon)\mathcal{Q}_{M=0}^0 \quad (10\cdot13)$$

where $\mathcal{Q}_{M=0}^{(\eta)} \equiv \bar{Q}\eta_{M=0}$ is the quadrupole moment associated with the outside nucleons. Eq. (10·13) means that the quantity ε defined by (10·12) describes the polarizability of the core by the outside nucleons.

§ 11. Normal vibrations of even-even spherical nuclei

The lower excited states of even-even spherical nuclei are described by Eq. (9·18). In order to investigate the energy spectrum, we shall transform the Hamiltonian (9·10) into that of the normal vibrations:

$$\mathcal{H}_{\text{coll.}} = \frac{1}{2}\sum_M \{|\dot{\alpha}_M^{(1)}|^2 + \omega_{(1)}^2|\alpha_M^{(1)}|^2 + |\dot{\alpha}_M^{(2)}|^2 + \omega_{(2)}^2|\alpha_M^{(2)}|^2\}. \quad (11\cdot1)$$

Here the normal coordinates $\alpha_M^{(1)}$ and $\alpha_M^{(2)}$ are defined through

$$\left.\begin{aligned}
\beta_M &= \frac{1}{\sqrt{I_\beta}}\sqrt{\frac{(\omega_\eta^2-\omega_{(1)}^2)}{(\omega_\eta^2-\omega_{(1)}^2)+(\omega_\beta^2-\omega_{(1)}^2)}}\cdot\alpha_M^{(1)} \\
&\quad + \frac{1}{\sqrt{I_\beta}}\sqrt{\frac{(\omega_{(2)}^2-\omega_\eta^2)}{(\omega_{(2)}^2-\omega_\eta^2)+(\omega_{(2)}^2-\omega_\beta^2)}}\cdot\alpha_M^{(2)}, \\
\eta_M &= \frac{1}{\sqrt{I_\eta}}\sqrt{\frac{(\omega_\beta^2-\omega_{(1)}^2)}{(\omega_\eta^2-\omega_{(1)}^2)+(\omega_\beta^2-\omega_{(1)}^2)}}\cdot\alpha_M^{(1)} \\
&\quad - \frac{1}{\sqrt{I_\eta}}\sqrt{\frac{(\omega_{(2)}^2-\omega_\beta^2)}{(\omega_{(2)}^2-\omega_\eta^2)+(\omega_{(2)}^2-\omega_\beta^2)}}\cdot\alpha_M^{(2)},
\end{aligned}\right\} \quad (11\cdot2)$$

while the corresponding eigenfrequencies are given by

$$\omega_{(1)}^2 = \frac{1}{2}(\omega_\beta^2+\omega_\eta^2) - \frac{1}{2}(\omega_\beta^2+\omega_\eta^2)$$

$$\times\left[1-\frac{4\omega_\beta^2}{(\omega_\beta^2+\omega_\eta^2)^2}\left\{\omega_\eta^2-\frac{G^{(0)}\Omega\Theta_n}{2\hbar^2\cdot\hbar\omega_\beta}\left[\sum_{NN'}\{a(NN')-b(NN')\}\right.\right.\right.$$

$$\left.\left.\left.\times(2u_N^2\,v_{N'}^2)(1-\delta_{NN'})\langle N_0N|g_2|N'N_0\rangle\right]^2\right\}\right]^{1/2}, \qquad (11\cdot3a)$$

$$\omega_{(2)}^2=\frac{1}{2}(\omega_\beta^2+\omega_\eta^2)+\frac{1}{2}(\omega_\beta^2+\omega_\eta^2)$$

$$\times\left[1-\frac{4\omega_\beta^2}{(\omega_\beta^2+\omega_\eta^2)^2}\left\{\omega_\eta^2-\frac{G^{(0)}\Omega\Theta_n}{2\hbar^2\cdot\hbar\omega_\beta}\left[\sum_{NN'}\{a(NN')-b(NN')\}\right.\right.\right.$$

$$\left.\left.\left.\times(2u_N^2\,v_{N'}^2)(1-\delta_{NN'})\langle N_0N|g_2|N'N_0\rangle\right]^2\right\}\right]^{1/2}. \qquad (11\cdot3b)$$

If $\hbar\omega_\beta\gg\hbar\omega_\eta$, then (11·3a) and (11·3b) become

$$\hbar\omega_{(1)}\simeq G^{(0)}\Omega\sqrt{1-\Theta_n/\Theta_{n_0}}, \qquad (11\cdot4a)$$

$$\hbar\omega_{(2)}\simeq\hbar\omega_\beta, \qquad (11\cdot4b)$$

where

$$\frac{1}{\Theta_{n_0}}=\frac{1}{G^{(0)}\Omega}\left[4\langle N_0N_0|g_2|N_0N_0\rangle+\frac{1}{2\hbar\omega_\beta}\left\{\sum_{NN'}\{a(NN')-b(NN')\}\right.\right.$$

$$\left.\left.\times(2u_N^2\,v_{N'}^2)(1-\delta_{NN'})\langle N_0N|g_2|N'N_0\rangle\right\}^2\right]$$

$$=\frac{1}{G^{(0)}\Omega}[4\langle N_0N_0|g_2|N_0N_0\rangle+\bar{Q}^{-2}\{\varepsilon/(1-\varepsilon)\}^2C_\beta\langle N_0|q_2|N_0\rangle^2]. \qquad (11\cdot5)$$

Eq. (11·4a) should be compared with Eq. (163) in Belyaev's paper,[4] which has been obtained by the use of the "cranking model" of Inglis.[5]

As has been pointed out by Belyaev,[4] the normal vibration of the first type ($\hbar\omega_{(1)}$) preserves the equilibrium relation between β_M and η_M. Indeed, the condition $\alpha_M^{(2)}=0$ can be rewritten as

$$\beta_M=\sqrt{\frac{I_\eta(\omega_\eta^2-\omega_{(1)}^2)}{I_\beta(\omega_\beta^2-\omega_{(1)}^2)}}\cdot\eta_M\simeq\frac{\kappa}{C_\beta}\eta_M. \qquad (11\cdot6)$$

With the aid of (10·9), (11·6) becomes

$$\alpha_M=\left(1+\sqrt{\frac{I_\eta(\omega_\eta^2-\omega_{(1)}^2)}{I_\beta(\omega_\beta^2-\omega_{(1)}^2)}}\right)\eta_M\simeq\{1/(1-\varepsilon)\}\eta_M \qquad (11\cdot7)$$

which is equivalent to (10·11). In the vibration of the first type the core thus adjusts itself adiabatically to the deformation of the outside nucleons.

By the use of (11·6), the normal coordinate of the first type, $\alpha_M^{(1)}$, becomes

$$\alpha_M^{(1)}=\sqrt{I_\eta}\sqrt{\frac{(\omega_\beta^2-\omega_{(1)}^2)+(\omega_\eta^2-\omega_{(1)}^2)}{(\omega_\beta^2-\omega_{(1)}^2)}}\cdot\eta_M\simeq\sqrt{I_\eta}\,\eta_M. \qquad (11\cdot8)$$

Using (11·7), we can rewrite (11·8) as

$$\alpha_M^{(1)}=\sqrt{I_\beta I_\eta}\cdot\frac{\sqrt{(\omega_\beta^2-\omega_{(1)}^2)+(\omega_\eta^2-\omega_{(1)}^2)}}{\sqrt{I_\beta(\omega_\beta^2-\omega_{(1)}^2)}+\sqrt{I_\eta(\omega_\eta^2-\omega_{(1)}^2)}}\cdot\alpha_M\simeq(1-\varepsilon)\sqrt{I_\eta}\alpha_M. \qquad (11\cdot9)$$

This indicates that the normal vibration of the first type, characterizing the lower excited states of even-even spherical nuclei, corresponds to that of α_M used in the Bohr-Mottelson model.[8] It should be noted, however, that for closed shell nuclei η_M vanishes as discussed in § 10 and the vibration of the first type disappears. In this case, therefore, the vibration of α_M becomes that of the second type ($\hbar\omega_\beta$).

§ 12. Determination of the inertial parameter and the surface tension parameter

The relation (11·9) uniquely determines the inertial parameter, B, and the surface tension parameter, C, for the quadrupole vibration of even-even nuclei. With the aid of (11·9), the Hamiltonian for the normal vibration of the first type,

$$\mathcal{H}_{\text{coll.}}^{(1)}=\frac{1}{2}\sum_M\{|\dot{\alpha}_M^{(1)}|^2+\omega_{(1)}^2|\alpha_M^{(1)}|^2\}, \qquad (12\cdot1)$$

can be rewritten as

$$\mathcal{H}_{\text{coll.}}^{(1)}=\sum_M\left\{\frac{1}{2B}|\pi_M|^2+\frac{1}{2}C|\alpha_M|^2\right\}, \qquad (12\cdot2)$$

where π_M is the momentum conjugate to α_M and satisfies $[\pi_M, \alpha_{M'}]=-i\hbar\delta_{MM'}$. B and C are the inertial parameter and the surface tension parameter respectively, and are uniquely determined as

$$B=I_\beta I_\eta\frac{(\omega_\beta^2-\omega_{(1)}^2)+(\omega_\eta^2-\omega_{(1)}^2)}{\{\sqrt{I_\beta(\omega_\beta^2-\omega_{(1)}^2)}+\sqrt{I_\eta(\omega_\eta^2-\omega_{(1)}^2)}\}^2}\simeq(1-\varepsilon)^2I_\eta, \qquad (12\cdot3)$$

$$C=I_\beta I_\eta\cdot\omega_{(1)}^2\frac{(\omega_\beta^2-\omega_{(1)}^2)+(\omega_\eta^2-\omega_{(1)}^2)}{\{\sqrt{I_\beta(\omega_\beta^2-\omega_{(1)}^2)}+\sqrt{I_\eta(\omega_\eta^2-\omega_{(1)}^2)}\}^2}\simeq(1-\varepsilon)^2I_\eta\omega_{(1)}^2. \qquad (12\cdot4)$$

Inserting (10·8) in (12·3), we get

$$B\simeq(1-\varepsilon)^2\frac{\hbar^2\bar{Q}^2}{G^{(0)}\Omega\Theta_n\langle N_0|q_2|N_0\rangle^2}$$

$$=\frac{9}{8}(1-\varepsilon)^2\frac{\hbar^2A^2R_0^4(2\Omega-1)(2\Omega+1)\Omega}{G^{(0)}\pi\langle n_0l_0|r^2|n_0l_0\rangle^2(\Omega-1)^2(\Omega+1)}\cdot\frac{1}{n(2\Omega-n)}, \qquad (12\cdot5)$$

where n is the number of nucleons in the N_0-shell and $\Omega=(2j_0+1)/2$ ((3·11)). Thus the ratio between (12·5) and $B_{\text{irrot.}}$ becomes

$$B/B_{\text{irrot.}}\simeq3(1-\varepsilon)^2\frac{\hbar^2AR_0^2}{G^{(0)}m\langle n_0l_0|r^2|n_0l_0\rangle^2}\cdot\frac{(2\Omega-1)(2\Omega+1)\Omega}{(\Omega-1)^2(\Omega+1)}\cdot\frac{1}{n(2\Omega-n)}, \qquad (12\cdot6)$$

where $B_{\text{irrot.}}$ is the inertial parameter for the oscillations of an irrotational liquid drop :[8]

$$B_{\text{irrot.}}=\frac{3}{8\pi}AmR_0^2 \quad (m: \text{ the nucleon mass}).$$

The formula (12·6) should be compared with Eq. (155) in Belyaev's paper,[4] which has been obtained by using the " cranking formula ".

Now let us remember that ε, the polarizability of the core by the outside nucleons, is expressed as

$$\begin{aligned}\varepsilon&=\frac{\kappa}{C_\beta+\kappa}\\&=\bar{Q}\langle N_0|q_2|N_0\rangle^{-1}\Big[\sum_{NN'}\{a(NN')-b(NN')\}(2u_N^2\,v_{N'}^2)(1-\delta_{NN'})\langle N_0\,N|q_2|N'\,N_0\rangle\Big]\\&\times\{\sqrt{2I_\beta\hbar\omega_\beta}\cdot\omega_\beta+\bar{Q}\langle N_0|q_2|N_0\rangle^{-1}\Big[\sum_{NN'}\{a(NN')-b(NN')\}(2u_N{}^2v_{N'}^2)\\&\qquad\times(1-\delta_{NN'})\langle N_0\,N|q_2|N'\,N_0\rangle\Big]\}^{-1}.\end{aligned}$$

This indicates that ε is determined mainly by the properties of the core and does not depend appreciably on the pairing interaction ($G^{(0)}$) and on the number of nucleons in the partially-filled N_0-shell. Neglecting such a dependence of ε, we can empirically estimate the value from the data of nuclei with one particle outside closed shells. The rough estimates of (12·6) with $\langle n_0 l_0|r^2|n_0 l_0\rangle\simeq\frac{3}{5}R_0^2$, $R_0=1.4A^{1/3}\times10^{-13}$ cm, $G^{(0)}=50\,\text{Mev}/A$* and $\varepsilon=0.8$** are given in Table I together with experimental values. Though the estimation is very rough, the calculated values are in qualitative agreement with observed trends.

With the aid of (12·5) and (11·4a), the surface tension parameter (12·4) can be written as

$$C\simeq(1-\varepsilon)^2\frac{\bar{Q}^2\cdot\overline{G^{(0)}}\,\Omega}{\langle N_0|q_2|N_0\rangle^2\theta_n}\{1-\theta_n/\theta_{n_0}\}, \tag{12·7}$$

where $\bar{Q}\equiv(3/\sqrt{5\pi})AR_0^2$, and $\langle N_0|q_2|N_0\rangle$ is given by (10·4). θ_{n_0} is given by (11·5) and represents a ratio between the strength of $G^{(0)}$ and of $G^{(2)}$. Formula (12·7) should be compared with Eq. (114) in Belyaev's paper.[4]

As $(1-\varepsilon)^2\bar{Q}^2\cdot\overline{G^{(0)}}\,\Omega/\langle N_0|q_2|N_0\rangle^2\theta_n>0$, the stability of the spherical shape of even-even nuclei is determined by the sign of $\{1-\theta_n/\theta_{n_0}\}$. If $\theta_{n_0}>1$, the nucleus

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* The value of $\overline{G^{(0)}}$ is determined approximately from the equation

$$\overline{G^{(0)}}\,\Omega\simeq P(j_0)_{\text{exp.}},$$

where $P(j_0)_{\text{exp.}}$ is the experimental value of pairing energy. In $1g_{9/2}$- and $2d_{5/2}$-shell, $P(j_0)_{\text{exp.}}\simeq 25(2j_0+1)/A$ Mev.

** The $E2$-transition rate in $_{82}\text{Pb}^{207}$ implies that the effective charge, $e_{\text{eff.}}\equiv\varepsilon/(1-\varepsilon)\cdot Z/A\cdot e$, of Pb^{207} is $1.1e$. From this fact, we get $\varepsilon=0.73$. Note that Pb^{207} is the nucleus with one neutron hole inside the ***double magic*** closed shell.

Table I. Inertial parameter of even-even nuclei

a) P. H. Stelson and T. K. McGowan, Phys. Rev. **110** (1958), 489

b) K. Alder, A. Bohr, T. Huus, B. R. Mottelson and A. Winther, Rev. Mod. Phys. **28** (1956), 432.

nucleus	proton configuration	neutron configuration	B/B_{irrot} (exp.)		$B/B_{\text{irrot.}}$ (cal.)
			Ref. a)	Ref. b)	
$_{32}Ge^{74}$	$(2p_{3/2})^4$: closed	$(1g_{9/2})^2$		10	14
$_{32}Ge^{76}$	$(2p_{3/2})^4$: closed	$(1g_{9/2})^4$		9.3	9.7
$_{42}Mo^{98}$	$(1g_{9/2})^2$	$(2d_{5/2})^6$: closed	13		20
$_{44}Ru^{100}$	$(1g_{9/2})^4$	$(2d_{5/2})^6$: closed	9.6		14
$_{46}Pd^{110}$	$(1g_{9/2})^6$	$(1d_{5/2})^6\,(1g_{7/2})^8$: closed	9.9	8.5	16
$_{48}Cd^{112}$	$(1g_{9/2})^8$	$(1d_{5/2})^6\,(1g_{7/2})^8$: closed	10.5	12	23
$_{48}Cd^{114}$	$(1g_{9/2})^8$	$(3s_{1/2})^2$: closed	10.5	11	23
$_{50}Sn^{116}$	$(1g_{9/2})^{10}$: closed	$(1h_{11/2})^2$	14		19
$_{50}Sn^{118}$	〃	$(1h_{11/2})^4$	13		12
$_{50}Sn^{120}$	〃	$(1h_{11/2})^6$	14		11
$_{50}Sn^{122}$	〃	$(1h_{11/2})^8$	13		13
$_{50}Sn^{124}$	〃	$(1h_{11/2})^{10}$	16		20

remains spherical for any occupation. If $\theta_{n_0}<\theta_n(\leq 1)$, however, the spherical nucleus is unstable and becomes the deformed nucleus. With the aid of (8·15a), the condition $\theta_{n_0}<\theta_n$ can be rewritten as

$$n_0<n<(2\Omega-n_0). \tag{12·8}$$

Here n_0 is the "characteristic number" of nucleons in the N_0-shell, which has been stressed by Weisskopf,[10] and is given by

$$n_0=\Omega(1-\sqrt{1-\theta_{n_0}}). \tag{12·9}$$

Thus a sharp transition from the spherical nuclei into the deformed nuclei occurs at the "characteristic number" n_0, provided that $\theta_{n_0}<1$.

§ 13. Concluding remarks

Assuming the possibility of the division, into three parts, of the effective interaction between particles as in Eq. (2·2), we have investigated the mechanism of collective motion in even-even spherical nuclei from the standpoint of particle excitations. In so far as the vibrational motion is concerned, the physical implication underlying the nuclear collective model proposed by Bohr and Mottelson[8] has been made clear.

Our method of description of the nuclear collective motion is quite different from that of the "cranking model".[5] The method is more fundamental than that of the "cranking model" in the sense that it is developed by using the effective interaction between particles explicitly. Indeed, with the aid of the method,

the various physical parameters used by Belyaev[4] can be derived uniquely from the "first" principle.

In this paper, we have neglected effects of the coupling between collective and intrinsic (particle) motions. Such effects will be considered in a subsequent paper. It is outside the scope of this paper and remains to be investigated to relate the effective interaction used in this paper with the nuclear force known from the two-body problems.

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Note added in proof: Very recently R. Arvieu and M. Véneroni made an attempt similar to ours [Comptes rendus **250** (1960), 992; preprint].

References

1) K. Ikeda, M. Kobayasi, T. Marumori, T. Shiozaki and S. Takagi, Prog. Theor. Phys. **22** (1959), 663.
2) M. Kobayasi and T. Marumori, Prog. Theor. Phys. **23** (1960), 387.
3) N. N. Bogoliubov, J.E.T.P., USSR, **34** (1958), 58, 73.
N. N. Bogoliubov, V. V. Tolmachev and D. V. Shirkov, "*A new method in the theory of superconductivity*", Izdat. Akad. Nauk. USSR (1958).
4) S. T. Belyaev, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **31** (1959) No. 11.
5) D. R. Inglis, Phys. Rev. **96** (1954), 1059; **97** (1955), 701; **103** (1956), 1786.
6) K. Sawada, Phys. Rev. **106** (1957), 372.
K. Sawada, K. A. Brueckner, N. Fukuda and R. Brout, Phys. Rev. **108** (1957), 507.
G. Wentzel, Phys. Rev. **108** (1957), 1593.
7) T. Marumori and E. Yamada, Prog. Theor. Phys. **13** (1955), 557.
S. Hayakawa and T. Marumori, Prog. Theor. Phys. **18** (1957), 396.
T. Miyazima and T. Tamura, Prog. Theor. Phys. **15** (1956), 255.
T. Nishiyama, Prog. Theor. Phys. **17** (1957), 711.
8) A. Bohr, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **26** (1952), No. 14.
A. Bohr and B. R. Mottelson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **27** (1953), No. 16.
9) A. Bohr, *Proceedings of the Paris Conference on Nuclear Structure* (1958).
B. R. Mottelson, in "deWitt: *Many Body Problem*" (John Wiley and Sons, New York, 1959), p. 283.
10) V. F. Weisskopf, Supple. Prog. Theor. Phys. No. **11** (1959), 52.
11) D. L. Hill and J. A. Wheeler, Phys. Rev. **89** (1953), 1102.
12) M. Nogami, Soryushironkenkyu (mimeographed circular in Japanese) **10** (1956), 600.
13) R. A. Ferrell, Phys. Rev. **107** (1957), 1631.
14) Y. Shono and H. Tanaka, Prog. Theor. Phys. **22** (1959), 177.
15) S. Hayakawa, Soryushironkenkyu (mimeographed circular in Japanese) **14** (1957), 18.
16) M. Nogami, lecture at Research Institute for Fundamental Physics, Kyoto University. (1958).
17) S. Takagi, Prog. Theor. Phys. **21** (1959), 174.
18) H. Ehrenreich and M. H. Cohen, Phys. Rev. **115** (1959), 786.
19) J. P. Elliott, Proc. Roy. Soc. **245** (1958), 128, 562.
20) S. A. Moszkowski, Phys. Rev. **110** (1958), 403.

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Electron-Electron Interaction in Hot Electron Problems

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The behaviours of warm and hot electrons in n-type germanium at 77°K are investigated theoretically. The following assumptions are adopted: (1) Electrons are not degenerate, (2) the shape of the energy surface is spherically symmetric, (3) electrons are scattered by acoustic and optical modes of vibrations, and by ionized impurity centers and (4) the electron-electron interaction is taken into account by the Fokker-Planck approximation. The results are compared with Gunn's recent experiments. We find that the Maxwellian distribution function with the electron temperature does not give a good agreement with experiment when the impurity density is low. On the other hand, the electron-electron interaction works decisively in reducing the electron distribution to the Maxwellian form when the impurity density is high. The critical density seems to lie near $10^{15}/\text{cm}^3$ at 77°K.

§ 1. Introduction

In the previous paper,[1] which will be referred to as I, we discussed the behaviours of the warm electrons in n-type germanium at room temperature by solving the Boltzmann equation approximately. The electron energy loss due to the interaction with the optical mode is found to be quite large as compared with that due to the acoustic scattering. Further, if the interaction constants between the conduction electron and the acoustic and optical vibrations are suitably chosen so as to fit observed values of the ohmic mobility, the agreement between theory and experiment concerning the warm electrons seems to be reasonably good. In the second paper,[2] which will be referred to as II, we extended the discussion to the hot electron in n-type germanium at room temperature.[3] The agreement between theory and experiment is again found to be reasonably good. In II we also discussed the anisotropy of the hot electron current and estimated the intensity of the intervalley scattering so as to fit the observed data of Sasaki and others.[4] The intensity of the intervalley scattering thus estimated is in good agreement with the result derived from the acousto-electric effect.[5] In these papers, however, we did not consider the effects of impurity scatterings and the electron-electron interaction, because these effects were supposed to be small at room temperature unless the impurity density is high.

Recently Gunn[6] made a very nice experiment. He observed the behaviours of the warm and hot electrons of n-type germanium doped by Sb at 77°K by changing the impurity concentration from 6.6×10^{13} to $2.9\times10^{16}\,\text{cm}^{-3}$. It is found that the effect of the impurity scattering on the non-ohmic current is remarkable

at that temperature, and some features of the current suggest the importance of the electron-electron interaction at highly doped samples.* Since these experiments are quite interesting and very important, we would like to analyse them theoretically and obtain some information concerning the effect of the electron-electron interaction on hot electron problems.

§ 2. Boltzmann equation

We shall derive the Boltzmann equation for hot electrons under the following assumptions: (1) Electrons are not degenerate, (2) the shape of the energy surface is spherically symmetric, (3) electrons are scattered by acoustic and optical modes of vibrations, and by ionized impurity centers and (4) the electron-electron interaction is taken into account by the Fokker-Planck approximation.

Let us denote the distribution function by

$$f(\boldsymbol{k}) = f_0(E) + K_z g(E) \tag{1}$$

where $\boldsymbol{k}$ and E are the wave number vector and energy of conduction electrons, respectively, and z is the direction of the applied field. Then, the Boltzmann equation is

$$\left[\frac{\partial f}{\partial t}\right] \equiv \left[\frac{\partial f}{\partial t}\right]_F + \left[\frac{\partial f}{\partial t}\right]_{ac} + \left[\frac{\partial f}{\partial t}\right]_{op} + \left[\frac{\partial f}{\partial t}\right]_{imp} + \left[\frac{\partial f}{\partial t}\right]_{ee} = 0. \tag{2}$$

The first term which represents the effect of the applied field is given by

$$\left[\frac{\partial f}{\partial t}\right]_F = \frac{eF}{\hbar}\left(g(E) + \frac{2}{3}E\frac{dg}{dE} + \frac{\hbar^2}{m}K_z\frac{df_0}{dE}\right). \tag{3}$$

As shown in papers I and II, the second and third terms are given by

$$\left[\frac{\partial f}{\partial t}\right]_{ac} = \frac{A}{\sqrt{E}}\left\{E^2\frac{d^2f_0}{dE^2} + \left(\frac{E^2}{kT} + 2E\right)\frac{df_0}{dE} + \frac{2E}{kT}f_0 - \frac{K_z}{2mc^2}Eg(E)\right\}, \tag{4}$$

and

$$\begin{aligned}\left[\frac{\partial f}{\partial t}\right]_{op} = \frac{B}{\hbar\omega\sqrt{E}}\Big\{&[E(E+\hbar\omega)]^{1/2}[f_0(E+\hbar\omega)\exp(\hbar\omega/kT) - f_0(E)] \\ &+ \epsilon(E)[E(E-\hbar\omega)]^{1/2}[f_0(E-\hbar\omega) - f_0(E)\exp(\hbar\omega/kT)] \\ &- K_z g(E)\left([E(E+\hbar\omega)]^{1/2} + \epsilon(E)\exp(\hbar\omega/kT)[E(E-\hbar\omega)]^{1/2}\right)\Big\},\end{aligned} \tag{5}$$

where m is the effective mass of the conduction electron, c is the velocity of sound waves, $\hbar\omega$ is the characteristic energy of an optical phonon and $\epsilon(E)$ is a step function equal to one for $E > \hbar\omega$ and equal to zero otherwise. The constant A is connected with the ohmic acoustic mobility μ_a by the following relation:

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* K. Seeger investigated similar problems and obtained the result which is qualitatively the same as Gunn's.[7]

$$A=\frac{8ec^2}{3\sqrt{\pi kT}}\frac{1}{\mu_a}, \tag{6}$$

and the constant B is connected with the rate of the optical scattering. Later we shall determine μ_a and B/A from experimental datas. The rate of scattering due to the ionized impurities is given by

$$\left[\frac{\partial f}{\partial t}\right]_{\text{imp}}=\frac{4e}{3m\sqrt{\pi kT}}\frac{6(kT)^2}{\mu_i E\sqrt{E}}K_x g(E), \tag{7}$$

where μ_i means the mobility when only the ionized impurity is taken into account as scatterers. We shall use the Brooks-Herring formula for the ionized impurity scattering. Finally, we consider the electron-electron interaction which works to reduce the electron distribution to the Maxwellian form with the electron temperature. It is, however, a rather complicated task to derive a general formula of the electron-electron interaction for hot electrons. In fact, Spitzer and Härm[8] evaluated the ohmic mobility of a completely ionized gas by solving the Fokker-Planck equation. The generalization of this classical method to the non-ohmic case is possible and now in progress. However, we adopt here a simplified Fokker-Planck equation[9]:

$$\left[\frac{\partial f}{\partial t}\right]_{ee}=\frac{e}{m\mu_{ee}}\frac{\partial}{\partial v}\left\{vf_0+\frac{kT_e}{m}\frac{df_0}{dv}\right\}, \tag{8}$$

or

$$\left[\frac{\partial f}{\partial t}\right]_{ee}=\frac{2e}{3m\mu_{ee}}\frac{1}{\sqrt{E}}\frac{d}{dE}\left\{E^{3/2}f_0+kT_eE^{3/2}\frac{df_0}{dE}\right\}, \tag{9}$$

where a quantity μ_{ee} is introduced phenomenologically as a measure of the strength of the electron-electron interaction and it has a dimension of the mobility, and the electron temperature T_e is defined as $(3/2)kT_e=\langle E\rangle_{Av}$=the average kinetic energy of the electron gas. Inserting (3), (4), (5), (7) and (9) into (2) and equating the terms of the same angular dependence we have a set of coupled equations:

$$\begin{aligned}&\frac{A}{\sqrt{E}}\left[\left[E^2\frac{d^2f_0}{dE^2}+\left(\frac{E^2}{kT}+2E\right)\frac{df_0}{dE}+\frac{2E}{kT}f_0\right]\right.\\&+\frac{B}{\hbar\omega\sqrt{E}}\left\{\sqrt{E(E+\hbar\omega)}\left[f_0(E+\hbar\omega)\exp(\hbar\omega/kT)-f_0(E)\right]\right.\\&\left.\left.+\epsilon(E)\sqrt{E(E-\hbar\omega)}\left[f_0(E-\hbar\omega)-f_0(E)\exp(\hbar\omega/kT)\right]\right\}\right.\\&\left.+\frac{2e}{3m\mu_{ee}A}\frac{d}{dE}\left(kT_eE^{3/2}\frac{df_0}{dE}+E^{3/2}f_0\right)\right]=\frac{eF}{\hbar}\left(g+\frac{2}{3}E\frac{dg}{dE}\right),\end{aligned} \tag{10}$$

and

$$\left\{\frac{A\sqrt{E}}{2mc^2}+\frac{B}{\hbar\omega}[\sqrt{E+\hbar\omega}+\epsilon(E)\exp(\hbar\omega/kT)\sqrt{E-\hbar\omega}]\right.$$
$$\left.+\frac{4}{3m\sqrt{\pi kT}}\frac{6(kT)^2}{\mu_i E\sqrt{E}}\right\}g(E)=\left(\frac{eF\hbar}{m}\right)\frac{df_0}{dE}. \tag{11}$$

Eliminating $g(E)$ from these equations and after some computations, we have an equation for $f_0(E)$:

$$F[f_0(x)]\equiv\frac{d}{dx}\left\{x^2\frac{df_0}{dx}+x^2f_0+rq^2\left[x^{3/2}\left(\frac{T_e}{T}\right)\frac{df_0}{dx}+x^{3/2}f_0\right]\right\}$$
$$+\left(\frac{B}{As}\right)\left\{\sqrt{x(x+s)}[f_0(x+s)e^s-f_0(x)]+\epsilon(x)\sqrt{x(x-s)}[f_0(x-s)-f(x)e^s]\right\}$$
$$+\frac{d}{dx}\left\{[px^3/(x^2+q^2)]\frac{df_0}{dx}\right\}=0, \tag{12}$$

where $x=E/kT$, $p=3\pi(\mu_aF)^2/16c^2$, $q^2=6\mu_a/\mu_i$, $s=\hbar\omega/kT$ and $r=(\sqrt{\pi}/24)(kT/mc^2)\times(\mu_i/\mu_{ee})$. In the following section we shall investigate a solution of the equation for some special cases.

§ 3. Warm electron in *n*-type germanium at 77°K

The main purpose of the present paper is to discuss Gunn's experiment on the warm electron in *n*-type germanium at 77°K. For this purpose it is convenient to introduce a function $\xi(x)$ defined by

$$f_0(x)=a(1+p\xi(x))\exp(-x), \tag{13}$$

where $f_0(x)$ is assumed to be properly normalized. The function $\xi(x)$ is regarded as being independent of F. Inserting (13) into (12) and retaining quantities up to the second order of F, we obtain the following equation to be satisfied by the function $\xi(x)$:

$$L[\xi(x)]\equiv\frac{d}{dx}\{x^2\xi'(x)e^{-x}\}+rq^2\frac{d}{dx}\left\{x^{3/2}e^{-x}\left(\xi'(x)-\frac{T_e-T}{Tp}\right)\right\}$$
$$+\frac{B}{As}\left\{\sqrt{x(x+s)}[\xi(x+s)-\xi(x)]e^{-x}+\epsilon(x)e^s\sqrt{x(x-s)}[\xi(x-s)-\xi(x)]e^{-x}\right\}$$
$$-\frac{d}{dx}\left[\frac{x^3}{x^2+q^2}e^{-x}\right]=0. \tag{14}$$

Here we regard the quantity $(T_e-T)/T$ as a second order quantity of F. Another part of the distribution function is simply written as

$$g(x)=\frac{eF\hbar}{m}\frac{x^2}{x^2+q^2}\frac{1}{\sqrt{x}}\frac{df_0}{dx}, \tag{15}$$

because the effect of the optical scattering on the mobility can be ignored at this low temperature.

First, let us consider an ideal case in which the impurity scattering can be ignored, then Eq. (14) is reduced to

$$L_0[\xi(x)]=\frac{d}{dx}\{x^2\xi'(x)e^{-x}\}+\frac{B}{As}\left\{\sqrt{x(x+s)}[\xi(x+s)-\xi(x)]e^{-x}\right.$$
$$\left.+\epsilon(x)e^{s}\sqrt{x(x-s)}[\xi(x-s)-\xi(x)]e^{-x}\right\}-\frac{d}{dx}\{xe^{-x}\}=0. \qquad (16)$$

This equation is identical with Eq. (19) in I, where we got an approximate solution of the equation by reducing it to a differential equation. This process may be regarded as a tolerable approximation at high temperatures, but not accurate at low temperatures. Here we shall adopt another approach. In paper II we showed that an integral

$$\int E\left[\frac{\partial f}{\partial t}\right]d\boldsymbol{p}\equiv\int E\left\{\left[\frac{\partial f}{\partial t}\right]_F+\left[\frac{\partial f}{\partial t}\right]_{ac}+\left[\frac{\partial f}{\partial t}\right]_{op}\right\}d\boldsymbol{p}=0 \qquad (17)$$

was equivalent to the condition of the energy conservation of the electron system:

$$\int E\left\{\left[\frac{\partial f}{\partial t}\right]_{ac}+\left[\frac{\partial f}{\partial t}\right]_{op}\right\}d\boldsymbol{p}=\sigma F^2, \qquad (18)$$

which has been used by Stratton[10] and others to discuss various hot electron problems. Since there is only one available condition here, the trial function $f_0(x)$ can contain only one adjustable parameter, that is, the electron temperature. However, the generalization of this approach to many adjustable parameters is quite easy. These conditions are

$$\int E^n\left[\frac{\partial f}{\partial t}\right]d\boldsymbol{p}=0 \quad (n=1, 2, \cdots s), \qquad (19)$$

which can determine s parameters in the trial distribution function $f_0(x;\ \alpha, \beta, \cdots)$.* In practice, we have used the following trial functions:

$$\left.\begin{aligned}&(\mathrm{I}) \quad \xi(x)=bx,\\ &(\mathrm{II}) \quad \xi(x)=bx+cx^2,\\ &(\mathrm{III}) \quad \xi(x)=bx+cx^2+dx^3,\\ &(\mathrm{IV}) \quad \xi(x)=bx+cx^2+dx^3+ex^4.\end{aligned}\right\} \qquad (20)$$

The coefficients b through e are independent of F and are determined by the conditions

$$\int_0^\infty x^n L_0[\xi(x)]dx=0. \quad (n=1, 2, 3, 4) \qquad (21)$$

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* This approach is derived also from a variational procedure.[11]

In these simple cases the above conditions are reduced to a set of coupled linear equations. We easily see that the assumption (I) in (20) is equivalent to taking

$$f_0(E)=a\exp(-E/kT_e)\fallingdotseq a\exp(-x)\left(1+\frac{T_e-T}{T}x\right), \tag{22}$$

because the parameter (T_e-T) is a quantity of the order of F^2. The constant a in (13) is determined by the normalization condition

$$\int_0^\infty f(x)\sqrt{x}\,dx=1, \tag{23}$$

and it becomes

$$a=\frac{2}{\sqrt{\pi}}\left(1-p\left[\frac{3}{2}b+\frac{15}{4}c+\frac{105}{8}d+\frac{945}{16}e\right]\right). \tag{24}$$

Next, the expression for the non-ohmic mobility is necessary. By using well-known procedures we obtain a formula:

$$\mu=\mu_a\left(1-p\left[\frac{1}{2}b+\frac{7}{4}c+\frac{57}{8}d+\frac{561}{16}e\right]\right). \tag{25}$$

If we denote the conductivity of the warm electrons as

$$\sigma=\sigma_0(1+\beta F^2), \tag{26}$$

then

$$\beta=-(2.544\times10^{-3})\left(\frac{1}{2}b+\frac{7}{4}c+\frac{57}{8}d+\frac{561}{16}e\right). \tag{27}$$

Before entering into the computation of β we must determine numerical values of the physical quantities. We choose: $c=5.4\times10^5$ cm sec and $\theta=\hbar\omega/k=400$°K. The acoustic mobility at 77°K is determined as 35500 cm^2/ volt sec from Morin's observation and the value of $\boldsymbol{J}$ defined by

$$\left(\frac{2mc^2}{\hbar\omega}\right)\frac{B}{A}=Js/(e^s-1) \tag{28}$$

is determined so as to fit the observed temperature depenpence of the mobility between 90°K and room temperature. The value of $\boldsymbol{J}$ is determined as 0.123. In order to determine the ratio (B/A) from $\boldsymbol{J}$ it is necessary to determine the effective mass m. We assume that it is determined by a relation $1/m=(1/3)(1/m_1+2/m_2)$ and the observed values of the cyclotron mass.* Then, the value of m is determined as 0.12 m_0 and the value of B/A at 77°K becomes 3.45. From

* It is, however, rather difficult to define the effective mass uniquely, because there is no general relation between the anisotropic masses in a real crystal and the idealized single mass m in (28). If we use, for example, a relation $m=(m_1\,m_2^2)^{1/3}$ instead of the previous one, then the value of the effective mass becomes 0.22 m_0.

these values we see that the field intensity which corresponds to $p=1$ is about 20 volt/cm. By using these values and (22) and (25) we compute the value of β for four trial functions shown in (20). The results are: $\beta_1=-0.30\times10^{-4}$, $\beta_2=-1.1\times10^{-4}$, $\beta_3=-1.5\times10^{-4}$ and $\beta_4=-2.0\times10^{-4}$. No experiments are available to test the theory.

Next, let us investigate the cases in which the impurity scattering is taken into account. Then, Eqs. (17), (25) and (27) are modified as

$$L_1[\xi(x)]\equiv\frac{d}{dx}\{x^2\xi'(x)e^{-x}\}+\frac{B}{As}\left\{\sqrt{x(x+s)}[\xi(x+s)-\xi(x)]e^{-x}\right.$$

$$\left.+\epsilon(x)e^{s}\sqrt{x(x-s)}[\xi(x-s)-\xi(x)]e^{-x}\right\}-\frac{d}{dx}\left\{\frac{x^3}{x^2+q^2}e^{-x}\right\}=0, \quad (29)$$

$$\mu/\mu_a=I_3\{1-pQ\} \quad (30)$$

and

$$\beta=-(2.544\times10^{-3})Q, \quad (31)$$

respectively, where

$$Q=\left(\frac{5}{2}-\frac{I_4}{I_3}\right)b+\left(\frac{15}{4}+\frac{2I_4}{I_3}-\frac{I_5}{I_3}\right)c+\left(\frac{105}{8}+\frac{3I_5}{I_3}-\frac{I_6}{I_3}\right)d$$

$$+\left(\frac{945}{16}+\frac{4I_6}{I_3}-\frac{I_7}{I_3}\right)e \quad (32)$$

and

$$I_n=\int_0^\infty x^n e^{-x}/(x^2+q^2)\cdot dx. \quad (33)$$

Again, we use the trial functions (20), and the coefficients b through e are determined by the conditions

$$\int_0^\infty x^n L_1[\xi(x)]dx=0. \quad (n=1, 2, 3, 4).$$

The results are shown in Fig. 1, where curve I corresponds to the trial function I and curve IV corresponds to the trial function IV in (20). Observed data of Gunn are also shown by crosses and lines. The theory explains the change of the sign of β with increasing impurity density.*

The sign of Q is mainly determined by that of the first term in (32). When q is very small, the value of I_4/I_3 is nearly equal to 2, while it approaches 4 when q becomes very large. There is a value of q in which the ratio I_4/I_3 is

* The computation shows that β is fairly sensitive to the impurity density. When $q=0.01$, which corresponds roughly to the impurity density of $8\times10^{12}\,\mathrm{cm}^{-3}$, the ohmic mobility is reduced only by 2% from the acoustic mobility, while the value of β becomes 0.88 of the ideal case.

5/2. The other terms in Q show a similar tendency, but their contribution to the sign of Q is much smaller. When the density of impurities is small, the mobility is mainly determined by the lattice scattering. Then, the rate of the lattice scattering increases with increasing electron temperature and the mobility becomes smaller with increasing applied field. That is the reason why the sign of β is negative in comparatively pure specimens. On the other hand, when the density of impurity ions is large, the mobility is mainly determined by the impurity scattering. Since the rate of this type of scattering decreases with increasing electron temperature, the sign of β is positive in the specimens where the impurity scattering is dominant. In the sample where the effects of the both types of scattering are in a balance the value of β is expected to be extremely small.

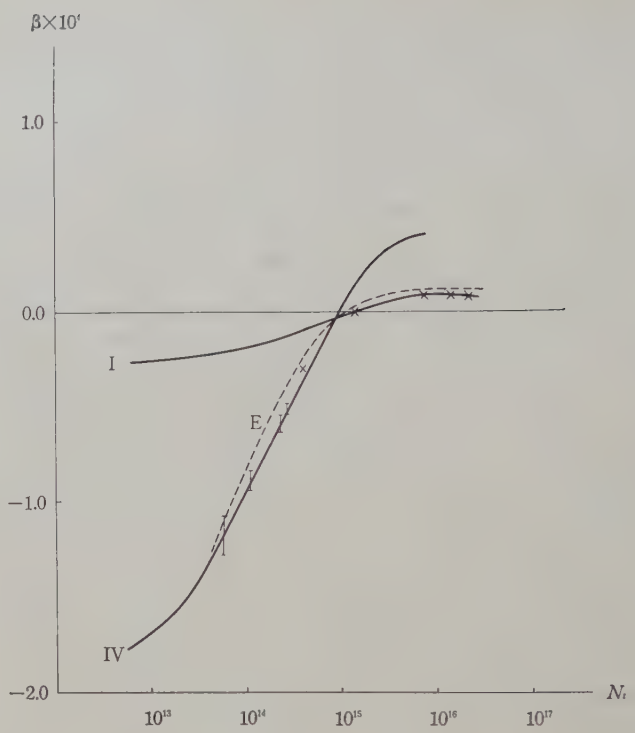

Fig. 1

By comparing the theory with experiment we see that the curve IV agrees well with the observation in the region of small impurity density. On the other hand, the agreement between curve I and the observations is found to be very poor. Although the numerical assignment of the effective mass is not unique and the accuracy of the approximation adopted is not quite clear, it is almost certain that the approximate distribution function $\exp(-E/kT_e)$ is not applicable in this region. In the region of high impurity density, that is, $N_i \gtrsim 10^{15}$, the situation becomes reverse. Curve I agrees quite well with the observations, while curve IV gives too large values to fit the observed data. In the latter the discrepancy is so remarkable, that there is little doubt in believing that the distribution function IV is inadequate for the cases of high impurity density. This situation gives a suggestion that the electron-electron interaction is so important in this region that the conduction electrons take approximately the Maxwellian distribution with an electron temperature.

In order to get a more reliable support for this speculation, let us proceed to the next approximation, in which the electron-electron interaction is explicitly taken into account. Then, we consider $L[\tilde{\xi}(x)]$ in (14) instead of (16) or (29) and

adopt the trial function IV in (20). The coefficients b through e are determined by the conditions

$$\int_0^\infty x^n L[\xi(x)]dx=0. \quad (n=1, 2, 3, 4.) \tag{34}$$

The non-ohmic mobility and β are assumed to be given by (30) and (31), respectively. In fact, the electron-electron interaction causes a slight change of the mobility by modifying the function I_3. However, since this effect seems to be comparatively small, we ignored it approximately. Unfortunately, Eq. (34) contains a phenomenological constant μ_{ee}. Here we regard it as an adjustable parameter and choose the value so as to get a good fit to the observations through all range of the impurity density. In Fig. 1 we show, as an example, the computed result with an assumption $\mu_{ee}=\mu_i$ (curve E). The value of $\boldsymbol{r}$ is found to be 28. We see that the agreement between theory and experiment is fairly good. Although it remains untouched to derive β from the first principle and compare it with experiment, the assumption that μ_{ee} is nearly equal to μ_i seems to be reasonable.* Therefore, we think that the electron-electron interaction works decisively in reducing the electron distribution to Maxwellian form with the electron temperature when the density of the conduction electrons is above $10^{15}/\text{cm}^3$ at 77°K.

At room temperature the ratio of μ_i and the ohmic mobility becomes about one hundred times larger than that at 77°K, so that the value of $\boldsymbol{rq}^2$ is expected to reduce about 1/25. Therefore, the impurity effect on β is expected to become conceivable at the impurity density of $10^{15}/\text{cm}^3$ and β may change the sign at the density of about $10^{17}/\text{cm}^3$. On the other hand, if such measurements as made by Gunn are performed at low temperature, 20°K~30°K, the mathematical treatment becomes much easier, because the effect of the optical scattering on the energy loss may be negligibly small, and then the distribution function is easily found to be

$$f_0(x)=a\{1+p\xi(x)\}\exp(-x), \tag{35}$$

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* Notes: (1) The electron temperature is easily evaluated as: $(T_e-T)/Tp=b+5c+210d/8+2520e/16$. (2) Since $(\partial f/\partial t)_{ee}=0$ is always satisfied by the function $f_0(E)=\exp(-E/kT_e)$, the terms in $(\partial f/\partial t)_{ee}$ contribute nothing to the coefficient of b, but they have finite contributions to the coefficients of c, d and e. Therefore, if rq^2 is sufficiently large, the coefficients of c, d and e in (34) become large as compared with those of b, and consequently the values of c, d and e are much reduced. (3) If we adopt the value of rq^2 as 28, the effect of the electron-electron interaction on β is rather considerable. Even when $N_i\sim1.5\times10^{14}$ and $3.8\times10^{14}/\text{cm}^3$, the interaction reduces the value of β from 8.2×10^{-4} to 6.4×10^{-4} in the former case and from 3.9×10^{-4} to 2.7×10^{-4} in the latter. (4) If we choose μ_{ee} sufficiently small as compared with μ_i, it leads to a good fit in the region of $N_i\gtrsim10^{15}$, but a poor fit in the region of $N_i\ll10^{15}$. On the contrary, a sufficiently large value of μ_{ee} leads to an opposite situation. Thus, we could not find a value of μ_{ee} that gives a very good fit for all range of the impurity density. It is not clear whether this comes from the oversimplified treatment of the electron-electron interaction mentioned previously, or other mechanisms, for example, inelastic impurity scatterings, work to reduce the value of β at high density.

$$\xi(x)=\int_0^x x^{3/2}dx/(x^2+q^2)(\sqrt{x}+r'q^2)+r'q^2\left(\frac{T_e-T}{Tp}\right)\int_0^x dx'(\sqrt{x}+r'q^2),$$

where $\boldsymbol{r}'$ is defined by $\boldsymbol{r}'=\boldsymbol{r}(n/N_i)$. Here n is the number of conduction electrons at that temperature. We see that the condition that the distribution function is approximately given by $\exp(-E/kT_e)$ is $\boldsymbol{r}'\boldsymbol{q}^2\gg 1$.

§ 4. Acoustic hot electron

Here we wish to discuss Gunn's experiments about the drift velocity of the hot electrons in n-type germanium at 77°K. In the sample of small impurity concentration ($N_i\sim 1.4\times 10^{14}\mathrm{cm}^{-3}$) the variation of the drift velocity with applied field exhibits a usual feature. On the contrary, the sample of large impurity concentration ($N_i\sim 2.9\times 10^{16}\mathrm{cm}^{-3}$) shows reverse characteristics. The drift velocity vs. the field relation begins to deviate upwards from the ohmic line and the drift velocity continues to be larger than the ohmic one up to the field intensity of about 10^3 volt/cm, where it crosses the ohmic line, and then it becomes smaller than the ohmic velocity. These features were well predicted by the theory. When the field is not very large, the impurity scattering performs a main role and it decreases gradually with increasing electron temperature. However, when the electron temperature becomes considerably large, the impurity scattering is almost dying out, and the lattice scattering becomes predominant. Further, we expected that the drift velocity of a less pure sample was always smaller than that of a purer sample, but the former would gradually converge to the latter with increasing electron temperature. Gunn's experiment, however, is against this prediction. According to his experiment the drift velocity vs. field curves cross each other at a certain value of the field intensity and above this critical field the drift velocity of a less pure sample becomes larger than that of a purer sample.*

In this section we shall investigate the possibility of the "crossing" for the simplified model, in which the optical scattering is ignored. Then, the fundamental equation (12) is reduced to a differential equation and its solution is given by

$$f_0(x)\sim\exp\left[-\int^x (y+rq^2\sqrt{y})\,dy\Big/\left\{y+\left(\frac{T_e}{T}\right)rq^2\sqrt{y}+\frac{py^2}{y^2+q^2}\right\}\right]. \qquad (36)$$

Here we insert the electron temperature into a logarithmic term in q, which is computed by a trial function: $f_0(E)=\exp(-E/kT_e)$. If we ignore the effect of the electron-electron interaction, then we have

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* In this low temperature the anisotropy of the hot electron current is expected to be fairly large. We do not know the details of the experiment concerning the direction of the applied field.

$$f_0(x) \sim \exp(-x)\,(x^2+px+q^2)^{p/2} \times \exp\left\{-\frac{p^2}{2}\tan^{-1}\left[x+\frac{p}{2}\Big/\left(q^2-\frac{p^2}{4}\right)^{1/2}\right]\Big/\left(q^2-\frac{p^2}{4}\right)^{1/2}\right\}, \quad \text{if } q^2>p^2/4, \tag{37}$$

and

$$f_0(x) \sim \exp(-x)\,(x^2+px+q^2)^{p/2} \times\left[\left\{x+\frac{p}{2}+\left(\frac{p^2}{4}-q^2\right)^{1/2}\right\}\Big/\left\{x+\frac{p}{2}-\left(\frac{p^2}{4}-q^2\right)^{1/2}\right\}\right]^{p^2/4(p^2/4-q^2)^{1/2}}, \quad \text{if } p^2/4>q^2. \tag{38}$$

Further, if we ignore q^2 in comparison with $p^2/4$ and px, then we have a previous result:

$$f_0(x) \sim (x+p)^p \exp(-x). \tag{39}$$

Therefore, we see that the distribution function is given by $\exp(-E/kT_e)$ when the impurity density is large, and it is given by $(E/kT+p)^p\exp(-E/kT)$ when the sample is pure. As shown in the previous paper[12] the conductivity of the acoustic hot electron is given by an approximate formula:

$$\sigma/\sigma_0=1.078\,p^{-1/4},$$

when the field is large. On the other hand, the distribution function $\exp(-E/kT_e)$ gives the value of the conductivity:

$$\sigma/\sigma_0=1.189\,p^{-1/4},$$

when the field is large. Thus, it is possible in the acoustic hot electrons that the drift velocity of the impure sample becomes by about 10% larger than that of the pure sample when the field intensity is sufficiently large.*

§ 5. Optical hot electron

The model that we considered in the previous section is too much simplified to apply to a real case, so we cannot put much confidence in the conclusion obtained there. It is, however, a rather laborious task to get a reliable distribution function in real cases, when the electron temperature is not very high as compared with the characteristic temperature. We suppose, however, that the distribution of the sample $N_i \sim 2.9\times10^{16}\text{cm}^{-3}$ is well represented by the simple function $\exp(-E/kT_e)$, because the electron-electron interaction is sufficiently strong there. This prediction is justified by comparing the computed value of the drift velocity with the

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* The electron-electron interaction works to reduce the mobility,[13] but this effect may be quite small, when the electron temperature is very high.

experiment. The agreement between theory and experiment is very good.* As a result of the computation we find that the impurity scattering is still effective even when the field intensity is above 10^3 volt/cm, and the drift velocity does not saturate at all. This behaviour agrees well with the observation. On the contrary, the computed drift velocity is in poor agreement with the experiment for the sample of $N_i \sim 1.4 \times 10^{14} \mathrm{cm}^{-3}$, when the exponential trial function is adopted. The discrepancy between the theoretical and experimental values amounts to $20\% \sim 30\%$ in the field intensity of $300 \sim 2000$ volt/cm. It is supposed that the electron-electron interaction is not strong enough to reduce the electron distribution to the Maxwellian form. If we use the gaussian function $f_0(E) = \exp[-E^2/(kT_e)^2]$ as a trial function, the agreement is slightly improved but is not yet satisfactory. (The discrepancy amounts to $10\% \sim 20\%$.) In another sample with the intermediate impurity density ($N_i \sim 7.8 \times 10^{15} \mathrm{cm}^{-3}$) the agreement is better. The difference between the observed and computed values is not over ten percent in the field intensity of $300 \sim 2000$ volt/cm when the exponential function is used as a trial distribution function. In this paper we did not make an effort to obtain an accurate distribution function for the sample of $N_i \sim 1.4 \times 10^{14} \mathrm{cm}^{-3}$, so we cannot prove the possibility of the "crossing" due to the electron-electron interaction. However, the analysis mentioned in this section clearly shows that the nature of the hot electrons depends on the impurity density, and there exists a qualitative difference between the sample of high and low impurity density. Thus, we see that the results obtained in this section are consistent with the conclusion in Section 3.

Finally, the author wishes to express his thanks to Mr. H. Yamagishi for his kind help in numerical work.

Appendix

The explicit expressions of Eqs. (21) are given as follows.

$$(\alpha_{ij}) \cdot \begin{pmatrix} s\,b \\ s^2 c \\ s^3 d \\ s^4 e \end{pmatrix} = \begin{pmatrix} 1/s^2 \\ 4/s^3 \\ 18/s^4 \\ 96/s^5 \end{pmatrix}$$

where

$$\alpha_{11} = 2/s^3 + (B/A) I_0/s^2,$$

$$\alpha_{12} = \alpha_{21} = 12/s^4 + (B/A)(2I_1 + sI_0)/s^3,$$

* Here we compare the computed values of the ratio of the drift velocity at some field intensity to the ohmic drift velocity at the same field intensity with the experiment, because the measured curve of $\mu_n(N_i)$ does not show a complete agreement with the calculated value from the Conwell-Weisskopf expression or the Brooks-Herring expression. The discrepancy is within 10 percent, but it makes impossible the comparison in the absolute value of the hot drift velocity.

$$\alpha_{13}=\alpha_{31}=72/s^5+(B/A)\,(I_2+3sI_1+s^2I_0)/s^4$$

$$\alpha_{14}=\alpha_{41}=480/s^6+(B/A)\,(4I_3+6sI_2+4s^2I_1+s^3I_0)/s^5,$$

$$\alpha_{22}=96/s^5+(B/A)\,(4I_2+4sI_1+s^2I_0)/s^4,$$

$$\alpha_{23}=\alpha_{32}=720/s^6+(B/A)\,(6I_3+9sI_2+5s^2I_1+s^3I_0)/s^5$$

$$\alpha_{24}=\alpha_{42}=5760/s^7+(B/A)\,(8I_4+16sI_3+14s^2I_2+6s^3I_1+s^4I_0)/s^6,$$

$$\alpha_{33}=8640/s^7+(B/A)\,(9I_4+18sI_3+15s^2I_2+6s^3I_1+s^4I_0)/s^6,$$

$$\alpha_{34}=\alpha_{43}=60480/s^8+(B/A)\,(12I_5+30sI_4+34s^2I_3+21s^3I_2+7s^4I_1+s^5I_0)/s^7,$$

$$\alpha_{44}=645120/s^9+(B/A)\,(16I_6+48sI_5+68s^2I_4+56s^3I_3+28s^4I_2+8s^5I_1+s^6I_0)/s^8,$$

and

$$I_n=\int_0^\infty x^n[x\,(x+s)]^{1/2}\exp(-x)\,dx.$$

In the first approximation the mobility is given by

$$\mu=\mu_a\left(1-\frac{p}{2}\left[2+\frac{Bs}{A}\int_0^\infty[x\,(x+s)]^{1/2}\exp(-x)\,dx\right]\right).$$

The same formula in II is in error by factor 2.

References

1) J. Yamashita, Phys. Rev. **111** (1958), 1529.
2) J. Yamashita and K. Inoue, J. Phys. Chem. Solids **12**(1959), 1.
3) See also E. Conwell, J. Phys. Chem. Solids **8** (1959), 234; T. N. Morgan, J. Phys. Chem. Solids **8** (1959), 245; H. Sato, J. Phys. Soc. Japan **14** (1959), 1275.
4) W. Sasaki, M. Shibuya and K. Mizuguchi, J. Phys. Soc. Japan **13** (1958), 456.
5) G. Weinreich, T. M. Sanders and H. G. White, Phys. Rev. **114** (1959), 33.
6) J. B. Gunn, J. Phys. Chem. Solids **8** (1959), 329.
7) K. Seeger, Phys. Rev. **114** (1959), 476; Zeits. f. Phys. **156** (1959), 582.
8) L. Spitzer and R. Härm, Phys. Rev. **89** (1953), 977.
9) R. Becker, *Theorie der Wärme* p. 272.
10) R. Stratton, Proc. Roy. Soc. London **A 242** (1957), 355.
11) I. Adawi, Phys. Rev. **115** (1959), 1152.
12) J. Yamashita and M. Watanabe, Prog. Theor. Phys. **12** (1954), 443.
13) M. S. Sodha and P. C. Eastman, Zeits. f. Phys. **150** (1958), 242.

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Photodisintegration of the Deuteron at High Energies

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The exchange current effects in the photodisintegration of the deuteron at high energies are discussed. Special consideration is given to (i) correlation of the exchange effects with the large direct E1 transition, whose behavior has been the object of the recent several investigations, (ii) consequences of gauge invariance for the electric transitions due to exchange effects, (iii) evaluation of the magnetic transition caused by the (33)-resonance in terms of the results of the Chew-Low theory. It is shown that exchange effects affect the electric dipole transition operator in a way consistent with the Siegert theorem. The resulting change in the electric dipole cross section is evaluated in the adiabatic approximation as well as in neglect of higher order retardation effect. It is appreciable only at photon energies above the pion threshold. Some characteristics of the (33)-resonance effect are revealed, although nucleon-recoil effects and uncertainty of nuclear force make it difficult to give any quantitative estimation. Finally the results obtained here are compared with experiment.

§ 1. Introduction

Recently the progress of knowledge about nuclear force has motivated several detailed reviews[1)-5)] on the photodisintegration of the deuteron at about 80 Mev photon energy. As the result it has become clear that the experimental data can be well explained in terms of E1 and E2 transitions alone, provided that we use a somewhat large value to the D-state probability of the deuteron as well as the phase shifts fit for the p-p scattering data to the final continuum states. It should be noted that any necessity to invoke specific exchange current effects is not felt.

Subsequently, the analysis is extended on the same line to the higher energy region.[6),7)] This shows that the transitions from the deuteron D-state make the actual E1 cross section considerably larger than the Schiff and Marshall-Guth estimation.[8)]

Since the contribution of the direct (i.e. not involving exchange current interaction) E1 transition is recognized to be remarkable even in the high energy region, it is desirable to investigate the effects of pion-exchange-current, taking into account their correlation with the direct transition.

In this paper we shall discuss the exchange current effects in the region of photon energy 100 Mev to 350 Mev, and see where the previous discussions on the exchange current interaction must be supplemented or amended by the existence of the large direct E1 transition. For simplicity our discussion is confined to the contribution of one-pion-exchange processes.

In § 2 we formulate the direct process in a summarized form. The formulation is also convenient for the description of exchange effects.

Then, in § 3, exchange effects on the electric transitions are discussed. Here attention is called to the Siegert theorem, which is proved under certain conditions,[9] but not so exhaustively as to include a complex case discussed here. In examining this point the form of interaction of electromagnetic field with matter is changed from the usual one by means of gauge transformation. Then unambiguous identification of proper exchange effects is achieved and the theorem is easily checked.

It should be noted that the usual form of the electromagnetic interaction often leads to violation of the Siegert theorem. The ensuing result is an overestimation of the exchange effects. In fact it must be ascribed to this fact that previously some authors[10),11] deduced rather large influence of the exchange current even at energies below the pion threshold.

The modification of the electric transition cross section due to exchange effects are evaluated. The cross section is somewhat increased at energies above the pion threshold. Note that this result is in contrast to the one obtained by Bruno-Depken ;[10] they find that the direct transition and the transition due to exchange effects interfere destructively.

In § 4 the magnetic transition through intermediate states which involve a resonant pion in the (33)-state is discussed. Although the effect has already been discussed by several authors,[12)–15] it is interesting to formulate it in terms of the result of the Chew-Low theory.[16] The transition is found to be seriously affected by nucleon-recoil effects and the behaviour of the Fourier transform of nuclear potential at large momenta. Therefore the results obtained are qualitative rather than quantitative.

§ 5 is devoted to discussion of our results. A brief remark on the angular distribution is also given.

Throughout this paper the final state interaction is neglected. Since the final nucleons have in most cases such high energy that the applicability of usual nuclear potential is very much questionable, we consider this approximation reasonable at least at the first step. Another approximation used here is that we consider only the E1, M1 and E2 transitions. This approximation is not so bad as we see it at first sight, because the multipole expansion turns out to correspond to an expansion in the ratio of the photon momentum to the relative momentum of the final nucleons rather than the ratio of the deuteron dimension to the photon wavelength.

§ 2. Direct processes

In the following we summarize the transition amplitudes for the direct process shown in Fig. 1 in a form suitable for the present discussion.[17] These expressions

turn out, in subsequent sections, to be useful to the classification of exchange effects in terms of different multipoles.

Let the transition operator T be the quantity whose matrix element referred to the deuteron state as the initial state and to a continuum state as the final state gives the transition amplitudes. For the direct process it is given by

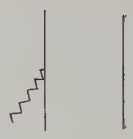

Fig. 1.
—— Nucleon
~~~~ Photon

$$T_{direct}(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}, \boldsymbol{x}) = \left(\frac{\boldsymbol{\tau}^{(1)}-\boldsymbol{\tau}^{(2)}}{2}\right)_3 \sqrt{\frac{2\pi}{\kappa}} \left\{ -\frac{e\kappa}{2\mu} i(\boldsymbol{\varepsilon}\cdot\boldsymbol{x}) \right.$$

$$\left. + \frac{e}{4M}(g_p - g_n) i(\boldsymbol{\sigma}^{(1)}-\boldsymbol{\sigma}^{(2)})\cdot\boldsymbol{\varepsilon}\times\boldsymbol{\kappa}) \right\}, \qquad (2\cdot1)$$

corresponding to the E1 and M1 transitions. Here $\boldsymbol{\varepsilon}$ and $\boldsymbol{\kappa}$ are the polarization vector and momenta of the incident quanta, $\boldsymbol{x}$ the relative distance between nucleons in units of the pion Compton wavelength μ^{-1} : $\boldsymbol{x}=\mu(\boldsymbol{r}_1-\boldsymbol{r}_2)$. $g_p=2.78$, $g_n=-1.91$ the nucleon magnetic moments. The other part of the notation is as usual.

In (2·1) we have already approximated the retardation factor $\exp\{\pm i(\boldsymbol{\kappa}\cdot\boldsymbol{r})/2\}$ by $1(\boldsymbol{r}\equiv\boldsymbol{r}_1-\boldsymbol{r}_2)$. As mentioned above, we use a plane wave $e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$ as the final state wave function. Therefore the above approximation implies to neglect $\kappa/2k$ compared with 1. As the ratio $\kappa/2k$ is rather small, for example, $\lesssim 0.3$ at $E_\gamma \approx 300$ Mev, this approximation is considered to be reasonable.

From (2·1) we calculate the transition amplitudes using the deuteron wave function obtained from the pion theoretical potential.[18] Then transition amplitudes into triplet odd states are given by*

$$\left\{\begin{aligned}
&\int \psi_f^{+} T_{direct} \psi_i \, d\boldsymbol{r} = g_t \left\{ A + iB(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)})\cdot\boldsymbol{\varepsilon}\times\boldsymbol{n}) \right. \\
&\qquad \left. + C[(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\varepsilon})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{n}) + (\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{n})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{\varepsilon})] + D(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{n})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{n}) \right\}, \\
&g_t = \left[\frac{e^2}{\hbar c}\cdot\frac{\hbar c \pi^2 E_\gamma}{k^2\mu^3}\right]^{1/2}, \\
&A = -\frac{(\boldsymbol{\varepsilon}\cdot\boldsymbol{n})}{3\sqrt{2}}\left(L_0 + \frac{3}{2}L_1 + \frac{7}{2}L_2 + \frac{3}{2}L_F\right), \\
&B = -\frac{1}{6\sqrt{2}}\left(L_0 + \frac{3}{2}L_1 - \frac{5}{2}L_2\right), \\
&C = \frac{1}{6\sqrt{2}}\left(L_0 - \frac{3}{2}L_1 + \frac{1}{2}L_2 - 3L_F\right), \\
&D = \frac{5}{2\sqrt{2}}(\boldsymbol{\varepsilon}\cdot\boldsymbol{n})L_F,
\end{aligned}\right. \qquad (2\cdot2)$$

* In the following expressions $\hbar$ and c are explicitly written for convenience' sake.
~~~~

$$\begin{cases} L_0=J_s-\sqrt{2}\,J_d, \quad L_1=J_s+\dfrac{1}{\sqrt{2}}J_d, \quad L_2=J_s-\dfrac{\sqrt{2}}{10}J_d, \\ J_s=\displaystyle\int_0^\infty dx\,x u(x)[zj_1(z)]_{z=kr}, \quad J_d=\int_0^\infty dx\,x w(x)[zj_1(z)]_{z=kr}, \\ L_F=\dfrac{3\sqrt{2}}{5}\displaystyle\int_0^\infty dx\,x w(x)[zj_3(z)]_{z=kr}. \end{cases}$$

Transition amplitudes into singlet even states read

$$\begin{cases} \int \psi_f^{+} T_{direct}\psi_i\,d\boldsymbol{r}=ig_s(\boldsymbol{\sigma}^{(1)}-\boldsymbol{\sigma}^{(2)})\cdot\left\{(\boldsymbol{\varepsilon}\times\boldsymbol{m})\left(M_0-\dfrac{M_2}{\sqrt{2}}\right)\right. \\ \qquad \left.+\dfrac{3}{\sqrt{2}}\boldsymbol{n}(\boldsymbol{\varepsilon}\times\boldsymbol{m}\cdot\boldsymbol{n})M_2\right\}, \\ g_s=\left[\dfrac{e^2}{\hbar c}\cdot\dfrac{c\hbar^3\pi^2}{2\mu Mk^3}\cdot\dfrac{E_\gamma}{Mc^2}\right]^{1/2}, \\ M_0=(g_p-g_n)\displaystyle\int_0^\infty dx\,u(x)[zj_0(z)]_{z=kr}, \\ M_2=(g_p-g_n)\displaystyle\int_0^\infty dx\,w(x)[zj_2(z)]_{z=kr}. \end{cases} \tag{2·3}$$

The resulting cross sections are as follows:

$$\begin{cases} \sigma_t=\dfrac{\pi}{27}\dfrac{e^2}{\hbar c}\dfrac{ME_\gamma}{\hbar^2 k}\dfrac{1}{\mu^3}\left\{L_0^2+3L_1^2+5L_2^2+\dfrac{15}{2}L_F^2\right\}, \\ \sigma_s=\dfrac{\pi}{3}\dfrac{e^2}{\hbar c}\dfrac{E_\gamma}{Mc^2}\dfrac{1}{k\mu}\{M_0^2+M_2^2\}. \end{cases} \tag{2·4}$$

Here $\hbar k$ is the relative momentum of the final nucleons, $E_\gamma=\hbar c\kappa$ the photon energy, $\boldsymbol{m}=\boldsymbol{\kappa}/\kappa$, $\boldsymbol{n}=\boldsymbol{k}/k$; $L_{0,1,2,F}$ the E1 transition amplitudes when the final states are ${}^3P_{0,1,2}$, ${}^3F_2$, respectively; $M_{0,2}$ the M1 transition amplitudes with the final states ${}^1S_0$, ${}^1D_2$, respectively. The factor g_t, g_s are the normalization factors, and the definition of the deuteron wave function is the same as in reference 18).

The contribution of the direct process to the cross section is shown in Fig. 2 (Curve I and Curve II). Our calculation of the E1 transition is different from that of Schiff and Marshall-Guth[8] only in the inclusion of the transitions from the D-state. But, as was discussed originally by Austern[19] and further by Hsieh[6] and Matsumoto,[7] the transitions from the D-state contribute essentially to the enhancement of direct E1 transition at high energies.

Some remarks are necessary about the M1 transition. We put $M_0=0$. Although this amplitude cannot be estimated properly without consideration of the

final state interaction, it will be unimportant anyway. On the other hand, the neglect of final state interaction in the calculation of M_2 is justified to some extent. In this case the effect of nuclear force is much weakened by the centrifugal force of D-state.

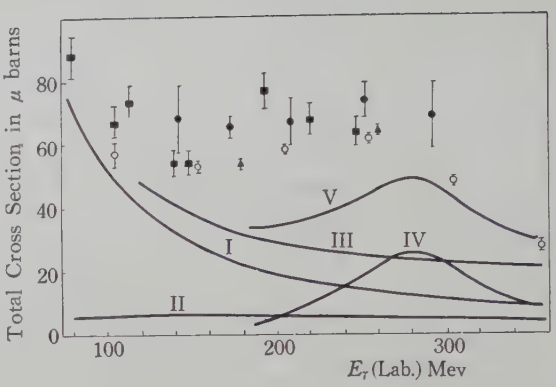

Fig. 2. Calculated contributions of various transitions to the total cross section. Curve I: direct E1 transition. Curve II: direct M1 transition. Curve III: electric transitions including pionic effects. Curve IV: magnetic transition associated with the (33)-resonance. Curve V: Curve III+IV. Experimental data are: Cornell ▲, Cal Tech ⊖, Illinois ■, Berkeley ◆.

§ 3. Exchange effects and the Siegert theorem

On account of gauge invariance the direct electric transitions and the exchange effects are closely related. It is therefore difficult to identify their respective contribution in a consistent way. In this section we investigate an equivalent operator to be added to the first term of (2·1) in order to involve transitions through intermediate states in which virtual pions are present. At this point we emphasize that our calculation ultimately leads to a result consistent with the Siegert theorem.

Since gauge invariance is critical for the discussion, our argument is compelled to stand on perturbation theory. Consequently the effect of (33)-resonance will be kept untouched in this section.

In order to assure gauge invariance we must consider the nucleon-recoil in the beginning. We start from the Schrödinger equation:

$$i\frac{\partial\psi}{\partial t}=H\{\mathbf{A}\}\psi \tag{3·1}$$

with $$H\{\mathbf{A}\}=H_0+H_{\pi N}+H_\gamma\{\mathbf{A}\},$$

where H_0 is the Hamiltonian of two free nucleons as well as free pion field, $H_{\pi N}$ the pion-nucleon interaction Hamiltonian, and $H_\gamma\{\mathbf{A}\}$ the interaction Hamiltonian referred to nucleons and pion field interacting with electromagnetic field. Explicitly the latter two have the form:

$$H_{\pi N}=\sum_{i=1,2}\sqrt{4\pi}(f/\mu)\,(\boldsymbol{\sigma}^{(i)}\cdot\boldsymbol{\nabla})\,(\boldsymbol{\tau}^{(i)}\cdot\boldsymbol{\phi})\,(\boldsymbol{r}_i), \tag{3·2}$$

$$H_\gamma\{\mathbf{A}\}=\sum_{i=1,2}\{-(e/2M)\,(1+\tau_3^{(i)})\,\mathbf{A}(\boldsymbol{r}_i)\cdot\boldsymbol{p}_i-\sqrt{4\pi}(ef/\mu)\,(\boldsymbol{\sigma}^{(i)}\cdot\mathbf{A}(\boldsymbol{r}_i))\,\epsilon_{3\alpha\beta}\tau_\alpha^{(i)}\phi_\beta(\boldsymbol{r}_i)\}$$

$$+e\,\epsilon_{3\alpha\beta}\int\phi_\alpha(\boldsymbol{r})\nabla\phi_\beta(\boldsymbol{r})\cdot\mathbf{A}(\boldsymbol{r})\,d\boldsymbol{r}.$$

The electromagnetic potential $\mathbf{A}(\boldsymbol{r}, t)$, corresponding to absorption of a photon with momenta $\boldsymbol{\kappa}$, is given by

$$\mathbf{A}(\boldsymbol{r}, t)=\sqrt{\frac{2\pi}{\kappa}}\,\boldsymbol{\varepsilon}\exp[i(\boldsymbol{\kappa}\cdot\boldsymbol{r}-\kappa t)]. \tag{3·3}$$

Owing to gauge invariance the Hamiltonian satisfies to the first order of e for arbitrary functions $\boldsymbol{\Xi}(\boldsymbol{r}, t)$ and $\Lambda(\boldsymbol{r}, t)$

$$H\{\boldsymbol{\Xi}+\operatorname{grad}\Lambda\}=e^{iD(t)}H\{\boldsymbol{\Xi}\}e^{-iD(t)}, \tag{3·4}$$

$$D(t)=\int\Big[\rho_\pi(\boldsymbol{r})+\sum_{i=1,2}\frac{1+\tau_3^{(i)}}{2}e\,\delta(\boldsymbol{r}-\boldsymbol{r}_i)\Big]\Lambda(\boldsymbol{r}, t)\,d\boldsymbol{r}.$$

Here $\rho_\pi(\boldsymbol{r})$ denotes the charge density operator of pion field.

The essential point of our procedure lies in changing a part of $H_\gamma\{\mathbf{A}\}$ by means of Eq. (3·4) and bringing it into a form to permit easy isolation of the transition operator for direct process.* For this purpose we expand $\mathbf{A}(\boldsymbol{r}, t)$ in terms of multipole radiations:[23]

$$\mathbf{A}(\boldsymbol{r}, t)=\sqrt{\frac{2\pi}{\kappa}}\,e^{-i\kappa t}\sum_{l=1}^{\infty}\sum_{m=-l}^{l}4\pi i^l\{\kappa^{-2}\operatorname{curl}[j_l(\kappa r)\boldsymbol{X}_{lm}(\boldsymbol{r})](\boldsymbol{\varepsilon}\cdot i\boldsymbol{\kappa}\times\boldsymbol{X}_{lm}(\boldsymbol{\kappa}))^*$$
$$+j_l(\kappa r)\boldsymbol{X}_{lm}(\boldsymbol{r})(\boldsymbol{\varepsilon}\cdot\boldsymbol{X}_{lm}(\boldsymbol{\kappa}))^*\}.$$

Then we divide each electric 2^l-pole field into two parts using

$$\operatorname{curl}[j_l(\kappa r)\boldsymbol{X}_{lm}(\boldsymbol{r})]=i[l(l+1)]^{-1/2}\{\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{r}-2)\phi_{lm}+\kappa^2\boldsymbol{r}\phi_{lm}\}, \tag{3·6}$$

where $$\phi_{lm}=j_l(\kappa r)\,Y_{lm}(\boldsymbol{r}).$$

We denote the sum of electric multipole fields arisen from the first term of (3·6) by $\operatorname{grad}\Lambda$ and the remaining part by $\boldsymbol{\Xi}$. Then

$$\mathbf{A}(\boldsymbol{r}, t)=\operatorname{grad}\Lambda(\boldsymbol{r}, t)+\boldsymbol{\Xi}(\boldsymbol{r}, t), \tag{3·7}$$

$$\Lambda(\boldsymbol{r}, t)=\sqrt{\frac{2\pi}{\kappa}}\,e^{-i\kappa t}\sum_{l=1}^{\infty}\sum_{m=-l}^{l}\frac{4\pi i^l}{\kappa^2\sqrt{l(l+1)}}(\boldsymbol{\varepsilon}\cdot\boldsymbol{\kappa}\times\boldsymbol{X}_{lm}(\boldsymbol{\kappa}))^*(\boldsymbol{\nabla}\cdot\boldsymbol{r}-2)\phi_{lm},$$

$$\boldsymbol{\Xi}(\boldsymbol{r}, t)=\sqrt{\frac{2\pi}{\kappa}}\,e^{-i\kappa t}\sum_{l=1}^{\infty}\sum_{m=-l}^{l}\frac{4\pi i^l}{\sqrt{l(l+1)}}\{(\boldsymbol{\varepsilon}\cdot\boldsymbol{\kappa}\times\boldsymbol{X}_{lm}(\boldsymbol{\kappa}))^*\boldsymbol{r}\phi_{lm}$$
$$+(\boldsymbol{\varepsilon}\cdot\boldsymbol{X}_{lm}(\boldsymbol{\kappa}))^*\mathbf{L}\phi_{lm}\}.$$

After substitution into the Hamiltonian and the gauge transformation

$$e^{-iD(t)}\psi(t)\to\psi(t)$$

with the aid of (3·4) we arrive at the Schrödinger equation equivalent to (3·1):

$$i\frac{\partial\psi}{\partial t}=\Big\{H_0+H_{\pi N}+\frac{\partial D}{\partial t}+H_\gamma(\boldsymbol{\Xi})\Big\}\psi. \tag{3·8}$$

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* Similar technique has been frequently used.[20),21),22]

Since the interaction terms are doubled, calculation is more complicated. Nevertheless, the form of electric multipole interactions presented above is the most convenient for discussing the Siegert theorem.

Now admittedly a precise evaluation of the transition operator involves serious difficulty because of the lack of a procedure to work with bound states in field theory. However, it seems plausible to assume that its approximate form may be found in the transition matrix for two nucleons going from one continuum state to another by absorbing a photon. This is somewhat justified by the fact that the deuteron binding energy is quite negligible compared with the photon energy considered here.

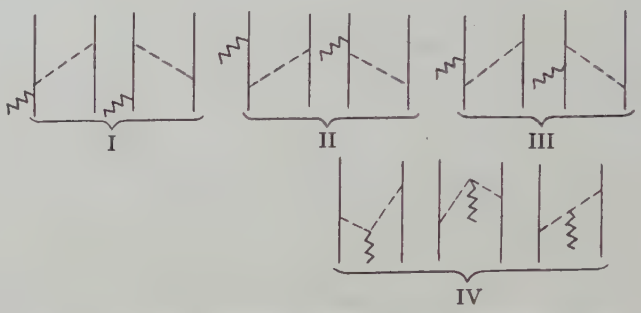

Fig. 3. f^2-order processes caused by the interaction $\partial D/\partial t$.

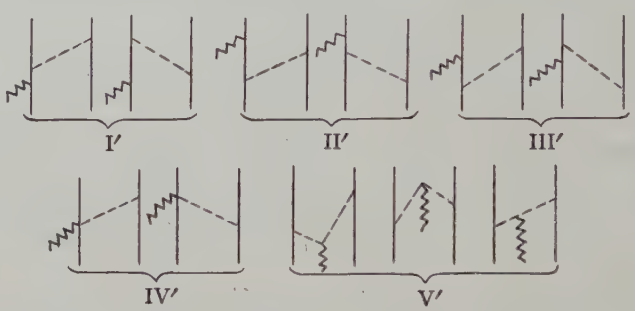

Fig. 4. f^2-order processes caused by the interaction $H_7\{\Xi\}$. Here only one extra graph, Graph IV′, is involved compared with Fig. 3.

In the following we write down the transition matrices for f^2-order processes shown in Figs. 3 and 4 under the following approximation: (i) We expand the energy denominators in the ratio of nucleon-recoil energy to pion energy ω_q, and omit the terms of the order of $1/M$,* (ii) We exploit the expansion in powers of $\kappa/2\omega_q$. This, roughly speaking, corresponds to the adiabatic approximation.

We denote the relative momenta of the nucleons in the initial and final state by $\boldsymbol{p}$ and $\boldsymbol{p}'$ respectively. Also we use the notation:

$$\begin{cases} \boldsymbol{q}_\pm = \boldsymbol{p}' - \boldsymbol{p} \pm \boldsymbol{\kappa}'/2, \quad E_p = p^2/2M, \\ V(\boldsymbol{q}) = -\dfrac{4\pi f^2}{\mu^2}(\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)})\dfrac{(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{q})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{q})}{\omega_q^2}, \\ L(\boldsymbol{q}) = \dfrac{2\pi f^2}{\mu^2}(\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)})\dfrac{(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{q})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{q})}{\omega_q^3}, \end{cases}$$

* Precisely, we define the mean recoil-energy in the initial and final state respectively, and neglect their variations. Thus we put $E_{p+\kappa'} \simeq E_p$, $E_{p'+\kappa'} \simeq E_{p'}$, etc., but $2(E_{p'} - E_p) \simeq \kappa$ in accordance with the energy conservation.

$$\left.\begin{aligned}
\Lambda(\boldsymbol{\kappa}') &= -\sqrt{\frac{2\pi}{\kappa}}\sum_l\sum_m \frac{(\boldsymbol{\varepsilon}\cdot\boldsymbol{\kappa}\times\boldsymbol{X}_{lm}(\boldsymbol{\kappa}))^*}{\kappa^2\sqrt{l(l+1)}}\int d\Omega_{\kappa''}\,Y_{lm}(\boldsymbol{\kappa}'') \\
&\quad\times\left(\boldsymbol{\kappa}'\cdot\frac{\partial}{\partial\boldsymbol{\kappa}'}+2\right)\delta(\boldsymbol{\kappa}''-\boldsymbol{\kappa}'), \\
\boldsymbol{\Xi}(\boldsymbol{\kappa}') &= \sqrt{\frac{2\pi}{\kappa}}\sum_l\sum_m \frac{i}{\sqrt{l(l+1)}}\left\{(\boldsymbol{\varepsilon}\cdot\boldsymbol{\kappa}\times\boldsymbol{X}_{lm}(\boldsymbol{\kappa}))^*\int d\Omega_{\kappa''}\,Y_{lm}(\boldsymbol{\kappa}'')\right. \\
&\quad\left.\times\frac{\partial}{\partial\boldsymbol{\kappa}'}\delta(\boldsymbol{\kappa}''-\boldsymbol{\kappa}')-(\boldsymbol{\varepsilon}\cdot\boldsymbol{X}_{lm}(\boldsymbol{\kappa}))^*\int d\Omega_{\kappa''}\,Y_{lm}(\boldsymbol{\kappa}'')\left[\boldsymbol{\kappa}'\times\frac{\partial}{\partial\boldsymbol{\kappa}'}\right]\delta(\boldsymbol{\kappa}''-\boldsymbol{\kappa}')\right\}.
\end{aligned}\right\} \tag{3·9}$$

$\Lambda(\boldsymbol{\kappa}')$, $\boldsymbol{\Xi}(\boldsymbol{\kappa}')$ are the Fourier transform of $\Lambda(\boldsymbol{r},t)$, $\boldsymbol{\Xi}(\boldsymbol{r},t)$. Note that in the above

$$|\boldsymbol{\kappa}''|=\kappa.$$

Then the transition matrices can be written in momentum representation*

$$\left.\begin{aligned}
T_{\mathrm{I}}(\varepsilon,\boldsymbol{\kappa};\boldsymbol{p},\boldsymbol{p}') &\simeq ie\kappa\int d^3\boldsymbol{\kappa}'\,\Lambda(\boldsymbol{\kappa}')\left[-\left\{\frac{V(\boldsymbol{q}_-)\frac{1+\tau_3^{(1)}}{2}}{\kappa+E_p-E_{p+\kappa'}+i\varepsilon}+\frac{V(\boldsymbol{q}_+)\frac{1+\tau_3^{(2)}}{2}}{\kappa+E_p-E_{p-\kappa'}+i\varepsilon}\right\}\right. \\
&\quad+\left\{L(\boldsymbol{q}_-)\frac{1+\tau_3^{(1)}}{2}+L(\boldsymbol{q}_+)\frac{1+\tau_3^{(2)}}{2}\right\}+\frac{\pi f^2\kappa}{\mu^2}(\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)}) \\
&\quad\left.\times\left\{\frac{\frac{1+\tau_3^{(1)}}{2}(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{q}_-)(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{q}_-)}{\omega^4 q_-}+\frac{\frac{1+\tau_3^{(2)}}{2}(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{q}_+)(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{q}_+)}{\omega^4 q_+}\right\}\right], \\
T_{\mathrm{II}}(\varepsilon,\boldsymbol{\kappa};\boldsymbol{p},\boldsymbol{p}') &\simeq ie\kappa\int d^3\boldsymbol{\kappa}'\,\Lambda(\boldsymbol{\kappa}')\left[-\left\{\frac{\frac{1+\tau_3^{(1)}}{2}V(\boldsymbol{q}_-)}{2E_p-2E_{p'-\kappa'/2}}+\frac{\frac{1+\tau_3^{(2)}}{2}V(\boldsymbol{q}_+)}{2E_p-2E_{p'+\kappa'/2}}\right\}\right. \\
&\quad+\left\{\frac{1+\tau_3^{(1)}}{2}L(\boldsymbol{q}_-)+\frac{1+\tau_3^{(2)}}{2}L(\boldsymbol{q}_+)\right\}-\frac{\pi f^2\kappa}{\mu^2} \\
&\quad\left.\times\left\{\frac{\frac{1+\tau_3^{(1)}}{2}(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{q}_-)(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{q}_-)}{\omega^4 q_-}+\frac{\frac{1+\tau_3^{(2)}}{2}(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{q}_+)(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{q}_+)}{\omega^4 q_+}\right\}(\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)})\right], \\
T_{\mathrm{III}}(\varepsilon,\boldsymbol{\kappa};\boldsymbol{p},\boldsymbol{p}') &\simeq -i\frac{2\pi ef^2\kappa}{\mu^2}\int d^3\boldsymbol{\kappa}'\,\Lambda(\boldsymbol{\kappa}')\left[-\frac{(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{q}_-)(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{q}_-)}{\omega^3 q_-}(\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)}+\tau_3^{(2)})\right. \\
&\quad\left.+\frac{(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{q}_+)(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{q}_+)}{\omega^3 q_+}(\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)}+\tau_3^{(1)})\right],
\end{aligned}\right\} \tag{3·10}$$

* Graphs I′, II′, III′ involve the effects of initial and final state interaction on the direct electric transitions which are associated with retardation effect and cannot be described in terms of nucleon coordinates alone. Such transitions are not essential for the present discussion and will not be considered here. See references 6), 21).

$$T_{\mathrm{IV}'+\mathrm{V}'}(\varepsilon, \boldsymbol{\kappa}; \boldsymbol{p}, \boldsymbol{p}') \simeq -i\frac{4\pi e f^2}{\mu^2}[\boldsymbol{\tau}^{(1)}\times\boldsymbol{\tau}^{(2)}]_3 \int d^3\boldsymbol{\kappa}'$$
$$\boldsymbol{\Xi}(\boldsymbol{\kappa}')\cdot\left[\boldsymbol{\sigma}^{(1)}\frac{\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{q}_-}{\omega^2_{q_-}}+\boldsymbol{\sigma}^{(2)}\frac{\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{q}_+}{\omega^2_{q_+}}-2\boldsymbol{q}\frac{(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{q}_+)(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{q}_-)}{\omega^2_{q_+}\omega^2_{q_-}}\right],$$
$$T_{\mathrm{IV}} \simeq T_{\mathrm{I}'} \simeq T_{\mathrm{II}'} \simeq T_{\mathrm{III}'} \simeq 0.$$

On examining the expressions (3·10) it is found that most part of the transition matrices must be discarded before including them into the transition operator. Thus the terms with $V(\boldsymbol{q}_\pm)$ must be discarded, because they represent corrections to initial and final wave functions, not to the transition operator itself. Also note that the $L(\boldsymbol{q}_\pm)$-terms are just cancelled by the contribution from graph III. Such cancellation has been expected from the physical meaning of these terms; they represent the effect of physical nucleons getting out of the clothes of pion cloud and going into interaction with electromagnetic field in the bare state.

The remainder can be regarded as the correction to the transition operator. We evaluate it approximately, by expanding $1/\omega_{q_\pm}$ as

$$\frac{1}{\omega_{q_\pm}} \simeq \frac{1}{\omega_q} \mp \frac{\boldsymbol{q}\cdot\boldsymbol{\kappa}}{2\omega_q{}^3} - \frac{\kappa^2}{8\omega_q{}^3} + \frac{3}{8}\frac{(\boldsymbol{q}\cdot\boldsymbol{\kappa})^2}{\omega_q{}^5} + \cdots.$$

Only the nonvanishing leading terms are retained here. Since the exchanged pion is shown to have a momentum close to k in most cases, this approximation again corresponds to the neglect of $\kappa/2k$ in comparison with 1.

We proceed to calculate the transition amplitudes from the corrections of the transition operator. We employ the Hulthén wave function to the deuteron:

$$\psi_i{}^m(\boldsymbol{r}) = \sqrt{\frac{\mu N}{4\pi}}\chi_m\frac{e^{-\alpha r}-e^{-\beta r}}{r}\ ;\quad \sqrt{N} = \left[\frac{2\alpha\beta(\alpha+\beta)}{\mu(\beta-\alpha)^2}\right]^{1/2}.$$

Thereby we have neglected the D-state contribution. In fact there is some reason to believe that the D-state is not so important in pion-exchange processes as in the direct process. In the latter the nucleons are necessarily at small distances in the deuteron, with a relative momentum comparable to the one in the final state. In the former, however, it is more favourable for the nucleons to be at large distances than at small distances owing to momentum transfer by the exchanged pion.

The exchange effect calculated under the above approximations modifies E1 partial-amplitudes as follows:

$$\Delta L_0 = \frac{\kappa f^2\sqrt{N}}{\pi}\int_\mu^{\bar{\omega}} d\omega\frac{q}{\omega^3}\left\{-8+4\frac{q^2(\omega^2+\mu^2)}{\omega^4}\right\}F_1,$$
$$\Delta L_1 = \frac{\kappa f^2\sqrt{N}}{\pi}\int_\mu^{\bar{\omega}} d\omega\frac{q}{\omega^3}\left\{7-4\frac{q^2(\omega^2+\mu^2)}{\omega^4}\right\}F_1,$$

$$\Delta L_2=\frac{\kappa f^2\sqrt{N}}{\pi}\int_{\mu}^{\bar{\omega}}d\omega\frac{q}{\omega^3}\left\{1-\frac{4}{5}\frac{q^2(\omega^2+\mu^2)}{\omega^4}\right\}F_1,$$

$$\Delta L_F=\frac{8\kappa f^2\sqrt{N}}{5\pi}\int_{\mu}^{\bar{\omega}}d\omega\frac{q^3(\omega^2+\mu^2)}{\omega^7}(5F_3-3F_1), \tag{3·11}$$

where $\omega^2=q^2+\mu^2$,

$$F_n(\omega,k)=\int_{-1}^{1}dx\,x^n\left(\frac{1}{l-x}-\frac{1}{l'-x}\right),$$

$$l=(q^2+k^2+\alpha^2)/2qk,\quad l'=(q^2+k^2+\beta^2)/2qk.$$

We take the cut-off energy $\bar{\omega}$ as 6μ. The result is not sensitive to the value of $\bar{\omega}$. The modification of the cross section due to the above corrections is shown in Fig. 2 (Curve III). Additional corrections to M1 transition amplitudes arise from $T_{\text{IV}'+\text{V}'}$. These are found to be unimportant compared with the direct M1 transition, hence we neglect them.

As is easily seen, exchange current corrections to the electric transition operator, as well as the corrections to the transition amplitudes, all involve κ as an extra multiplicative factor in contrast to the corresponding quantities referred to the direct process. Therefore these corrections will be negligible in the low energy limit. This means that the Siegert theorem holds in our result.

§ 4. (33)-resonance effect on the magnetic transition

In the preceding section we took out of consideration the interaction of nucleons with photons by means of the proper spin magnetic moment. Actually, several authors[11)–15)] have pointed out that near $E_\gamma=250$ Mev an important contribution to the cross section comes from the process in which a nucleon absorbs photons through magnetic interaction and emits a (33)-resonant pion in the intermediate state. According to Austern[12)]—the argument below also confirms this point—this process leads mainly to ^1D-state. Consequently the resulting cross section simply adds to that of the electric transitions which is predominant at the lower energies. Since the effect of the electric transition is found to continue to be large even at high energies, the magnitude of (33)-resonance effect must be reinvestigated. Note that our calculation is a little different from that of the previous authors in applying the result of the Chew-Low theory.

The processes concerned are shown in Fig. 5. Each graph has a vertex part where a nucleon i ($i=1$, 2) absorbs a photon and emits a resonance pion into

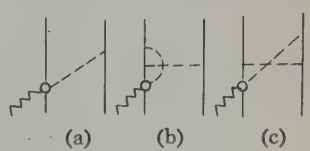

Fig. 5. Circles on nucleon lines represent the vertex emitting a resonant pion.

the (33)-state. It is reasonable that we use the following matrix element for photo-pion production obtained in the Chew-Low theory to express this vertex part:

$$\frac{\pi}{f}\frac{e}{M}(g_p-g_n)\left(\delta_{\alpha 3}-\frac{1}{3}\tau_\alpha^{(i)}\tau_3^{(i)}\right)\frac{2\boldsymbol{q}\cdot\boldsymbol{\kappa}\times\boldsymbol{\varepsilon}-i\boldsymbol{\sigma}^{(i)}\cdot\boldsymbol{q}\times(\boldsymbol{\kappa}\times\boldsymbol{\varepsilon})}{(4\omega_q\kappa)^{1/2}}$$

$$\times\frac{e^{i\delta_{33}}\sin\delta_{33}}{q^3}e^{i(\boldsymbol{\kappa}-\boldsymbol{q})\cdot\boldsymbol{r}_i}. \qquad (4\cdot1)$$

In (4·1) the necessity of the phase factor $\exp[i(\boldsymbol{\kappa}-\boldsymbol{q})\cdot\boldsymbol{r}_i]$ is well known in the case of two-nucleon system.[24] Absorption of pions, of course, occurs by means of $H_{\pi N}$ in (3·2). In the process of Fig. 5 (b), (c) there is an interaction via nuclear force before the absorption of the resonant pion. As far as the following approximations are adopted, it is shown that only the tensor force is important.*

If one neglects $\kappa/2k$ in comparison with 1, the contributions from each process modify the M1 amplitude M_2 as follows:

$$\Delta M_{2,a}=-\frac{\mu\sqrt{N}}{\sqrt{2}\pi}(g_p-g_n)\int_\mu^{\bar{\omega}}d\omega\frac{q}{\kappa-\omega+i\varepsilon}\left[\frac{e^{i\delta_{33}}\sin\delta_{33}}{q^2}\right]\left(F_2-\frac{1}{3}F_0\right)(\omega,k),$$

$$\Delta M_{2,b}=-\frac{\mu\gamma\sqrt{N}}{\sqrt{2}\pi}(g_p-g_n)\int_\mu^{\bar{\omega}}d\omega\,\omega\{U_E(q)-U_O(q)\}\left(F_2-\frac{1}{3}F_0\right)(\omega,k), \qquad (4\cdot2)$$

$$\Delta M_{2,c}=-\frac{\mu\gamma\sqrt{N}}{\sqrt{2}\pi}(g_p-g_n)\int_\mu^{\infty}d\omega\,\omega\{U_E(q)+U_O(q)\}\left(F_2-\frac{1}{3}F_0\right)(\omega,k).$$

Here

$$\gamma=\int\frac{d\boldsymbol{q}}{(2\pi)^3}\frac{e^{i\delta_{33}}\sin\delta_{33}}{q\omega_q(\kappa-\omega_q+i\varepsilon)^2}, \qquad (4\cdot3)$$

and $U_E(q)$ ($U_O(q)$) is related to the tensor force potential $V_E(12)$ ($V_O(12)$) in the triplet even (odd) states by

$$V(12)=\frac{1}{(2\pi)^3}\int d\boldsymbol{q}\,e^{i\boldsymbol{q}\cdot\boldsymbol{r}}\left(3\frac{(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{q})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{q})}{q^2}-\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)}\right)U(q). \qquad (4\cdot4)$$

The contribution from the process of Fig. 5 (c) has been evaluated in neglect of the momentum transfer by the resonant pion in contrast to that by the nuclear-force-pion. The magnitude of the latter is $\sim k$ and rather significant. In any case a precise evaluation of this contribution is difficult and here the rough estimation is enough.

Now we note that in the process of Fig. 5 (a) there is a tendency to suppress and smooth out the resonance effect in comparison with the other two. Since

* The L-S force is found to cause the E1 and M2 transitions ${}^3S_1\rightarrow{}^3P$. Here we do not consider this transition on account of indefinite characters of the L-S force.

the resonant pion has a momentum below one half the relative momentum of final nucleons, the transition due to exchange of the resonant pion is possible only when the initial two nucleons have a certain amount of momentum. Then the suppression occurs, because the probability that two nucleons in the deuteron have such a large momentum as required above is small.

$\Delta M_{2,a}$ is roughly equal in magnitude, and unfavoured in phase, to the M1 amplitude for the direct process. Therefore, in the following we shall neglect both of them.

Turning to the process of Fig. 5 (b), (c) we first evaluate γ. The quantity is related to the duration of the intermediate state with a resonant pion in the (33)-state. In fact γ has the form $\sim 1/(\kappa-\omega_0+i\Gamma/2)$, where ω_0 is the resonance energy and $\Gamma/2$ the half width. To show this we write

$$\gamma = \frac{1}{2\pi^2}\int_{\mu}^{\bar{\omega}} d\omega \left[\frac{\sin^2\delta_{33}}{q^2}\right]\frac{q^2\cot\delta_{33}+iq^2}{(\kappa-\omega+i\varepsilon)^2},$$

and approximate as

$$q^{-2}\sin^2\delta_{33} \simeq \zeta\Gamma^2/4[(\omega-\omega_0)^2+(\Gamma/2)^2],$$

$$q^2\cot\delta_{33} \simeq q^3\cot\delta_{33}/\omega \simeq (3\mu^2/4\omega_0 f^2)(\omega_0-\omega).$$

In the narrow resonance width we can extend the range of the ω-integration from $-\infty$ to $+\infty$ without serious error. After integration we further put $\Gamma/2 \approx 2f^2q_0^3/3\mu^2 \approx 2f^2q_0^2\omega_0/3\mu^2$, $i(\Gamma/2)/[\kappa-\omega_0+i\Gamma/2]^2 \approx 1/[\kappa-\omega_0+i\Gamma/2]$, obtaining

$$\gamma = \frac{3\zeta q_0^2}{4\pi}\,\frac{1-i2\omega_0\Gamma/3q_0^2}{\kappa-\omega_0+i\Gamma/2}. \qquad (4\cdot5)$$

From this result it is evident that the cross section of the magnetic transition shows a characteristic resonance behaviour. But it is hard to estimate the magnitude.

The above calculation is made on the assumption that the nucleons are completely at rest in the intermediate state. This seems to be incorrect. Since the nucleons obtain large momenta after interaction via nuclear force, we must consider their kinetic energies which nearly amount to the incident photon energy. In Eq. (4·3) the energy denominator $(\kappa-\omega+i\varepsilon)^{-2}$ must now be replaced by $-[\omega(\kappa-\omega+i\varepsilon)]^{-1}$. It follows that although the κ-dependence of γ is still given by Eq. (4·5), the whole expression is multiplied by a factor of magnitude $\sim\Gamma/2\omega_0$.

The magnetic transition is also affected by another uncertain factor. In (4·2) a main contribution to the integral comes from the region $q \approx k$. $U_K(q)$ for such a large q involves much ambiguity due to the unknown property of nuclear force at small distances.

Now, to clarify the situation we introduce a parameter η and multiply ΔM_2

by η with γ given by Eq. (4·5). η is considered to incorporate the recoil effects and the error arisen from the uncertainty of nuclear potential. We inquire how η should be taken to be consistent with experiment.

Investigation is made on the two potential models: One is the one-pion-exchange tensor potential

$$(\mu f^2/3)(\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)})S_{12}(1+3/x+3/x^2)e^{-x}/x,$$

and the other the phenomenological potential obtained by Gammel-Thaler.[25]

The G-T tensor potential in the triplet even states can be written, the inner repulsive core being neglected,

$$V_E(12)=-S_{12}V_{T0}e^{-\lambda x}/x,$$

where the parameters are $\lambda=1.70$, $V_{T0}=150\sim100$ Mev; the depth must be taken smaller as the energy increases.

In the G-T potential we get a magnetic transition cross section in rough agreement with experiment by putting $V_{T0}=100$ Mev and $\eta=0.20$ (see Curve IV in Fig. 2; other parameters are set as $\zeta=0.30\mu^{-2}$, $\omega_0=2.0\mu$, $\Gamma/2=0.32\mu$). In order to get the same cross section by using the one-pion-exchange potential, we must put $\eta=0.09$. Although these values are not far from $\Gamma/2\omega_0$, a precise evaluation of the (33)-resonance effect seems to be difficult until further development of theory and experiment are reached.

§ 5. Discussion

An important conclusion obtained in this paper is that the direct transition operator

$$-i\left(\frac{\boldsymbol{\tau}^{(1)}-\boldsymbol{\tau}^{(2)}}{2}\right)_3\sqrt{\frac{2\pi}{\kappa}}\frac{e\kappa}{2\mu}(\boldsymbol{\varepsilon}\cdot\boldsymbol{x}) \qquad (5\cdot1)$$

is little affected by the effect of exchange current, especially at energies below the pion threshold. The conclusion is consistent with the recent analysis that the transition operator (5·1) alone is sufficient for explaining the experimental data at photon energy 80 Mev. A large mesonic effect to electric transition claimed by some authors[10),11)] is due to confusion of corrections to the initial and final wave function with those to the transition operator itself and to inclusion of those terms into the transition operator which would actually be cancelled out from gauge invariance. In the above it has been checked by means of direct calculations that the exchange current effect yields corrections consistent with the Siegert theorem, provided that due consideration is taken to gauge invariance.*

Though the actual calculation is confined to one-pion-exchange processes, the

* The possibility that exchange effects may affect even the long-wavelength, electric dipole transition operator (contrary to the Siegert theorem) is discussed by L. L. Foldy.[26]

above remark also may be applicable to two- or more-pion-exchange processes. Thus, Zachariasen's analysis of two-pion-exchange processes[15] will not be correct. (The treatment of one-pion-exchange processes is similar to ours.) He finds that the effect is large and gives a profound depression to the cross section at $E_\gamma \approx 130$ Mev, which contradicts experiment. But the transition operator used there involves superfluous terms discussed above, and it is expected that the actual effect of two-pion-exchange is appreciably smaller than his estimate.

There remains much ambiguity about the (33)-resonance effect to magnetic transition. But the following consideration shows that the resonance effect is perhaps responsible for only one half or less of the cross section near $E_\gamma = 250$ Mev. First, we cannot assume too remarkable resonance effect without having an unwanted depression of the cross section below the resonance energy.[12)-15)] Second, the observed angular distribution above $E_\gamma \approx 250$ Mev appreciably deviates from the form $\sim 2+3\sin^2\theta$ which is expected under the condition of the predominant resonance effect. Finally, the electric transition can account for about half the observed cross section in this energy region. The electric transition occurs mainly through the (5·1) transition operator owing to the Siegert theorem, and the estimation of electric transition is considered rather reliable.

Our cross section is smaller than the observed one, especially near $E_\gamma \approx 200$ Mev. In the course of calculation various effects have been omitted. Of these the final state interaction, if considered, would probably result in an increase of the electric cross section by an amount of about 20%, judging from the analysis at $E_\gamma = 80$ Mev. Retardation effects may also increase the cross section.[6),7)]

Finally, we briefly discuss the angular distribution. So far we have retained only E1 and M1 transitions. However, these alone are unable to account for the remarkable forward-backward asymmetry observed in the high energy region. As is well known, the retardation effect in the direct transition process induces the E2 transition ${}^3S_1 \to {}^3D$, and can explain the asymmetry, at least, at low energies. Let us consider this transition by adding the following term to $\int \psi_f^+ T_{direct} \psi_i \, d\boldsymbol{r}$ in (2·2) :

$$\begin{cases} & -g_t(\boldsymbol{\varepsilon}\cdot\boldsymbol{n})(\boldsymbol{m}\cdot\boldsymbol{n})L_q, \\ \text{with} & L_q = (1/2\sqrt{2})(\kappa/\mu)\int_0^\infty dx\, x^2 u(x)[z j_2(z)]_{z=kr}. \end{cases} \tag{5·2}$$

Then the differential cross section can be written as[17]

$$\frac{d\sigma}{d\Omega} = a + b\sin^2\theta(1+2\beta\cos\theta).$$

In Fig. 6 the angular distributions calculated at $E_\gamma = 150$, 200, 250 Mev are given. Corresponding experimental data are also indicated. On comparison we see that both the isotropic part a and the coefficient β are required to be larger.

As for the isotropic part, it is not so unreasonable to assume that the situation will be improved by considering the final state interaction. In fact the analysis at 80 Mev shows that the final state interaction makes the isotropic part considerably larger.

The coefficient β, on the other hand, is rather involved. It is shown that generally the terms with odd powers of $\cos\theta$ in the angular distribution come from the interference between two kinds of transitions in which the final states have I-spin 0 and 1 respectively. Since the deuteron has I-spin 0, the transition into a continuum state with I-spin 0 cannot be caused by the interaction of pion current with electromagnetic field. Thus, in order to explain the large asymmetry mentioned above it is necessary to study the nucleon-current interaction in detail, taking account of accompanying retardation effects. This is a complicated problem, and we hope to discuss this point in another occasion.

In conclusion the author wishes to express his sincere thanks to Professor K. Nakabayasi for kind interest and encouragement throughout the work. He is also much indebted to Professor I. Sato for valuable discussions.

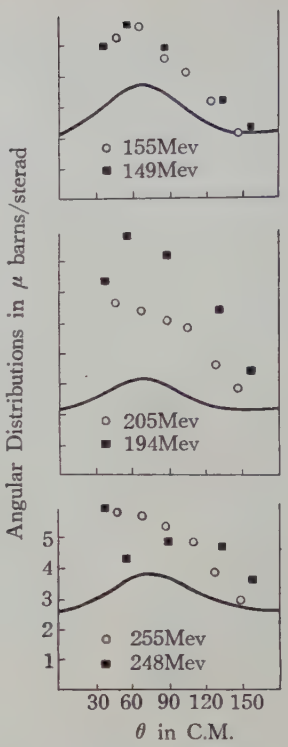

Fig. 6. Calculated angular distributions at $E_\gamma = 150$, 200, 250 Mev. Experimental data are: Cal Tech[28] ○, Illinois[29] ■.

Note added in proof : After the manuscript had been submitted, it was called to the author's attention that a similar analysis was done by L. D. Pearlstein and A. Klein: Phys. Rev. **118** (1960), 193. These authors obtained a result like ours on the exchange effects in the electric transition in a quite different manipulation. Their conclusions on the (33)-resonance effect, on the other hand, are somewhat different from ours.

References

1) J. J. de Swart and R. E. Marshak, Phys. Rev. **111** (1958), 272.
2) J. Iwadare and M. Matsumoto, to be published.
3) A. F. Nicholson and G. E. Brown, Proc. Phys. Soc. **73** (1959), 221.
4) W. Zernik, M. L. Rustgi, and G. Breit, Phys. Rev. **114** (1959), 1358.
5) J. J. de Swart and R. E. Marshak, Physica **25** (1959), 1001.
6) S. Hsieh, Prog. Theor. Phys. **21** (1959), 585.
7) M. Matsumoto, Prog. Theor. Phys. **23** (1960), 188.
8) L. I. Schiff, Phys. Rev. **78** (1950), 733.
J. E. Marshall and E. Guth, Phys. Rev. **78** (1950), 738.
9) For example, see R. G. Sachs, *Nuclear Theory* (Addison-Wesley Press, Cambridge, 1953), p. 241.

10) B. Bruno and S. Depken, Ark. Fysik **6** (1953), 177.
11) R. R. Wilson, Phys. Rev. **104** (1956), 218.
12) N. Austern, Phys. Rev. **100** (1955), 1522.
13) D. Ito, M. Ono, T. Kato and Y. Takahashi, Prog. Theor. Phys. **15** (1956), 74.
14) R. Suzuki, Prog Theor. Phys. **15** (1956), 536.
15) F. Zachariasen, Phys. Rev. **101** (1956), 371.
16) G. F. Chew and F. E. Low, Phys. Rev. **101** (1956), 1579.
17) For example, see J. Iwadare, S. Otsuki, M. Sano, S. Takagi and W. Watari, Prog. Theor. Phys. **16** (1956), 658; L. Hulthén and M. Sugawara, *Handbuch der Physik*, *XXXIX* (Springer Verlag, Berlin, 1957), 97.
18) J. Iwadare, S. Otsuki, R. Tamagaki and W. Watari, Supplement Prog. Theor. Phys. No. 3 (1956), 32.
19) N. Austern, Phys. Rev. **108** (1957), 973.
20) R. G. Sachs and N. Austern, Phys. Rev. **81** (1951), 705.
21) J. G. Brennan and R. G. Sachs, Phys. Rev. **88** (1952), 824.
22) R. H. Capps and W. G. Holladay, Phys. Rev. **99** (1955), 931.
23) J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952), Appendix B.
24) See, for example, H. Miyazawa, Phys. Rev. **101** (1956), 1564.
25) J. L. Gammel and R. M. Thaler, Phys. Rev. **107** (1957), 1337.
26) L. L. Foldy, Phys. Rev. **92** (1953), 178.
27) J. Keck, R. M. Littauer, G. K. O'Neill, A. M. Perry and W. M. Woodward, Phys. Rev. **93** (1954), 827.
28) J. C. Keck and A. V. Tollestrup, Phys. Rev. **101** (1956), 360.
29) E. A. Whalin, B.D. Schriever and A. O. Hanson, Phys. Rev. **101** (1956), 377.
30) D. R. Dixon and K. C. Bandtel, Phys. Rev. **104** (1956), 1730.

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Coupling of Dirac Particle and Gravitational Field

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By using legs connection we investigate the general relativistic Dirac equation which is invariant under the general coordinate transformations and spin transformations. We obtain the four-dimensional representation for the generalized Dirac matrices by introducing the six legs and the concept of sedenion conjugate. It is shown that Fermi-Fermi interaction term must be added to the original Lagrangian density, if the legs connection is regarded as independent field quantity. The electromagnetic field is also introduced in connection with the complex legs. Finally, we consider the relation between the spin transformation and Pauli-Gürsey type transformation.

§ 1. Introduction and summary

In the last few years many people have made attempt to unify all the elementary particles and their interactions. In connection with this, there remains an important problem of investigating the internal correlation between the space-time and iso-spin space. The fact that parity and strangeness are conserved in the strong interaction and that neither of them are conserved in the weak interaction may be considered to suggest that the space-time and iso-spin space are not completely independent. The author conjectures that iso-spin space may be amalgamated in space-time structure at least in the case of lepton where iso-spin has no definite meaning in contrast to the case of baryon. Originally, the reality of iso-spin itself is not a matter, but the symmetrical properties derived from this space play an important role. Therefore the role of the iso-space may be reduced to the algebraic manipulation. In fact, Heisenberg and his collaborators[1] have attempted to introduce the same freedom as iso-spin from the standpoint of the spinor unified theory by postulating that their non-linear equation is invariant under the Pauli-Gürsey transformation.[2] The non-linearity of the spinor equation will have some connection with the space-time structure as the non-linearity appears in the general relativity. As an example, Gürsey[3] has obtained the non-linear spinor equation different from Heisenberg's one on the assumption that the equation is invariant under the conformal transformation. The aim of this paper is to investigate the relationship among the non-linear character of the spinor equation, Pauli-Gürsey transformation and the general covariance or spin transformation. We also speculate that our legs connection will play the role of a link between the weak

interactions and gravitational interactions.

We shall start with the generalized Dirac equation in curved space-time

$$\gamma^p(\partial_p-\Gamma_p)\psi=0 \qquad (1\cdot1)^*$$

where Γ_p is the spin-affine connection and its general form is given by Green[4] except for its spur. Green has investigated this equation in order to assert that the observed failure of parity conservation is reconciled with the invariance properties postulated by general relativity. Further he identified some terms of Γ_p with the electromagnetic potential, etc. As will be mentioned in § 2, it seems to us that there is no necessary reason for such identification. Furthermore, to carry out this identification we must introduce the complex legs and add the electromagnetic interaction term artificially. Since the introduction of the complex legs complicates the problem, we use the real legs in § 3 and § 4.

Under a general transformation of coordinates $x_r\to{}'x_r$, Eq. (1·1) is invariant if Dirac spinor ψ and Dirac matrices γ^p transform as

$$\psi\to{}'\psi=\left|\frac{\partial x}{\partial x'}\right|^{1/2}\psi, \qquad \gamma^p\to{}'\gamma^p=\gamma^r\frac{\partial' x^p}{\partial x^r}. \qquad (1\cdot2)$$

In addition, another type of invariance is possible for what may be called spin transformation which leaves the functional dependence of the metric tensor on the coordinates unaltered,

$$\psi\to{}'\psi=S^{-1}\psi, \quad \gamma^p\to{}'\gamma^p=S^{-1}\gamma^pS. \qquad (1\cdot3)$$

Spin transformation is not necessarily related to a change of coordinates. For the special coordinate transformations which do not change the functional dependence of the metric tensor on the coordinates, i.e. such that ${}'g^{rs}({}'x)=g^{rs}({}'x)$, we can find a corresponding spin transformation so that the two transformations together leave the Dirac matrices unchanged

$$\psi(x)\to S\psi(x)\left|\frac{\partial x}{\partial' x}\right|^{1/2}, \quad \gamma^p(x)\to\gamma^p({}'x)=S\gamma^r(x)S^{-1}\frac{\partial' x^p}{\partial x^r}.$$

Such transformations play an important role in the case of special relativity. On the contrary, in the curved space-time, the spin transformation (1·3) has no connection with the coordinate transformation. The requirement of invariancy under the spin transformations will then introduce the new entities. The general form of spin transformations S can be expressed in terms of constant Dirac matrices $\gamma_{(\omega)}$ $(\omega=1, 2, 3, 4, 5)$

$$S=S_A+S_B$$

where

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* In what follows, the small Latin and small Greek indices represent quantities with respect to the space-time coordinates system and to the legs system respectively. We use the units $\hbar=c=1$.

$$S_A=1+(1/2)\sum_{\alpha,\beta=1}^{4} v^{\alpha\beta}[\gamma_{(\alpha)},\gamma_{(\beta)}]+v^5\gamma_{(5)},$$

$$S_B=\sum_{\alpha=1}^{4} v^{5\alpha}\gamma_{(5)}\gamma_{(\alpha)}+\sum_{\alpha=1}^{4} v^{\alpha}\gamma_{(\alpha)} \tag{1·4}$$

in which the coefficients v^{ω} and $v^{\omega\alpha}$ depend on the coordinates and $\gamma_{(\omega)}$ satisfy $\gamma_{(1)}^2=\gamma_{(2)}^2=\gamma_{(3)}^2=-\gamma_{(4)}^2=\gamma_{(5)}^2=1$. The indices in brackets are used to indicate explicitly quantities with respect to the legs system. The form of S shows that the spin transformation under which the theory is invariant is isomorphic with the group of rotations in an abstract 6-dimensional space. This corresponds to the result of Finkelstein[5] who showed that the group of generally covariant coordinate transformations may be mapped on the orthogonal group in six dimensions by introducing the line coordinates. Corresponding to the 6-dimensional abstract space, in § 3 we attach six legs at each point of the curved space-time. If Eq. (1·1) is obtained from the requirement that the action integral is invariant under the general 6-dimensional rotation, Γ_p should be of the form[6]

$$\Gamma_p=(1/2)\sum_{\mu,\nu=1}^{6} u_p^{(\mu\nu)}\ T_{(\mu\nu)} \tag{1·5}$$

where $T_{(\mu\nu)}$ is the representation of the generator of the six-dimensional rotation for ψ. The $u_p^{(\mu\nu)}$ has the meaning of the legs connection and depends the coordinates. Since we start with the massless Dirac equation, it should be sufficient to use the four-dimensional representation. In § 3, using the legs connection $u_p^{(\mu\nu)}$ and the concept of sedenion conjugate, we obtain a four-dimensional representation of Eq. (1·1).

In our formalism the legs connection $u_p^{(\mu\nu)}$ plays an important role. Therefore, it is natural to consider $u_p^{(\mu\nu)}$ as independent field variables. In § 4, we extend Weyl's four legs formalism[7] to the six legs one. If we vary the Lagrangian with respect to $u_p^{(\mu\nu)}$, we obtain an equation which is different from the one defining the legs connection. To circumvent this unsatisfactory situation we must alter the definition of $u_p^{(\mu\nu)}$, and also add the Fermi-Fermi interaction term to the Lagrangian in order to satisfy the integrability condition for six legs. Our added non-linear term has more general form than those of Heisenberg[1] or Kita.[8] It is remarkable that the non-linear term is obtained naturally by taking $u_p^{(\mu\nu)}$'s as independent field quantities. In order to introduce the coupling constant larger than the gravitation constant, we attempt to introduce the vector field expressed in terms of six legs $h_{(\mu)}^{A}$ and $u_p^{(\mu\nu)}$.

As already mentioned, if we want to have the imaginary spur part in Γ_p, we must use complex six legs. We see then that the space of the complex legs fields is that of Einstein-Schrödinger theory.[9] In the case of four legs, Sciama[10] proposed a geometrical theory of the electromagnetic field by using the complex four legs. He reached the interesting consequences that the charges of all bosons must be integer multiples of some basic charge, and that the direct interaction between a

particle and the electromagnetic field must be invariant under space and time reflections. If one uses Sciama's electromagnetic field, it is easily seen that one cannot obtain non-vanishing electromagnetic effects. To obtain the electromagnetic effect we must again introduce the electromagnetic potential artificially as in the case treated by Green. In § 5, it will be shown that the electromagnetic potential defined by Green is introduced through the consideration on the variational equation with respect to the complex six legs connection.

If we use Kramer's representation of $\gamma_{(\alpha)}$, the spin transformation S_A in (4·1) is of the from $\begin{pmatrix} \times & \times & 0 & \\ \times & \times & & \\ & & \times & \times \\ 0 & & \times & \times \end{pmatrix}$ and S_B $\begin{pmatrix} & & \times & \times \\ & 0 & \times & \times \\ \times & \times & & \\ \times & \times & 0 & \end{pmatrix}$. A general Lorentztransformation in vector space can correspond to S_A. In § 6, it will be shown that the generalized Pauli-Gürsey transformation corresponds to the spin transformation S_B.

§ 2. Preliminaries: generalized Dirac equation in the general relativity

In this section we shall sketch the outline of the general relativistic Dirac equation which is invariant under the general coordinate transformations and spin transformations, following the line of Green's formalism.[4] Since Green's formalism contains few unsatisfactory points,* slight modifications will be made. Throughout this paper the space-time coordinates $x^p(x^1, x^2, x^3, x^4=t)$ are used and the familiar summation-rule concerning the super- and sub-scripts in tensor calculus is applied to every kind of indices.

The γ^p in (1·1) are those four-four matrix functions which satisfy the relations

$$\{\gamma^p, \gamma^q\} = \gamma^p \gamma^q + \gamma^q \gamma^p = 2g^{pq}$$

where g^{pq} are defined by the relation $g^{pq}g_{qr}=\delta^p_r$ in which g_{qr} are non-singular metric field. We start with the postulate that the derivatives of matrix $\gamma_q = g_{qr}\gamma^r$ have a general form

$$\partial_p \gamma_q = E^s_{qp}\gamma_s + E^5_{qp}\gamma_5 + E^s_p \gamma_{qs} + E^5_p \gamma_{q5} \qquad (2\cdot1)$$

where E^5_{pq}'s are tensors depending on the coordinates, with the exception of E^s_{pq} which is affinity, and

$$\gamma_{pq} = \frac{1}{2}[\gamma_p, \gamma_q], \quad \gamma_{p5} = \gamma_p \gamma_5, \quad \gamma_5 = \frac{1}{4!}\varepsilon^{pqrs}\gamma_p \gamma_q \gamma_r \gamma_s, \quad \varepsilon^{pqrs} = iE_{pqrs}/\sqrt{-g} \qquad (2\cdot2)$$

in which g means the determinant of g_{pq}, and E_{pqrs} is the Levi-Civita symbol whose non-vanishing components are those for which all four indices are different and are equal to 1 or -1 according as (p, q, r, s) is an even or an odd permutation of (1, 2, 3, 4). Here we use the different quantities from Green's. The relation

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* For instance, the second equation (26) and that of (34) in his paper are incompatible with the third equation (27) and that of (33), respectively.

between our quantities and those of Green is as follows:

$$1=\gamma_5^2=(\text{“}\gamma_5^{2}\text{”})/g,\ E_{pq}^{5}=\sqrt{g}\ \text{“}E_{pq}^{5}\text{”},\ E_q^5=\sqrt{g}\ \text{“}E_p^5\text{”} \tag{2·3}$$

where “ ” means the quantities of Green. The other quantities are same as those of Green.

The coefficients E_{qp}^{s}, E_{qp}^{5}, E_p^s and E_p^5 which appear in (2·1) cannot be assigned arbitrarily, but must satisfy a set of identities in order that (2·1) be integrable. These identities are

$$\begin{aligned}
&\partial_{[q}E_{rp]}^{s}+E_{r[p}^{t}E_{tq]}^{s}+E_{r[p}^{5}E_{5q]}^{s}+E_{r[p}^{6}E_{6q]}^{s}=0\\
&\partial_{[q}E_{rp]}^{5}+E_{r[p}^{t}E_{tq]}^{5}+E_{r[p}^{6}E_{6q]}^{5}=0\\
&\partial_{[q}E_{p]}^{s}+E_{[p}^{t}E_{tq]}^{s}+E_{[p}^{5}E_{5q]}^{s}=0\\
&\partial_{[q}E_{p]}^{5}+E_{[p}^{t}E_{tq]}^{5}=0
\end{aligned} \tag{2·4}$$

where

$$A_{\cdot[p}B_{\cdot q]}=(1/2)[A_{\cdot p}B_{\cdot q}-A_{\cdot q}B_{\cdot p}],\ E_{pq}^{6}=g_{ps}E_q^s$$
$$E_{5q}^{s}=-g^{rs}E_{sp}^{5},\ E_{5p}^{6}=E_{6p}^{5}=E_p^5,\ E_{6p}^{s}=E_p^s.$$

The spin affine connection Γ_p is determined with the aid of Eq. (2·1) and

$$\partial_p\gamma_q=\Gamma_{qp}^{s}\gamma_s+[\Gamma_p,\gamma_q]$$

in which

$$\begin{aligned}
&\Gamma_{qp}^{s}=(1/2)g^{sr}(\partial_p g_{qr}+\partial_q g_{pr}-\partial_r g_{qp}),\\
&\Gamma_p=-(1/2)\{(1/2)\,C_{rp}^{t}\gamma^r\gamma_t+E_{rp}^{5}\gamma^r\gamma_5+E_p^t\gamma_t+E_p^5\gamma_5\}
\end{aligned} \tag{2·5}$$

where

$$C_{rp}^{t}\equiv E_{rp}^{t}-\Gamma_{rp}^{t},\quad C_{rp}^{t}g_{ts}=-C_{sp}^{t}g_{tr}. \tag{2·6}$$

Here we emphasize that spin affine connection Γ_p can be expressed without use of E_{rp}^{t}, etc., in the following form,

$$\Gamma_p=(1/4)g^{rs}\Gamma_{rp}^{t}\gamma_{st}-(1/16)\{\gamma^r\partial_p\gamma_r+\gamma^5\partial_p\gamma_5-(1/2)\gamma^{sr}\partial_p\gamma_{sr}-(1/2)\gamma^{5r}\partial_p\gamma_{5r}\}+(1/8)\Gamma_{rp}^{r} \tag{2·7}$$

by a straightforward, though tedius, calculation. The above Γ_p is the most general form of the usual traceless Γ_p' [11]

$$\Gamma_p'=-(1/8)(\partial_p\gamma_{rs}+\partial_r g_{sp}-\partial_s g_{rp})\gamma^{rs}. \tag{2·8}$$

If the spin affine connection (2·5) is substituted into the generalized Dirac equation

$$\gamma^p(\partial_p-\Gamma_p)\psi=0, \tag{2·9}$$

(2·9) reduces to

$$\gamma^p\partial_p\psi-(1/2)[g^{pq}C_{pq}^{r}\gamma_r+g^{pq}E_{pq}^{5}\gamma_5+E_p^p+\{E_p^s g^{pr}+(1/2)E_{pq}^{5}\varepsilon^{pqrs}\}\gamma_{rs}$$

$$+\{E_p^5 g^{ps}-(1/2)g_{rt}C_{pq}^{t}\varepsilon^{pqrs}\}\gamma_{s5}]\psi=0. \qquad (2\cdot10)$$

Green interpreted that the imaginary part of $g^{pq}C_{pq}^{r}$ has the character of an electromagnetic field, that the imaginary part of $g^{pq}E_{pq}^{5}$ has the character of a pseudo-scalar meson field, and that the real part of E_p^p acts as a mass field.*

Of course, there is no necessary reason to accept this interpretation. To ensure that the Lagrangian density be real, $(\partial_p-\Gamma_p)$ must satisfy the following equation,

$$\partial_p\eta+\eta\Gamma_p+\Gamma_p^{+}\eta=0, \qquad (2\cdot11)$$

where η is an hermitian matrix defining the adjoint spinor $\bar{\psi}=\psi^{+}\eta$ and satisfying $\eta\gamma_p+\gamma_p^{+}\eta=0$.* The equation (2·11) can also be written in the form

$$\partial_p\eta=\eta[(1/2)I_{rp}^{s}\gamma^{r}{}_{s}+I_{rp}^{5}\gamma^{r}\gamma_5+I_p^{r}\gamma_r+I_p^{5}\gamma_5] \qquad (2\cdot11')$$

in which I_{rp}^{s}, I_{rp}^{5}, I_p^{r} and I_p^{s} represent i times the imaginary parts of C_{rp}^{s}, E_{rp}^{5}, E_p^{r} and E_p^{5}, respectively. Since the definition of the adjoint field depends on I_{rp}^{s}, etc., it will be difficult to interpret that $g^{pr}I_{pr}^{s}$ and $g^{pr}I_{pr}^{5}$ have the usual properties of the electromagnetic field and pseudo-scalar meson field. Each term in (2·7) has not any physical meaning independently of each other and it will be difficult to separate the irreducible entities. In fact, we can show that the electromagnetic interaction term does not appear in the Lagrangian density for Dirac particle, though it appears in (2·10). (See the last equation of (4·1) below.) In order to introduce the electromagnetic effect we must then add the interaction term. On the contrary, as the real part of E_p^p does not enter into (2·11′), we shall interpret it as a mass field.

Green has also constructed an explicit representation of the γ_p by introducing the six legs and by using the 8–8 constant mutually anticommuting matrices. But, as we start with 4–4 Dirac matrix and Dirac equation without a constant mass term, it is natural to use the four-dimensional representation instead of eight dimensional one. In the next section, this problem will be discussed in detail by introducing "sedenion conjugate".

§ 3. Introduction of legs connection and four-dimensional representation of γ_p

We shall first construct a representation of the matrices whose derivatives are defined by (2·1).

If we introduce β^{AB} such as

$$\beta^{p6}=-\beta^{6p}=\gamma^{p}, \quad \beta_{p6}=-\gamma_p, \quad \beta^{56}=-\beta^{65}=-\beta_{56}=\gamma^{5}$$

$$\beta^{p5}=\gamma^{p5}, \quad \beta^{pq}=\gamma^{pq},$$

the spin affine connection Γ_p given by (2·7) can be written in the form

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* γ_p is taken so that $i\bar{\psi}\gamma_p\psi$ is real.

$$\Gamma_p = (1/4) g^{rs} \Gamma^t_{rp} \gamma_{st} + (1/32) \sum_{A,B=1}^{6} \beta^{AB} \partial_p \beta_{AB} + (1/8) \Gamma^r_{rp}. \tag{3·1}$$

To construct the four-dimensional representation of γ_p, it is instructive to remember the two component theory of the general relativistic Dirac equation.[12] As will be easily seen, the usual spin affine connection defined by (2·8) is expressed in the two component theory as

$$\Gamma_p' = -(1/8)(\partial_p \sigma_{rs} - \partial_r g_{sp} + \partial_s g_{rp}) \sigma^{rs} \tag{3·2}$$

where

$$\sigma^{rs} = (1/2)(\tilde{\sigma}^r \sigma^s - \tilde{\sigma}^s \sigma^r), \quad \sigma^r = e^{r(\alpha)} \sigma_{(\alpha)}, \quad \tilde{\sigma}^r = e^{r(\alpha)} \tilde{\sigma}_{(\alpha)}$$

in which $\tilde{\sigma}_{(\alpha)}$ means the "quaternion conjugate" of $\sigma_{(\alpha)}$ i.e. $\tilde{\sigma}_{(4)} = \sigma_{(4)} = -E_2$, $\tilde{\sigma}_{(\gamma)} = -\sigma_{(\gamma)}$ for $\gamma = 1, 2, 3$. Here and in what follows we use the Greek letters from the latter part of the alphabet for the range 1, 2, 3, 4, 5, 6, α and β for the range 1, 2, 3, 4, and γ for the range 1, 2, 3. $\sigma_{(\alpha)}$ and four legs $e^r_{(\alpha)}$ satisfy the following relations,

$$e^{r(\alpha)} e^s_{(\alpha)} = g^{rs}, \quad e^s_{(\alpha)} e_{s(\beta)} = \eta_{(\alpha\beta)},$$

$$\tilde{\sigma}_{(\alpha)} \sigma_{(\beta)} + \tilde{\sigma}_{(\beta)} \sigma_{(\alpha)} = -2\eta_{(\alpha\beta)},$$

$$\eta_{(11)} = \eta_{(22)} = \eta_{(33)} = -\eta_{(44)} = 1.$$

Since we take $x^4 = t$ and $\eta_{(44)} = -1$, it should be noted that our $\sigma_{(4)}$ plays the role of $\sigma_{(0)}$ in the usual formalism.

To achieve the four-dimensional represention of Eq. (3·1) impels us to introduce "sedenion conjugate" defined by

$$\tilde{\gamma}_{(6)} = \gamma_{(6)} = -E_4, \quad \tilde{\gamma}_{(\mu)} = -\gamma_{(\mu)} \text{ for } \mu = 1, 2, 3, 4, 5 \tag{3·3}$$

where

$$\gamma_{(\gamma)} = \rho_2 \sigma_{(\gamma)}, \quad i\gamma_{(4)} = \rho_3, \quad \gamma_{(5)} = i\gamma_{(1)}\gamma_{(2)}\gamma_{(3)}\gamma_{(4)} = -\rho_1 \tag{3·4}$$

in which ρ_γ, $\sigma_{(\gamma)}$ are Pauli matrices. These matrices therefore satisfy

$$\tilde{\gamma}_{(\mu)} \gamma_{(\nu)} + \tilde{\gamma}_{(\nu)} \gamma_{(\mu)} = -2\eta_{(\mu\nu)} \tag{3·5}$$

where

$$\eta_{(11)} = \eta_{(22)} = \eta_{(33)} = -\eta_{(44)} = \eta_{(55)} = -\eta_{(66)} = 1. \tag{3·6}$$

The legs indices are lowered and raised by means of this $\eta_{(\mu\nu)}$ and $\eta^{(\mu\nu)} (= \eta_{(\mu\nu)})$ respectively. Further, we put

$$\gamma_{(\mu\nu)} = (1/2)[\tilde{\gamma}_{(\mu)} \gamma_{(\nu)} - \tilde{\gamma}_{(\nu)} \gamma_{(\mu)}]. \tag{3·7}$$

One can verify the following relations which are useful for later considerations,

$$[\gamma_{(\mu\nu)}, \gamma_{(\rho\sigma)}] = \eta_{(\mu\rho)}\gamma_{(\nu\sigma)} - \eta_{(\nu\rho)}\gamma_{(\mu\sigma)} + \eta_{(\mu\sigma)}\gamma_{(\rho\nu)} - \eta_{(\nu\sigma)}\gamma_{(\rho\mu)}, \tag{3·8}$$

$$\{\gamma_{(\mu\nu)}, \gamma_{(\rho\sigma)}\} = 2(\eta_{(\mu\sigma)}\eta_{(\nu\rho)} - \eta_{(\nu\sigma)}\eta_{(\mu\rho)}) + \varepsilon_{(\mu\nu\rho\sigma\xi\eta)}\gamma^{(\xi\eta)} \tag{3·9}$$

where $\varepsilon_{(\mu\nu\rho\sigma\xi\eta)}$ is the antisymmetrical tensor of the sixth rank and has the value $\varepsilon_{(123456)}=i$.

As mentioned in the last section, the appearance of complex coefficients E^r_{pq}, etc., makes it complicated to construct η and to interpret their physical meanings. Therefore, we shall first start with the real six legs $h^A_{(\mu)}(A=1, 2, 3, 4, 5, 6)$ defined by

$$h^{A(\mu)}\, h^B{}_{(\mu)}=g^{AB}, \quad h^A{}_{(\mu)}\, h_{A(\nu)}=\eta_{(\mu\nu)} \qquad (3\cdot10)$$

where

$$g^{AB}=g^{pq} \quad \text{for } A, B=1, 2, 3, 4$$

$$g^{p5}=g^{p6}=0, \quad g^{55}=-g^{66}=1.$$

We shall now introduce the legs connection $u_p^{(\mu\nu)}$. For this purpose we transfer in parallel the legs at one point P to a neighbouring point P' by means of affine connection Γ^r_{pq} of the Riemannian space. These transferred legs will in general differ infinitesimally from the local legs at P'. It will be a simple assumption about this difference that it consists of an infinitesimal rotation in the six-dimensional pseudo-Euclidean space

$$L^{(\mu\nu)}=\delta^{(\mu\nu)}+u_p{}^{(\mu\nu)}dx^p$$

where dx^p is the displacement PP'. The $u_p^{(\mu\nu)}$ is the legs connection we are seeking for. Accordingly, one finds

$$\partial_q h^{p(\mu)}+\Gamma^p_{rq}\, h^{r(\mu)}+u_q{}^{(\mu\nu)}h^p{}_{(\nu)}=0 \qquad (3\cdot11)$$

$$\partial_q h^{A(\mu)}+u_q{}^{(\mu\nu)}h^A{}_{(\nu)}=0 \quad \text{for } A=5, 6. \qquad (3\cdot12)$$

We can find a representation of γ^p with the hel pof the real six legs $h^A_{(\mu)}$ and $\gamma^{(\mu\nu)}$. Let

$$\gamma^p=h^p{}_{(\mu)}\, h^6{}_{(\nu)}\, \gamma^{(\mu\nu)}. \qquad (3\cdot13)$$

From (3·9) and (3·10) it follows that

$$\{\gamma^p, \gamma^q\}=2g^{pq} \qquad (3\cdot14)$$

and other required anticommutation relations.

Also, by virture of (3·11) and (3·12), one obtains

$$\partial_q \gamma_p-\Gamma^r_{pq}\gamma_r+h^A{}_{(\mu)}\, u_q{}^{(\mu\nu)}h_{p(\nu)}\tilde{\alpha}_{[A}\alpha^{6]}+h^A{}_{(\mu)}u_q{}^{(\mu\nu)}\, h^6{}_{(\nu)}\tilde{\alpha}_{[p}\,\alpha_{A]}=0 \qquad (3\cdot15)$$

where

$$\tilde{\alpha}^A=h^{A(\alpha)}\,\tilde{\gamma}_{(\alpha)}.$$

Putting

$$C^r_{\cdot pq}=-h^r{}_{(\mu)}\, h_{p(\nu)}\, u_q{}^{(\mu\nu)}, \quad E^s_{\cdot pq}=-h^5{}_{(\mu)}h_{p(\nu)}u_q{}^{(\mu\nu)},$$

$$E^r_q=E^r_{\cdot 6q}=-h^r{}_{(\mu)}\, h^6{}_{(\nu)}\, u_q{}^{(\mu\nu)}=h^r{}_{(\mu)}\, h_{6(\nu)}\, u_q{}^{(\mu\nu)}, \qquad (3\cdot16)$$

$$E^5_q = E^5_{\cdot 6q} = -h^5{}_{(\mu)}\, h^6{}_{(\nu)}\, u_q{}^{(\mu\nu)} = h^5{}_{(\mu)}\, h_{6(\nu)}\, u_q{}^{(\mu\nu)},$$

we obtain (2·1), i.e.

$$\partial_q \gamma_p = E^s_{pq}\gamma_s + E^5_{pq}\gamma_5 + E^s_q \gamma_{ps} + E^5_q \gamma_{p5}.$$

Furthermore, it is a matter of course that the spin affine connection Γ_p defined by (2·5) takes a form of

$$\Gamma_p = (1/2)\, u_p{}^{(\mu\nu)} \gamma_{(\mu\nu)} \tag{3·17}$$

by virtue of (3·16), as it should be.

Finally, we shall show that our C^r_{pq}, E^5_{pq}, etc., satisfy the integrability condition (2·4). The integrability condition of (3·11) and (3·12) is

$$\partial_{[p}\, u_{q]}{}^{(\mu\nu)} + \eta_{(\rho\sigma)}\, u_{[p}{}^{(\mu\rho)}\, u_{q]}{}^{(\sigma\nu)} = (1/2)\, R^t_{\cdot spq}\, h_t{}^{(\nu)}\, h^{s(\mu)} \tag{3·18}$$

where $R^t_{\cdot spq}$ is the Riemann-Christoffel curvature tensor. Of course, in the case of the integrability condition of (3·12), the right-hand side of (3·18) must be put to equal to zero, because the condition has a meaning only when it operates on $h^A_{(\mu)}$ ($A=5$, 6). Multiplying $h_{s'(\mu)} h'^{}_{(\nu)}$ and contracting with regard to μ and ν, we obtain

$$C^t_{s[q;p]} + C^l_{s[p}\, C^t_{lq]} + E^5_{s[p}\, E^t_{5q]} + E^6_{s[p}\, E^t_{6q]} = (1/2)\, R^t_{\cdot spq} \tag{3·19}$$

after putting $s'=s$, $t'=t$, where $;p$ means the covariant derivatives with respect to the Christoffel affinity Γ^r_{pq}.

Similarly,

$$\begin{aligned} &E^5_{s[p;q]} + C^l_{s[p}\, E^5_{lq]} + E^6_{s[p}\, E^5_{6q]} = 0 \\ &E^6_{s[p;q]} + C^l_{s[p}\, E^6_{lq]} + E^5_{s[p}\, E^b_{5]}{}^{b} = 0 \\ &E^5_{[p;q]} + E^l_{[p}\, E^5_{lq]} = 0. \end{aligned} \tag{3·19′}$$

Eqs. (3·19) and (3·19′) are nothing but Eq. (2·4).

§ 4. Field equation derived from the variation of legs connection

From the analysis in the last section, we see that the legs connection $u^{(\mu\nu)}_p$ plays an important role in the presence of matter field. We shall therefore take $h^A_{(\mu)}$ and $u^{(\mu\nu)}_p$ as independent field variables. Taking the variation with regard to $h^p_{(\mu)}$, we get the usual equation for the gravitational field. Then, we shall consider only the variation with respect to $u^{(\mu\nu)}_p$.

We start with the Lagrangian

$$L = \int \sqrt{-g}\, R d\tau + \int \sqrt{-g}\, L_0 d\tau + \int \sqrt{-g}\, L_{int} d\tau$$

where

$$\begin{aligned} R &= 2h^p{}_{(\mu)} h^q{}_{(\nu)} \{\partial_{[p}\, u_{q]}{}^{(\mu\nu)} + \eta_{(\rho\sigma)}\, u_{[p}{}^{(\mu\rho)} u_{q]}{}^{(\sigma\nu)}\} \\ L_0 &= \kappa \{\bar{\psi}\, h^p{}_{(\mu)}\, h^6{}_{(\nu)}\, \gamma^{(\mu\nu)}\, \partial_p \psi - \partial_p \bar{\psi}\, h^p{}_{(\mu)}\, h^6{}_{(\nu)}\, \gamma^{(\mu\nu)}\, \psi\} \end{aligned} \tag{4·1}$$

$$L_{int}=-(\kappa/2)\bar{\psi}\{\gamma^{(\mu\nu)},\gamma^{(\rho\sigma)}\}\psi\, h^{p}_{(\mu)}\, h^{\beta}_{(\nu)}\, u_{p(\rho\sigma)}$$
$$=\kappa u_{p}^{(\mu\nu)}\{h^{p}_{(\mu)}h^{\beta}_{(\nu)}-h^{p}_{(\nu)}\, h^{\beta}_{(\mu)}\}\bar{\psi}\psi-(\kappa/2)\varepsilon_{(\mu\nu\rho\sigma\xi\eta)}h^{p(\mu)}\, h^{\beta(\nu)}u_{p}^{(\rho\sigma)}\bar{\psi}\gamma^{(\xi\eta)}\psi$$

in which κ is Einstein's gravitational constant and $\bar{\psi}=\psi^{+}\eta$ with $\eta=i\gamma_{(4)}$ which results from the assumption of the real property of $h^{A}_{(\mu)}$ and $u_{p(\mu\nu)}$.

The variation with regard to $u_{p}^{(\mu\nu)}$ leads to the following relations:

$$h^{s}_{(\mu)}\,\partial_{s}h^{p}_{(\nu)}-h^{s}_{(\nu)}\partial_{s}h^{p}_{(\mu)}=h^{s}_{(\nu)}u_{s(\mu\rho)}h^{p(\rho)}-h^{s}_{(\mu)}u_{s(\nu\rho)}h^{p(\rho)}$$
$$+\kappa(h^{p}_{(\mu)}h^{\beta}_{(\nu)}-h^{p}_{(\nu)}h^{\beta}_{(\mu)})\bar{\psi}\psi-(\kappa/2)\varepsilon_{(\mu\nu\rho\sigma\xi\eta)}h^{p(\rho)}\, h^{\beta(\sigma)}\bar{\psi}\gamma^{(\xi\eta)}\psi. \qquad (4\cdot2)$$

On the other hand, by eliminating Γ^{p}_{rq} from Eq. (3·11), we directly get

$$h^{s}_{(\mu)}\partial_{s}h^{p}_{(\nu)}-h^{s}_{(\nu)}\partial_{s}h^{p}_{(\mu)}=h^{s}_{(\nu)}u_{s(\mu\rho)}h^{p(\rho)}-h^{s}_{(\mu)}u_{s(\nu\rho)}h^{p(\rho)}, \qquad (4\cdot3)$$

instead of (4·2). Thus, between (4·2) and (4·3), there appears a difference by the terms which contain spinor field. Accordingly, to remove this disaccordance one must modify Eq. (3·11) defining legs connection.

As a simple modification, we postulate that

$$\partial_{q}h^{p(\mu)}+\Gamma^{p}_{rq}h^{r(\mu)}+u_{q}^{(\mu\nu)}h^{p}_{(\nu)}+o_{q}^{(\mu\nu)}h^{p}_{(\nu)}=0 \qquad (4\cdot4)$$

with

$$o_{q}^{(\mu\nu)}=-\kappa(h^{\beta(\mu)}h_{q}^{(\nu)}-h^{\beta(\nu)}h_{q}^{(\mu)})\bar{\psi}\psi$$
$$+(\kappa/4)\varepsilon^{(\mu\nu\rho\sigma\xi\eta)}h_{q(\rho)}h^{\beta}_{(\sigma)}\bar{\psi}\gamma_{(\xi\eta)}\psi. \qquad (4\cdot5)$$

It is readily shown that if we eliminate Γ^{p}_{rq} from (4·4) the relation (4·2) is derived. Similarly, eliminating $u_{q}^{(\mu\nu)}$ and $o_{q}^{(\mu\nu)}$ we obtain

$$\partial_{q}g^{rs}+\Gamma^{s}_{qt}g^{rt}+\Gamma^{r}_{tq}g^{ts}=0.$$

As we now regard Eq. (4·4) as defining the legs connection, the integrability condition (3·18) must be modified. Since the new condition has a very complicated form, we do not give it here. Multiplying this condition by $h^{p}_{(\mu)}h^{q}_{(\nu)}$ and contracting with regard to p, q and μ, ν, we get the following rather simple expression by a straightforward but tedious calculation,

$$R=2h^{p}_{(\mu)}h^{q}_{(\nu)}\{\partial_{[p}u_{q]}^{(\mu\nu)}+\eta_{(\rho\sigma)}u_{[p}^{(\mu\rho)}u_{q]}^{(\sigma\nu)}\}$$
$$-12\kappa^{2}\bar{\psi}\psi\cdot\bar{\psi}\psi+(\kappa^{2}/16)\varepsilon^{(\mu\omega\rho\sigma\xi\eta)}\varepsilon^{(\omega'\nu\rho'\sigma'\xi'\eta')} \qquad (4\cdot6)$$
$$\times\eta_{(\omega\omega')}h^{q}_{(\nu)}h_{q(\rho)}h^{\nu}_{(\mu)}h_{p(\rho')}h^{\beta}_{(\sigma)}h^{\beta}_{(\sigma')}\bar{\psi}\gamma_{(\xi\eta)}\psi\cdot\bar{\psi}\gamma_{(\xi'\eta')}\psi.$$

Further, it should be noticed that the interaction term in (4·1) must also be modified due to the addition of $o_{p}^{(\mu\nu)}$ to the legs connection $u_{p}^{(\mu\nu)}$. This additonal interaction term is

$$L_{int}(\mathrm{add})=-8\kappa^{2}\bar{\psi}\psi\cdot\bar{\psi}\psi-(\kappa^{2}/8)\varepsilon^{(\mu\nu\rho\sigma\xi\eta)}\,\varepsilon^{(\mu'\nu'\rho'\sigma'\xi'\eta')}\eta_{(\rho\rho')}\,\eta_{(\sigma\sigma')}\times$$
$$\times h^{p}_{(\mu)}\,h_{p(\mu')}\,h^{\beta}_{(\nu)}h^{\beta}_{(\nu')}\bar{\psi}\gamma_{(\xi\eta)}\psi\cdot\bar{\psi}\gamma_{(\xi'\eta')}\psi. \qquad (4\cdot7)$$

Thus, by taking into account Eqs. (4·6) and (4·7), our Lagrangian reduces to

$$L=\int\sqrt{-g}\,Rd\tau+\int\sqrt{-g}\,L_0d\tau+\int\sqrt{-g}\,L_{int}d\tau+F \tag{4·8}$$

with

$$F=-20\kappa^2\,\bar{\psi}\psi\cdot\bar{\psi}\psi+F' \tag{4·9}$$

in which

$$F'=-\frac{\kappa^2}{16}\varepsilon^{(\mu\nu\rho\sigma\xi\eta)}\,\varepsilon^{(\mu'\nu'\rho'\sigma'\xi'\eta')}\eta_{(\mu\mu')}\eta_{(\nu\nu')}\eta_{(\rho\rho')}h^6_{(\sigma)}\,h^6_{(\sigma')}\bar{\psi}\gamma_{(\xi\eta)}\psi\cdot\bar{\psi}\gamma_{(\xi'\eta')}\psi$$

$$=-\frac{3}{4}\kappa^2\{\bar{\psi}\gamma^{(\xi\eta)}\psi\cdot\bar{\psi}\gamma_{(\xi\eta)}\psi+2h^{6(\xi)}\,h^6_{(\xi')}\bar{\psi}\gamma_{(\xi\eta)}\psi\cdot\bar{\psi}\gamma^{(\xi'\eta)}\psi\}. \tag{4·9′}$$

F can be expressed also in the form

$$F=-20\kappa^2\bar{\psi}\psi\cdot\bar{\psi}\psi-(3/4)\kappa^2\sum_{A,B=1}^{5}\bar{\psi}\gamma_{AB}\psi\cdot\bar{\psi}\gamma^{AB}\psi. \tag{4·9″}$$*

Further, if we use the identities among $\bar{\psi}\gamma_{(\xi\eta)}\psi$'s in the c-number theory,[13] (4·9) reduces to a simple form

$$F=-17\kappa^2\bar{\psi}\psi\cdot\bar{\psi}\psi. \tag{4·9‴}$$

In other words, it can be said that our result (4·9) is an expansion of $-17\kappa^2\bar{\psi}\psi\cdot\bar{\psi}\psi$ in terms of $\bar{\psi}\gamma_{(\xi\eta)}\psi$.

Recently, Kita[8] has advocated that $sp\Gamma_p$ can include the terms $\bar{\psi}\gamma_p\psi$ and $\bar{\psi}\gamma_p\gamma_5\psi$ from the viewpoint of the spinor unified theory. Our result shows that this proposition is a necessary conclusion based on the consideration that $u_p^{(\mu\nu)}$ should be independent field variables subject to the variational principle. Moreover, our result involves the most general type of bilinear forms of $\bar{\psi}$ and ψ.

So far we have considered only one sort of Dirac particle. In the case where two Dirac particles described by ψ and φ coexist, the following Fermi-Fermi interaction term F_{int} is added to (4·8),

$$F_{int}=-20\kappa^2\{\bar{\psi}\psi\cdot\bar{\varphi}\varphi\}-\frac{3}{4}\kappa^2[\{\bar{\psi}\gamma^{(\xi\eta)}\psi,\ \bar{\varphi}\gamma_{(\xi\eta)}\varphi\}+2h^{6(\xi)}h^6_{(\xi')}\{\bar{\psi}\gamma_{(\xi\eta)}\psi,\ \bar{\varphi}\gamma^{(\xi'\eta)}\varphi\}]. \tag{4·10}$$

If we interpret the above result in the usual four legs formalism by regarding $h^6_{(5)}$ and $h^6_{(6)}$ as scalar quantities, the last term in (4·10) contains the parity non-conserving terms.

The coupling constant of the above Fermi-Fermi interaction is very small. In order to introduce the coupling constant larger than the gravitational constant, we shall try to introduce electromagnetic-type and *ps-ps* meson-type interaction term. As a simple interaction term, we take

$$\widetilde{L}_{int}=i\tilde{\kappa}h_A^{(\rho)}h^{6(\sigma)}\bar{\psi}\gamma_{(\rho\sigma)}\psi\{h^p_{(\mu)}\,u_p^{(\mu\nu)}\,h^A_{(\nu)}-h^p_{(\nu)}\,u_p^{(\mu\nu)}h^A_{(\mu)}\}$$

---

* In the four legs formalism, we obtain Weyl's result, $F_w=-(3/2)\kappa^2\bar{\psi}\gamma^{(ab)}\psi\cdot\bar{\psi}\gamma_{(ab)}\psi$.

$$=-2i\tilde{\kappa}h^{6(\sigma)}\bar{\psi}\gamma_{(\rho\sigma)}\psi\cdot u_p^{(\rho\nu)}\,h^p_{(\nu)} \tag{4·11}$$

where $\tilde{\kappa}$ is a new interaction constant which need not be equal to κ. Here, i must be multiplied in order to guarantee the hermitian property of the interaction Lagrangian density, because we have used the real legs $h^A_{(\mu)}$ and real legs connection $u_p^{(\mu\nu)}$. At first sight, the introduction of the large coupling constant $\tilde{\kappa}$ raises a question whether the new interaction term (4·11) gives a large gravitational effect between the Dirac particles. However, we can show that it gives the second order gravitational effect, as seen from the linearized theory of gravitaion where the term containing $u_p^{(\mu\nu)}$ gives the higher order effect.

Putting

$$h^q_{(\mu)}u_q^{(\mu\nu)}h^p_{(\nu)}=g^{rs}\,C^p_{rs}=V^p$$
$$h^q_{(\mu)}\,u_q^{(\mu\nu)}\,h^5_{(\nu)}=g^{rs}\,E^5_{rs}=\phi,$$

we can express (4·11) as

$$\widetilde{L}_{int}=2i\tilde{\kappa}\bar{\psi}\gamma_p\psi\cdot V^p+2i\tilde{\kappa}\bar{\psi}\gamma_5\psi\cdot\phi. \tag{4·11′}$$

It should be noted that our procedure of introduction of V^p and ϕ is different from Green's.

The introduction of new interaction term (4·11) involves the modification of Eq. (4·4) defining $u_p^{(\mu\nu)}$ by similar arguments. (4·11) should be modified as

$$\partial_q\,h^{p(\mu)}+\Gamma^p_{rq}\,h^{r(\mu)}+u_q^{(\mu\nu)}\,h^p_{(\nu)}+o_q^{(\mu\nu)}\,h^p_{(\nu)}+\tilde{o}_q^{(\mu\nu)}\,h^p_{(\nu)}=0 \tag{4·12}$$

with

$$\tilde{o}_q^{(\mu\nu)}=-i\tilde{\kappa}(h_q^{(\mu)}\,h^6_{(\rho)}\bar{\psi}\gamma^{(\nu\rho)}\psi-h_q^{(\nu)}\,h^6_{(\rho)}\bar{\psi}\gamma^{(\mu\rho)}\psi).$$

The additive interaction term $\widetilde{F}$ which corresponds to (4·8) can also be found as

$$\widetilde{F}=4\tilde{\kappa}^2\,h^5_{(\rho)}\,h^6_{(\sigma)}\,\bar{\psi}\gamma^{(\rho\sigma)}\psi\cdot h^5_{(\rho')}\,h^6_{(\sigma')}\,\bar{\psi}\gamma^{(\rho'\sigma')}\psi, \tag{4·13}$$

in the derivation of which we have taken into account the relation $\partial_p(\sqrt{-g}\,h^p_{(\rho)}h^6_{(\sigma)}\bar{\psi}\gamma^{(\rho\sigma)}\psi)=0$.

The above result (4·13) may be considered to suggest that the vector field V^p and pseudoscalar field ϕ play a role of intermediate boson whose second order effects are usual weak interactions.

§ 5. Extension to the complex legs

We have seen that E^p_{qr}, etc., are real quantities so far as real legs are adopted. In order to obtain the complex E^p_{qr} etc., we must introduce complex legs $h^p_{(\mu)}$ at each point of space. This implies that the metric tensor g^{ij} of the space is no longer real symmetric tensor of Riemannian geometry, but it becomes complex and hermitian.

On the analogy of the complex four legs formalism of Sciama,[10] we shall

introduce the complex six legs $h^A_{(\mu)}$, hermitian connection Γ^p_{qr} and antihermitian legs connection $u_q^{(\mu\nu)}$. They are connected by the equations similar to (3·11) and (3·12) :

$$\partial_q h^{p(\mu)} + \Gamma^p_{rq} h^{r(\mu)} + u_q^{(\mu\nu)} h^p_{(\nu)} = 0 \tag{5·1}$$

$$\partial_q h^{A(\mu)} + u_q^{(\mu\nu)} h^A_{(\nu)} = 0 \quad A = 5, 6. \tag{5·2}$$

The metric tensor is defined by

$$g^{ij} = h^{i(\mu)} h^{j*}_{(\mu)}, \quad g^{55} = -g^{66} = 1$$

$$g^{i5} = h^{i(\mu)} h^{5*}_{(\mu)} = 0, \quad g^{i6} = h^{i(\mu)} h^{6*}_{(\mu)} = 0. \tag{5·3}$$

Using the antihermitian property of $u_q^{(\mu\nu)}$, we can eliminate it and obtain

$$g^{i\ j}_{+-;k} = \partial_k g^{ij} + g^{sj} \Gamma^i_{sk} + g^{is} \Gamma^j_{ks} = 0 \tag{5·4}$$

which is the well-known equation in Einstein's unified theory.[9]

We can also eliminate Γ^p_{qr} by using its hermitian property

$$h^{p(\rho)} h^{q*}_{(\mu)} u_{q(\nu\rho)} + h^{p*(\rho)} h^q_{(\nu)} u_{q(\rho\mu)} = h^q_{(\nu)} \partial_q h^{p*}_{(\mu)} - h^{q*}_{(\mu)} \partial_q h^p_{(\nu)}. \tag{5·5}$$

We shall now consider variational equation with respect to $u_q^{(\mu\nu)}$. As a Lagrangian for the gravitational field without Dirac particle, we take

$$L = \int 2\sqrt{-g}\, h^p_{(\mu)} h^{q*}_{(\nu)} (\partial_{[p} u_{q]}^{(\mu\nu)} + \eta_{(\rho\rho')} u_{[p}^{(\mu\rho)} u_{q]}^{(\rho'\nu)})\, d\tau. \tag{5·6}$$

The legs connection $u_q^{(\mu\nu)}$ can be decomposed into irreducible constituents

$$u_q^{(\mu\nu)} = u_q^{\underset{\vee}{(\mu\nu)}} + u_q^{(1)\underline{(\mu\nu)}} + u_q^{(2)\underline{\underline{(\mu\nu)}}} \tag{5·7}$$

where

$$u_q^{(1)\underline{(\mu\nu)}} = (1/6)\eta^{(\mu\nu)} u_q^{\underline{(\rho\rho)}} \text{ and } u_q^{(2)\underline{\underline{(\mu\nu)}}} = u_q^{\underline{(\mu\nu)}} - u_q^{(1)\underline{(\mu\nu)}}$$

in which $u_q^{\underline{(\rho\rho)}} = \eta_{(\rho\sigma)} u_q^{\underline{(\rho\sigma)}}$ and $-$ or $\vee$ under a couple of indices means symmetric or antisymmetric part of that couple. For the sake of simplicity, we omit here the spurless part $u_q^{(2)\underline{\underline{(\mu\nu)}}}$. The variation with respect to $u_{q\underset{\vee}{(\mu\nu)}}$ leads to

$$\begin{aligned}
&[-h^s_{(\nu)} (\partial_s h^{p*}_{(\mu)} + \Gamma^{p*}_{rs} h^{r*}_{(\mu)} + u^*_{s(\mu\rho)} h^{p(\rho)*}) + h^{s*}_{(\mu)} (\partial_s h^p_{(\nu)} + \Gamma^p_{rs} h^r_{(\nu)} + u_{s(\nu\rho)} h^{p(\rho)}) \\
&- (\partial_s h^s_{(\nu)} + \Gamma^r_{sr} h^s_{(\nu)} + u_{s(\nu\rho)} h^{s(\rho)}) h^{p*}_{(\mu)} + (\partial_s h^{s*}_{(\mu)} + \Gamma^{r*}_{sr} h^{s*}_{(\mu)} + u^*_{s(\mu\rho)} h^{s(\rho)*}) h^p_{(\nu)} \\
&+ \Gamma^{\ r}_{\underset{\vee}{sr}} h^s_{(\nu)} h^{p*}_{(\mu)} + \Gamma^{\ r}_{\underset{\vee}{sr}} h^{s*}_{(\mu)} h^p_{(\nu)}]
\end{aligned} \tag{5·8}$$

$$-[\nu \text{ and } \mu \text{ interchanged}] = 0.$$

If $\Gamma = (\Gamma^{\ r}_{\underset{\vee}{sr}} h^s_{(\nu)} h^{p*}_{(\mu)} + \Gamma^{\ r}_{\underset{\vee}{sr}} h^{s*}_{(\mu)} h^p_{(\nu)}) - (\nu \text{ and } \mu \text{ interchanged})$ were absent, (5·8) would imply that (5·5) holds. Therefore it can be said that Eq. (5·1) is derived from the variation principle. However, this is not the case.

The simplest way to remove Γ is to put $\Gamma^{\ r}_{\underset{\vee}{sr}} = 0$ as was done by Einstein.[9] This implies also that $\partial_s(\boldsymbol{g}^{is}_{\vee}) = 0$, where $\boldsymbol{g}^{is}_{\vee} = \sqrt{-g}\, g^{is}_{\vee}$, as easily seen from (5·4).

$\partial_s(\boldsymbol{g}^{rs}_{\vee})=0$ is nothing but the equation derived from the Lagrangian (5·6) by variation with respect to $u_q^{(\underline{\rho\rho})}$. Sciama interpreted that $F_{pq}=\partial_p u_q^{(\underline{\rho\rho})}-\partial_q u_p^{(\underline{\rho\rho})}=-\partial_p \Gamma^r_{\underset{\vee}{rq}}+\partial_q \Gamma^r_{\underset{\vee}{rp}}$ (see (5·10) below) represents the electromagnetic field. Then the vanishing of $\Gamma^r_{\underset{\vee}{rq}}$ means that there are no electromagnetic fields in the Einstein theory. Accordingly, this procedure may be not appropriate.

Therefore, in order to obtain non-vanishing electromagnetic effect, we shall now eliminate the term Γ by adding

$$L_{add}=-\int\sqrt{-g}\,\Gamma^r_{\underset{\vee}{sr}}u_p^{(\underline{\mu\nu})}(h^s_{(\mu)}h^{p*}_{(\nu)}+h^p_{(\mu)}h^{s*}_{(\nu)})d\tau \tag{5·9}$$

to Lagrangian (5·6).

From the relation (5·1) we have

$$\Gamma^r_{\underset{\vee}{rp}}=-u_p^{(\underline{\rho\rho})}-i\ \mathrm{Im}\ \partial_p\log\varepsilon \tag{5·10}$$

where

$$\partial_p\log\varepsilon=h_r^{(\rho)}\,\partial_p h^r_{(\rho)}.$$

If we add $\frac{1}{6}\int\sqrt{-g}\ u_p^{(\underline{\rho\rho})}u_q^{(\underline{\sigma\sigma})}g^{pq}d\tau$ to the Lagrangian density, the variation with respect to $u_r^{(\underline{\rho\rho})}$ yields

$$-(1/3)\partial_s(\boldsymbol{g}^{rs}_{\vee})+\sqrt{-g}\,u_q^{(\mu\nu)}(h^r_{(\mu)}h^{q*}_{(\nu)}+h^q_{(\mu)}h^{r*}_{(\nu)})=0 \tag{5·11}$$

which implies

$$\partial_s(\sqrt{-g}\,A^s)=0 \tag{5·12}$$

with

$$A^s=(1/2)\mathrm{Im}(u_q^{(\mu\nu)}h^s_{(\mu)}h^{q*}_{(\nu)}).$$

(5·12) is the Lorentz condition for the vector potential A^s in the curved space. This suggests that we can interpret A^s tentatively as the electromagnetic four-potential. As mentioned in § 2, Green has interpreted that

$$\widetilde{A}^s=(1/2)\mathrm{Im}\,(g^{pq}\,C^s_{pq})$$

has the character of an electromagnetic four-potential. Using (3·16) and (5·3), we can easily show that A^s is nothing but $\widetilde{A}^s$ artificially introduced by Green. It is interesting that our electromagnetic potential based on the complex legs is the same as Green's.

There remains the problem of fomulating the interaction of Dirac particle and complex legs connection in the hermitian and complex metric space. As seen from Eq. (2·11), in the case of complex legs the interaction term $\bar{\psi}(\gamma^p\Gamma_p+\Gamma_p\hat{\gamma}^p)\psi$ is not hermitian and η has a complicated form. Then the discussion in the last section does not hold without any radical modification, because one must add a non-hermitian additive interaction Lagrangian to get rid of the discrepancy con-

cerning the variation equation and (5·1). Therefore for the Dirac particle we shall use the physical real legs $H^{i}_{(\mu)}$ defined by

$$h^{i(\mu)}\, h^{j*}_{(\mu)} + h^{i*}_{(\mu)}\, h^{j(\mu)} = 2\mathrm{Re}\; g^{ij} = 2H^{i}_{(\mu)}\, H^{j(\mu)}. \tag{5·13}$$

The electromagnetic interaction term is

$$L^{int}_{E.M.} = (1/2)\bar{\psi} H^{(\mu)}_{s}\, H^{s(\nu)}\, \gamma_{(\mu\nu)}\psi \cdot \mathrm{Im}(u_{p}^{(\rho\sigma)}\, h^{s}_{(\rho)}\, h^{p*}_{(\sigma)}). \tag{5·14}$$

The discussion concerning this problem will be made on another occasion.

§ 6. The relation between the spin transformation and Pauli transformation

To discuss Pauli-Gürsey type transformation, it may be appropriate to start with the Gürsey's representation of spinor equation. The use of Gürsey's representation is not essential but instructive. Gürsey introduced the 2×2 wave matrix $\Psi(\psi)$ for ψ

$$\Psi = \Psi(\psi) = (\psi_{+},\ (\psi_{c})_{+}) \tag{6·1}$$

where $\psi_{+} = (1+\rho_1)/2\cdot\psi$ and ψ_c is the charge conjugate of ψ in the usual theory. From the above definition, it is easily seen that

$$\begin{aligned}
&\Psi(i\psi) = i\Psi(\psi)\tau_3, \quad \Psi(\sigma_{(\gamma)}\psi) = \sigma_{(\gamma)}\Psi(\psi)\tau_3, \\
&\Psi(\rho_1\psi) = \Psi(\psi)\tau_3, \ \Psi(\rho_2\psi) = i\tilde{\Psi}^{+}(\psi), \quad \Psi(\rho_3\psi) = \tilde{\Psi}^{+}(\psi), \\
&\Psi(\bar{\psi}) = \tilde{\Psi}(\psi), \qquad \Psi(\psi_c) = \Psi(\psi)\tau_1
\end{aligned} \tag{6·2}$$

where τ_γ is a new set of Pauli matrices and $\tilde{\Psi}$ means the quaternion conjugate of $\Psi = \sigma^{(\alpha)}\psi_\alpha$, namely $\tilde{\Psi} = -\sigma^{(1)}\psi_1 - \sigma^{(2)}\psi_2 - \sigma^{(3)}\psi_3 + \sigma^{(4)}\psi_4$ in which $\sigma^{(4)} = E_2$.* In what follows, $\sigma^{(\gamma)}$ operates on Ψ from the left while τ_γ from the right.

We shall first express $\Psi(\Gamma_p\psi)$ in which Γ_p is the spin affine connection defined by (3·17), i.e. $\Gamma_p = (1/2)u_p^{(\mu\nu)}\gamma_{(\mu\nu)}$. Using the relation (6·2), we get

$$\begin{aligned}
\Psi(\Gamma_p\psi) = (1/2)\, &u_p^{(\alpha\beta)}\, \tilde{\sigma}_{(\alpha)}\sigma_{(\beta)}\Psi + u_p^{(56)}\,\Psi\tau_3 \\
&+ iu_p^{(\alpha 6)}\tilde{\sigma}_{(\alpha)}\tilde{\Psi}^{+}\tau_3 + iu_p^{(\alpha 5)}\tilde{\sigma}_{(\alpha)}\tilde{\Psi}^{+}.
\end{aligned} \tag{6·3}$$

On the right-hand side of (6·3) appear both Ψ and $\tilde{\Psi}^{+}$. We shall then express the third and fourth terms in terms of Ψ. For this purpose it is convenient to introduce the fundamental basic vector defined by

$$\begin{aligned}
\rho e_\alpha^{(1)} &= (1/2)sp(\Psi^{+}\sigma_{(\alpha)}\Psi\tau^1) = (i/2)(\bar{\psi}_c\gamma_{(\alpha)}\psi + \bar{\psi}\gamma_{(\alpha)}\psi_c) \\
\rho e_\alpha^{(2)} &= (1/2)sp(\Psi^{+}\sigma_{(\alpha)}\Psi\tau^2) = (1/2)(\bar{\psi}_c\gamma_{(\alpha)}\psi - \bar{\psi}\gamma_{(\alpha)}\psi_c) \\
\rho e_\alpha^{(3)} &= (1/2)sp(\Psi^{+}\sigma_{(\alpha)}\Psi\tau^3) = i\bar{\psi}\gamma_{(5\alpha)}\psi
\end{aligned} \tag{6·4}$$

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* In this section, we use the small Greek indices also to represent quantities with respect to the flat space-time coordinate system where $h^{p}_{(\alpha)} = \delta^{p}_{\alpha}$. See, for instance, Eq. (6·7) below.

$$\rho e_\alpha^{(4)} = (1/2)\, sp(\Psi^+ \sigma_{(\alpha)} \Psi \tau^4) = i\bar{\psi}\gamma_{(\alpha)}\psi$$

where

$$\rho^2 = \Omega^2 + \hat{\Omega}^2$$

with $\Omega = \frac{1}{2}\, sp\,\mathrm{Re}(\widetilde{\Psi}\Psi) = \bar{\psi}\psi$, $\hat{\Omega} = \frac{1}{2}\, sp\,\mathrm{Im}(\widetilde{\Psi}\Psi) = i\bar{\psi}\gamma_{(5)}\psi$. $e^\alpha_{(\beta)}$ satisfies the relations

$$e^\alpha_{(\beta)} e_{\alpha(\beta')} = \eta_{(\beta\beta')}, \quad e_{\alpha(\beta)}\, e_{\alpha'}{}^{(\beta)} = \eta_{\alpha\alpha'}. \tag{6·5}$$

Using the relation for any 2×2 matrix M

$$M = -(1/2)\, sp(M\tilde{\tau}_\beta)\tau^\beta = -(1/2)\, sp(M\tau_\beta)\tilde{\tau}^\beta,$$

we find

$$\tilde{\sigma}^{(\alpha)}\widetilde{\Psi}^+ \tau_3 = -(1/2)\, sp(\widetilde{\Psi}\tilde{\sigma}^{(\alpha)}\widetilde{\Psi}^+\tilde{\tau}^\beta)\tau_\beta\tau_{3'}\,(\widetilde{\Psi}\Psi) = -(1/2)\, sp(\Psi^+\sigma^{(\alpha)}\Psi\tau_\beta)\Psi\tau^\beta\tau_{3'}/(\widetilde{\Psi}\Psi)$$

$$= -\frac{\rho}{\Omega + i\hat{\Omega}}\Psi e^\alpha_{(\beta)}\tau^\beta\tau_3. \tag{6·6}$$

In the Appendix, we interpret the meaning of the above equation by expressing it in terms of the usual spinor. Eq. (6·6) shows that we are inclined to consider the spin affine connection $u_p^{(\alpha\beta)}\gamma_{(\alpha)}$ as a spin connection in the τ space. We shall now show that such an interpretation is possible.

The Lagrangian density for Dirac particle without mass in Minkowski space

$$L_E = (1/2)(\bar{\psi}\gamma^{(\alpha)}\partial_\alpha\psi - \bar{\psi}_c\gamma^{(\alpha)}\partial_\alpha\psi_c) \tag{6·7}$$

can be expressed by Gürsey's wave matrix

$$L_E = (1/2)\, sp(\Psi^+\gamma^{(\alpha)}\partial_\alpha\Psi\tau_3). \tag{6·7'}$$

The above density is invariant under the following rotation in τ-space

$$\Psi' = \Psi \exp\{(i\tau^\alpha\theta_\alpha)\tau^3\} \tag{6·8}$$

or

$$\Psi' = \Psi \exp(\tau^1\bar{\theta}_1 + \tau^2\bar{\theta}_2 + \tau^3\bar{\theta}_3 + i\tau^4\bar{\theta}_4) \tag{6·8'}$$

where θ_α, $\bar{\theta}_\alpha$ are real parameters. The transformation (6·8') can be decomposed into two transformations

$$\Psi' = \Psi \exp(i\,\bar{\theta}_4) \tag{6·9}$$

and

$$\Psi' = \Psi \exp(i\tau^3\bar{\theta}_3 + \tau^2\bar{\theta}_2 + \tau^1\bar{\theta}_1). \tag{6·10}$$

These equations are expressed in ψ representation as

$$\psi' = \exp(-i\gamma_{(5)}\bar{\theta}_4)\psi \tag{6·9'}$$

and

$$\psi' = a\psi + b\psi_c, \quad \bar{\psi}' = a^*\bar{\psi} + b^*\bar{\psi}_c, \quad |a|^2 - |b|^2 = 1. \tag{6·10'}$$

The transformation (6·10) is isomorph to the rotation in the three-dimensional Lorentz space. If the spin affine connection is $\Gamma_p=\frac{1}{2}(\frac{1}{2}C_{rp}^{t}\gamma^{r}\gamma_t+E_{rp}^{5}\gamma^{r}\gamma_5)$ instead of (2·5), γ^p is expressed as

$$\gamma^p=\sum_{\mu=1}^{5}h_{(\mu)}^{p}\gamma^{(\mu)}$$

linearly by five legs contrary to (3·13) where it is expressed quadratically by six legs. It seems interesting to us that the spin affine connection containing (6·6) brings about the non-linear character of representation and also introduces the rotation in the three-dimensional Lorentz space.

Let us replace the transformation (6·8) with the wider one derived from replacing the constant parameter $\bar{\theta}_\alpha$ by a set of arbitrary function of coordinates $\bar{\theta}_\alpha(x)$. By postulating the invariance of Lagrangian under this wider transformations, a new field $E_p^\beta(x)$ is introduced through the expression[6]

$$\nabla_p\Psi=\partial_p\Psi-i\Psi\tau_\beta\tau_3E_p^\beta(x). \tag{6·11}$$

Thus we have seen that the legs connection $u_p^{(\alpha 6)}$ is related to the above $E_p^\beta(x)$ by

$$u_p^{(\alpha 6)}=-\frac{\Omega+i\hat{\Omega}}{\rho}e_{(\beta)}^{\alpha}E_p^\beta(x) \tag{6·12}$$

in terms of ρ, Ω, $\hat{\Omega}$ and fundamental basic vectors $e_\alpha^{(\beta)}$.

Following the same procedure as above, we relate the legs connection $u_p^{(\alpha 5)}$ appearing in the fourth term of (6·3) to $F_p^\beta(x)$ by

$$u_p^{(\alpha 5)}=-\frac{\Omega+i\hat{\Omega}}{\rho}e_{(\beta)}^{\alpha}F_p^\beta(x). \tag{6·13}$$

In this case, $F_p^\beta(x)$ is introduced through the expression

$$\nabla_p\Psi=\partial_p\Psi-i\Psi\tau_\beta F_p^\beta(x) \tag{6·14}$$

by requiring that the Lagrangian density

$$L_F=(1/2)(\bar{\psi}\gamma^{(5\alpha)}\partial_\alpha\psi+\bar{\psi}_c\gamma^{(5\alpha)}\partial_\alpha\psi_c)=(1/2)sp(\Psi^{+}\sigma^{(\alpha)}\partial_\alpha\Psi) \tag{6·15}$$

is invariant under the transformation

$$\Psi'=\Psi\exp(i\tau^\alpha\theta_\alpha(x)). \tag{6·16}$$

When the parameters θ_α do not depend on the coordinates, the transformation (6·16) is nothing but the Pauli transformation

$$\psi'=\exp(-i\gamma_{(5)}\theta_4)\psi,$$

$$\psi'=c\psi+d\gamma_{(5)}\psi_c,\ \bar{\psi}'=c^*\bar{\psi}-d^*\bar{\psi}\gamma_{(5)},\ |c|^2+|d|^2=1. \tag{6·16′}$$

In this case, the second transformation of (6·16′) is isomorph to a rotation in the three-dimensional Euclidean space. Thus we have shown that each of spin affine connections $u_p^{(\alpha 6)}\gamma_{(\alpha)}$ and $u_p^{(\alpha 5)}\gamma_{(\alpha 5)}$ has an intimate relation with Pauli-like

transformation.

Unfortunately, there is no Lagrangian density left invariant under the combination of two transformations (6·8′) and (6·16), though they preserve the field equation invariant. Therefore, in order to obtain the Lagrangian density which is invariant under the transformation corresponding to the spin affine connection $u_p^{(\alpha 6)}\gamma_{(\alpha)}+u_p^{(\alpha 5)}\gamma_{(\alpha 5)}$, we are impelled to regard ψ as an eight-component spinor and to generalize the Pauli transformation.

We introduce 8×8 matrices

$$\Gamma^{(\alpha)}=i\gamma^{(5)}\gamma^{(\alpha)}\omega_1,\ \Gamma^{(5)}=\omega_3,\ \Gamma^{(6)}=i\omega_2 \tag{6·17}$$

where ω_γ is a set of Pauli matrices. The adjoint spinor $\overline{\varphi}$ and the charge conjugate φ_c of φ are defined by

$$\overline{\varphi}=i\varphi^+\Gamma^{(4)}\Gamma^{(6)}=-\varphi^+\gamma^{(5)}\gamma^{(4)}\omega_3, \tag{6·18}$$

$$\varphi_c=C\times\omega_2\overline{\varphi}^T.$$

We shall take the Lagrangian density

$$L=(1/2)[\overline{\varphi}\Gamma^{(6)}\Gamma^{(\alpha)}\partial_\alpha\varphi+\overline{\varphi}_c\Gamma^{(6)}\Gamma^{(\alpha)}\partial_\alpha\varphi_c] \tag{6·19}$$

as a generalization of (6·7) and (6·15). The transformations corresponding to $u_{p(\alpha 5)}\Gamma^{(\alpha)}\Gamma^{(5)}+u_{p(\alpha 6)}\Gamma^{(\alpha)}\Gamma^{(6)}$ are

$$\begin{aligned}\varphi'&=(A\omega_3+B\omega_2)\varphi+(C\omega_2+D\omega_3)\gamma_{(5)}\varphi_c\\ \overline{\varphi}'&=\overline{\varphi}(A^*\omega_3-B^*\omega_2)+\overline{\varphi}_c\gamma_{(5)}(C^*\omega_2-D^*\omega_3)\end{aligned} \tag{6·20}$$

with

$$|A|^2+|B|^2+|C|^2+|D|^2=1$$

and

$$\varphi'=\exp(i\gamma_{(5)}\theta)\varphi. \tag{6·20′}$$

It is evident that these transformations with eight parameters preserve the Lagrangian density (6·19) invariant. Our eight-dimensional representation is necessary only when it is regarded that the spin transformation corresponds to a product of the usual Lorentz transformation and the generalized Pauli transformation under which the Lagrangian density is invariant. Therefore, the introduction of an eight spinor does not contradict with the discussion in § 3 where we represent γ^μ by 4×4 matrices.

§ 7. Concluding remarks

Though we have restricted the problems to *c*-number theory, the discussion in § 6 will be applicable also to *q*-number theory. By using Gürsey representation, the transition from *c*-number theory to *q*-number one is performed by interchanging τ_3 and τ_4. For instance, while $i\overline{\psi}\gamma^{(\alpha)}\psi$ is represented as $\frac{1}{2}sp(\Psi^+\sigma^{(\alpha)}\Psi)$ in *c*-number theory, it is represented as $\frac{1}{2}sp(\Psi^+\sigma^{(\alpha)}\Psi\tau_3)$ in *q*-number theory. But, if we treat

q-number theory, there arises a very difficult problem concerning the quantization of general relativity. This is one of the reasons why we do not enter into q-number theory. However, our treatment, in which the legs connection $u_p^{(\mu\nu)}$ is regarded as an independent field quantity, will be useful for the discussion of quantization. Arnowitt and Deser[14] have shown that the Palatini formulation of the Einstein theory may circumvent the difficulty in quantizing the gravitational field. This technique, by making the metric tensor and the affinity independent field variables, will be useful only in the case where the Dirac particle is absent. From the Lagrangian density (4·1) we see that our field variables are suitable to quantize the interaction between the gravitational field and Dirac particle.

Finally, we must mention the question whether the requirement of the general covariance is necessary in the region of atomic scale or not. Even when taking into account the gravitational effect, many people have usually considered that it is sufficient to use Newtonian gravitational theory, because it represents a very good approximation for all gravitational phenomena inside the solar system. The most elementary effect which represents a crucial test of the general relativity is the gravitational shift of spectral lines, which is a direct consequence of the principle of equivalence. By using the Mössbauer effect of recoilless nuclear resonance absorption of gamma ray in Fe^{57}, Cranshaw et al.[15] have recently found the line shift for the height difference (12.5 meters) available to us inside the laboratory. If we want to obtain the value of Einstein's gravitational red shift by solving the Dirac equation in the linearized gravitational theory such as Birkoff's, some rather artificial technique is necessary as shown by Moshinsky.[16] Therefore the recent experiment seems to imply that it may be most natural to require the general covariance of the theory also in the region of atomic scale.

Appendix

We shall derive the relation (6·6) in the usual ψ representation to clarify the meaning of (6·6).

First we show that $\gamma_{(\alpha)}\psi$ is uniquely expressed by a linear combination of ψ, $\gamma_{(5)}\psi$, ψ_c, $\gamma_{(5)}\psi_c$ in the form

$$\gamma_{(\alpha)}\psi = w\psi + x\gamma_{(5)}\psi + y\psi_c + z\gamma_{(5)}\psi_c. \qquad (A\cdot 1)$$

In order that the above equation can be solved with respect to w, x, y, z the determinant of the matrix

$$X = \begin{pmatrix} \psi_1 & (\gamma_{(5)}\psi)_1 & (\psi_c)_1 & (\gamma_{(5)}\psi_c)_1 \\ \psi_2 & (\gamma_{(5)}\psi)_2 & (\psi_c)_2 & (\gamma_{(5)}\psi_c)_2 \\ \psi_3 & (\gamma_{(5)}\psi)_3 & (\psi_c)_3 & (\gamma_{(5)}\psi_c)_3 \\ \psi_4 & (\gamma_{(5)}\psi)_4 & (\psi_c)_4 & (\gamma_{(5)}\psi_c)_4 \end{pmatrix}$$

must not be equal to zero. The determinant of the matrix X can be calculated

as follows. Using the charge conjugation operator C, we have

$$X^TC^{-1}X=\begin{pmatrix} \psi^T C^{-1}\psi & \psi^T C^{-1}\gamma_{(5)}\psi & \psi^T C^{-1}\psi_c & \psi^T C^{-1}\gamma_{(5)}\psi_c \\ \psi^T\gamma_{(5)}^T C^{-1}\psi & \psi^T\gamma_{(5)}{}^T C^{-1}\gamma_{(5)}\psi & \psi^T\gamma_{(5)}^T C^{-1}\psi_c & \psi^T\gamma_{(5)}{}^T C^{-1}\gamma_{(5)}\psi_c \\ \psi_c^T C^{-1}\psi & \psi_c^T C^{-1}\gamma_{(5)}\psi & \psi_c^T C^{-1}\psi_c & \psi_c^T C^{-1}\gamma_{(5)}\psi_c \\ \psi_c^T\gamma_{(5)}{}^T C^{-1}\psi & \psi_c^T\gamma_{(5)}{}^T C^{-1}\gamma_{(5)}\psi & \psi_c^T\gamma_{(5)}^T C^{-1}\psi_c & \psi_c^T\gamma_{(5)}{}^T C^{-1}\gamma_{(5)}\psi_c. \end{pmatrix}.$$

Adopting the antisymmetric nature of C^{-1} and $C^{-1}\gamma_{(5)}$, and taking the determinant of the matrix $X^TC^{-1}X$, we get

$$|X^TC^{-1}X|=|C^{-1}||X|^2=\begin{vmatrix} 0 & 0 & \bar\psi\psi & \bar\psi\gamma_{(5)}\psi \\ 0 & 0 & \bar\psi\gamma_{(5)}\psi & \bar\psi\psi \\ -\bar\psi\psi & -\bar\psi\gamma_{(5)}\psi & 0 & 0 \\ -\bar\psi\gamma_{(5)}\psi & -\bar\psi\psi & 0 & 0 \end{vmatrix}$$

$$=-\rho^2=-(\Omega^2+\hat\Omega^2)=-[(\bar\psi\psi)^2-(\bar\psi\gamma_{(5)}\psi)^2]=-\bar\psi(1-\gamma_{(5)})\psi\cdot\bar\psi(1+\gamma_{(5)})\psi.$$

Since $|C^{-1}|\neq 0$, X does not vanish provided that

$$\rho^2=\Omega^2+\hat\Omega^2\neq 0. \tag{A·2}$$

Therefore, if $\rho^2\neq 0$, we can solve (A·1) with respect to w, x, y, z. The result is as follows:

$$\gamma_{(\alpha)}\psi=(1/\rho)[\{(-i\Omega e_\alpha{}^{(4)}+\hat\Omega e_\alpha{}^{(3)})+(\hat\Omega e_\alpha{}^{(4)}-i\Omega e_\alpha{}^{(3)})\gamma_{(5)}\}\psi$$
$$-(\Omega+i\hat\Omega\gamma_{(5)})(e_\alpha{}^{(2)}-ie_\alpha{}^{(1)})\psi_c] \tag{A·3}$$

where $e_\alpha{}^{(\beta)}$ is given by (6·4).

Similarly,

$$\gamma_{(\alpha)}\gamma_{(5)}\psi_c=(1/\rho)[-(i\Omega e_\alpha{}^{(4)}+\hat\Omega e_\alpha{}^{(3)})\gamma_{(5)}\psi_c+(\hat\Omega e_\alpha{}^{(4)}+i\Omega e_\alpha{}^{(3)})\psi_c$$
$$+(e_\alpha{}^{(2)}+ie_\alpha{}^{(1)})(i\hat\Omega+\Omega\gamma_{(5)})\psi] \tag{A·4}$$

$$\gamma_{(\alpha)}\psi_c=(1/\rho)[(i\Omega e_\alpha{}^{(4)}+\hat\Omega e_\alpha{}^{(3)})\psi_c-(\hat\Omega e_\alpha{}^{(4)}+i\Omega e_\alpha{}^{(3)})\gamma_{(5)}\psi_c$$
$$-(e_\alpha{}^{(2)}+ie_\alpha{}^{(1)})(i\hat\Omega+\Omega\gamma_{(5)})\gamma_{(5)}\psi] \tag{A·5}$$

$$\gamma_{(\alpha)}\gamma_{(5)}\psi=(1/\rho)[(i\Omega e_\alpha{}^{(4)}-\hat\Omega e_\alpha{}^{(3)})\gamma_{(5)}\psi-(\hat\Omega e_\alpha{}^{(4)}-i\Omega e_\alpha{}^{(3)})\psi$$
$$+(e_\alpha^{(2)}-ie_\alpha{}^{(1)})(i\hat\Omega+\Omega\gamma_{(5)})\psi_c]. \tag{A·6}$$

By combining Eqs. (A·3)–(A·6), new sets of Pauli matrices are introduced. As an example, putting

$$\Psi_I=\begin{pmatrix}\psi\\ \gamma_{(5)}\psi_c\end{pmatrix}, \tag{A·7}$$

we obtain from (A·3) and (A·4)

$$\gamma_{(\alpha)}\Psi_I=(1/\rho)[(-i\Omega+\hat\Omega\gamma_{(5)})e_\alpha{}^{(4)}+(\hat\Omega-i\Omega\gamma_{(5)})e_\alpha{}^{(\gamma)}\xi_\gamma]\Psi_I \tag{A·8}$$

where ξ_γ is a set of Pauli matrices. Therefore, $\gamma_{(\alpha)}$ is isomorph to a rotation with the generators ξ_γ and $\gamma_{(5)}\xi_\gamma$.

Further, if we put

$$\Psi_{II}=\begin{pmatrix}\gamma_{(5)}\psi_c\\ \psi\\ \psi_c\\ \gamma_{(5)}\psi\end{pmatrix} \quad \text{and} \quad \Psi_{III}=\begin{pmatrix}\gamma_{(5)}\psi_c\\ -\psi\\ \psi_c\\ \gamma_{(5)}\psi\end{pmatrix}, \tag{A·9}$$

$$\gamma_{(5)}\gamma_{(\alpha)}\Psi_{II}=(1/\rho)i[-i(\Omega\omega_2+\hat{\Omega}\omega_3)e_\alpha^{(4)}+(\hat{\Omega}\omega_2-\Omega\omega_3)e_\alpha^{(7)}\xi_\gamma]\Psi_{II} \tag{A·10}$$

and

$$\begin{aligned}\gamma_{(5)}\gamma_{(\alpha)}\Psi_{III}=(1/\rho)i[&-\omega_3(i\hat{\Omega}e_\alpha^{(4)}+\Omega e_\alpha^{(3)}\xi_3)+\Omega(e_\alpha^{(1)}\xi_1+e_\alpha^{(2)}\xi_2)\\ &-\omega_2(i\Omega e_\alpha^{(4)}\xi_3-\hat{\Omega}e_\alpha^{(3)})+\hat{\Omega}\omega_1(e_\alpha^{(1)}\xi_2-e_\alpha^{(2)}\xi_1)]\Psi_{III}\end{aligned} \tag{A·11}$$

where ω_γ is a set of Pauli matrices.

If we express the transformation (6·20) in terms of 2-4 wave matrix, it has a similar form to (A·11).

References

1) H. P. Dürr, W. Heisenberg, H. Mitter, S. Schlieder and K. Yamazaki, Z. Naturforsch. **14** (1959), 441.
2) W. Pauli, Nuovo Cimento **6** (1957), 204.
F. Gürsey, Nuovo Cimento **7** (1958), 204.
3) F. Gürsey, Nuovo Cimento **6** (1957), 988.
4) H. S. Green, Nuclear Phys. **7** (1958), 373.
5) R. Finkelstein, Phys. Rev. **110** (1958), 1200.
6) R. Utiyama, Phys. Rev. **101** (1956), 1957.
7) H. Weyl, Z. Physik **56** (1929), 330.
H.Weyl, Phys. Rev. **77** (1950), 699.
8) H. Kita, Suppl, Prog. Theor. Phys. No. **9** (1959), 5.
9) A. Einstein, Rev. Mod. Phys. **20** (1948), 35.
10) D. W. Sciama, Nuovo Cimento **8** (1958), 417.
11) For instance, O. Klein, Arkiv Mat. Astro. Fys., Stockholm **25A** (1957).
12) H. Jehle, Z. Physik **100** (1936), 702.
P. G. Bergmann, Phys. Rev. **107** (1957), 624.
13) For instance, T. Takabayasi, Suppl., Prog. Theor. Phys. No. **4** (1957).
14) R. Arnowitt and S. Deser, Phys. Rev. **113** (1959), 745; **116** (1959), 1322.
15) T. E. Cranshaw, J. P. Schiffer and A. B. Whitehead, Phys. Rev. Letters **4** (1960), 163.
16) M. Moshinsky, Phys. Rev. **80** (1950), 514.

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Information about the Proton-Proton Scattering Matrix from Coulomb Interference*

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It is shown that a measurement of the p-p scattering parameters at very small angles and angles close to π where coulomb interference is important will directly determine the coefficients of the scattering matrix. Since the coefficients are known functions of the phase shifts, such measurements may rule out some of the phase shift solutions and may provide a unique set of phase shifts.

§ 1. Introduction

The phase shift analysis of the 310 Mev proton-proton scattering experiment of Chamberlain et al.[1] carried out by Stapp, Ypsilantis and Metropolis[2] led to 5 phase shift solutions after excluding 3 solutions which did not fit the production of pions in the process $p+p\rightarrow d+\pi^+$. The experiment[1] consisted in the measurement of five scattering parameters I_0, P, D, R and A. As Puzikov, Rydin and Smorodinsky[3] have shown, because of the identity of particles it is necessary to make measurements only in the angular range 0 to $\pi/2$ and if the unitarity of the S-matrix is used, only five scattering parameters need be measured at a given energy. It has been further pointed out by Marshak[4] that the S-matrix will be uniquely determined only if at least one of the experiments involves scatterings which are not in the same plane.

With the existing experimental data at 310 Mev, it is almost possible to obtain a unique phase shift solution. The modified analysis of the p-p scattering data carried out by Moravcsik et al.[5),6)] in which the partial waves G and higher are calculated with the one pion exchange potential indicates that since the Stapp solutions 1 and 2 converge respectively to solutions 3 and 4, one is left essentially with only two solutions (it being possible to discard solution 6 through a measurement of C_{KP}). It is now of interest to discriminate between solutions 1 and 2 especially since Gammel and Thaler[7] have shown that solution 1 is derivable from a local potential while solution 2 has non-local potential characteristics. In this

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note we have indicated how it may be possible to discriminate among solutions 1 and 2 by a measurement of the scattering parameters in the small angle region where the coulomb interference effects dominate.

§ 2. Nucleon-nucleon scattering

The scattering matrix for nucleon-nucleon scattering which is charge symmetric and invariant with respect to rotation, parity and time reversal may be written as follows,*

$$M(\theta)=A+C(\sigma_{1n}+\sigma_{2n})+B\sigma_{1n}\sigma_{2n}+\frac{1}{2}G(\sigma_{1K}\sigma_{2K}+\sigma_{1P}\sigma_{2P})+\frac{1}{2}H(\sigma_{1K}\sigma_{2K}-\sigma_{1P}\sigma_{2P}) \tag{1}$$

where $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$ are the spins of nucleons 1 and 2;

$$\sigma_{1n}=\boldsymbol{\sigma}_1\cdot\boldsymbol{n} \text{ etc., } \boldsymbol{n}=\frac{\boldsymbol{k}_{\text{in}}\times\boldsymbol{k}_{\text{out}}}{|\boldsymbol{k}_{\text{in}}\times\boldsymbol{k}_{\text{out}}|},\ \boldsymbol{P}=\frac{\boldsymbol{k}_{\text{in}}+\boldsymbol{k}_{\text{out}}}{|\boldsymbol{k}_{\text{in}}+\boldsymbol{k}_{\text{out}}|},\ \boldsymbol{K}=\frac{\boldsymbol{k}_{\text{out}}-\boldsymbol{k}_{\text{in}}}{|\boldsymbol{k}_{\text{out}}-\boldsymbol{k}_{\text{in}}|} \tag{2}$$

and M, A, C, B, G, H are functions of energy and angle. The scattering parameters are then:[8]

$$I_0=|A|^2+|B|^2+2|C|^2+\frac{1}{2}|G|^2+\frac{1}{2}|H|^2 \tag{3}$$

$$I_0P=2\mathrm{Re}\,C^*(A+B) \tag{4}$$

$$I_0(1-D)=|G|^2+|H|^2 \tag{5}$$

$$I_0R=\{|A|^2-|B|^2+\mathrm{Re}G^*H\}\cos\frac{\theta}{2}+2\,\mathrm{Im}\,C^*(A-B)\sin\frac{\theta}{2} \tag{6}$$

$$I_0A=-\{|A|^2-|B|^2+\mathrm{Re}G^*H\}\sin\frac{\theta}{2}+2\,\mathrm{Im}\,C^*(A-B)\cos\frac{\theta}{2} \tag{7}$$

$$I_0R'=\{|A|^2-|B|^2-\mathrm{Re}G^*H\}\sin\frac{\theta}{2}-2\,\mathrm{Im}\,C^*(A-B)\cos\frac{\theta}{2} \tag{8}$$

$$I_0A'=\{|A|^2-|B|^2-\mathrm{Re}G^*H\}\cos\frac{\theta}{2}+2\,\mathrm{Im}\,C^*(A-B)\sin\frac{\theta}{2} \tag{9}$$

$$\frac{1}{2}I_0C_{nn}=\mathrm{Re}\,A^*B+|C|^2-\frac{1}{4}|G|^2+\frac{1}{4}|H|^2 \tag{10}$$

---

* If the M-matrix is written in the form

$$M(\theta)=bS+c(\sigma_{1n}+\sigma_{2n})+\tfrac{1}{2}g(\sigma_{1K}\sigma_{2K}+\sigma_{1P}\sigma_{2P})T+\tfrac{1}{2}h(\sigma_{1K}\sigma_{2K}-\sigma_{1P}\sigma_{2P})T+n\sigma_{1n}\sigma_{2n}T$$

where $S=\frac{1}{4}(1-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)$, $T=\frac{1}{4}(3+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)$,

then, using the relations

$$\sigma_{1n}\sigma_{2n}T=\sigma_{1n}\sigma_{2n}+S \text{ and } \sigma_{1P}\sigma_{2P}+\sigma_{1K}\sigma_{2K}+\sigma_{1n}\sigma_{2n}=\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2,$$

it is easy to show that

$$n=A+B,\ g=2A+G,\ b=A-B-G,\ c=C,\ h=H.$$

$$I_0 C_{KP} = -2\,\mathrm{Im}\,C^* H \tag{11}$$

$$\frac{1}{2} I_0 (C_{PP} + C_{KK}) = \mathrm{Re}\,G^* (A - B) \tag{12}$$

$$\frac{1}{2} I_0 (C_{PP} - C_{KK}) = -\mathrm{Re}\,H^* (A + B). \tag{13}$$

§ 3. Coulomb interference

The coefficients A, B, etc., in the scattering matrix for proton-proton scattering are sums of coulomb and nuclear parts, to be denoted by the suffices c and n respectively. The expressions for the coulomb parts can be obtained by comparing the coefficients of the coulomb scattering matrix M_c with Eq. (1). The antisymmetrized coulomb matrix M_c is given in the non-relativistic approximation by

$$M_c = \{f_c(\theta) - f_c(\pi-\theta)\} \frac{(3+\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)}{4} + \{f_c(\theta) + f_c(\pi-\theta)\} \frac{(1-\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)}{4} \tag{14}$$

where

$$f_c(\theta) = \frac{-\eta}{k(1-\cos\theta)} \exp\left[-i\eta \ln \frac{1}{2}(1-\cos\theta)\right] \tag{15}$$

with $\eta = e^2/\hbar v$, v being the velocity in the laboratory system and k, θ are the momentum and scattering angle in the c.m. system. Thus from (1) and (14), we obtain

$$A_c = f_c(\theta) - \frac{1}{2} f_c(\pi-\theta), \quad B_c = -\frac{1}{2} f_c(\pi-\theta), \tag{16}$$

$$G_c = -f_c(\pi-\theta), \qquad C_c = 0 = H_c,$$

so that $A(\theta) = A_n(\theta) + A_c(\theta)$, and so on. (17)

The angle θ_0 below which coulomb effects become important can be estimated, say, from the equality $A_n(\theta) = A_c(\theta)$. Bethe[8] gives the value of $\mathrm{Re}\,A_n(0) = (1/4k) \times (3.05)$ for p-p scattering at 313 Mev in the laboratory. θ_0 may then be estimated from the equation $3.05/(4k) \approx \eta \Big/ \left(k \frac{\theta_0^2}{2}\right)$ where $\eta = 0.011$ at 313 Mev, thus giving $\theta_0 \sim 10°$.

For small angles $\theta = \epsilon$, we will denote $A_n(\epsilon)$ by $A_<$ and $A_n(\pi-\epsilon)$ by $A_>$, and so on. It should be noted that A_c is important both for ϵ and $\pi-\epsilon$ while B_c and G_c are important only for $\pi-\epsilon$.

$C_<$:

We can determine $C_<$ from the equations

$$I_0 P = 2\,\mathrm{Re}\,C^* (A+B) \tag{4}$$

$$I_0 R \sin\frac{\theta}{2} + I_0 A \cos\frac{\theta}{2} = 2\,\mathrm{Im}\,C^* (A-B). \tag{18}$$

Writing these equations in full,

$$I_0 P = 2\,\mathrm{Re}\,C_n^*(A_n+B_n) + 2\{C_{nR}(A_c+B_c)_R + C_{nI}(A_c+B_c)_I\}$$

$$I_0 R \sin\frac{\theta}{2} + I_0 A \cos\frac{\theta}{2} = 2\,\mathrm{Im}\,C_n^*(A_n-B_n) + 2\{-C_{nI}(A_c-B_c)_R + C_{nR}(A_c-B_c)_I\}$$

where R stands for real part and I for the imaginary part. Eliminating C_{nR} from the above two equations, we obtain the following equation which determines C_{nI}

$$I_0 P(A_c-B_c)_I - \left\{I_0 R \sin\frac{\theta}{2} + I_0 A \cos\frac{\theta}{2}\right\}(A_c+B_c)_R$$
$$= 2\,\mathrm{Re}\,C_n^*(A_n+B_n)\cdot\mathrm{Im}(A_c-B_c) - 2\,\mathrm{Im}\,C_n^*(A_n-B_n)\cdot\mathrm{Re}(A_c+B_c)$$
$$+ 2C_{nI}\{|A_c|^2 - |B_c|^2\}. \qquad (19)$$

For $\theta=\epsilon$ the last term on the R.H.S. dominates and so we can determine $C_{nI<}$ by ignoring the other terms. Similarly, we can obtain $C_{nR<}$. We cannot obtain $C_{nI>}$ from Eq. (19) since at $\theta=\pi-\epsilon$ the last two terms on the R.H.S. of Eq. (19) are equally important. In fact $C_>$ cannot be determined immediately, since, except for the equation for I_0P, C does not occur multiplied by A, B, or G. Note that in the combination $A-B$, $f_c(\pi-\theta)$ drops out so that A_c-B_c is important only for $\theta=\epsilon$.

$H_>$:

The equations

$$I_0 A + I_0 R' = -2\,\mathrm{Re}\,G^* H \sin\frac{\theta}{2}, \qquad (20)$$

$$\frac{1}{2} I_0 (C_{KK} - C_{PP}) = \mathrm{Re}\,H^*(A+B) \qquad (13)$$

will determine $H_>$. In order to determine $H_<$, we require another equation involving H^*A besides Eq. (13). Since no such combination occurs it is not possible to obtain $H_<$ from coulomb interference. However, since H is symmetric about $\pi/2$, $H_<=H_>$.

$C_>$:

Using $H_>$, we can obtain $C_>$ from Eq. (4) for I_0P, and Eq. (11) for

$$I_0 C_{KP} = -2\,\mathrm{Im}\,C^* H.$$

Also since C is symmetric about $\pi/2$, we should have $C_>=C_<$.

$A_<$:

$A_<$ is overdetermined. It can be obtained from any two equations for I_0, I_0R, I_0A, I_0R', I_0A' but not from the combinations (I_0R, I_0A') and (I_0A, I_0R') because they yield the same equation in the limit $\theta=\epsilon$. The best determination seems to be from (I_0, I_0R) or I_0 and

$$I_0(R+A)\cos\frac{\theta}{2}+I_0R'\left(\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)+\frac{1}{2}I_0(1+D)=2|A|^2+2|C|^2. \quad (21)$$

$B_<$:

$B_<$ could be determined from Eq. (10) for $(1/2)I_0C_{nn}$ and another equation in which A^*B occurs. Since there is no other such relation, $B_<$ cannot be determined from coulomb interference. However, if $A_<$, $C_<$, $G_<$ and $H_<$ are known, $B_<$ can be obtained from this equation.

$A_>$ and $B_>$:

From Eqs. (6) to (9) for I_0R, I_0A, I_0R' and I_0A', we can solve for $A_>$ and $B_>$, having already determined $H_>$.

$G_<$:

$G_<$ could be obtained from Eq. (12) for $(1/2)I_0(C_{PP}+C_{KK})$ and another relation containing G^*A. Since there is no such relation, it is not possible to obtain $G_<$ directly. However, since $H_<$ can be determined from the equality $H_<=H_>$, we can obtain $G_<$ from Eq. (6) for I_0D and Eq. (12) for $(1/2)I_0(C_{PP}+C_{KK})$.

$G_>$:

Eqs. (5) and (10) for $I_0(1-D)$ and $(1/2)I_0C_{nn}$ will give $G_>$, having determined $A_>$, $B_>$, $H_>$.

From the above analysis, it follows that a determination of the coefficients occurring in the scattering matrix for p-p scattering is possible at small c.m. angles $\theta=\epsilon$ and at large angles $\theta=\pi-\epsilon$ from a consideration of the interference between the coulomb and nuclear parts of the coefficients in the scattering parameters. Thus the nuclear parts of the coefficients A, B, C, G and H at ϵ and $\pi-\epsilon$ can be obtained from a measurement of the scattering parameters at ϵ and $\pi-\epsilon$. If a phase shift analysis is carried out, these coefficients can be evaluated from the phase shifts also and a comparison with their values obtained from experiment would enable us to reject some of the phase shift solutions.

In Figs. 1–5, we have plotted the real and imaginary parts of the coefficients A, B, C, G, and H for the modified Stapp solutions 1 and 2 as predicted by Moravcsik et al.[6] The solutions predict different values of the coefficients and if experiments can be performed in the small angle region where coulomb interference is important, it may establish a preference for one solution compared to the other. It may be noted

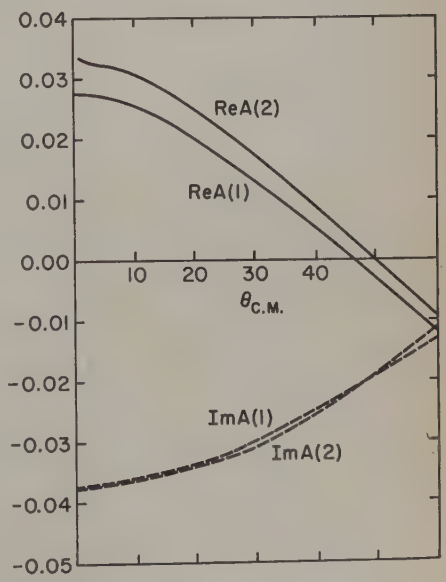

Fig. 1. Real and imaginary parts of the coefficient A for modified Stapp solutions 1 and 2.

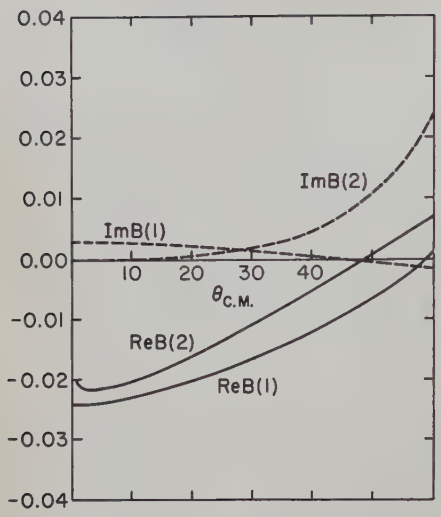

Fig. 2. Real and imaginary parts of the coefficient B for modified Stapp solutions 1 and 2.

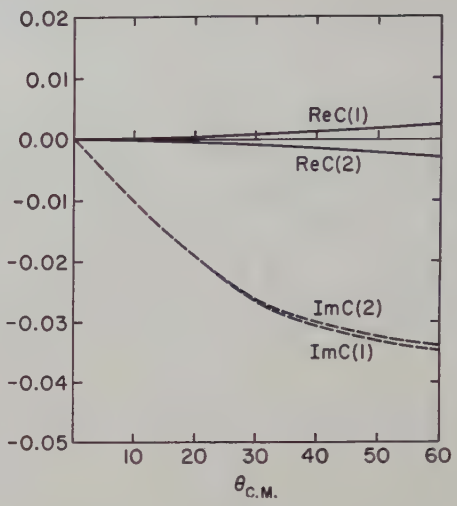

Fig. 3. Real and imaginary parts of the coefficient C for modified Stapp solutions 1 and 2.

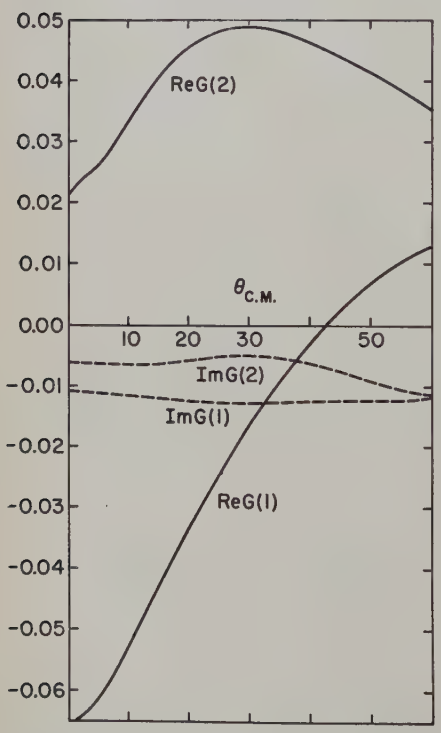

Fig. 4. Real and imaginary parts of the coefficient G for modified Stapp solutions 1 and 2.

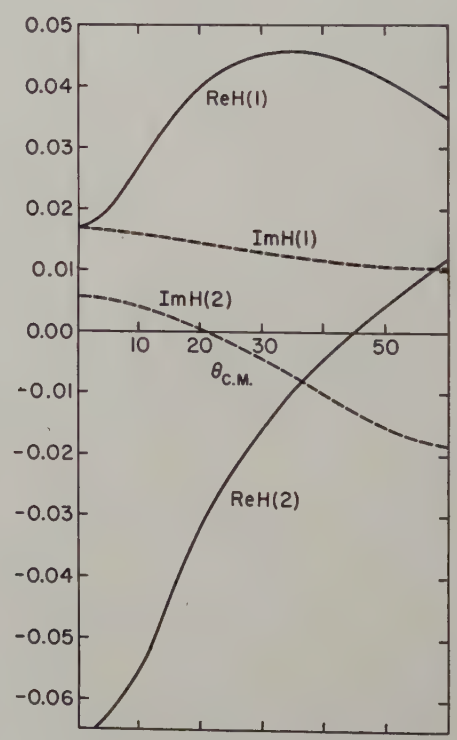

Fig. 5. Real and imaginary parts of the coefficient H for modified Stapp solutions 1 and 2.

from the curves that while the coefficients A, B and C do not differ appreciably for solutions 1 and 2, the coefficients G and H have widely different values for the two solutions. The coefficient $H_n(\pi-\epsilon)=H_n(\epsilon)$ can be obtained from a measurement of $I_0(A+R')$ [or $I_0(R-A')$], Eq. (20), and $I_0(C_{KK}-C_{PP})$, Eq. (13), at angles close to π and then $G(\epsilon)$ be obtained from $I_0(1-D)$, Eq. (5), and $I_0(C_{KK}+C_{PP})$, Eq. (12), at small angles. Thus a measurement of these scattering parameters may exclude one of the solutions.

With regard to performing the experiments, it is to be noted that measurements become extremely difficult at very small angles or close to π. It may be possible to measure the triple scattering parameters at such angles. However, the feasibility of measuring the spin correlation parameters at these angles is doubtful on account of the fact that one of the protons has very small energy. Therefore, if one is to avoid the use of the spin correlation parameters one should work with Eqs. (3) to (9) only. Then the only equations available for the determination of G and H are Eq. (5) and Eq. (20) which, however, are not sufficient to determine the real and the imaginary parts of G and H separately. One may, however, still be able to discriminate between solutions 1 and 2 since a measurement of $I_0 \times (1-D)$ and $I_0(R'+A)/\sin(\theta/2)$ at $\pi-\epsilon$ would respectively determine the experimental values of $2\mathrm{Re}\, G_c^* G_n$ and $-2\mathrm{Re}\, G_c^* H_n$ at $\pi-\epsilon$, and a comparison of these values with those predicted for the phase shift solutions 1 and 2 can then be made.

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References

1) Chamberlain, Segre, Tripp, Wiegand and Ypsilantis, Phys. Rev. **105** (1957), 288.
2) Stapp, Ypsilantis and Metropolis, Phys. Rev. **105** (1957), 302.
3) Puzikov, Rydin and Smorodinsky, Soviet Phys. JETP, **5** (1957), 489.
4) R. E. Marshak, London Conf. on Nuclear Forces and the Few-Nucleon Problem, July 1959, Proc. to be published by Pergamon Press.
5) M. J. Moravcsik, University of California, Radiation Laboratory, Report UCRL-5317-T, August 1958 (unpublished). Cziffra, MacGregor, Moravcsik and Stapp, Phys. Rev. **114** (1959), 880.
6) MacGregor, Moravcsik and Stapp, Phys. Rev. **116** (1959), 1248.
7) J. Gammel and R. Thaler, same as reference 4).
8) H. A. Bethe, Annals of Phys. **3** (1958), 190.

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Electromagnetic Structure of the Nucleon. V*

——*Numerical Results of Three-Pion-State Contributions*——

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Numerical results of the 3π-state contributions to the isoscalar part of the electromagnetic structure of the nucleon are presented in the no-loop approximation of relativistic perturbation theory. They are compared with the previous results obtained on other approximations.

In our previous papers,[1] we have calculated the 3π-state contributions to the isoscalar part of the electromagnetic structure of the nucleon by means of relativistic perturbation theory. But, since the obtained results contain eight-fold integrals, we are not capable of their precise numerical evaluations at least at the present stage of our ability. Instead, we calculate the spectral functions in the no-loop approximation,** $\alpha_1^S(m^2)$ and $\alpha_2^S(m^2)$ (cf. II(6·26) or III[2](2·1)), which are given by four-fold integrals. For $m^2\to\infty$, these approximate functions behave like

$$\begin{aligned}\alpha_1^S(m^2) &\sim \frac{3}{32\pi^3}\left(\frac{g^2}{4\pi}\right)^3\frac{m^2}{M^2}\left[\frac{1}{3}\log\frac{m^2}{M^2}-\frac{5}{4}\right],\\ \alpha_2^S(m^2) &\sim \frac{1}{64\pi^3}\left(\frac{g^2}{4\pi}\right)^3\frac{m^2}{M^2},\end{aligned}\tag{1}$$

contrary to the original functions which tend to zero. But for $m\lesssim M$ this approximation is very good as was shown in II, § 6–3, and at the threshold both expressions almost coincide as was stated in III, § 4.

The bold solid lines in Fig. 1 show the spectral functions in the no-loop approximation. For convenience of comparison, we display their static limits, $\lim \alpha_1^S(m^2)$ and $\lim \alpha_2^S(m^2)$ (cf. III(2·6)), and the lowest order contributions from the nucleon current, $\alpha_1^{S(N\bar{N})}(m^2)$ and $-\alpha_2^{S(N\bar{N})}(m^2)$ (cf. I(3·12)). From Fig. 1 we find the following facts:

i) $\alpha_1^S(m^2)$ and $\alpha_2^S(m^2)$ are positive definite and monotonously increasing, and quite small for $m\lesssim M$.

ii) The static limits are always larger than the relativistic results. Recoil effect is very large.

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** These have been investigated also by Kato[3] by means of dispersion theoretical approach (in Born term).

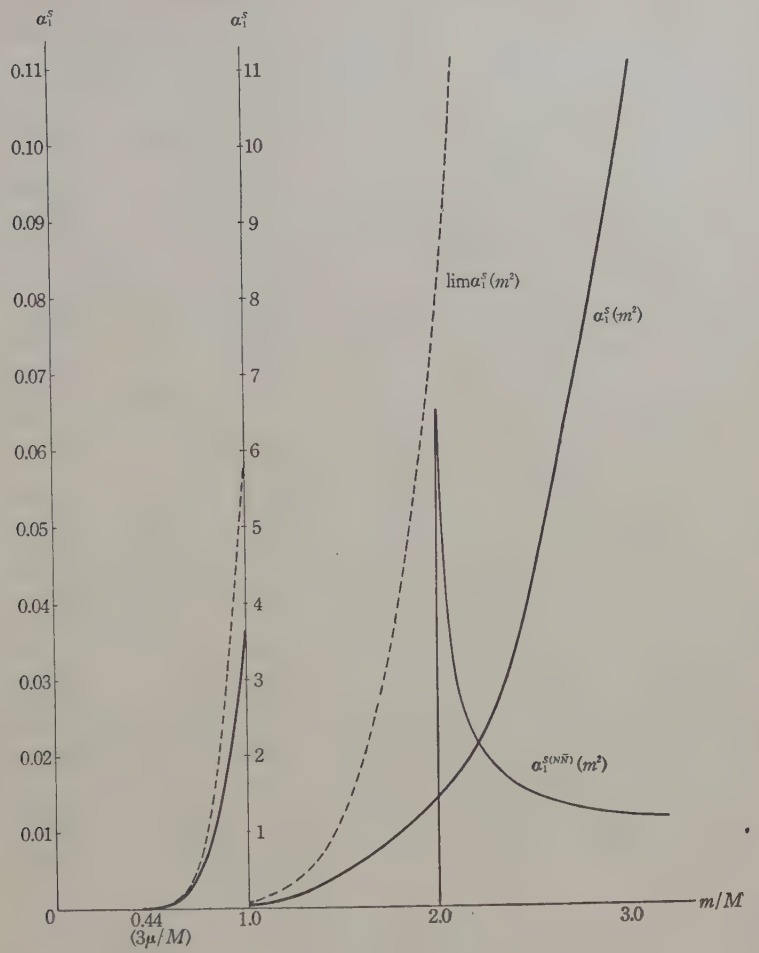

Fig. 1

iii) The spectral functions of the nucleon current have extremely sharp maxima very near the threshold ($m_{\max}=2.007M$).*

Table I shows charge and a.m.m. and their r^2-moments calculated by $\alpha_1{}^S(m^2)$ and $\alpha_2{}^S(m^2)$ with square cutoff $m\leqq m_c$. The charge mean square radius, $\langle r^2\rangle_1{}^S$, is very small for $m_c\approx 2M$. However, our previous value for $\langle r^2\rangle_1{}^S$ which was estimated in a very crude way from the original eight-fold integral was unexpectedly large. But we should not immediately conclude that the apparent disagreement between both results is due to the no-loop approximation, because the different cutoff methods used in these calculations may yield apparently qualitatively different results as was noticed in III, § 3–2 for the static limits. Therefore, to

* Of course $\alpha_1{}^{S(N\bar{N})}=\alpha_2{}^{S(N\bar{N})}=0$ at the threshold $m=2M$.

Table I

	unit	$m_c=M$	$m_c=1.5M$	$m_c=2M$	$m_c=2.5M$	$m_c=3M$
Q^S	e	0.010	0.14	0.66	2.0	4.6
$\langle r^2\rangle_1{}^S$	10^{-26} cm^2	0.0081	0.048	0.14	0.27	0.45
μ^S	$e/2M$	0.020	0.22	0.83	2.1	4.2
$\mu^S\langle r^2\rangle_2{}^S$	$e/2M\cdot10^{-26}$ cm^2	0.0068	0.038	0.089	0.15	0.23

make comparison easy, we calculate the r^2-moments, *etc.*, in the no-loop approximation with square cutoff $r\geqq r_c$ in the distribution functions according to the following formulae (cf. II(4·13)) :

$$Q^S=\int_{(3\mu)^2}^{m_c^2}\frac{\alpha_1{}^S(m^2)}{m^2}(1+mr_c)e^{-mr_c}dm^2,$$
$$\frac{1}{2}\langle r^2\rangle_1{}^S=6\int_{(3\mu)^2}^{m_c^2}\frac{\alpha_2{}^S(m^2)}{m^4}\left\{1+mr_c+\frac{(mr_c)^2}{2}+\frac{(mr_c)^3}{6}\right\}e^{-mr_c}dm^2,\ \textit{etc.} \tag{2}$$

These calculated results are summarized in Table II. Since the spectral functions in the no-loop approximation are not calculated for $m>3M$, these values yet should not be compared with those in Table III of I. In the calculation in I, § 5–1, as the integrals are non-vanishing only for $m=\kappa(x)$, the square cutoff in the original spectral functions is carried out by restricting the integration domain to the region where $\kappa(x)<m_c$. Table III shows estimated cutoff values (for $m_c=3M$) from the previous 100 samples selected at random. One must not take these values seriously since the above estimation is very crude. But from the comparison of

Table II

	unit	$r_c=1/M$			$r_c=2/M$		
		$m_c=2M$	$m_c=2.5M$	$m_c=3M$	$m_c=2M$	$m_c=2.5M$	$m_c=3M$
Q^S	e	0.33	0.77	1.1	0.11	0.18	0.25
$\langle r^2\rangle_1{}^S$	10^{-26} cm^2	0.12	0.23	0.36	0.082	0.13	0.16
μ^S	$e/2M$	0.42	0.84	1.3	0.14	0.22	0.27
$\mu^S\langle r^2\rangle_2{}^S$	$e/2M\cdot10^{-26}$ cm^2	0.083	0.13	0.19	0.056	0.079	0.093

Table III

	unit	$r_c=1/M,\ \kappa<3M$	$r_c=2/M,\ \kappa<3M$
Q^S	e	3.3	0.87
$\langle r^2\rangle_1{}^S$	10^{-26} cm^2	0.99	0.52
μ^S	$e/2M$	10.7	1.4
$\mu^S\langle r^2\rangle_2{}^S$	$e/2M\cdot10^{-26}$ cm^2	1.3	0.49

Table II with Table III we may conjecture some features of the differences between the original spectral functions and those in the no-loop approximation. The original charge spectral function will probably have a maximum at $m<3M$ whose value is larger than the corresponding value of that in the no-loop approximation, and decrease and become negative. The original a.m.m. spectral function (for which the values in Table III are less reliable) will be rather large up to $m\approx 3M$ and nearly vanish for larger m (cf. Table III of I). We may thus expect that the original spectral functions become large more quickly (with respect to m) than those in the no-loop approximation, thanks to the contribution from the closed loop.

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References

1) K. Hiida, N. Nakanishi, Y. Nogami and M. Uehara, Prog. Theor. Phys. **22** (1959), 247, 351. These are referred to as I and II, respectively.
2) K. Hiida and N. Nakanishi, Prog. Theor. Phys. **22** (1959), 863 (referred to as III).
3) M. Kato, private communication.

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Collective Excitation of Nuclear Matter

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Collective excitations of nuclear matter which have their origin in the short-range interaction between nucleons in nucleus are studied in order to correlate strongly excited level groups in nuclear reactions of relatively high energy with nuclear structure. Making use of an effective inter-nucleon potential which is determined from the analysis of pairing energies of various nuclei, the collective excitation energies are estimated in one particle-hole pair approximation of the new Tamm-Dancoff method.

Three stable modes of excitations are possible, that is, spin wave, isospin wave and coupled spin-isospin wave. Of these three modes the spin wave is the lowest exciting wave, the coupled spin-isospin wave is the second and the isospin wave is the highest. Each of them has its own critical momentum above which they become unstable.

The strengths of coupling of these exciting waves with particle motion are also estimated. The higher exciting wave has the stronger coupling with particle motion and therefore the larger width of excitation energy.

These exciting mechanisms would have some connection with the experimental results on inelastic scattering of nucleons and photonuclear reactions.

§ 1. Introduction

At present the aspect of the nuclear excitations of relatively high energy is becoming clear from the experiments on the inelastic scattering of high energy nucleons, the photonuclear reactions and other nuclear phenomena. These nuclear excitations are closely connected with the short-range interaction of nucleons in atomic nucleus. Therefore they afford a clue to investigation of the structure of nucleus. It may be expected that the collective excitations of nucleus which have their origin in the short-range interaction between nucleons in nucleus might be found in the excitations of relatively high energy.

As the first step of the investigation we here adopt the Fermi gas model of nucleus and consider the collective excitations of nuclear matter. Here, of course, the important finite nature of nucleus is not taken into account, but our results will be qualitatively correct at high energies.

In § 2 we derive the eigenvalue equation for collective excitation which results from one particle-hole pair state in the lowest approximation of the new Tamm-Dancoff method.[1] Similar equations were studied by Glassgold, Heckrotte and Watson,[2] but they took only terms in the interaction Hamiltonian which correspond to that which Sawada[3] took in the problem of plasma oscillation of the electron gas at high density. Though we work with one particle-hole pair ap-

proximation, our treatment is more accurate than theirs. The interaction potential between a particle and a hole changes its sign, when we compare it with the potential between particles, in a similar way to the particle-anti-particle interaction potential.[4]

In § 3 the eigenvalue equation derived in § 2 is solved, using an effective residual interaction between nucleons in nucleus which Nomoto[5] obtained from the analysis of pairing energies of various nuclei. Then it follows that three stable modes of exciting waves are possible, that is, spin wave, isospin wave[6] and coupled spin-isospin wave. Of these three modes the spin wave is the lowest exciting wave, the coupled spin-isospin wave is the second and the isospin wave is the highest. These waves have excitation energies linear in momentum when their momenta are small compared with the Fermi momentum of nucleus. Each of them has its own critical momentum above which they become unstable. These mechanisms of nuclear excitations would have some bearing on several resonance peaks which have been observed in the experiments on the inelastic scattering of high energy nucleons,[7] photonuclear reactions[8),9] and inelastic scattering of electrons.[8]

In § 4 we study the properly normalized wave functions of the above collective excitations. Then the coupling strengths of these exciting waves with particle motion are estimated. The results show that the higher exciting wave has the stronger coupling with particle motion. Therefore the width of the excitation energy is largest for the isospin wave.

In this paper we take the infinite Fermi gas model and neglect the important finite nature of nucleus. Also there are ambiguities in the effective residual interaction. Therefore we do not attempt to compare the results with experimental results quantitatively. It will be very interesting to investigate the excitation mechanisms treated in this paper using shell model wave functions for actual nuclei and compare with the experimental results on the aforementioned nuclear reactions. These problems will be treated in a forthcoming paper.

§ 2. Eigenvalue equation for one particle-hole pair state

We begin with the following nuclear Hamiltonian,

$$H=H_0+H_I, \tag{1}*$$

$$H_0=-\frac{1}{2M}\int d\boldsymbol{x}\,\psi^*(\boldsymbol{x})\nabla^2\psi(\boldsymbol{x}),$$

$$H_I=\int d\boldsymbol{x}_1 d\boldsymbol{x}_2\psi^*(\boldsymbol{x}_2)\,\psi^*(\boldsymbol{x}_1)\,V(\boldsymbol{x}_1,\,\boldsymbol{x}_2)\,\psi(\boldsymbol{x}_1)\,\psi(\boldsymbol{x}_2),$$

where M is the effective mass of nucleon in nucleus, $\psi(\boldsymbol{x})$ the wave function of nucleon which includes both spin and isospin, and V the effective residual inter-

* We use the natural unit $\hbar=c=1$.

action potential between nucleons in nucleus. V is generally a nonlocal potential but we assume it to be a local central one and neglect the effect of the tensor force. Then V is written in the form

$$V(\boldsymbol{x}_1,\ \boldsymbol{x}_2)=\sum_{i=1}^{4}\boldsymbol{\Gamma}_i^{(1)}\cdot\boldsymbol{\Gamma}_i^{(2)}\,V_i(|\boldsymbol{x}_1-\boldsymbol{x}_2|), \tag{2}$$

$$\boldsymbol{\Gamma}_1=1,\ \boldsymbol{\Gamma}_2=\boldsymbol{\sigma},\ \boldsymbol{\Gamma}_3=\boldsymbol{\tau},\ \boldsymbol{\Gamma}_4=\boldsymbol{\tau}\boldsymbol{\sigma},$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ are the spin and the isospin operators for nucleon respectively, and we use the usual matrix representations for them.

We introduce the Fourier transforms of ψ, ψ^* and V as follows,

$$\psi(\boldsymbol{x})=\frac{1}{(2\pi)^{3/2}}\sum_{\rho=1}^{4}\left(\int_{|p|>p_F}d\boldsymbol{p}\,e^{i\boldsymbol{p}\cdot\boldsymbol{x}}u_\rho\,a_\rho(\boldsymbol{p})+\int_{|p|<p_F}d\boldsymbol{p}\,e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\tau_2\sigma_2 v_\rho^{*T}b_\rho^*(\boldsymbol{p})\right),$$

$$\psi^*(\boldsymbol{x})=\frac{1}{(2\pi)^{3/2}}\sum_{\rho=1}^{4}\left(\int_{|p|>p_F}d\boldsymbol{p}\,e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}u_\rho^*\,a_\rho^*(\boldsymbol{p})+\int_{|p|<p_F}d\boldsymbol{p}\,e^{i\boldsymbol{p}\cdot\boldsymbol{x}}v_\rho^{\,T}\tau_2\sigma_2 b_\rho(\boldsymbol{p})\right), \tag{3}^*$$

$$V_i(|\boldsymbol{x}_1-\boldsymbol{x}_2|)=\frac{1}{(2\pi)^3}\int d\boldsymbol{k}\,e^{i\boldsymbol{k}\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2)}\,V_i(k),$$

where $u_\rho(\rho=1,\,2,\,3,\,4)$ are the spin and isospin wave functions for nuclear particles which are in the spin and isospin state ρ and can be represented by column matrices. v_ρ is defined by

$$v_\rho=\tau_2\sigma_2 u_{-\rho}^{*T} \tag{4}$$

and represents the spin and isospin wave function for the hole. $\tau_2\sigma_2$ expresses the fact that the spin and isospin of a particle and a hole are reversed to each other. This matrix is analogous to the charge conjugation matrix in positron theory. $a_\rho(\boldsymbol{p})$ and $a_\rho^*(\boldsymbol{p})$ respectively annihilate and create a particle with momentum $\boldsymbol{p}$ and spin and isospin ρ. These quantities are defined only for $p>p_F$. Similarly, $b_\rho(\boldsymbol{p})$ and $b_\rho^*(\boldsymbol{p})$ are annihilation and creation operators for holes within the Fermi sea, being defined only for $p<p_F$. These operators satisfy the well-known anticommutation relations

$$\{a_\rho^*(\boldsymbol{p}),\ a_{\rho'}(\boldsymbol{p}')\}=\delta_{\rho\rho'}\delta(\boldsymbol{p}-\boldsymbol{p}'),\ \{b_\rho^*(\boldsymbol{p}),\ b_{\rho'}(\boldsymbol{p}')\}=\delta_{\rho\rho'}\delta(\boldsymbol{p}-\boldsymbol{p}'). \tag{5}$$

In the following we use for convenience sake the following notation:

$$\begin{aligned}&\sum_\rho a_\rho(\boldsymbol{p})\,(u_\rho)_\alpha\to a_\alpha(\boldsymbol{p}),\quad \sum_\rho a_\rho^*(\boldsymbol{p})\,(u_\rho^*)_\alpha\to a_\alpha^*(\boldsymbol{p}),\\ &\sum_\rho b_\rho(\boldsymbol{p})\,(v_\rho)_\alpha\to b_\alpha(\boldsymbol{p}),\quad \sum_\rho b_\rho^*(\boldsymbol{p})\,(v_\rho^*)_\alpha\to b_\alpha^*(\boldsymbol{p}).\end{aligned} \tag{6}$$

a_α, a_α^*, b_α and b_α^* satisfy the same anticommutation relations as (5). $a(\boldsymbol{p})$ and $b(\boldsymbol{p})$ are column matrices, $a^*(\boldsymbol{p})$ and $b^*(\boldsymbol{p})$ row matrices respectively.

We investigate in this paper the collective excitations of nuclear matter in one

* Asterisk denotes Hermite conjugate and T transposed.

particle-hole pair approximation. Therefore, when we introduce Fourier transforms (3) into (1), various terms appear, but we leave only the following terms,

$$H'=H_0'+H_I^{(1)}+H_I^{(2)}, \tag{7}$$

$$H_0'=\frac{1}{2M}\int_{|p|>p_F} d\boldsymbol{p}\, a^*(\boldsymbol{p})p^2 a(\boldsymbol{p})-\frac{1}{2M}\int_{|p|<p_F} d\boldsymbol{p}\, b^*(\boldsymbol{p})p^2 b(\boldsymbol{p}),$$

$$\begin{aligned}H_I^{(1)}=&-\frac{1}{(2\pi)^3}\sum_{i=1}^{4}\int d\boldsymbol{p}_1' d\boldsymbol{p}_2' d\boldsymbol{p}_1 d\boldsymbol{p}_2 d\boldsymbol{k} V_i(k)\delta(\boldsymbol{p}_1'-\boldsymbol{p}_1-\boldsymbol{k})\delta(\boldsymbol{p}_2'-\boldsymbol{p}_2+\boldsymbol{k})\\ &\times(a^*(\boldsymbol{p}_1')\boldsymbol{\Gamma}_i\tau_2\sigma_2 b^*(-\boldsymbol{p}_1)+b(-\boldsymbol{p}_1')\tau_2\sigma_2\boldsymbol{\Gamma}_i a(\boldsymbol{p}_1))\cdot\\ &\times(a^*(\boldsymbol{p}_2')\boldsymbol{\Gamma}_i\tau_2\sigma_2 b^*(-\boldsymbol{p}_2)+b(-\boldsymbol{p}_2')\tau_2\sigma_2\boldsymbol{\Gamma}_i a(\boldsymbol{p}_2)),\end{aligned}$$

$$\begin{aligned}H_I^{(2)}=&-\frac{2}{(2\pi)^3}\sum_{i=1}^{4}\int d\boldsymbol{p}_1' d\boldsymbol{p}_2' d\boldsymbol{p}_1 d\boldsymbol{p}_2 d\boldsymbol{k} V_i(k)\delta(\boldsymbol{p}_1'-\boldsymbol{p}_1-\boldsymbol{k})\delta(\boldsymbol{p}_2'-\boldsymbol{p}_2+\boldsymbol{k})\\ &\times(a^*(\boldsymbol{p}_1')b^*(-\boldsymbol{p}_2)\boldsymbol{\Gamma}_i^{(1)}\cdot\boldsymbol{\Gamma}_i'^{(2)}b(-\boldsymbol{p}_2')a(\boldsymbol{p}_1)),\end{aligned}$$

where

$$\boldsymbol{\Gamma}_1'=-\boldsymbol{\Gamma}_1,\ \boldsymbol{\Gamma}_2'=\boldsymbol{\Gamma}_2,\ \boldsymbol{\Gamma}_3'=\boldsymbol{\Gamma}_3,\ \boldsymbol{\Gamma}_4'=-\boldsymbol{\Gamma}_4 \tag{8}$$

and we used the relations

$$\tau_2{}^T=-\tau_2,\ \sigma_2{}^T=-\sigma_2,\ \sigma_2\boldsymbol{\sigma}\sigma_2=-\boldsymbol{\sigma}^T,\ \tau_2\boldsymbol{\tau}\tau_2=-\boldsymbol{\tau}^T. \tag{9}$$

$H_I^{(1)}$ is the Hamiltonian which Sawada[3] took in the problem of plasma oscillation of the electron gas, where the long-range Coulomb force acts. In our problem we are concerned with short-range nuclear forces and $H_I^{(2)}$ gives comparable contributions with $H_I^{(1)}$. If we compare $H_I^{(2)}$ with the ordinary interaction between particles, we can see from (8) that the sign of V_1 and V_4 is reversed. This situation is analogous to the interaction potential between a nucleon and an anti-nucleon.[4] Glassgold, Heckrotte and Watson[2] studied a similar problem, but they did not take $H_I^{(2)}$.

Next we derive the eigenvalue equation for one particle-hole pair state using the new Tamm-Dancoff method[1] in the lowest approximation. Though we are uncertain how good one pair approximation is for short-range interaction, we suppose that one pair state plays an important part at any rate in the energy region considered here and our results will be correct at least qualitatively. From the Schroedinger equation

$$(E-H_0'-H_I^{(1)}-H_I^{(2)})\Psi=0,$$

we get the following equations for the one pair amplitudes,

$$\begin{aligned}&(\epsilon+E_{q_1}-E_{q_2})(\Psi_0^* b_\beta^*(\boldsymbol{q}_2)a_\alpha^*(\boldsymbol{q}_1)\Psi)=(\Psi_0^*[b_\beta^*(\boldsymbol{q}_2)a_\alpha^*(\boldsymbol{q}_1),\ H_I^{(1)}+H_I^{(2)}]\Psi),\\ &(\epsilon+E_{q_1}-E_{q_2})(\Psi_0^* a_\beta(-\boldsymbol{q}_2)b_\alpha(-\boldsymbol{q}_1)\Psi)=(\Psi_0^*[a_\beta(-\boldsymbol{q}_2)b_\alpha(-\boldsymbol{q}_1),\ H_I^{(1)}+H_I^{(2)}]\Psi),\end{aligned} \tag{10}$$

where $\epsilon=E-E_0$, $E_q=(1/2M)\cdot\boldsymbol{q}^2$, Ψ_0 is the ground state and E_0 its energy.

Calculating the commutators on the right-hand side of (10) and leaving only the one pair terms, we obtain

$$[b_\beta^*(\boldsymbol{q}_2)a_\alpha^*(\boldsymbol{q}_1),\ H_I^{(1)}]=-\frac{2}{(2\pi)^3}\sum_i\int d\boldsymbol{k}V_i(|\boldsymbol{q}_1+\boldsymbol{q}_2|)\,(\tau_2\sigma_2\boldsymbol{\Gamma}_i)_{\beta\alpha}$$
$$\times Sp[\boldsymbol{\Gamma}_i\tau_2\sigma_2 b^*(-\boldsymbol{k}+\boldsymbol{q}_1+\boldsymbol{q}_2)a^*(\boldsymbol{k})+\tau_2\sigma_2\boldsymbol{\Gamma}_i a(\boldsymbol{k}-\boldsymbol{q}_1-\boldsymbol{q}_2)b(-\boldsymbol{k})]$$
$$+\frac{2}{(2\pi)^3}\sum_i\int d\boldsymbol{k}V_i(k)\,(\tau_2\sigma_2\boldsymbol{\Gamma}_i a(-\boldsymbol{q}_2+\boldsymbol{k})b(-\boldsymbol{q}_1-\boldsymbol{k})\tau_2\sigma_2\boldsymbol{\Gamma}_i)_{\beta\alpha}, \qquad (11)$$

$$[b_\beta^*(\boldsymbol{q}_2)a_\alpha^*(\boldsymbol{q}_1),\ H_I^{(2)}]=-\frac{2}{(2\pi)^3}\sum_i\int d\boldsymbol{k}V_i(k)\,(b^*(\boldsymbol{q}_2-\boldsymbol{k})\boldsymbol{\Gamma}_i')_\beta(a^*(\boldsymbol{q}_1+\boldsymbol{k})\boldsymbol{\Gamma}_i)_\alpha.$$

In the right-hand side of the above formulas, b^*a^* and ab are regarded as matrices with 4 rows and 4 columns (density matrix) and Sp denotes summation with respect to spin and isospin. Similarly, we have

$$[a_\beta(-\boldsymbol{q}_2)b_\alpha(-\boldsymbol{q}_1),\ H_I^{(1)}]=\frac{2}{(2\pi)^3}\sum_i\int d\boldsymbol{k}V_i(|\boldsymbol{q}_1+\boldsymbol{q}_2|)\,(\boldsymbol{\Gamma}_i\tau_2\sigma_2)_{\beta\alpha}$$
$$\times Sp[\boldsymbol{\Gamma}_i\tau_2\sigma_2 b^*(-\boldsymbol{k}+\boldsymbol{q}_1+\boldsymbol{q}_2)a^*(\boldsymbol{k})+\tau_2\sigma_2\boldsymbol{\Gamma}_i a(\boldsymbol{k}-\boldsymbol{q}_1-\boldsymbol{q}_2)b(-\boldsymbol{k})]$$
$$-\frac{2}{(2\pi)^3}\sum_i\int d\boldsymbol{k}V_i(k)\,(\boldsymbol{\Gamma}_i\tau_2\sigma_2 b^*(\boldsymbol{q}_2-\boldsymbol{k})a^*(\boldsymbol{q}_1+\boldsymbol{k})\boldsymbol{\Gamma}_i\tau_2\sigma_2)_{\beta\alpha}, \qquad (12)$$

$$[a_\beta(-\boldsymbol{q}_2)b_\alpha(-\boldsymbol{q}_1),\ H_I^{(2)}]=\frac{2}{(2\pi)^3}\sum_i\int d\boldsymbol{k}V_i(k)\,(\boldsymbol{\Gamma}_i a(-\boldsymbol{q}_2+\boldsymbol{k}))_\beta(\boldsymbol{\Gamma}_i' b(-\boldsymbol{q}_1-\boldsymbol{k}))_\alpha.$$

Then we decompose the one pair amplitudes into independent components with respect to spin and isospin as follows,

$$(\Psi_0^* b_\beta^*(\boldsymbol{q}_2)a_\alpha^*(\boldsymbol{q}_1)\Psi)=(\tau_2\sigma_2\sum_{i=1}^4 f_i(\boldsymbol{q}_2,\ \boldsymbol{q}_1)\boldsymbol{\Gamma}_i)_{\beta\alpha},$$
$$(\Psi_0^* a_\beta(-\boldsymbol{q}_2)b_\alpha(-\boldsymbol{q}_1)\Psi)=(\sum_{i=1}^4 g_i(-\boldsymbol{q}_2,\ -\boldsymbol{q}_1)\boldsymbol{\Gamma}_i\tau_2\sigma_2)_{\beta\alpha}. \qquad (13)$$

Here f_1 and g_1 represent the amplitude with spin 0 and isospin 0, f_2 and g_2 spin 1 and isospin 0, f_3 and g_3 spin 0 and isospin 1 and f_4 and g_4 spin 1 and isospin 1 respectively. If we insert (11) and (12) into the right-hand side of (10) and use Eqs. (8), (9) and (13) and the anticommutation relations between σ's and τ's and then compare the coefficients of the independent components on the both sides, we obtain the following equations,

$$(\epsilon_i+E_{q_1}-E_{q_2})f_i(\boldsymbol{q}_2,\ \boldsymbol{q}_1)=-\frac{8}{(2\pi)^3}\int d\boldsymbol{k}V_i(|\boldsymbol{q}_1+\boldsymbol{q}_2|)\,(f_i(-\boldsymbol{k}+\boldsymbol{q}_1+\boldsymbol{q}_2,\ \boldsymbol{k})$$
$$+g_i(\boldsymbol{k}-\boldsymbol{q}_1-\boldsymbol{q}_2,\ -\boldsymbol{k}))+\frac{2}{(2\pi)^3}\int d\boldsymbol{k}V_i'(k)\,(f_i(\boldsymbol{q}_2-\boldsymbol{k},\ \boldsymbol{q}_1+\boldsymbol{k})$$
$$+g_i(-\boldsymbol{q}_2+\boldsymbol{k},\ -\boldsymbol{q}_1-\boldsymbol{k})),\quad |\boldsymbol{q}_1|>p_F,\ |\boldsymbol{q}_2|<p_F; \qquad (14)$$

$$(\epsilon_i+E_{q_1}-E_{q_2})\,g_i(-\boldsymbol{q}_2,\ -\boldsymbol{q}_1)=\frac{8}{(2\pi)^3}\int d\boldsymbol{k}V_i(|\boldsymbol{q}_1+\boldsymbol{q}_2|)\,(f_i(-\boldsymbol{k}+\boldsymbol{q}_1+\boldsymbol{q}_2,\ \boldsymbol{k})$$
$$+g_i(\boldsymbol{k}-\boldsymbol{q}_1-\boldsymbol{q}_2,\ -\boldsymbol{k}))-\frac{2}{(2\pi)^3}\int d\boldsymbol{k}V_i'(k)\,(f_i(\boldsymbol{q}_2-\boldsymbol{k},\ \boldsymbol{q}_1+\boldsymbol{k})$$
$$+g_i(-\boldsymbol{q}_2+\boldsymbol{k},\ -\boldsymbol{q}_1-\boldsymbol{k})),\quad |\boldsymbol{q}_1|<p_F,\ |\boldsymbol{q}_2|>p_F;$$

where

$$\begin{aligned}V_1'&=V_1+3V_2+3V_3+9V_4,\\ V_2'&=V_1-V_2+3V_3-3V_4,\\ V_3'&=V_1+3V_2-V_3-3V_4,\\ V_4'&=V_1-V_2-V_3+V_4.\end{aligned}\tag{15}$$

(14) is the eigenvalue equation for one pair state in our approximation.

§ 3. Estimation of excitation energy

In this section we solve the integral equation (14) in the preceding section to estimate the energy of the collective excitations. As is well known the nuclear potential between free nucleons is very strong and singular and produce strong correlation effects between nucleons in nucleus. This real inter-nucleon potential cannot be used in our equations as it stands. After dealing with the strong correlation effects properly and separating the self-consistent part, we should use the remaining interaction between nucleons in nucleus in our formulas. Instead of this we use the effective residual interaction potential determined from the analysis of the pairing energies of various nuclei. Nomoto[5] obtained from the analysis of the pairing energies of even-even and odd mass number nuclei the following inter-nucleon potential,

$$V_{ij}=8\pi r_0^3\left[V_t\frac{3+\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j}{4}+V_s\frac{1-\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j}{4}\right]\delta(\boldsymbol{x}_i-\boldsymbol{x}_j),\tag{16}$$

$$V_t=-(25.4\pm6.7)\ \text{Mev},\quad V_s=-(14.2\pm3.1)\ \text{Mev},$$

where r_0 is connected with the nuclear radius R as $R=r_0A^{1/3}$. If the effective residual interactions are sufficiently short-ranged and weak in the odd states, they can be approximately represented by the above form. Since the delta-function interactions act only in relative s states, we can write (16) in the following second quantized form,

$$V_{ij}=\frac{8\pi r_0^3}{2}\int d\boldsymbol{x}_i d\boldsymbol{x}_j\psi^*(\boldsymbol{x}_j)\psi^*(\boldsymbol{x}_i)\left[V_t\frac{3+\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j}{4}\frac{1-\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j}{4}\right.$$
$$\left.+V_s\frac{1-\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j}{4}\frac{3+\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j}{4}\right]\delta(\boldsymbol{x}_i-\boldsymbol{x}_j)\psi(\boldsymbol{x}_i)\psi(\boldsymbol{x}_j).\tag{16'}$$

In this case V_i and V_i' in the preceding section become as follows,

$$\begin{aligned} V_1&=\frac{\pi r_0^3}{4}(3V_t+3V_s)<0,\\ V_2&=\frac{\pi r_0^3}{4}(V_t-3V_s)>0,\\ V_3&=\frac{\pi r_0^3}{4}(-3V_t+V_s)>0,\\ V_4&=\frac{\pi r_0^3}{4}(-V_t-V_s)>0, \end{aligned} \tag{17}$$

and

$$V_1'=-4V_1,\ V_2'=-4V_2,\ V_3'=-4V_3,\ V_4'=-4V_4. \tag{18}$$

Therefore (14) is written as

$$\begin{aligned} &(\epsilon_i+E_{q_1}-E_{q_2})f_i(\boldsymbol{q}_2,\ \boldsymbol{q}_1)=-\frac{8}{(2\pi)^3}V_i\Big[\int d\boldsymbol{k}(f_i(-\boldsymbol{k}+\boldsymbol{q}_1+\boldsymbol{q}_2,\ \boldsymbol{k})\\ &+g_i(\boldsymbol{k}-\boldsymbol{q}_1-\boldsymbol{q}_2,\ -\boldsymbol{k}))+\int d\boldsymbol{k}(f_i(\boldsymbol{q}_2-\boldsymbol{k},\ \boldsymbol{q}_1+\boldsymbol{k})+g_i(-\boldsymbol{q}_2+\boldsymbol{k},\ -\boldsymbol{q}_1-\boldsymbol{k}))\Big],\\ &\qquad |\boldsymbol{q}_2|<p_F,\ |\boldsymbol{q}_1|>p_F;\\ &(\epsilon_i+E_{q_1}-E_{q_2})g_i(-\boldsymbol{q}_2,\ -\boldsymbol{q}_1)=\frac{8}{(2\pi)^3}V_i\Big[\int d\boldsymbol{k}(f_i(-\boldsymbol{k}+\boldsymbol{q}_1+\boldsymbol{q}_2,\ \boldsymbol{k})\\ &+g_i(\boldsymbol{k}-\boldsymbol{q}_1-\boldsymbol{q}_2,\ -\boldsymbol{k}))+\int d\boldsymbol{k}(f_i(\boldsymbol{q}_2-\boldsymbol{k},\ \boldsymbol{q}_1+\boldsymbol{k})+g_i(-\boldsymbol{q}_2+\boldsymbol{k},\ -\boldsymbol{q}_1-\boldsymbol{k}))\Big],\\ &\qquad |\boldsymbol{q}_2|>p_F,\ |\boldsymbol{q}_1|<p_F. \end{aligned} \tag{19}$$

If we change integration variable from $\boldsymbol{k}$ to $\boldsymbol{k}'=\boldsymbol{k}-\boldsymbol{q}_1$, in the first integral of (19), the first integral becomes equal to the second. That is, in the present case the part which was neglected in reference 2) gives the same contributions as the part considered there. Generally, $H_I^{(2)}$ will contribute comparably with $H_I^{(1)}$. Separating the centre of mass motion, i.e. putting

$$\boldsymbol{q}_1=\boldsymbol{p}+\frac{\boldsymbol{K}}{2},\quad \boldsymbol{q}_2=-\boldsymbol{p}+\frac{\boldsymbol{K}}{2}$$

and changing the integration variable from $\boldsymbol{k}$ to $\boldsymbol{p}'=\boldsymbol{p}+\boldsymbol{k}$ in the second integral of (19), we obtain

$$\begin{aligned} &\Big(\epsilon_i+\frac{\boldsymbol{K}\cdot\boldsymbol{p}}{M}\Big)f_i\Big(-\boldsymbol{p}+\frac{\boldsymbol{K}}{2},\ \boldsymbol{p}+\frac{\boldsymbol{K}}{2}\Big)=-\frac{16}{(2\pi)^3}V_i\int d\boldsymbol{p}'\Big(f_i\Big(-\boldsymbol{p}'+\frac{\boldsymbol{K}}{2},\ \boldsymbol{p}'+\frac{\boldsymbol{K}}{2}\Big)\\ &+g_i\Big(\boldsymbol{p}'-\frac{\boldsymbol{K}}{2},\ -\boldsymbol{p}'-\frac{\boldsymbol{K}}{2}\Big)\Big),\qquad \Big|-\boldsymbol{p}+\frac{\boldsymbol{K}}{2}\Big|<p_F,\ \Big|\boldsymbol{p}+\frac{\boldsymbol{K}}{2}\Big|>p_F, \end{aligned} \tag{20}$$

$$\left(\epsilon_i+\frac{\boldsymbol{K}\cdot\boldsymbol{p}}{M}\right)g_i\left(\boldsymbol{p}-\frac{\boldsymbol{K}}{2},\ -\boldsymbol{p}-\frac{\boldsymbol{K}}{2}\right)=\frac{16}{(2\pi)^3}V_i\int d\boldsymbol{p}'\left(f_i\left(-\boldsymbol{p}'+\frac{\boldsymbol{K}}{2},\ \boldsymbol{p}'+\frac{\boldsymbol{K}}{2}\right)\right.$$
$$\left.+g_i\left(\boldsymbol{p}'-\frac{\boldsymbol{K}}{2},\ -\boldsymbol{p}'-\frac{\boldsymbol{K}}{2}\right)\right),\qquad \left|-\boldsymbol{p}+\frac{\boldsymbol{K}}{2}\right|>p_F,\ \left|\boldsymbol{p}+\frac{\boldsymbol{K}}{2}\right|<p_F.$$

From (20) we get the following eigenvalue equation in the same way as Sawada,[3]

$$1=\frac{16}{(2\pi)^3}V_i\left(\int_{\substack{|-p+K/2|>p_F\\|p+K/2|<p_F}}d\boldsymbol{p}\frac{1}{\epsilon_i+(\boldsymbol{K}\cdot\boldsymbol{p}/M)}-\int_{\substack{|-p+K/2|<p_F\\|p+K/2|>p_F}}d\boldsymbol{p}\frac{1}{\epsilon_i+(\boldsymbol{K}\cdot\boldsymbol{p}/M)}\right).\tag{21}$$

As Sawada et al. discussed, this eigenvalue equation has two types of solutions, i.e. the continuum solutions which correspond to the free pair excitation and a discrete solution which represents a collective excitation. Integrating Eq. (21), we have the relationship between the collective energy ϵ_i and V_i,

$$1=\frac{2M^3}{\pi^2K^3}V_i\left[\left\{\left(\epsilon_i+\frac{K^2}{2M}\right)^2-\left(\frac{Kp_F}{M}\right)^2\right\}\log\frac{\epsilon_i+(Kp_F/M)+(K^2/2M)}{\epsilon_i-(Kp_F/M)+(K^2/2M)}\right.$$
$$\left.+\left\{\left(\epsilon_i-\frac{K^2}{2M}\right)^2-\left(\frac{Kp_F}{M}\right)^2\right\}\log\frac{\epsilon_i-(Kp_F/M)-(K^2/2M)}{\epsilon_i+(Kp_F/M)-(K^2/2M)}-2\frac{K^3p_F}{M^2}\right].\tag{22}$$

This equation has a stable real solution ϵ_i for positive V_i, but has only an unstable solution for negative V_i. If we take the interaction potential (16), as seen from (17), three stable modes of collective excitations are possible. When the momentum K is small compared with the Fermi momentum p_F, that is, $K^2/2M$ is small compared with Kp_F/M, we can expand (22) with respect to $K^2/2M$. Then we have

$$1=\frac{4M^2V_i}{\pi^2K}\left\{\epsilon_i\log\frac{\epsilon_i+(Kp_F/M)}{\epsilon_i-(Kp_F/M)}-2\frac{Kp_F}{M}\right\}.\tag{22'}$$

(22′) has a solution ϵ_i linear in K, i.e.

$$\epsilon_i=u_iK,$$

where u_i satisfies the equation

$$1=\frac{4M^2V_i}{\pi^2}\left\{u_i\log\frac{u_i+(p_F/M)}{u_i-(p_F/M)}-2\frac{p_F}{M}\right\}.\tag{22''}$$

In the case of plasma oscillation, where the long-range Coulomb force acts, the plasmon energy has the classical limit when $K\to0$, but in our case, where short-range potential concerns, the collective excitation energy tends to zero in the limit $K\to0$.

On the other hand, we obtain from (22) the values of $K_{\max}$ and $\epsilon_{i\max}$ at which

the collective solutions cross over into the continuum solution. This occurs at $\epsilon_{i\,\max}=K_{\max}^2/2M+K_{\max}p_F/M$, which gives

$$1=\frac{4Mp_FV_i}{\pi^2}\left\{\left(1+\frac{K_{\max}}{2p_F}\right)\log\left(1+\frac{2p_F}{K_{\max}}\right)-1\right\}. \tag{23}$$

For momenta above this value, the collective solutions become unstable.

Taking the values of constants, $Mc^2=940$ Mev, $r_0=\hbar/\mu c$ (π-meson Compton wavelength) and $p_F=(9\pi/8)^{1/3}\cdot\hbar/r_0$, Eqs. (22″) and (23) are solved numerically. The results are listed in the following table:

	mode 2	mode 3	mode 4
u	0.233	0.29	0.26
$K_{\max}/p_F$	0.070	0.90	0.44
$\epsilon_{\max}$ (Mev)	3.50	63.0	25.9

Mode 2 corresponds to the spin wave, mode 3 to the isospin wave and mode 4 to the coupled spin-isospin wave. To represent these circumstances clearly, the excitation energies of collective modes and the maximum energies of free pair excitation for a given value of K are plotted in Fig. 1. As is evident from Fig. 1, the spin wave is the lowest exciting wave, the coupled spin-isospin wave is the next, and the isospin wave is the highest. There is another interesting feature that according to the value of K the number of the stable collective modes varies from one to three.

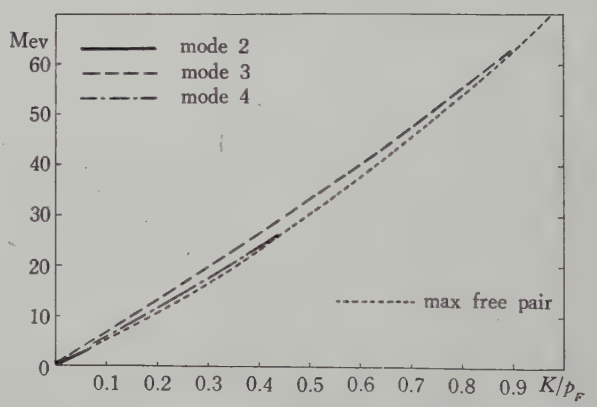

Fig. 1. Collective excitation energies and maximum energies of free pair excitation

The values of ϵ_i for a large nucleus can be estimated, by taking $K=\hbar\pi/R$ and $R\sim 6\hbar/\mu c$ and using $\epsilon_i=u_iK$, and we have

$$\epsilon_2=17,\ \epsilon_3=21,\ \epsilon_4=19\ \text{(Mev)}.$$

These figures show the order of magnitude only qualitatively.

As V_1 is negative, mode 1 is always unstable. This means that our approximation is not good for mode 1, i.e. in this case the finite nature of nucleus becomes important or the terms in H_I other than $H_I^{(1)}$ and $H_I^{(2)}$ are not negligible. Therefore we should take some other approximation method. However, mode 1

would be connected with low excitation close to the ground state and even if we treat it properly, other three stable modes which have relatively high excitation energies would be influenced little.

These mechanisms of nuclear excitations would have some connection with the several resonance peaks which have been observed in the experiments on the inelastic scattering of high energy nucleons,[7] photonuclear reactions[8),9)] and inelastic scattering of electrons.[8] It will be very interesting to investigate the excitation mechanisms treated in this paper in detail for actual nuclei using shell model wave functions and compare with the experimental results on the aforementioned nuclear reactions. These problems will be treated in a forthcoming paper.

§ 4. Coupling of collective modes with particle motion

First we determine the wave function of the collective mode in the preceding section. Multiplying $\boldsymbol{\Gamma}_i\tau_2\sigma_2$ on the first formula of (13) from the left and taking spur, we have

$$f_i(\boldsymbol{q}_2,\ \boldsymbol{q}_1)=-\frac{1}{4}(\Psi_0^*\, a^*(\boldsymbol{q}_1)\boldsymbol{\Gamma}_i\tau_2\sigma_2 b^*(\boldsymbol{q}_2)\Psi_i). \tag{24}$$

Similarly, from the second formula of (13)

$$g_i(-\boldsymbol{q}_2,\ -\boldsymbol{q}_1)=-\frac{1}{4}(\Psi_0^*\, b(-\boldsymbol{q}_1)\tau_2\sigma_2\boldsymbol{\Gamma}_i a(-\boldsymbol{q}_2)\Psi_i), \tag{25}$$

hence in our approximation Ψ_i can be written in the following form,

$$\begin{aligned}&\Psi_i(\boldsymbol{K})=A_i^*(\boldsymbol{K})\Psi_0,\\ &A_i^*(\boldsymbol{K})=\int d\boldsymbol{p}\Big[\alpha_p\, a^*\Big(\boldsymbol{p}+\frac{\boldsymbol{K}}{2}\Big)\boldsymbol{\Gamma}_i\tau_2\sigma_2 b^*\Big(-\boldsymbol{p}+\frac{\boldsymbol{K}}{2}\Big)\\ &\qquad\qquad+\beta_p b\Big(-\boldsymbol{p}-\frac{\boldsymbol{K}}{2}\Big)\tau_2\sigma_2\boldsymbol{\Gamma}_i a\Big(\boldsymbol{p}-\frac{\boldsymbol{K}}{2}\Big)\Big].\end{aligned} \tag{26}$$

α_p and β_p must be chosen so that $\Psi_i(\boldsymbol{K})$ becomes a properly normalized eigenfunction of the Hamiltonian, that is,

$$\begin{aligned}&(H_0'+H_1')\Psi_i(\boldsymbol{K})=(E_0+\epsilon_i)\Psi_i(\boldsymbol{K}),\\ &(H_0'+H_1')\Psi_0=E_0\Psi_0,\end{aligned}$$

where

$$H_1'=H_1^{(1)}+H_1^{(2)}.$$

This equation further can be written as an operator equation

$$[H_0'+H_1',\ A_i^*(\boldsymbol{K})]=\epsilon_i A_i^*(\boldsymbol{K}). \tag{27}$$

Taking the Hermite conjugate of the above equation,

$$[H_0'+H_I',\ A_i(\boldsymbol{K})]=-\epsilon_i A_i(\boldsymbol{K}). \tag{28}$$

From this equation we have

$$(H_0'+H_I')A_i(\boldsymbol{K})\Psi_0=(E_0-\epsilon_i)A_i(\boldsymbol{K})\Psi_0.$$

Since Ψ_0 is the ground state of the lowest energy, this equation can be satisfied only if

$$A_i(\boldsymbol{K})\Psi_0=0.$$

Using this condition we can fix the normalization as follows,

$$\begin{aligned}(\Psi_i^*(\boldsymbol{K})\Psi_j(\boldsymbol{K}'))&=(\Psi_0^* A_i(\boldsymbol{K})A_j^*(\boldsymbol{K}')\Psi_0)=(\Psi_0^*[A_i(\boldsymbol{K}),\ A_j^*(\boldsymbol{K}')]\Psi_0)\\&=4\delta_{ij}\delta(\boldsymbol{K}-\boldsymbol{K}')\int d\boldsymbol{p}(|\alpha_p|^2-|\beta_p|^2),\end{aligned} \tag{29}$$

so that

$$4\int d\boldsymbol{p}(|\alpha_p|^2-|\beta_p|^2)=1. \tag{30}$$

The coefficients α_p and β_p can be determined by taking account of commutation relations (11) and (12), Eq. (27) and the eigenvalue equation. Making use of this result, $\Psi_i(\boldsymbol{K})$ can be written as

$$\begin{aligned}\Psi_i(\boldsymbol{K})=N_{i\boldsymbol{K}}\int d\boldsymbol{p}\Bigg(&\frac{a^*(p+\boldsymbol{K}/2)\boldsymbol{\Gamma}_i\tau_2\sigma_2 b^*(-\boldsymbol{p}+\boldsymbol{K}/2)}{\epsilon_i-E_{p+\boldsymbol{K}/2}+E_{-p+\boldsymbol{K}/2}}\\&+\frac{b(-\boldsymbol{p}-\boldsymbol{K}/2)\tau_2\sigma_2\boldsymbol{\Gamma}_i a(\boldsymbol{p}-\boldsymbol{K}/2)}{\epsilon_i-E_{p+\boldsymbol{K}/2}+E_{-p+\boldsymbol{K}/2}}\Bigg)\Psi_0,\end{aligned} \tag{31}$$

where $N_{i\boldsymbol{K}}$ is given on account of (30) by the solution of the following equation,

$$\begin{aligned}1=4N_{i\boldsymbol{K}}^2\Bigg(&\int\limits_{\substack{|p+\boldsymbol{K}/2|>p_F\\|-p+\boldsymbol{K}/2|<p_F}} d\boldsymbol{p}\frac{1}{(\epsilon_i-E_{p+\boldsymbol{K}/2}+E_{-p+\boldsymbol{K}/2})^2}\\&-\int\limits_{\substack{|p+\boldsymbol{K}/2|<p_F\\|-p+\boldsymbol{K}/2|>p_F}} d\boldsymbol{p}\frac{1}{(\epsilon_i-E_{p+\boldsymbol{K}/2}+E_{-p+\boldsymbol{K}/2})^2}\Bigg).\end{aligned} \tag{32}$$

Carrying out the integration, we have

$$\begin{aligned}1=\frac{8\pi M^3}{K^3}N_{i\boldsymbol{K}}^2\Bigg[&\left(\epsilon_i-\frac{K^2}{2M}\right)\log\frac{\epsilon_i+(Kp_F/M)-(K^2/2M)}{\epsilon_i-(Kp_F/M)-(K^2/2M)}\\&+\left(\epsilon_i+\frac{K^2}{2M}\right)\log\frac{\epsilon_i-(Kp_F/M)+(K^2/2M)}{\epsilon_i+(Kp_F/M)+(K^2/2M)}\Bigg].\end{aligned} \tag{33}$$

Expanding with respect to $K^2/2M$ as before,

$$1=\frac{8\pi M^2}{K}N_{i\boldsymbol{K}}^2\left(\log\frac{\epsilon_i-(Kp_F/M)}{\epsilon_i+(Kp_F/M)}+\frac{2\epsilon_i(Kp_F/M)}{\epsilon_i^2-(Kp_F/M)^2}\right). \tag{34}$$

If we introduce the approximate solution $\epsilon_i = u_i K$ into the above equation, we get

$$\frac{1}{N_{iK}^2} = \frac{8\pi M^2}{K}\left(\log\frac{u_i-(p_F/M)}{u_i+(p_F/M)} + \frac{2u_i(p_F/M)}{u_i^2-(p_F/M)^2}\right) \equiv \frac{M^2}{K}\frac{c_i^2}{u_i}, \quad (34')$$

where the utmost right is the definition of c_i. Thus in our approximation N_{iK} is determined from (34′) as

$$N_{iK} = \frac{\sqrt{\epsilon_i}}{Mc_i}. \quad (35)$$

From the numerical values of u_i obtained in the preceding section, c_i's are numerically

$$c_2 = 14.1, \quad c_3 = 3.76, \quad c_4 = 5.51. \quad (36)$$

Combining these results, we find

$$A_i^*(\boldsymbol{K}) = \frac{\sqrt{\epsilon_i}}{Mc_i}\int d\boldsymbol{p}\left(\frac{a^*(\boldsymbol{p}+\boldsymbol{K}/2)\boldsymbol{\Gamma}_i\tau_2\sigma_2 b^*(-\boldsymbol{p}+\boldsymbol{K}/2)}{\epsilon_i - E_{p+K/2}+E_{-p+K/2}}\right.$$
$$\left. + \frac{b(-\boldsymbol{p}-\boldsymbol{K}/2)\tau_2\sigma_2\boldsymbol{\Gamma}_i a(\boldsymbol{p}-\boldsymbol{K}/2)}{\epsilon_i - E_{p+K/2}+E_{-p+K/2}}\right), \quad (37)$$
$$A_i(\boldsymbol{K}) = \frac{\sqrt{\epsilon_i}}{Mc_i}\int d\boldsymbol{p}\left(\frac{a^*(\boldsymbol{p}-\boldsymbol{K}/2)\boldsymbol{\Gamma}_i\tau_2\sigma_2 b^*(-\boldsymbol{p}-\boldsymbol{K}/2)}{\epsilon_i - E_{p+K/2}+E_{-p+K/2}}\right.$$
$$\left. + \frac{b(-\boldsymbol{p}+\boldsymbol{K}/2)\tau_2\sigma_2\boldsymbol{\Gamma}_i a(\boldsymbol{p}+\boldsymbol{K}/2)}{\epsilon_i - E_{p+K/2}+E_{-p+K/2}}\right).$$

Next we calculate the matrix element for the creation of a quantum of the collective excitation wave arising from its coupling with particle motion. Denoting two one particle states by Ψ_1 and Ψ_1', this is evaluated in one pair approximation as follows,

$$\langle \Psi_1'^* A_i(\boldsymbol{K}) H_I \Psi_1\rangle \simeq \langle \Psi_1'^*[A_i(\boldsymbol{K}),\ H_I]\Psi_1\rangle \simeq \langle \Psi_1'^*[A_i(\boldsymbol{K}),\ H_I^{(3)}]\Psi_1\rangle, \quad (38)$$

where $H_I^{(3)}$ is given by

$$H_I^{(3)} = \frac{2}{(2\pi)^3}\sum_i \int d\boldsymbol{p}_1' d\boldsymbol{p}_2' d\boldsymbol{p}_1 d\boldsymbol{p}_2 d\boldsymbol{k} V_i(k)\delta(\boldsymbol{p}_1'-\boldsymbol{p}_1-\boldsymbol{k})\delta(\boldsymbol{p}_2'-\boldsymbol{p}_2+\boldsymbol{k})$$
$$\times (a^*(\boldsymbol{p}_2')\boldsymbol{\Gamma}_i\tau_2\sigma_2 b^*(-\boldsymbol{p}_2)+b(-\boldsymbol{p}_2')\tau_2\sigma_2\boldsymbol{\Gamma}_i a(\boldsymbol{p}_2))(a^*(\boldsymbol{p}_1')\boldsymbol{\Gamma}_i a(\boldsymbol{p}_1)). \quad (39)$$

Similarly, the matrix element for the annihilation is estimated as

$$\langle \Psi_1'^* H_I A_i^* \Psi_1\rangle \simeq -\langle \Psi_1'^*[A_i^*(\boldsymbol{K}),\ H_I]\Psi_1\rangle \simeq -\langle \Psi_1'^*[A_i^*(\boldsymbol{K}),\ H_I^{(3)}]\Psi_1\rangle. \quad (40)$$

Using (37) for $A_i^*(\boldsymbol{K})$, the above commutator becomes

$$[A_i^*(\boldsymbol{K}),\ H_I^{(3)}] = \frac{2}{(2\pi)^3}\frac{\sqrt{\epsilon_i}}{Mc_i}\int d\boldsymbol{p}_1 d\boldsymbol{p}_2\delta(\boldsymbol{p}_1-\boldsymbol{p}_2-\boldsymbol{K})(a^*(\boldsymbol{p}_1)\boldsymbol{\Gamma}_i a(\boldsymbol{p}_2))$$

$$\times\Big(\int_{\substack{|p+K/2|>p_F\\|-p+K/2|<p_F}} dp-\int_{\substack{|p+K/2|<p_F\\|-p+K/2|>p_F}} dp\Big)\frac{1}{\epsilon_i-E_{p+K/2}+E_{-p+K/2}}$$

$$\times\Big(V_i'\Big(\Big|p_2+\frac{K}{2}-p\Big|\Big)-4V_i(K)\Big). \tag{41}$$

Similarly,

$$[A_i(K),\ H_I^{(3)}]=\frac{2}{(2\pi)^3}\frac{\sqrt{\epsilon_i}}{Mc_i}\int dp_1dp_2\delta(p_1-p_2+K)(a^*(p_1)\Gamma_i a(p_2))$$

$$\times\Big(\int_{\substack{|p+K/2|<p_F\\|-p+K/2|>p_F}} dp-\int_{\substack{|p+K/2|>p_F\\|-p+K/2|<p_F}} dp\Big)\frac{1}{\epsilon_i-E_{p+K/2}+E_{-p+K/2}}$$

$$\times\Big(V_i'\Big(\Big|p_2-\frac{K}{2}-p\Big|\Big)-4V_i(K)\Big). \tag{42}$$

If we take the interaction (16), making use of (18) and (21) we have

$$[A_i^*(K),\ H_I^{(3)}]=-\frac{\sqrt{\epsilon_i}}{Mc_i}\int dp_1dp_2\delta(p_1-p_2-K)(a^*(p_1)\Gamma_i a(p_2)) \tag{41'}$$

and

$$[A_i(K),\ H_I^{(3)}]=\frac{\sqrt{\epsilon_i}}{Mc_i}\int dp_1dp_2\delta(p_1-p_2+K)(a^*(p_1)\Gamma_i a(p_2)). \tag{42'}$$

In our approximation, in which

$$[A_i(K),\ A_j^*(K')]=\delta_{ij}\delta(K-K'),$$

the effective interaction Hamiltonian between the collective exciting wave and particle becomes

$$H_{\text{eff}}\simeq\sum_i\frac{1}{Mc_i}\int dp_1dp_2dK\sqrt{\epsilon_i}(\psi^*(p_1)\Gamma_i\psi(p_2))$$

$$\times(A_i(K)\delta(p_1-p_2-K)+A_i^*(K)\delta(p_1-p_2+K)). \tag{43}$$

As can be seen from this expression, the coupling strength is larger for the isospin wave than for the coupled spin-isospin wave and spin wave. Therefore the isospin wave has the largest width of excitation energy.

§ 5. Concluding remarks

In the preceding sections we have studied the collective excitations of nuclear matter which have their origin in the short-range interactions between nucleons in nucleus and found that three stable modes of collective excitations are possible and that the higher exciting wave has the stronger coupling with particle motion and therefore the larger width of excitation energy.

Here we speculate the connection between these exciting modes and the experimentally found levels. The isospin wave will correspond to the giant resonance states in photonuclear reactions as Ferentz, Gell-Mann and Pines[6] discussed first. As isospin wave has the largest coupling with particle motion, this mode will be observed most easily. Other two modes have weak coupling compared with isospin wave. Particularly, spin wave has the smallest coupling and only small stable region as seen from Fig. 1. Therefore in order to find these modes experimentally, accurate measurements will be required. Mode 1 becomes unstable in our approximation; however, if we treat it properly, mode 1 would be connected with low excitations close to the ground state.

An example which shows the existence of the coupled spin-isospin wave and the spin wave might be found in C^{12}. In the experiment on the inelastic scattering of protons on C^{12}, Tyrén and Maris[10] found several strongly excited levels. Among these the upper three levels have excitation energies 20 Mev, 15.0 Mev and 12.6 Mev respectively. The 20 Mev level corresponds to the giant resonance states. The 15 Mev level has spin and parity $1+$ and isospin 1. This level was also observed in the experiment of the inelastic scattering of electrons.[8] Therefore this level will correspond to the coupled spin-isospin wave. According to the analysis by Sakamoto[11] on the basis of impulse approximation, the 12.6 Mev level has spin and parity $1+$ and isospin 0. This level would probably correspond to the spin wave considered in this paper. Of course these are only speculations. It will be very interesting to investigate the excitation mechanisms treated in this paper in detail for actual nuclei using shell model wave functions.

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References

1) F. J. Dyson, Phys. Rev. **91** (1953), 1543.
2) A. E. Glassgold, W. Heckrotte and K. M. Watson, Ann. Phys. **6** (1959), 1.
3) K. Sawada, Phys. Rev. **106** (1957), 372.
K. Sawada, K. A. Brueckner, N. Fukuda and R. Brout, Phys. Rev. **108** (1957), 507.
4) J. Iwadare and S. Hatano, Prog. Theor. Phys. **15** (1956), 185.
5) M. Nomoto, Prog. Theor. Phys. **18** (1957), 483.
6) M. Ferentz, M. Gell-Mann and D. Pines, Phys. Rev. **92** (1953), 836.
7) H. Tyrén and Th. A. J. Maris, Nuclear Phys. **7** (1958), 24. Their previous works are cited in this paper.
8) M. V. Mihailović, G. Pregl, G. Kernel and M. Kregar, Phys. Rev. **114** (1959), 1621.
W. C. Barber and F. E. Gudden, Phys. Rev. Letters **3** (1959), 219.
9) G. E. Brown and M. Bolsterli, Phys. Rev. Letters **3** (1959), 472.
10) H. Tyrén and Th. A. J. Maris, Nuclear Phys. **4** (1957), 637.
11) Y. Sakamoto, Prog. Theor. Phys. **23** (1960), 1205.

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Quadrupole Interaction in Crystals

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The quadrupole array for the lowest energy of quadrupole interaction in crystals is obtained by generalizing Luttinger and Tisza's theory of dipole interaction. The theory is presented for the two kinds of problems: the simplest case of quadrupole pertaining to a doubly degenerate orbital e_g and the other general cases both for the cubic crystals. In the former case, the quadrupole interaction is written in terms of fictitious spins, whence we can get the solution of lowest energy classically, serving to obtain some informations of the orbital ordering in compounds with Mn^{3+}, Cu^{2+} and Cr^{2+}. In the latter cases, the five components of the quadrupole are totally effective, leading to a rather complicated problem. The classical solution is, however, easily obtained, though somewhat complicated. The quadrupole array of lowest energy in face-centered cubic lattice proves to be realized in molecular crystals N_2, N_2O, CO_2 and CO, which is accompanied with large quadrupole but without or with small dipole moment.

§ 1. Introduction

In this paper, we shall look for the quadrupole array in a cubic crystal which brings about the lowest energy of quadrupole interactions. There are some interesting applications of the results obtained.

Firstly, the crystal structures and the configuration of molecular orientation for many molecular crystals have been found by the *X*-ray analysis. It is known that molecular crystals composed of linear molecules N_2, N_2O, CO_2 and CO show the same type of ordering of molecular orientation at low temperatures, being crystallized in F.C.C. lattice.** Between two molecules, there are three or more kinds of anisotropic interactions in general, that is, the valence, van der Waals and multipole interactions. We shall consider here the non-polar molecule with considerable magnitude of quadrupole moment. Therefore the quadrupole-quadrupole interaction will be much effective. It is known that between two hydrogen molecules the directional dependence of both the van der Waals and valence forces are considerably smaller than that of the quadrupole interactions, which proves to be predominantly effective in causing the orientational ordering of ortho-hydrogen.[1] Due to this situation, we are inclined to think that the orientational ordering of molecules in the crystals mentioned will mainly come from the molecular quadru-

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** Hereafter S.C., B.C.C. and F.C.C. lattices mean simple cubic, body-centered cubic and face-centered cubic lattices respectively.

pole interaction. In fact, we shall see later that the molecules N_2, N_2O, CO_2 and CO have such magnitude of quadrupole moment that the orientational ordering of molecule in crystal is explained satisfactorily from its quadrupole interaction. Moreover, our predicted array proves to be just in the actual situation.

Secondly, there have appeared a theory of magnetic anisotropy related to the cobaltous and ferrous oxides, where Co^{2+} and Fe^{2+} in cubic field have a large orbital momentum unquenched, hence a large multipole interaction results.[2] The study of preferential array of quadrupole may give some informations about such problem. Now the magnetic property associated with the *d*-electron states in the metal ions of perovskite type compound $LaMnO_3$ and deformed perovskite type MnF_3 has been one of the recent interesting problems.[3] Their magnetic structure is said to be the "A"-type structure in which spins in a *c*-plane are coupled ferromagnetically and spins of two adjacent planes are coupled antiferromagnetically. The magnetic ordering of these compounds should be explained by assuming appropriate ordering of e_g-orbitals. It is well-known that a five-fold degenerate *d*-level in cubic crystalline field splits into a doubly degenerate e_g-level and a triply degenerate t_{2g}-level. Since the level splitting is of the order of magnitude of $10^4\,cm^{-1}$, we have only to take up the interaction within the lowest manifold. In the case of Mn^{3+} ion, e_g-level is lower, because the electron configuration of Mn^{3+}, $(3d)^4$, is equivalent to the ion with a *d*-hole in the Hund lowest state, hence the level order is reversed. In the above compounds, the orbital ordering will be caused at first by the interaction between the displacement of anion and e_g-electron but the effect of quadrupole interaction of e_g-electron which will be treated in Sections 3 and 4 is not so small. Therefore it seems to be worthwhile to find the ordering of e_g-orbitals with the lowest energy.

In the present paper, we shall look for the quadrupole array with the lowest energy in cubic crystal, following Luttinger and Tisza's theory of dipole interaction.[4] They divided the lattice into $8p$ simple cubic sublattices ($p=1$ for S.C. lattice, $p=2$ for B.C.C. lattice and $p=4$ for F.C.C. lattice), where in each sublattice all dipoles are parallel, hence any state of dipole array should be represented by a point in $24p$ dimensional vector space. Under the condition that a vector representing any dipole array should be confined to the constant dipole surface $\sum_{i=1}^{8p}\sum_{\alpha=x}^{z}\mu_{i\alpha}^2=\text{const}$, they reduced the problem of obtaining the dipole array with minimum energy to an eigen-value problem. That is, they looked for the array vector which is parallel to the dipolar field vector generated from the original array, and obtained the lowest array consistent with the condition $\sum_{\alpha=x}^{z}\mu_{i\alpha}^2=\mu^2$, $i=1, \cdots, 8$. Here μ_α denotes the α-th component of dipole whose magnitude is equal to μ.

The quadrupole interaction takes a form (7) bilinear with respect to the five components of quadrupoles belonging to two interacting atoms (or molecules). A quadrupole localized at lattice point is considered as a vector whose components are given by

$$z_1=\frac{\sqrt{3}}{2}(\xi^2-\eta^2),\quad z_2=\frac{1}{2}(2\zeta^2-\xi^2-\eta^2),$$
$$z_3=\sqrt{3}\,\eta\zeta,\quad z_4=\sqrt{3}\,\zeta\xi,\quad z_5=\sqrt{3}\,\xi\eta. \tag{1}$$

Here ξ, η and ζ are Cartesian coordinates whose origin is at each lattice point. If we assume the vector (ξ, η, ζ) to be confined to the surface of the sphere with unit radius, the condition

$$\xi^2+\eta^2+\zeta^2=1 \tag{2}$$

leads to the condition

$$\sum_{i=1}^{5} z_i^2=1. \tag{3}$$

However, not all of z_i's are independent, because the components of a real quadrupole (1) are subjected only to two independent variables. Therefore a five-dimensional vector $\boldsymbol{z}(z_1, \cdots, z_5)$ subjected to (3) but free from (2) does not always correspond to a real quadrupole. But, in order to seek the quadrupole array of the lowest energy, we shall first solve our problem within a rather extended manifold of the fictitious quadrupole just mentioned. The procedure to be followed is in complete parallel with Luttinger and Tisza's method.

If the eigen-vector belonging to the lowest eigen-value represents a real quadrupole or if a linear combination of vectors which are degenerate to the lowest value represents a real quadrupole, that is all right. As we shall see later, however, in S.C. and B.C.C. lattices this is not the case and only in the F.C.C. lattice in which the actual problem is included we shall see such a situation to be realized.

There is a case in which the problem is greatly simplified. It appears in the quadrupole interaction between d-electrons in the e_g-state. Actually, if we represent the five components of a quadrupole by e_g-orbital:

$$\psi_1=\frac{\sqrt{3}}{2}(\xi^2-\eta^2),\quad \psi_2=\frac{1}{2}(2\zeta^2-\xi^2-\eta^2), \tag{4}$$

we have

$$z_1=-\frac{2}{7}\overline{r^2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad z_2=-\frac{2}{7}\overline{r^2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{5}$$

but the matrix components of z_3, z_4 and z_5 vanish identically, where $\overline{r^2}$ denotes averaged square of radius of electron coordinate. Using the Pauli spin matrix we may write them as

$$z_1=-\frac{2}{7}\overline{r^2}\sigma_x,\quad z_2=-\frac{2}{7}\overline{r^2}\sigma_z. \tag{6}$$

In this formalism, the procedure is very similar to that for the dipole array and

the exact classical solution is straightforwardly obtained for the three lattices considered.

§ 2. Quadrupole interaction

The quadrupole interaction between two quadrupoles is written in a usual form:

$$V_{q-q}=\frac{3e^2}{4R^5}r_1^2 r_2^2 f(\theta_1, \varphi_1; \theta_2, \varphi_2), \tag{7}$$

$$f(\theta_1, \varphi_1; \theta_2, \varphi_2)=8P_2(\cos\theta_1)P_2(\cos\theta_2) \\ -16\sin\theta_1\cos\theta_1\sin\theta_2\cos\theta_2\cos(\varphi_1-\varphi_2)+\sin^2\theta_1\sin^2\theta_2\cos 2(\varphi_1-\varphi_2). \tag{8}$$

Here $(r_i, \theta_i, \varphi_i)$ denote the polar coordinates of the position vector of electron whose origin is at the center of gravity of charge distribution localized around the i-th quadrupole, and the polar axis is taken to be a line connecting two centers of quadrupoles whose distance is equal to $\boldsymbol{R}$ (see Fig. 1).

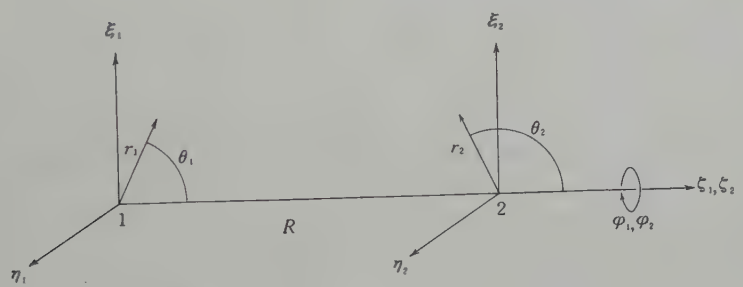


Fig. 1. The definitions of $(r_i, \theta_i, \varphi_i)$ and $\boldsymbol{R}$.

If we take our coordinates system so as to coincide with three principal axes of cubic crystal, the quadrupole interaction (7) is transformed into

$$V_{q-q}=\frac{3}{4}e^2\sum_{\mu,\nu=1}^{5} z_\mu(1)\,V_{\mu\nu}(\boldsymbol{R})\,z_\nu(2) \tag{9}$$

where $z_\mu(i)$ denotes the μ-th component of the i-th quadrupole and $V_{\mu\nu}(\boldsymbol{R})$'s are listed in Table I.

The next step is to construct a matrix of V_{q-q} with respect to the lowest degenerate states. The simplest case is concerned with doubly degenerate e_g-orbital (Γ_3 manifold in Bethe's notation), where (6) may be substituted for z.

In general cases, the matrix representation can be constructed with the help of operator equivalence method,[5] that is,

$$z_1=c_J\frac{\sqrt{3}}{2}(J_x^2-J_y^2), \quad z_2=c_J\frac{1}{2}\cdot(2J_z^2-J_x^2-J_y^2),\cdots, \tag{10}$$

Table I. Expressions for $V_{\mu\nu}(\boldsymbol{R})\times R^5$. (θ, ϕ) denote the polar angles of the direction $\boldsymbol{R}$.

μ	ν	$P_4^0(\cos\theta)$	$P_4^2(\cos\theta)$		$P_4^4(\cos\theta)$	
			$\cos 2\phi$	$\sin 2\phi$	$\cos 4\phi$	$\sin 4\phi$
1	1	2/3			1/36	
1	2		$1/3\sqrt{3}$			
1	5					1/36
2	2	4				
2	5			$1/3\sqrt{3}$		
3	3	−8/3	−2/9			
3	4			−2/9		
4	4	−8/3	2/9			
5	5	2/3				−1/36

μ	ν	$P_4^1(\cos\theta)$		$P_4^3(\cos\theta)$	
		$\cos\phi$	$\sin\phi$	$\cos 3\phi$	$\sin 3\phi$
1	3		1/3		1/18
1	4	−1/3		1/18	
2	3		$2/\sqrt{3}$		
2	4	$2/\sqrt{3}$			
3	5	−1/3		−1/18	
4	5		1/3		1/18

where c_J is dependent only on quantum number J of degenerate manifold. In these cases, the classical treatment of quadrupole interaction is to consider the energy (9) with quadrupole normalized in the sense of (3).

In such a way, the energy of the quadrupole interaction in crystal is written as

$$\mathrm{W}=\sum_{j>k}\sum_{\mu,\nu} z_\mu{}^j F_{\mu\nu}(\boldsymbol{R}_j-\boldsymbol{R}_k) z_\nu{}^k \qquad (11)$$

in general cases, where $F_{\mu\nu}(\boldsymbol{R}_j-\boldsymbol{R}_k)$ is equal to $V_{\mu\nu}(\boldsymbol{R})\times(3e^2\mu_Q^2/4)$ and z_μ is now the component of normalized quadrupole. In the following, we shall measure the quadrupole energy W in the units of $(3e^2\mu_Q^2/4a^5)$ where a denotes the lattice constant. $F_{\mu\nu}(\boldsymbol{R}_j-\boldsymbol{R}_k)$ are symmetric with respect to suffices μ and ν.

The physical meaning of $F_{\mu\nu}(\boldsymbol{R}_j-\boldsymbol{R}_k)$ is given by stating that the μ-th component of quadrupole field at the j-th lattice point is equal to $F_{\mu\nu}(\boldsymbol{R}_j-\boldsymbol{R}_k)$ if we put a unit fictitious quadrupole $z_\nu=1$ at the k-th lattice point.

§ 3. The e_g-orbital array in the S.C. lattice

Let us first study the array of fictitious spin associated with e_g-orbital in the S.C. lattice. The quadrupole energy (9) is then written as

$$W=\sum_{j>k}\boldsymbol{\sigma}_j\cdot\boldsymbol{F}(\boldsymbol{R}_j-\boldsymbol{R}_k)\cdot\boldsymbol{\sigma}_k \tag{12}$$

by substitution of (6) in (9), where the components of a dyadic $\boldsymbol{F}(\boldsymbol{R})$ are given by

$$\begin{aligned} &F_{xx}(\boldsymbol{R})=F_{11}(\boldsymbol{R}),\quad F_{zz}(\boldsymbol{R})=F_{22}(\boldsymbol{R}),\\ &F_{xz}(\boldsymbol{R})=F_{zx}(\boldsymbol{R})=F_{12}(\boldsymbol{R}), \end{aligned} \tag{13}$$

and the other components vanish identically.

We shall classically look for the lowest array of spin vector which is doubly periodic. Then we shall divide S.C. lattice into eight S.C. sublattices generated from eight lattice points:

$$\begin{aligned} &(0\ 0\ 0),\quad (1\ 0\ 0),\quad (0\ 1\ 0),\quad (0\ 0\ 1),\\ &(0\ 1\ 1),\quad (1\ 0\ 1),\quad (1\ 1\ 0)\ \text{and}\ (1\ 1\ 1), \end{aligned} \tag{14}$$

in which all spins are parallel within each sublattice. Thus we have only to assign the spin orientation on the eight lattice points mentioned above in order to designate spin array in the crystal. If we assume a tentative array, it causes a field at, say, the k-th lattice point. Further, if the spin at this lattice point is parallel to the field generated there and the situation is the same in the whole lattice, the system will have a certain amount of efficient gain or loss of quadrupole energy. Following Luttinger and Tisza, we shall refer below to such array as characteristic array. The spin array with maximum gain or loss of quadrupole energy should naturally be included in the whole assembly of characteristic arrays.

Let us then find characteristic array $\boldsymbol{\Sigma}$, which is designated by 2×8 components inclusive of eight spins because the y-component of $\boldsymbol{\sigma}$ disappears throughout eight sublattices. The characteristic array is thus obtained from an eigen-value problem:

$$\boldsymbol{F}\cdot\boldsymbol{\Sigma}=f\boldsymbol{\Sigma}. \tag{15}$$

Here the field operator $\boldsymbol{F}$ is a square matrix of sixteen dimension, whose component $F_{\mu\nu}^{pq}$, $p, q=1, 2, \cdots, 8$; $\mu, \nu=1, 2$ gives the μ-th component of the quadrupole field acting on a lattice point belonging to the p-th sublattice when we put spin along the ν-th direction on the q-th sublattice. Since the magnitude of $\boldsymbol{\Sigma}$ is given by

$$(\boldsymbol{\Sigma},\ \boldsymbol{\Sigma})=\sum_{i=1}^{8}(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_i)=8, \tag{16}$$

the eigen-value of (15), f, just corresponds to twice as much as the quadrupole energy per atom, $1/16\cdot(\boldsymbol{\Sigma}, \boldsymbol{F}\cdot\boldsymbol{\Sigma})$. Here the classical magnitude of fictitious spin is assumed to be equal to unity.

To obtain characteristic vector $\boldsymbol{\Sigma}$, it is necessary to see the structure of field operator. It is a cyclic matrix with respect to p and q, whose component can be obtained with the help of Table I. Components $F_{\mu\nu}^{pp}$'s, $\mu, \nu=1, 2$, are simply written as follows:

$$F_{11}^{pp} \equiv G_{11}(000) = \frac{1}{192} \cdot \sum_{\substack{l_1 l_2 l_3 \\ \neq -\infty}}^{\infty} [19(l_1^4+l_2^4)+4l_3^4-102l_1^2l_2^2-12l_3^2(l_1^2+l_2^2)]/l^9,$$

$$F_{12}^{pp} \equiv G_{12}(000) = 5/64\sqrt{3} \cdot \sum_{\substack{l_1 l_2 l_3 \\ \neq -\infty}}^{\infty} (l_1^2-l_2^2)(6l_3^2-l_1^2-l_2^2)/l^9, \tag{17}$$

$$F_{22}^{pp} \equiv G_{22}(000) = 1/64 \cdot \sum_{\substack{l_1 l_2 l_3 \\ \neq -\infty}}^{\infty} [3(l_1^4+l_2^4)+8l_3^4-24l_3^2(l_1^2+l_2^2)+6l_1^2l_2^2]/l^9,$$

where $l^2=l_1^2+l_2^2+l_3^2$. The other components $F_{\mu\nu}^{pq}$, $p \neq q$, are also obtained from replacing l_1, l_2 and l_3 in (17) by $l_1-(u_1^p-u_1^q)/2$, $l_2-(u_2^p-u_2^q)/2$ and $l_3-(u_3^p-u_3^q)/2$ respectively, in which (u_1^p, u_2^p, u_3^p) denotes the lattice point generating the p-th S.C. sublattice as given by (14). According to this definition, we may put

$$F_{\mu\nu}^{pq} = G_{\mu\nu}\left(\frac{u_1^p-u_1^q}{2}, \frac{u_2^p-u_2^q}{2}, \frac{u_3^p-u_3^q}{2}\right), \tag{18}$$

whose numerical values are tabulated in Table II (see the Appendix).

Table II. The numerical values of $G_{\mu\nu}(u_1 u_2 u_3)$.

u_1	u_2	u_3	$G_{11}(\boldsymbol{u})$	$G_{22}(\boldsymbol{u})$	$G_{12}(\boldsymbol{u})$
0	0	0	0.39	0.39	0
1/2	0	0	6.32	3.12	−2.77
0	1/2	0	6.32	3.12	2.77
0	0	1/2	1.52	7.91	0
0	1/2	1/2	0.29	−1.16	−1.26
1/2	0	1/2	0.29	−1.16	1.26
1/2	1/2	0	−1.88	1.02	0
1/2	1/2	1/2	−0.78	−0.78	0

The solution of an eigen-value equation (15) is much simplified if we make use of the cyclic nature of $F_{\mu\nu}^{pq}$. Without inquiring into the mathematical details, we see that the quadrupole field has the same translational symmetry as that of the spin array, hence the spin arrays of different symmetry cannot combine with each other. Taking this situation into account, let us construct a vector $\boldsymbol{\Sigma}_\mu(\boldsymbol{\lambda})$ by

$$\boldsymbol{\Sigma}_\mu(\boldsymbol{\lambda}) = \sum_{u^p} \exp(i\boldsymbol{\lambda}\cdot\boldsymbol{u}^p)\boldsymbol{e}_\mu^p \tag{19}$$

where $\boldsymbol{e}_\mu^p$ refers to a unit vector which represents the p-th spin along the μ-th direction in which the other spins have zero component. Since there are 8×2 independent vectors $\boldsymbol{e}_\mu^p$, there are 8×2 independent $\boldsymbol{\Sigma}_\mu(\boldsymbol{\lambda})$, which we choose as

$$\boldsymbol{\lambda}=(0\,0\,0),\ (\pi\,0\,0),\ (0\,\pi\,0),\ (0\,0\,\pi),$$
$$(0\,\pi\,\pi),\ (\pi\,0\,\pi),\ (\pi\,\pi\,0) \text{ and } (\pi\,\pi\,\pi). \tag{20}$$

The basic arrays thus generated are shown in Fig. 2.

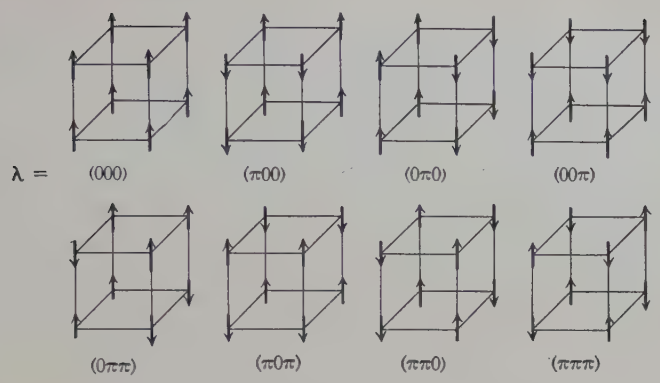


Fig. 2. Basic arrays for $\Sigma_z(\boldsymbol{\lambda})$.

In such a way, the eigen-value equation (15) becomes

$$\boldsymbol{F}(\boldsymbol{\lambda})\cdot\boldsymbol{\Sigma}(\boldsymbol{\lambda})=f\boldsymbol{\Sigma}(\boldsymbol{\lambda}). \tag{21}$$

Here $\boldsymbol{\Sigma}(\boldsymbol{\lambda})=(\boldsymbol{\Sigma}_x(\boldsymbol{\lambda}),\boldsymbol{\Sigma}_z(\boldsymbol{\lambda}))$ and the components of $\boldsymbol{F}(\boldsymbol{\lambda})$ are given by

$$F_{\mu\nu}(\boldsymbol{\lambda})=\sum_p G_{\mu\nu}\left(\frac{u_1^p}{2},\frac{u_2^p}{2},\frac{u_3^p}{2}\right)\exp(i\boldsymbol{\lambda}\cdot\boldsymbol{u}^p), \tag{22}$$

whose numerical values are obtained by the aid of Table II and given in Table III.

Due to the cubic symmetry of the lattice, it will be sufficient to search for the characteristic arrays with $\boldsymbol{\lambda}=(0\,0\,0)$, $(0\,0\,\pi)$, $(\pi\,\pi\,0)$ and $(\pi\,\pi\,\pi)$, which are just given by $\boldsymbol{\Sigma}_\mu(\boldsymbol{\lambda})$'s because $\boldsymbol{\Sigma}_x$ and $\boldsymbol{\Sigma}_z$ with those $\boldsymbol{\lambda}$'s do not combine with each other as can be seen in Table III. The lowest energy states can be realized by $\boldsymbol{\Sigma}_x(\pi\,\pi\,0)$, $\boldsymbol{\Sigma}_z(\pi\,\pi\,\pi)$ and $\boldsymbol{\Sigma}_x(\pi\,\pi\,\pi)$. The other arrays give essentially the same pattern.

Table III. The numerical values of $F_{\mu\nu}(\boldsymbol{\lambda})$.

$\boldsymbol{\lambda}(\lambda_1\,\lambda_2\,\lambda_3)$	$F_{11}(\boldsymbol{\lambda})$	$F_{22}(\boldsymbol{\lambda})$	$F_{12}(\boldsymbol{\lambda})$
0 0 0	12.45	12.45 ($\equiv f_1$)	0
0 0 π	9.81 ($\equiv f_3$)	2.83 ($\equiv f_2$)	0
π π 0	−13.97 ($\equiv f_5$)	4.62 ($\equiv f_4$)	0
π π π	−14.28	−14.28 ($\equiv f_6$)	0

In order to derive the orbital array of e_g from the array of spin orientation, we may use the well-known spinor transformation. When the fictitious spin points the direction at angle θ from z-axis, the parallel and antiparallel states of spin along this direction are respectively transformed into

$$\psi_1' = \cos\frac{\theta}{2}\psi_1 + \sin\frac{\theta}{2}\psi_2,$$
$$\psi_2' = -\sin\frac{\theta}{2}\psi_1 + \cos\frac{\theta}{2}\psi_2, \tag{23}$$

in which ψ_1 and ψ_2 are doubly degenerate orbitals of e_g as given by (4). We shall discuss below the orbital array for respective cases mentioned above.

(i) $\boldsymbol{\lambda} = (0\,0\,0)$

This ferromagnetic solution corresponds to the uniform orbital array in which ψ_1(or ψ_2) is put on the whole lattice. Both solutions are degenerate with a high field value $f=12.45$.

(ii) $\boldsymbol{\lambda} = (0\,0\,\pi)$

We have the spin array in which spins are parallel among lattice points on the xy-plane but antiparallel between two adjacent xy-planes. An array with spin along the z-axis gives $f=2.83$, while another array with spin along the x-axis gives a higher value $f=9.87$. Both arrays have high energy.

(iii) $\boldsymbol{\lambda} = (\pi\,\pi\,0)$

The field value is given by $f=4.62$ or -13.97 according as spins are along the z- or x-axis. The spin array is such that the neighbouring spins along the z-axis are parallel but those along the x-axis and y-axis antiparallel. The orbital array with lower energy is obtained by putting

$$\psi_1' = \frac{1}{2\sqrt{2}}[(\sqrt{3}-1)\xi^2 - (\sqrt{3}+1)\eta^2 + 2\zeta^2],$$
$$\psi_2' = \frac{1}{2\sqrt{2}}[(\sqrt{3}+1)\xi^2 - (\sqrt{3}-1)\eta^2 - 2\zeta^2], \tag{24}$$

on the lattice, as can be seen in Fig. 3.

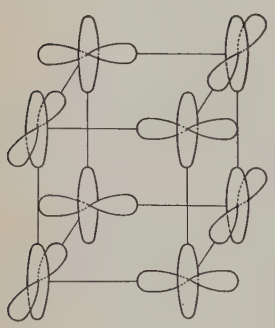
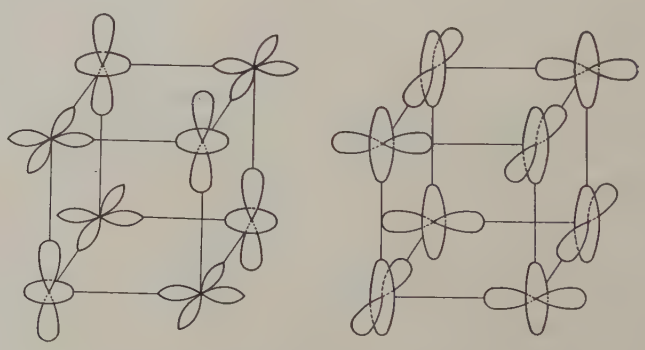


Fig. 3. Orbital array with $f=-13.97$.



(a) (b)

Fig. 4. Orbital arrays being degenerate with $f=-14.28$.

(iv) $\boldsymbol{\lambda} = (\pi\,\pi\,\pi)$

Spins along the x- or z-axis are surrounded by antiparallel spins along the

same axis. Both arrays are degenerate with $f=-14.28$, whose orbital analogues are illustrated in Fig. 4.

§ 4. The e_g-orbital array in the B.C.C. and F.C.C. lattices

(A) *The e_g-orbital array in the B.C.C. lattice*

Dividing this lattice into S.C. sublattices, for each S.C. lattice we have the characteristic array, which are respectively designated by $\boldsymbol{\Sigma}_\mu^{\,1}(\boldsymbol{\lambda})$ and $\boldsymbol{\Sigma}_\mu^{\,2}(\boldsymbol{\lambda})$, in cases of $\boldsymbol{\lambda}=(0\,0\,0)$, $(0\,0\,\pi)$, $(\pi\,\pi\,0)$ and $(\pi\,\pi\,\pi)$. In these cases, which are all worth knowing, characteristic arrays will be obtained within a subspace subtended by $\boldsymbol{\Sigma}^1(\boldsymbol{\lambda})$ and $\boldsymbol{\Sigma}^2(\boldsymbol{\lambda})$. In order to obtain these arrays, it will be necessary to know the quadrupole field on the 1st S.C. lattice coming from $\boldsymbol{\Sigma}^2(\boldsymbol{\lambda})$. The μ-th component of the field is given by

$$F_{\mu\nu}^{12}(\boldsymbol{\lambda}) \equiv \sum_{u^p} G_{\mu\nu}\left(\frac{1}{4}+\frac{u_1^p}{2}, \frac{1}{4}+\frac{u_2^p}{2}, \frac{1}{4}+\frac{u_3^p}{2}\right) \exp(i\boldsymbol{\lambda}\cdot\boldsymbol{u}^p), \qquad (25)$$

which proves to vanish identically except a particular array with $\boldsymbol{\lambda}=(0\,0\,0)$ because we have

$$G_{\mu\nu}\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)=G_{\mu\nu}\left(\frac{3}{4}, \frac{1}{4}, \frac{1}{4}\right)=\cdots, \qquad (26)$$

for $\mu, \nu=1, 2$. We can therefore see that there is no coupling between $\boldsymbol{\Sigma}^1(\boldsymbol{\lambda})$ and $\boldsymbol{\Sigma}^2(\boldsymbol{\lambda})$ in the case with $\boldsymbol{\lambda}\neq 0$, in which the quadrupole energy per atom takes the same value as given in § 3.

Let us then consider the particular case $\boldsymbol{\lambda}=(0\,0\,0)$. Using the following relations

$$\begin{aligned} &G_{12}\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)=0, \\ &G_{11}\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)=G_{22}\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)=-3.10, \end{aligned} \qquad (27)$$

we have a field matrix

$$\begin{matrix} \boldsymbol{\Sigma}_\mu^{\,1}(0) \\ \boldsymbol{\Sigma}_\mu^{\,2}(0) \end{matrix} \begin{bmatrix} f_1 & f_{11} \\ f_{11} & f_1 \end{bmatrix}, \qquad (28)$$

where $f_1=12.45$ and $f_{11}=8G_{11}(\frac{1}{4},\frac{1}{4},\frac{1}{4})=-24.80$. Eigenvalues of the matrix are easily obtained as $f_1\pm f_{11}$ equal to 37.25 or -12.35, which are higher in energy than f-values found in the array with $\boldsymbol{\lambda}=(\pi\,\pi\,0)$ or $(\pi\,\pi\,\pi)$. From this analysis we find that in the lowest array of e_g-orbital there is no correlation between corner lattice and body-centered one in B.C.C. lattice.

(B) *The e_g-orbital array in the F.C.C. lattice*

The lattice is decomposed into four S.C. lattices, characterized by the following:

	corner	yz-face	zx-face	xy-face
generating point	(0 0 0)	(0 1/2 1/2)	(1/2 0 1/2)	(1/2 1/2 0)
label	1	2	3	4
characteristic array	$\Sigma^1(\boldsymbol{\lambda})$	$\Sigma^2(\boldsymbol{\lambda})$	$\Sigma^3(\boldsymbol{\lambda})$	$\Sigma^4(\boldsymbol{\lambda})$

In parallel with (25), we construct the field matrix. For example,

$$\sum_{\boldsymbol{u}^p} G_{\mu\nu}\left(\frac{u_1^p}{2}, \frac{1}{4}+\frac{u_2^p}{2}, \frac{1}{4}+\frac{u_3^p}{2}\right)\exp(-i\boldsymbol{\lambda}\cdot\boldsymbol{u}^p) \tag{29}$$

gives the matrix component between $\boldsymbol{\Sigma}_\mu^1(\boldsymbol{\lambda})$ and $\boldsymbol{\Sigma}_\nu^2(\boldsymbol{\lambda})$, which is denoted by $F_{\mu\nu}^{12}(\boldsymbol{\lambda})$.

A symmetric property,

$$\begin{aligned} &F_{\mu\nu}^{12}(\boldsymbol{\lambda})=F_{\mu\nu}^{34}(\boldsymbol{\lambda}),\ F_{\mu\nu}^{13}(\boldsymbol{\lambda})=F_{\mu\nu}^{24}(\boldsymbol{\lambda}),\\ &F_{\mu\nu}^{14}(\boldsymbol{\lambda})=F_{\mu\nu}^{23}(\boldsymbol{\lambda}), \end{aligned} \tag{30}$$

will be easily seen. The numerical values of $G_{\mu\nu}(u_1u_2u_3)$ necessary for our problem are tabulated in Table IV.

Table IV. Numerical values of $G_{\mu\nu}(u_1u_2u_3)$ necessary for the F.C.C. problem.

$(u_1u_2u_3)$	$G_{11}(\boldsymbol{u})$	$G_{22}(\boldsymbol{u})$	$G_{12}(\boldsymbol{u})$
0 1/4 1/4	2.96	−8.93	−10.30
1/2 1/4 1/4	−0.65	−0.42	0.20
1/4 1/4 0	−14.85	8.90	0
1/4 1/4 1/2	−0.30	−0.76	0

We shall now discuss the characteristic arrays with respective $\boldsymbol{\lambda}$.

(i) $\boldsymbol{\lambda}=(0\,0\,0)$

The field matrix is written in the following form:

$$\begin{array}{c} \\ \boldsymbol{\Sigma}_x^1 \\ \boldsymbol{\Sigma}_x^2 \\ \boldsymbol{\Sigma}_x^3 \\ \boldsymbol{\Sigma}_x^4 \\ \boldsymbol{\Sigma}_z^1 \\ \boldsymbol{\Sigma}_z^2 \\ \boldsymbol{\Sigma}_z^3 \\ \boldsymbol{\Sigma}_z^4 \end{array} \begin{array}{c} \begin{array}{cccccccc} \boldsymbol{\Sigma}_x^1 & \boldsymbol{\Sigma}_x^2 & \boldsymbol{\Sigma}_x^3 & \boldsymbol{\Sigma}_x^4 & \boldsymbol{\Sigma}_z^1 & \boldsymbol{\Sigma}_z^2 & \boldsymbol{\Sigma}_z^3 & \boldsymbol{\Sigma}_z^4 \end{array} \\ \begin{pmatrix} f & a & a & b & 0 & e & -e & 0 \\ a & f & b & a & e & 0 & 0 & -e \\ a & b & f & a & -e & 0 & 0 & e \\ b & a & a & f & 0 & -e & e & 0 \\ 0 & e & -e & 0 & f & c & c & d \\ e & 0 & 0 & -e & c & f & d & c \\ -e & 0 & 0 & e & c & d & f & c \\ 0 & -e & e & 0 & d & c & c & f \end{pmatrix} \end{array} . \tag{31}$$

In order to construct the characteristic array, we shall introduce the following four arrays:

$$\begin{aligned}
&[\Sigma_\mu^1,\ \Sigma_\mu^2,\ \Sigma_\mu^3,\ \Sigma_\mu^4]\equiv S_\mu^1,\\
&[\Sigma_\mu^1,\ -\Sigma_\mu^2,\ -\Sigma_\mu^3,\ \Sigma_\mu^4]\equiv S_\mu^2,\\
&[\Sigma_\mu^1,\ -\Sigma_\mu^2,\ \Sigma_\mu^3,\ -\Sigma_\mu^4]\equiv S_\mu^3,\\
&[\Sigma_\mu^1,\ \Sigma_\mu^2,\ -\Sigma_\mu^3,\ -\Sigma_\mu^4]\equiv S_\mu^4.
\end{aligned} \tag{32}$$

Basing on these arrays, we may largely factorize the field matrix. We have

$$\begin{matrix} S_x^3 \\ S_z^3 \end{matrix}\begin{bmatrix} f_1-b & -2e \\ -2e & f_1-d \end{bmatrix},\quad \begin{matrix} S_x^4 \\ S_z^4 \end{matrix}\begin{bmatrix} f_1-b & 2e \\ 2e & f_1-d \end{bmatrix} \tag{33}$$

and the other S_μ^ν's give characteristic array, whose eigenvalue is found to be f_1+2a+b for S_x^1, f_1+2c+d for S_z^1, f_1-2a+b for S_x^2 and f_1-2c+d for S_z^4. If we use the numerical values $a=9.25$, $b=-60.60$, $c=-37.40$, $d=32.56$, $e=-40.40$ and $f\equiv f_1=12.45$, which are calculated with the help of Table IV and later referred to a, b, c, d and e as g_{11}, φ_{11}, g_{22}, φ_{22} and φ_{12} in the order, the lowest array proves to be given by S_x^2 with energy $f_1-2a+b=-66.65$. This lowest energy is three-fold degenerate, in which the other two arrays are included in S^3 and S^4 but these are crystallographically equivalent to S_x^2. The orbital analogue of array S_x^2 can be seen in Fig. 5.

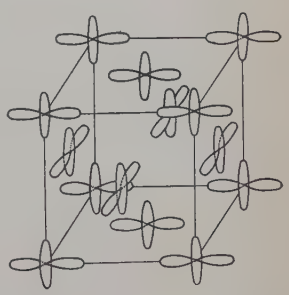


Fig. 5. The lowest orbital array in F. C. C. lattice.

(ii) $\boldsymbol{\lambda}=(0\,0\,\pi)$

The field matrix bringing the lowest energy is given by

$$\begin{matrix} \Sigma_x^1 \\ \Sigma_x^2 \end{matrix}\begin{bmatrix} f_3 & \varphi'_{11} \\ \varphi'_{11} & f_3 \end{bmatrix} \tag{34}$$

and another similar matrix concerned with Σ_x^2 and Σ_x^3. The lowest characteristic array is then given by a doubly degenerate set of $[\Sigma_x^1, \Sigma_x^2, \Sigma_x^3, \Sigma_x^4]$, and $[\Sigma_x^1, -\Sigma_x^2, -\Sigma_x^3, \Sigma_x^4]$ whose eigenvalue is equal to $f_3+\varphi'_{11}=-48.36$, since $\varphi'_{11}=-58.17$. In the other words, an xy-plane inclusive of lattice point of two sublattices 1 and 4 sit down cannot be correlated with adjacent xy-planes inclusive of those of sublattices 2 and 3 sit down.

(iii) $\boldsymbol{\lambda}=(\pi\,\pi\,0)$, $(\pi\,\pi\,\pi)$

In these cases, there is no coupling between different sublattices. The lowest value is given by $f_6=-14.28$.

The characteristic array with the lowest energy is then included in $\boldsymbol{\lambda}=(0\,0\,0)$. It may be concluded that in F.C.C. lattice an energy gain due to the interaction between sublattices is much larger than a loss due to the self-energy. On the other hand, in B.C.C. lattice it is not the case.

§ 5. The quadrupole array in the S.C. lattice

If we consider a quadrupole as a fictitious vector of five dimensions, $\boldsymbol{Z}$, nor-

malized in the sense of (3), the procedure for obtaining the quadrupole array is straightforward.

Let us now generalize the matrix component of field operator, $F_{\mu\nu}^{pq}$, given by (18), to the general quadrupole with $\mu, \nu=1, \cdots, 5$. The new components of $F_{\mu\nu}^{pq}$ are also defined by $G_{\mu\nu}((u^p-u^q)/2)$. These are obtained with the help of Table I. Writing them formally in a simple case, we have

$$
\begin{aligned}
G_{13}(0\,0\,0) &= \frac{5}{96}\sum_{\substack{l_1 l_2 l_3 \\ \neq -\infty}}^{\infty} l_2 l_3 (9l_1^2-5l_2^2+2l_3^2)/l^9, \\
G_{15}(0\,0\,0) &= \frac{35}{96}\sum_{\substack{l_1 l_2 l_3 \\ \neq -\infty}}^{\infty} l_1 l_2 (l_1^2-l_2^2)/l^9, \\
G_{23}(0\,0\,0) &= \frac{5}{32\sqrt{3}}\sum_{\substack{l_1 l_2 l_3 \\ \neq -\infty}}^{\infty} l_2 l_3 (4l_3^2-3l_1^2-3l_2^2)/l^9, \\
G_{25}(0\,0\,0) &= -\frac{5}{32\sqrt{3}}\sum_{\substack{l_1 l_2 l_3 \\ \neq -\infty}}^{\infty} l_1 l_2 (6l_3^2-l_1^2-l_2^2)/l^9, \\
G_{33}(0\,0\,0) &= \frac{2}{96}\sum_{\substack{l_1 l_2 l_3 \\ \neq -\infty}}^{\infty} [l_1^4-4(l_2^4+l_3^4)-3l_1^2(l_2^2+l_3^2)+27l_2^2 l_3^2]/l^9.
\end{aligned}
\tag{35}
$$

General expression for $G_{\mu\nu}(\boldsymbol{u})$ is derived from the above expression by substituting l_1-u_1, l_2-u_2 and l_3-u_3 for l_1, l_2 and l_3 respectively.

The other $G_{\mu\nu}(\boldsymbol{u})$'s are the following:

$$
\begin{aligned}
&G_{14}(u_1u_2u_3)=G_{13}(u_2u_1u_3), \\
&G_{24}(u_1u_2u_3)=G_{23}(u_2u_1u_3), \\
&G_{33}(u_1u_2u_3)=G_{44}(u_2u_1u_3)=G_{55}(u_3u_2u_1), \\
&G_{34}(u_1u_2u_3)=G_{35}(u_1u_3u_2)=G_{45}(u_2u_1u_3)=-\frac{2\sqrt{3}}{3}G_{25}(u_1u_2u_3).
\end{aligned}
\tag{36}
$$

In S.C. lattice, $z_\mu^p(\mu=3, 4, 5)$, designating the array of fictitious quadrupole of z_μ put on the p-th sublattice, does not couple with the other components of quadrupole, as can easily be seen from inspection of (35). Characteristic array is therefore obtained as

$$
\begin{gathered}
\boldsymbol{Z}_\mu(\boldsymbol{\lambda}) = \sum_{\boldsymbol{u}^p} \exp(i\boldsymbol{\lambda}\cdot\boldsymbol{u}^p)\boldsymbol{z}_\mu^{\,p}; \\
\mu=3, 4, 5,
\end{gathered}
\tag{37}
$$

where $\boldsymbol{z}_\mu^p$ denotes a vector analogous to $\boldsymbol{e}_\mu^p$ in (19). The corresponding eigenvalue of the field is given by $F_{\mu\mu}(\boldsymbol{\lambda})$ of (22), which in turn is obtained from $G_{\mu\mu}(\boldsymbol{u})$. The value for $G_{\mu\mu}(\boldsymbol{u})$ for $\mu=3, 4, 5$ can be derived from $G_{\mu\nu}(\boldsymbol{u})$ for $\mu, \nu=1, 2$ (Appendix), and $F_{\mu\mu}(\boldsymbol{\lambda})$'s thus estimated are tabulated in Table V. The other components of field matrix are essentially given by those in Table III.

Thus, the lowest energy of fictitious quadrupole proves to be given by $\boldsymbol{Z}_1(\pi\,\pi\,\pi)$ and $\boldsymbol{Z}_2(\pi\,\pi\,\pi)$ or almost by $\boldsymbol{Z}_1(\pi\,\pi\,0)$, whose eigenvalue $f_6=-14.28$ or $f_5=-13.97$ is considerably lower than -8.87 associated with $\boldsymbol{Z}_5(0\,0\,\pi)$. But, these characteristic arrays are all fictitious, because they are not consistent with (1) on the surface of unit sphere.

Although we treat the quadrupole as a fictitious vector, whole assembly of characteristic arrays obtained forms a complete set, whence any array of real quadrupole with double periodicity can be constructed by a linear combination. Let an array of real quadrupole be $\boldsymbol{Z}$:

$$\boldsymbol{Z}=\sum_{j,\lambda} a_{j\lambda}\boldsymbol{Z}_j(\boldsymbol{\lambda}), \tag{38}$$

where $\Sigma|a_{j\lambda}|^2=1$. The energy, $1/2\cdot f$, associated with this array is then obtained by

$$f=\sum_{j,\lambda}|a_{j\lambda}|^2 f_j(\boldsymbol{\lambda}), \tag{39}$$

where $f_j(\boldsymbol{\lambda})$ denotes the eigenvalue of $\boldsymbol{F}$ belonging to $\boldsymbol{Z}_j(\boldsymbol{\lambda})$. This relation can be easily proved with the help of

$$f=(1/8)\,(\boldsymbol{Z},\ \boldsymbol{FZ}).$$

Table V. The numerical values of $F_{\mu\mu}(\boldsymbol{\lambda})$ for $\mu=3$, 4 and 5. The other values omitted can be obtained from appropriate cyclic permutation.

$\lambda_1\,\lambda_2\,\lambda_3$	$F_{33}(\boldsymbol{\lambda})$	$F_{44}(\boldsymbol{\lambda})$	$F_{55}(\boldsymbol{\lambda})$
0 0 0	−8.30	−8.30	−8.30
0 0 π	−1.88	−1.88	−8.87
π π 0	−3.08	−3.08	15.51
π π π	9.52	9.52	9.52

Due to the cubic symmetry, a real quadrupole will probably be parallel to one of (0 0 1), (1 1 0), (1 1 1) and the other cubically equivalent directions in the lowest state. The real quadrupole parallel to (0 0 1) is simply represented by $z_2=1$ and $z_1=z_3=z_4=z_5=0$, which proves to correspond to $\zeta=1$ and $\xi=\eta=0$ if we use (1). It may be noted that $z_2=-1$ with $z_1=z_3=z_4=z_5=0$ is a fictitious quadrupole. If we take new coordinate system $(\xi',\ \eta',\ \zeta')$ whose ζ'-axis is parallel to (1 1 0), the real quadrupole with $z_2'=1$ and $z_1'=z_3'=z_4'=z_5'=0$ is also parallel to (1 1 0), where z's denote the quadrupole components in the new coordinate system. Using the transformation formula $\zeta'=1/\sqrt{2}\cdot(\xi+\eta)$, we have

$$z_2'=-\left(\frac{1}{2}z_2+\frac{\sqrt{3}}{2}z_5\right).$$

In the same way, the real quadrupole parallel to (1 1 1) proves to be represented by the state in which

$$z_2''=1/\sqrt{3}\cdot(z_3+z_4+z_5)=1$$

and all of the other z_j'''s are equal to zero where z_j'''s denote the quadrupole components in the coordinate system with ζ''-axis parallel to (1 1 1). The real quadrupole parallel to one of the other three equivalent directions: (−1 1 1), (1 −1 1) and (1 1 −1) is represented by the state in which one of

$$\frac{1}{\sqrt{3}}(z_3-z_4-z_5),\quad \frac{1}{\sqrt{3}}(-z_3+z_4-z_5)\quad \text{and}\quad \frac{1}{\sqrt{3}}(-z_3-z_4+z_5)$$

is equal to unity where the other four independent components of quadrupole can always make zero.

In such manner, we construct array of real quadrupole and examine the field value associated. Some results with lower energy are given in Table VI, whose patterns can be seen in Fig. 6.

Table VI. Some arrays of real quadrupole in S. C. lattice.

array	field value	figure 6
$(1/\sqrt{3})[\boldsymbol{Z}_3(0\,0\,0)+\boldsymbol{Z}_4(0\,0\,0)+\boldsymbol{Z}_5(0\,0\,0)]$;	−8.30	(a)
$(1/\sqrt{3})[\boldsymbol{Z}_3(\pi\,0\,0)+\boldsymbol{Z}_4(\pi\,0\,0)+\boldsymbol{Z}_5(0\,0\,0)]$;	−6.35	(b)
$-[(1/2)\cdot\boldsymbol{Z}_2(0\,0\,0)+(\sqrt{3}/2)\cdot\boldsymbol{Z}_5(0\,0\,\pi)]$;	−3.56	(c)
$-[(\sqrt{3}/2)\cdot\boldsymbol{Z}_1(\pi\,\pi\,0)+(1/2)\cdot\boldsymbol{Z}_2(0\,0\,0)]$;	−7.37	(d)

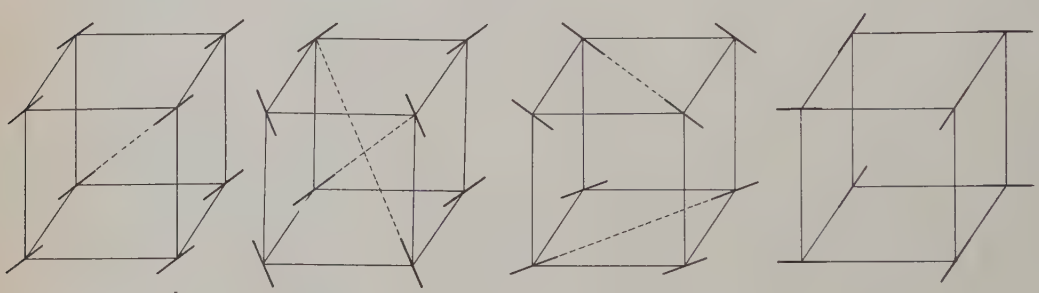


Fig. 6. Quadrupole arrays given in Table VI.

The lowest array will be $(1/\sqrt{3})\cdot(\boldsymbol{Z}_3+\boldsymbol{Z}_4+\boldsymbol{Z}_5)$ with $\lambda=(0\,0\,0)$, an array in which all quadrupoles are directed along a trigonal axis of S.C. crystal. Of course, there are four arrays of such type, corresponding to four trigonal axes of cubic crystal.

§ 6. The quadrupole array in the B.C.C. and F.C.C. lattices

The procedure is somewhat similar to that in S.C. lattice, and we may decompose B.C.C. or F.C.C. lattice into two or four S.C. sublattices and construct

the field matrix with respect to S.C. characteristic arrays of fictitious quadrupole, as can be seen in Table VIII. The numerical values are obtained from $G_{\mu\nu}(u)$'s whose independent quantities are tabulated in Table VII. The other $G_{\mu\nu}(u)$'s can be derived from relations given in the Appendix.

We shall below discuss the quadrupole array in B.C.C. and F.C.C. lattices separately by use of the field matrix given in Table VIII.

(A) *Quadrupole array in B.C.C. lattice*

As can be seen in Table VII, two S.C. characteristic arrays with $\boldsymbol{\lambda}=(0\,0\,\pi)$

Table VII. Numerical values of $G_{\mu\nu}(u)$.

u_1 u_2 u_3	$G_{33}(u)$	$G_{18}(u)$
1/4 1/4 1/4	2.07	2.36
1/4 1/4 0	−5.93	0
1/4 1/4 1/2	0.51	0
0 1/4 1/4	17.84	−6.00
1/2 1/4 1/4	0.05	0.87

Table VIII A. Quadrupole field at body-, yz face-, zx face- and xy face-centered points, coming from a characteristic array on the corner lattice.

λ	array	field at B.C. pt.	field at F.C. pt.		
			yz-face	zx-face	xy-face
(0 0 0)	$\mathbf{Z}_1$	$f_{11}\mathbf{Z}_1$	$g_{11}\mathbf{Z}_1+g_{12}\mathbf{Z}_2$	$h_{11}\mathbf{Z}_1+h_{12}\mathbf{Z}_2$	$\varphi_{11}\mathbf{Z}_1$
	$\mathbf{Z}_2$	$f_{22}\mathbf{Z}_2$	$g_{22}\mathbf{Z}_2+g_{21}\mathbf{Z}_1$	$h_{22}\mathbf{Z}_2+h_{21}\mathbf{Z}_1$	$\varphi_{22}\mathbf{Z}_2$
	$\mathbf{Z}_3$	$f_{33}\mathbf{Z}_3$	$g_{33}\mathbf{Z}_3$	$h_{33}\mathbf{Z}_3$	$\varphi_{33}\mathbf{Z}_3$
	$\mathbf{Z}_4$	$f_{44}\mathbf{Z}_4$	$g_{44}\mathbf{Z}_4$	$h_{44}\mathbf{Z}_4$	$\varphi_{44}\mathbf{Z}_4$
	$\mathbf{Z}_5$	$f_{55}\mathbf{Z}_5$	$g_{55}\mathbf{Z}_5$	$h_{55}\mathbf{Z}_5$	$\varphi_{55}\mathbf{Z}_5$
(0 0 π)	$\mathbf{Z}_1$				$\varphi_{11}'\mathbf{Z}_1$
	$\mathbf{Z}_2$				$\varphi_{22}'\mathbf{Z}_2$
	$\mathbf{Z}_3$				$\varphi_{33}'\mathbf{Z}_3$
	$\mathbf{Z}_4$				$\varphi_{44}'\mathbf{Z}_4$
	$\mathbf{Z}_5$				$\varphi_{55}'\mathbf{Z}_5$
(π π 0)	$\mathbf{Z}_1$				
	$\mathbf{Z}_2$	$f_{25}\mathbf{Z}_5$			$\varphi_{25}\mathbf{Z}_5$
	$\mathbf{Z}_3$	$f_{34}\mathbf{Z}_4$			$\varphi_{34}\mathbf{Z}_4$
	$\mathbf{Z}_4$	$f_{43}\mathbf{Z}_3$			$\varphi_{43}\mathbf{Z}_3$
	$\mathbf{Z}_5$	$f_{52}\mathbf{Z}_2$			$\varphi_{52}\mathbf{Z}_2$
(π π π)	$\mathbf{Z}_1$		$g_{13}\mathbf{Z}_3$	$h_{14}\mathbf{Z}_4$	
	$\mathbf{Z}_2$		$g_{23}\mathbf{Z}_3$	$h_{24}\mathbf{Z}_4$	$\varphi_{25}'\mathbf{Z}_5$
	$\mathbf{Z}_3$		$g_{31}\mathbf{Z}_1+g_{32}\mathbf{Z}_2$	$h_{35}\mathbf{Z}_5$	$\varphi_{34}'\mathbf{Z}_4$
	$\mathbf{Z}_4$		$g_{45}\mathbf{Z}_5$	$h_{41}\mathbf{Z}_1+h_{42}\mathbf{Z}_2$	$\varphi_{43}'\mathbf{Z}_3$
	$\mathbf{Z}_5$		$g_{54}\mathbf{Z}_4$	$h_{53}\mathbf{Z}_3$	$\phi_{52}'\mathbf{Z}_2$

Table VIII B. Numerical values for the matrix component.

component	value	component	value
f_{11}, f_{22}	−24.81	φ_{11}'	−58.17
f_{33}, f_{44}, f_{55}	16.54	φ_{22}'	38.66
f_{25}	−21.82	$\varphi_{33}', \varphi_{44}'$	−25.78
f_{34}	25.19	φ_{55}'	70.56
g_{11}, h_{11}	9.25	φ_{25}	24.02
g_{22}, h_{22}	−37.40	φ_{34}	−27.70
$g_{12}, -h_{12}$	−40.40		
$g_{33}, h_{44}, \varphi_{55}$	72.19	$g_{13}, -h_{14}$	−27.36
$g_{44}, h_{33}, \varphi_{33}$	−21.70	$g_{23}, h_{24}, 1/2\cdot\varphi_{25}'$	15.79
$g_{55}, h_{55}, \varphi_{44}$		$g_{45}, h_{35}, \varphi_{34}'$	−36.40
φ_{11}	−60.60		
φ_{22}	32.56		

and $(\pi\,\pi\,\pi)$ do not couple with each other, hence the energy value is given by a self-energy of S.C. characteristic array. It will be sufficient to look into the arrays with $\boldsymbol{\lambda}=(0\,0\,0)$ and $(\pi\,\pi\,0)$.

(i) $\boldsymbol{\lambda}=(0\,0\,0)$

Field matrix is very similar to (28). The characteristic arrays and their field values are as follows:

$$\begin{aligned}
&[\boldsymbol{Z}_1, \boldsymbol{Z}_1], [\boldsymbol{Z}_2, \boldsymbol{Z}_2]: && f=-12.35,\\
&[\boldsymbol{Z}_1, -\boldsymbol{Z}_1], [\boldsymbol{Z}_2, -\boldsymbol{Z}_2]: && 37.25,\\
&[\boldsymbol{Z}_3, \boldsymbol{Z}_3], [\boldsymbol{Z}_4, \boldsymbol{Z}_4], [\boldsymbol{Z}_5, \boldsymbol{Z}_5]: && 8.23,\\
&[\boldsymbol{Z}_3, -\boldsymbol{Z}_3], [\boldsymbol{Z}_4, -\boldsymbol{Z}_4], [\boldsymbol{Z}_5, -\boldsymbol{Z}_5]: && -24.8.
\end{aligned} \tag{40}$$

$[\boldsymbol{Z}_2, \boldsymbol{Z}_2]$ gives real quadrupole (Fig. 7a), but the other characteristic arrays are fictitious. By use of their linear combination, we shall look for the array of real quadrupole.

$$-(\sqrt{3}/2\cdot[\boldsymbol{Z}_5, -\boldsymbol{Z}_5]+1/2\cdot[\boldsymbol{Z}_2, \boldsymbol{Z}_2]) \tag{41}$$

is a real quadrupole, which has field value equal to −21.7 and will be the lowest in B.C.C. lattice (Fig. 7b).

Another real quadrupole array which gives rather low field value −13.8 is constructed as

$$1/\sqrt{3}\cdot([\boldsymbol{Z}_3, -\boldsymbol{Z}_3]+[\boldsymbol{Z}_4, -\boldsymbol{Z}_4]+[\boldsymbol{Z}_5, \boldsymbol{Z}_5]), \tag{42}$$

which can be seen in Fig. 7c.

(ii) $\boldsymbol{\lambda}=(\pi\,\pi\,0)$

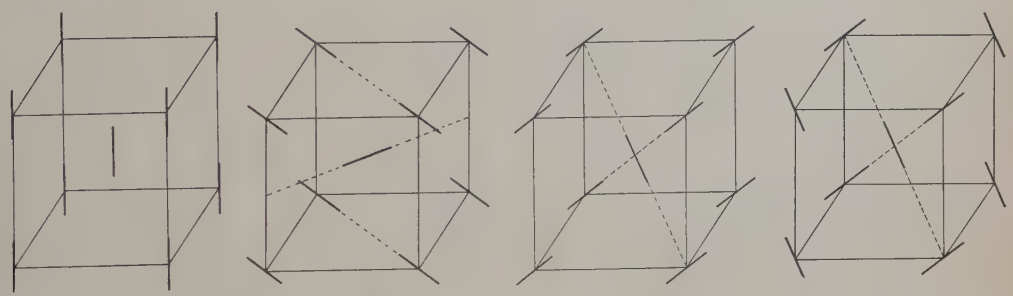
Fictitious arrays of lower field value are

$$\begin{aligned}&[\boldsymbol{Z}_1, \boldsymbol{Z}_1], [\boldsymbol{Z}_1, -\boldsymbol{Z}_1]: \quad f=-14.2,\\ &[\boldsymbol{Z}_3, -\boldsymbol{Z}_4], [\boldsymbol{Z}_4, -\boldsymbol{Z}_3]: \quad -28.3.\end{aligned} \tag{43}$$

A real quadrupole array constructed from them is

$$\begin{aligned}&1/\sqrt{3}\,([\boldsymbol{Z}_3, -\boldsymbol{Z}_4]+[\boldsymbol{Z}_4, -\boldsymbol{Z}_3])\ (\pi\,\pi\,0)\\ &+1/\sqrt{3}\,[\boldsymbol{Z}_5, \boldsymbol{Z}_5]\ (0\,0\,0),\end{aligned} \tag{44}$$

whose field value is found to be -16.1. This array is illustrated in Fig. 7d.



(a) (b) (c) (d)

Fig. 7. Quadrupole arrays in B.C.C. lattice

(B) *Quadrupole array in F.C.C. lattice*

(i) $\boldsymbol{\lambda}=(0\,0\,0)$

While $\boldsymbol{Z}_1$ and $\boldsymbol{Z}_2$ couple with each other, each of $\boldsymbol{Z}_3$, $\boldsymbol{Z}_4$ and $\boldsymbol{Z}_5$ does not couple with any different type of $\boldsymbol{Z}_j$ respectively. The characteristic arrays have a similar form to (32). For example,

$$\begin{aligned}&[\boldsymbol{Z}_3, \boldsymbol{Z}_3, \boldsymbol{Z}_3, \boldsymbol{Z}_3], : \quad f=20.5,\\ &[\boldsymbol{Z}_3, -\boldsymbol{Z}_3, -\boldsymbol{Z}_3, \boldsymbol{Z}_3], : \quad -80.5,\\ &[\boldsymbol{Z}_3, -\boldsymbol{Z}_3, \boldsymbol{Z}_3, -\boldsymbol{Z}_3], : \quad -80.5,\\ &[\boldsymbol{Z}_3, \boldsymbol{Z}_3, -\boldsymbol{Z}_3, -\boldsymbol{Z}_3], : \quad 107.3.\end{aligned} \tag{45}$$

In such a way, we may find the lowest array of two-fold degeneracy from respective $\boldsymbol{Z}_i$, $i=1, 2$ and 3. Then we designate them by

$$\begin{aligned}&\boldsymbol{X}_1=[\boldsymbol{Z}_3, -\boldsymbol{Z}_3, -\boldsymbol{Z}_3, \boldsymbol{Z}_3],\\ &\boldsymbol{X}_2=[\boldsymbol{Z}_4, \boldsymbol{Z}_4, -\boldsymbol{Z}_4, -\boldsymbol{Z}_4],\\ &\boldsymbol{X}_3=[\boldsymbol{Z}_5, -\boldsymbol{Z}_5, \boldsymbol{Z}_5, -\boldsymbol{Z}_5],\end{aligned} \tag{46}$$

$$\begin{aligned}&\boldsymbol{Y}_1=[\boldsymbol{Z}_3, -\boldsymbol{Z}_3, \boldsymbol{Z}_3, -\boldsymbol{Z}_3],\\ &\boldsymbol{Y}_2=[\boldsymbol{Z}_4, -\boldsymbol{Z}_4, -\boldsymbol{Z}_4, \boldsymbol{Z}_4],\\ &\boldsymbol{Y}_3=[\boldsymbol{Z}_5, \boldsymbol{Z}_5, -\boldsymbol{Z}_5, -\boldsymbol{Z}_5].\end{aligned} \tag{47}$$

We can construct real quadrupole arrays by using this degenerate set. They are designated by

$$1/\sqrt{3}\,(\boldsymbol{X}_1+\boldsymbol{X}_2+\boldsymbol{X}_3),\quad 1/\sqrt{3}\,(\boldsymbol{Y}_1+\boldsymbol{Y}_2+\boldsymbol{Y}_3), \tag{48}$$

whose field value is of course equal to -80.5. This array can be seen in Fig. 8. On the other hand, the lowest characteristic array constructed by $\boldsymbol{Z}_1$ and $\boldsymbol{Z}_2$ has been obtained in § 4 (B). This array, being now fictitious, has brought a field value equal to -66.65, but any real quadrupole array constructed by using it is of higher energy.

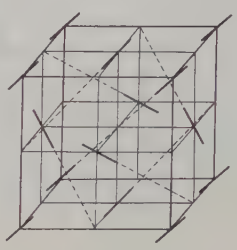


Fig. 8. The lowest quadrupole array in F.C.C. lattice.

(ii) $\boldsymbol{\lambda}=(0\,0\,\pi)$

The lowest arrays are

$$[\boldsymbol{Z}_5,\ -\boldsymbol{Z}_5,\ -\boldsymbol{Z}_5,\ \boldsymbol{Z}_5],\ [\boldsymbol{Z}_5,\ -\boldsymbol{Z}_5,\ \boldsymbol{Z}_5,\ -\boldsymbol{Z}_5],$$

whose field value is found to be -79.4. Each of them should be combined with the other characteristic array with higher energy in order to construct any real array, resulting in a promotion of energy. Some field values of array are given by -48.4, -37.4, -27.7, etc.

(iii) $\boldsymbol{\lambda}=(\pi\,\pi\,0)$, $(\pi\,\pi\,\pi)$

Field values are given by $(-60.7,\ -50.3,\ -14.0,\ 25.2,\ 66.5)$ for $(\pi\,\pi\,0)$ and $(-37.6,\ -12.8,\ 3.6,\ 26.2,\ 27.7)$ for $(\pi\,\pi\,\pi)$, which are of no interest.

We may finally conclude that the array (48) leads to the exact lowest field value in F.C.C. lattice, which is considerably lower than the corresponding value in § 4 (B).

§ 7. Quadrupole interaction in magnetic compounds and molecular crystals

(A) *The orbital ordering of localized magnetic electron*

Although the quadrupole interaction is not very effective, our result seems to give some insights into the preferential ordering of *d*-orbitals which are two-fold (e_g) or three-fold (t_{2g}) degenerate under a crystalline field of cubic symmetry.

Let us first look into the stabilizing character of t_{2g}-orbital in S.C., B.C.C. and F.C.C. lattices. Since we have done the corresponding study about e_g-orbital in §§ 3 and 4, we may then compare the results for both types of orbital. Three orbitals of t_{2g} are written as

$$\phi_1=\sqrt{3}\,\eta\zeta,\quad \phi_2=\sqrt{3}\,\zeta\xi,\quad \phi_3=\sqrt{3}\,\xi\eta. \tag{49}$$

By using them, the quadrupole components are represented by (10) with $c_J=-\frac{2}{7}\cdot\overline{r^2}$, where J_x, J_y and J_z should be understood to be the components of orbital angular momentum with quantum number equal to unity. An oriented quadrupole corresponds to a state in which z_2 has the maximum magnitude along the orientation

axis, say, the z-axis, hence a state with the eigenvalue of J_z equal to zero. Since z_2 equals -1 in this state, the classical treatment of the t_{2g}-problem is just given by that in §§ 5 and 6 with the same magnitude of quadrupole moment, $\mu_Q(=-\frac{2}{7}\cdot\overline{r^2})$, as in the e_g-problem.

In such a way, we shall compare the ordering energy of t_{2g}-orbital, which the results obtained in §§ 5 and 6 give, with that of e_g-orbital obtained in §§ 3 and 4. We shall recapitulate the results for respective lattice in Table IX. As can be seen in this table, the quadrupole energy associated with e_g is much stabilized in S.C. lattice than that with t_{2g}, but the situation is reversed in B.C.C. and F.C.C. lattices.

As mentioned in § 1, the ordering of e_g-orbital plays the important role in order to understand the magnetic structure of $LaMnO_3$ and MnF_3. In these compounds, the magnetic lattice is essentially S.C., where an orbital pattern Fig. 4(b)

Table IX. The lowest array for e_g- and t_{2g}-orbitals.

orbital		S.C.	B.C.C.	F.C.C.
e_g	array	Fig. 4	Fig. 4	Fig. 5
	field value	−14.28	−14.28	−66.5
t_{2g}	array	Fig. 6 (a)	Fig. 7 (b)	Fig. 8
	field value	−8.30	−21.7	−80.5

seems to be realized. The reason why the pattern which the compounds mentioned choose is not the type of Fig. 4(a) but that of Fig. 4(b) should be interpreted in terms of a certain kind of the Jahn-Teller distortion. In actuality, the field value for $LaMnO_3$ proves to be about 200°K in unit of the Boltzmann constant, hence the excitation energy necessary to reverse a fictitious spin is estimated to be about 400°K. It is rather small compared with the observed transition temperature ~780°K, where the e_g-orbital ordering is considered to melt. But we will not further inquire into this problem in the present paper.

We shall here note another point. Any pattern which is not constructed by one of the lowest patterns or linear combination of them will always be higher in quadrupole energy. For example, we shall discuss the energy associated with an orbital array as shown in Fig. 9(a), where two non-orthogonal orbitals,

$$1/2\cdot(2\xi^2-\eta^2-\zeta^2),\quad 1/2\cdot(2\eta^2-\xi^2-\zeta^2), \tag{50}$$

are mounted on lattice points as shown. These orbitals will be written as

$$\frac{\sqrt{3}}{2}\psi_1-\frac{1}{2}\psi_2,\quad -\frac{\sqrt{3}}{2}\psi_1-\frac{1}{2}\psi_2. \tag{51}$$

Comparing (51) with (23), we shall obtain the spin pattern in which two kinds of spins, pointing the directions at angles $-\pi/3$ and $\pi/3$ from the z-axis in the zx-plane, align alternatively in lattice (Fig. 9(b)). This array may be con-

sidered as a superposition of two arrays, in which one is an array with ferromagnetic spin along the z-axis and another with antiferromagnetic array, $\boldsymbol{\lambda}=(\pi\,\pi\,\pi)$, of spin along the x-axis. The coefficients of linear combination proves to be $1/2$ and $\sqrt{3}/2$ for respective array. Therefore we obtain the corresponding field value as

$$[(\sqrt{3}/2)^2(-14.28)+(1/2)^2\cdot(12\cdot 45)]=-7.60$$

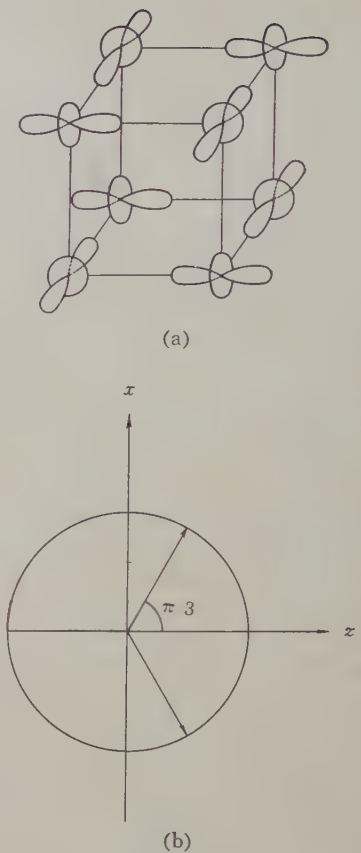
with the help of (39) and Table III.

(B) *The orientational ordering in molecular crystal*

We shall discuss some aspects of crystal structure observed in molecular crystal. Although our results are to be applied to molecular crystals composed of molecules with large quadrupole moment in cubic crystal, we shall also refer further to crystals of the other types.[6]

(i) *B.C.C. lattice*

The lowest pattern is of the type Fig. 7 (b), which will be supposed to be distorted tetragonally. Such crystal structure is found in fluorides and oxides of metals, which is ionic crystal and is well-known as rutile or SnO_2-type structure.



(a)

(b)

Fig. 9. An orbital array (a) and its spin analogue (b).

Another pattern of the type Fig. 7 (a), being rather low in energy, is found in hydrogen cyanide HCN, which is distorted tetragonally.[7] Since this molecule has large electrical dipole moment ($\mu=2.65\times10^{-18}$ e.s.u.),[8] our prediction of crystal structure fails naturally. But quadrupole moment of HCN is also large ($\mu_Q=0.775\times10^{-16}$ cm^2).[9]

(ii) *F.C.C. lattice*

Some molecular crystals composed of non-polar molecules with appreciable magnitude of quadrupole moment are known and are listed in Table X. The quadrupole energy per molecule, W_Q, is estimated as one half of the field value, -80.5, multiplied by $3e^2\mu_Q^2/4a^5$. In these crystals listed, crystal lattice is of the same structure at low temperatures, which is just the lowest pattern predicted in Fig. 8. We shall note some features observed for respective crystals.

Nitrogen (N_2).[10] The orientational ordering breaks down at transition point 34.5°K, which is comparable with the quadrupole energy per molecule in unit of the Boltzmann constant k.

Nitric oxide (N_2O). N_2O molecule is of the form N–N–O, hence probably has very small dipole moment. Due to its large quadrupole moment, the quadru-

Table X. Lattice constant (a), transition temperature (T_λ), melting point (T_M), quadrupole moment (μ_Q) and quadrupole energy per molecule (W_Q) of molecular crystals of solid nitrogen type crystallized in F.C.C. lattice.[13),14)]

molecular crystal	$a\times10^8$ cm	T_λ°K	T_M°K	$\mu_Q\times10^{16}$ cm^2	W_Q°K
N_2	5.56	34.5	63.08	0.155	20
N_2O	5.66		182.3	0.455	127
CO_2	5.58		190	0.32	94
CO	5.63	64.55	68.1	0.165	24

pole energy is comparable with kT of melting temperature. We suppose that this will be one of the reasons why the orientational order persists up to the melting point in this crystal.

Carbon dioxide (CO_2).[11)] CO_2 molecule is of the form O–C–O, hence without dipole moment. The situation will be similar to that in nitric oxide.

Carbon monoxide (CO).[12)] CO molecule has small dipole moment $\mu=0.1\times10^{-18}$ e.s.u.[8)] and therefore we suppose that quadrupole effect is much more effective than dipole effect in solid state. The transition point of CO crystal is, however, higher than that of N_2 crystal. This will probably be due to dipole-quadrupole interaction between CO molecules.

Another interesting feature observed for N_2O and CO crystals is that there remains a further disorder concerned with such direction as N–N–O or C–O within the pattern of nitrogen type at very low temperatures. This is supposed to be due to the situation that their directions are determined by small dipolar effect.

Acknowledgement

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Appendix

Some relations between $G_{\mu\nu}(u)$'s

The following symmetries can be easily established by inspecting (17): $G_{\mu\mu}(u_1u_2u_3)$'s are symmetric with respect to permutation between u_1 and u_2, while $G_{12}(u_1u_2u_3)$ is antisymmetric with respect to the same permutation. And we can see further relations:

$$\begin{aligned}
&G_{11}(u_1u_2u_3)+G_{11}(u_2u_1u_3)+G_{11}(u_3u_1u_2)\\
&\quad=G_{22}(u_1u_2u_3)+G_{22}(u_2u_3u_1)+G_{22}(u_3u_1u_2),\\
&G_{11}(u_1u_2u_3)-G_{11}(u_3u_2u_1)=G_{22}(u_3u_2u_1)-G_{22}(u_1u_2u_3),\\
&\quad G_{12}(vuu)=\sqrt{3}/4\cdot[G_{22}(uuv)-G_{11}(uuv)].
\end{aligned}\qquad (A\cdot1)$$

Using these relations, we can derive G_{12} and G_{22} from G_{11}, for example,

$$G_{22}(uuv)=1/3\cdot[4G_{11}(vuu)-G_{11}(uuv)],$$
$$G_{22}(vuu)=1/3\cdot[G_{11}(vuu)+2G_{11}(uuv)]. \qquad (A\cdot 2)$$

We shall note some useful relations for $G_{\mu\nu}$'s, $\mu, \nu=1, \cdots, 5$. If we compare G_{33}, of (35) with $G_{\mu\nu}$ of (17), a relation

$$G_{33}(u_1u_2u_3)=-\frac{2}{3}G_{22}(u_1u_2u_3)-\frac{2}{\sqrt{3}}G_{12}(u_1\bar{u}_2u_3) \qquad (A\cdot 3)$$

is established, hence G_{33}, G_{44} and G_{55}, can be derived from G_{11}, G_{22} and G_{12} with the help of (36). Other relations are as follows:

$$G_{23}(vuu)=1/2\cdot G_{25}(uuv)=-1/\sqrt{3}\cdot G_{13}(vuu),$$
$$G_{34}(uuv)=4/3\cdot G_{21}(vuu), \qquad (A\cdot 4)$$
$$G_{33}(uuv)=G_{33}(uvu)=-2/3\cdot G_{11}(uuv),$$

which are obtained in the same way.

References

1) T. Nakamura, Prog. Theor. Phys. **14** (1955), 135.
2) J. Kanamori, Prog. Theor. Phys. **17** (1957), 177.
3) E. O. Wollan and W. C. Koehler, Phys. Rev. **100** (1955), 545.
   E. O. Wollan, H. R. Child, W. C. Koehler and M. K. Wilkinson, Phys. Rev. **112** (1958), 1132.
   J. B. Goodenough, Phys. Rev. **100** (1955), 564.
4) J. M. Luttinger and L. Tisza, Phys. Rev. **70** (1946), 954.
5) K. W. H. Stevens, Proc. Phys. Soc. A **65** (1952), 209.
6) R. W. G. Wyckoff, *Crystal Structure* (Inter Science Publishers, Inc., New York, 1948).
7) W. J. Dumage and W. N. Lipscomb, Acta Cryst. **4** (1951), 330.
8) P. Debye, *Polar Molecules* (Chemical Catalog Co., New York, 1929).
9) J. O. Hirschfelder, C. F. Curtiss and R. B. Bird, *Molecular Theory of Gases and Liquids* (John Wiley & Sons, Inc., New York, 1954).
10) L. Vegard, Z. Physik **58** (1929), 497.
11) J. de Smedt and W. H. Keesom, Proc. Acad. Sci. Amsterdam **27** (1924), 839.
12) L. Vegard, Z. Physik **61** (1930), 185.
13) K. Clusius, Z. physik. Chem. B **3** (1929), 41.
14) J. O. Cloyton and W. F. Giauque, J. Am. Chem. Soc. **54** (1942), 2610.

Letters to the Editor

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Influence of Interelectronic Collision on Hall Mobility in Non-polar Semiconductor

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December 26, 1959

Recently Blatt[1] calculated drift and Hall mobilities by using the method of partial wave.

As he did not take the effect of interelectronic collision into account, we here try to estimate the magnitude of correction due to this effect.

By the method of Keyes[2] we can write the mobility ratio in place of the usual representation, that is,

$$\frac{\mu_H}{\mu}=\frac{\langle v^2\rangle\langle v\tau_p^2\rangle}{(\langle v^2\tau_p\rangle)^2}$$

where v=velocity, and τ=relaxation time,

$$\frac{\mu_H}{\mu}=\frac{\langle v^2(\tau_p/(\tau_p+\tau_1))^2\rangle(\langle v^2/(\tau_p+\tau_1)\rangle)^2}{\langle v^2/(\tau_p+\tau_1)^2\rangle(\langle v^2\tau_p/(\tau_p+\tau_1)\rangle)^2}.$$

In the case of acoustic phonon scattering when τ_1(=interelectronic relaxation time) goes to 0, which means ideally strong interelectronic collision

$$\frac{\mu_H}{\mu}=\frac{(\langle 4v^2/\tau_p\rangle)^2}{\langle v^2/\tau_p\rangle\langle v^2\rangle}=\frac{128}{45\pi}=0.90.$$

And in the limit of $\tau_1\to\infty$ which means no interelectronic collision

$$\frac{\mu_H}{\mu}=\frac{\langle v^2\rangle\langle v^2\tau_p^2\rangle}{(\langle v^2\tau_p\rangle)^2}=\frac{3\pi}{8}=1.18$$

because $\tau_p=27\pi u_0^2\hbar^4M/2VC^2kTm\cdot 1/p=A/p$ where the notation is same as that in the previous paper.[3] We can calculate the temperature variation of mobility ratio.

$$\frac{\mu_H}{\mu}=\frac{1}{\tau^2}\frac{A^2}{m^2}\left\langle\frac{v^2}{(v+B)^2}\right\rangle\Big/\left\langle\frac{v^4}{(v+B)^2}\right\rangle$$

where $B=A/m\tau_1$.

Thus

$$\left\langle\frac{v^2}{(v+B)^2}\right\rangle=1-2BC\left(\frac{2\pi kT}{m}\right)^2-2B^3CF\left(\frac{m}{2kT},B\right)-B^2C\left(\frac{kT}{m}\right)+B^3\left\{1+\frac{m}{kT}e^{mB^2/2kT}\times\mathrm{Erfc}\left(\sqrt{\frac{m}{2kT}},B\right)\right\}$$

and

$$\left\langle\frac{v^4}{(v+B)^2}\right\rangle=\frac{3kT}{m}-2B\frac{2\sqrt{2kT}}{\sqrt{\pi m}}+3B^2-5B^3C\left(\frac{2\pi kT}{m}\right)^{1/2}-5B^5CF\left(\frac{m}{2kT},B\right)+B^5\left\{1+\frac{m}{kT}e^{mB^2/2kT}\right.$$

$$\times \mathrm{Erfc}\left(\sqrt{\frac{m}{2kT}}, B\right)\Big\}.$$

where

$$F\left(\frac{m}{2kT}, B\right)=\int_0^\infty \frac{e^{-(mx^2/2kT)}}{(x+B)}\,dx$$

and

$$C=4\pi\left(\frac{m}{2\pi kT}\right)^{3/2}.$$

In the case of impurity scattering,

$$\tau_p=\frac{3\kappa^2 p^3}{2\pi m N_1 e^4 \ln(1+(2\kappa E/e^2 N_1^{1/3})^2)}.$$

But analytically exact calculation is hardly possible. At high temperature

$$\tau_p\simeq\frac{3\kappa^2 p^3}{2\pi m N_1 e^2 \ln(1+(3\kappa kT/e^2 N_1^{1/3})^2)},$$

and in this case $\mu_H/\mu=315\pi/512=1.93$ if $\tau_1=\infty$. If $\tau'=0$,

$$\frac{\mu_H}{\mu}=\frac{(\langle 1/v\rangle)^2}{\langle 1/v^4\rangle\langle v^2\rangle}=0.$$

Blatt[1] showed that $\tau_p\simeq A'E$ in the neighbor of medium temperature which changes from 20°K to 100°K with impurity concentrations. In this case $\mu_H/\mu=7/5$ if $\tau_1=\infty$, and $\mu_H/\mu=1/3$ if $\tau_1=0$. At low temperature, $\tau_p\simeq 3/2\pi N_1^{1/3}v$. In this case,

$$\mu_H/\mu=3\pi/8 \text{ if } \tau_1=\infty,$$
$$\mu_H/\mu=128/45\pi=0.90 \text{ if } \tau_1=0.$$

If we adopt

$$\tau_1=\frac{2^{5/2}\kappa^2\hbar^3}{\pi^3 m e^4 \ln(1+(9\kappa^2 kT\hbar^2/2\pi m e^4)e^{2\varepsilon_0/3kT})}$$

for the range of impurity scattering which means no compensation,

$$\frac{\mu_H}{\mu}=\frac{3\pi}{8} \quad \text{at} \quad T=0.$$

At high temperature interelectronic collision is negligible, so we can obtain

$$\frac{\mu_H}{\mu}=\frac{315\pi}{512} \quad \text{at} \quad T=\infty.$$

These behavior is illustrated by the figure.

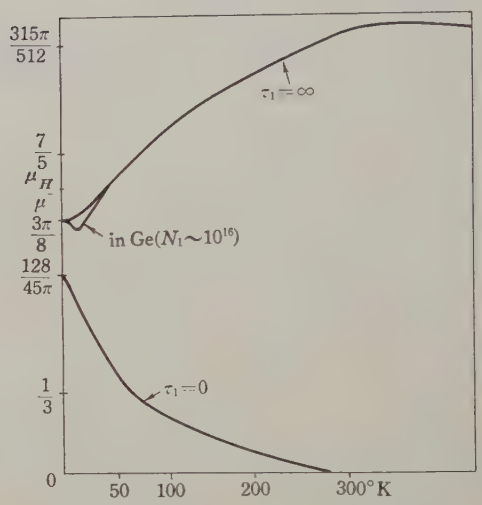

Fig. 1. Temperature variation of mobility ratio in the range of impurity scattering

In this way we are led to the conclusion that the correction due to the interelectronic collision is very small in magnitude.

1) F. J. Blatt, Phys. Rev. **105** (1957), 1203; Jour. Phys. Chem. Solid **1** (1957), 262.
2) R. W. Keyes, Jour, Phys. Chem. Solid **6** (1958), 1.
3) Y. Uchiyama, Prog. Theor. Phys. **21** (1959), 939.

de Haas-van Alphen Effect and Coulomb Interaction

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April 25, 1960

Using the formalism of quantum statistical mechanical perturbation due to Matsubara[1] and others, we have investigated the effect of the Coulomb interaction between electrons on the orbital diamagnetism of electron gas. The propagator of the electron which is introduced by Matsubara is given by

$$\left.\begin{aligned} S(x', x) &= \sum_r f_r u_r(\boldsymbol{x}') u_r^*(\boldsymbol{x}) e^{-\varepsilon_r \tau}, \\ &\qquad \tau = t' - t < 0 \\ &= \sum_r (f_r - 1) u_r(\boldsymbol{x}') u_r(\boldsymbol{x}) e^{-\varepsilon_r \tau}, \\ &\qquad \tau > 0 \end{aligned}\right\} \tag{1}$$

where f_r is the Fermi distribution function and $u_r(\boldsymbol{x})$ is the eigen-function of energy ε_r; t is the fourth variable corresponding to the inverse temperature. We have chosen for $u_r(\boldsymbol{x})$ the so-called Landau function, i.e. the eigen-function of the free electron in a constant magnetic field, and calculated the above propagator by using the technique similar to that used by Sondheimer and Wilson[2] in the calculation of the density matrix of free electron gas under magnetic field. The result can be written as

$$S(x', x) = S^{(1)}(x', x) + S^{(2)}(x', x) + S^{(3)}(x', x) \tag{2}$$

$$S^{(1)}(x', x) = e(\boldsymbol{x}', \boldsymbol{x}) \zeta^{3/2} e^{-\zeta\tau} \frac{1}{2\pi i} \times \int_{c_0} \frac{1}{u - \zeta\tau} \frac{1}{u^{3/2}} \exp\left(u - \frac{D}{u}\right) du,$$

$$e(\boldsymbol{x}', \boldsymbol{x}) = -\left(\frac{2\pi m}{\hbar^2}\right)^{3/2} \times \exp\left(-\frac{1}{2} i\alpha^2 (x' + x)(y' - y)\right),$$

$$\alpha^2 = \frac{eH}{c\hbar}, \quad D = \frac{1}{4} k^2 R^2,$$

$$R^2 = (x' - x)^2 + (y' - y)^2 + (z' - z)^2, \tag{3}$$

$$S^{(2)}(x', x) = e(\boldsymbol{x}', \boldsymbol{x}) \zeta^{3/2} e^{-\zeta\tau} \frac{1}{2\pi i} \times \int \frac{1}{u - \zeta\tau} \frac{1}{u^{3/2}} \exp\left(u - \frac{D}{u}\right) \times (e^{-(1/4)\cdot\mu_0 H/\zeta \cdot R'^2 k^2} - 1) du,$$

$$\mu_0 = \frac{e\hbar}{2m^* c}, \quad (m^* = \text{effective mass}),$$

$$R'^2 = (x' - x)^2 + (y' - y)^2, \tag{4}$$

$$S^{(3)}(x', x) = e(\boldsymbol{x}', \boldsymbol{x}) (-2(\mu_0 H)^{3/2} e^{-\zeta\tau}) \times J_0(kR') \times \sum_{r=1}^{\infty} \frac{(-1)^r}{(r\pi)^{3/2}} \frac{1}{1 + (\mu_0 H\tau / r\pi)^2} \times \frac{r\pi^2 \kappa T / \mu_0 H}{\sinh(r\pi^2 \kappa T / \mu_0 H)} \times \left\{ \sin\left(\frac{r\pi\zeta}{\mu_0 H} + \frac{\mu_0 H k^2 (z - z')^2}{4\zeta r\pi} - \frac{\pi}{4}\right) + \frac{\mu_0 H\tau}{r\pi} \cos\left(\frac{r\pi\zeta}{\mu_0 H} + \frac{\mu_0 H k^2 (z - z')^2}{4\zeta r\pi} - \frac{\pi}{4}\right) \right\}, \tag{5}$$

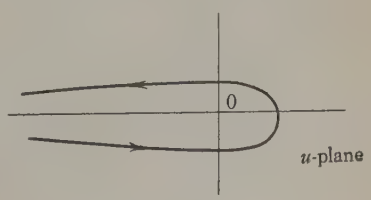


Fig. 1.

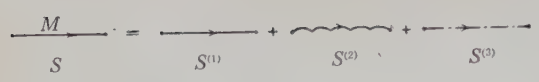


Fig. 2.

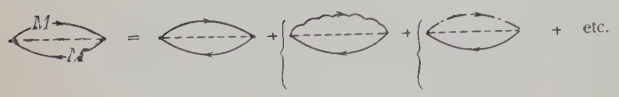
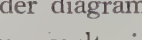


Fig. 3.

where the magnetic field is taken in z-direction; $\zeta=\hbar^2k^2/2m^*$ and κ is the Boltzmann constant; C_0 means the path of the contour integration shown in Fig. 1; J_0 is the Bessel function.

Using this propagator, we can investigate the effect of the Coulomb interaction on magnetic behaviours of electron gas by the diagramatical method used by various authors.[3] In carrying out those analysis, it will be convenient to decompose the electron line into three lines as in Fig. 2 corresponding to $S^{(1)}$, $S^{(2)}$ and $S^{(3)}$, where by ⟶M⟶ we represent the total electron line under magnetic field. Consequently, from one ordinary diagram, we obtain several new diagrams of the same form. (See, e.g. Fig. 3.) Roughly speaking, the contributions from diagrams composed only of the line ⟶ do not differ from those of the corresponding zero field diagrams except the difference between the diagrams which contain the lines ⟶ and ⇝ contribute, for example, to the effect of Coulomb interaction on Landau diamagnetism which has recently been treated by Kanazawa and Matsudaira[5] and the diagrams which contain the line ⟶·⟶ will give the contributions on the oscillatory behaviour of magnetic properties. We have investigated in detail the contribution from the diagram to the magnetic susceptibility at 0°K and evaluated the shift of phase constant of the de Haas-van Alphen oscillation. We drop the diagrams of the form , since they give the higher order terms in H. We have not succeeded in taking into account the contributions from the higher order diagrams such as  etc., so our result gives only the effect of the exchange interaction. The inclusion of those higher order diagrams, that is, the correlation effect, may probably decrease the effect in our results. Dresselhaus[6] has already treated the exchange effect, but we think our method will be more powerful and our results are clearer than his results. Retaining only the lowest order term in H, we get

$$\chi_{osc}^{(4)}=-\frac{\nu N_0}{V_A}\left(\frac{2}{\pi}\right)^{1/2}(\mu_0{}^0)^{1/2}$$
$$\times\left(\frac{e^4m}{2\hbar^2}\right)^{1/2}\frac{1}{H^{3/2}}\left\{\cos\left(\frac{\pi\zeta}{\mu_0H}-\frac{\pi}{4}\right)\right.$$
$$\left.\times C(p)-\sin\left(\frac{\pi\zeta}{\mu_0H}-\frac{\pi}{4}\right)S(p)\right\}$$

$$\left.\begin{matrix}C(p)\\S(p)\end{matrix}\right\}=\int_0^\infty\frac{dx}{x^{1/2}}J_{3/2}(x)$$
$$\times\int_0^1 dz\,J_0(x\sqrt{1-z^2})\left\{\begin{matrix}\cos(px^2z^2)\\\sin(px^2z^2)\end{matrix}\right.$$

$$p=\frac{\mu_0H}{4\pi\zeta} \qquad (6)$$

where V_A is atomic volume, ν is the number of free electron per atom, N_0 is Avogadro number and $\mu_0{}^0$ is the Bohr

magneton which contains true electronic mass m. The integrals $C(p)$ and $S(p)$ are evaluated numerically. Adding this $\chi_{osc}^{(1)}$ and $\chi_{osc}^{(0)}$ of the free electron gas,[2] we can estimate graphically the change of the phase constant of the oscillation. The results are plotted as the function of r_s ($4\pi r_s^3 a_0^3/3 = V_A/N_0$, a_0: Bohr radius). (Fig. 4). We have taken tentatively the values $m^*/m = 10^{-6}$ and

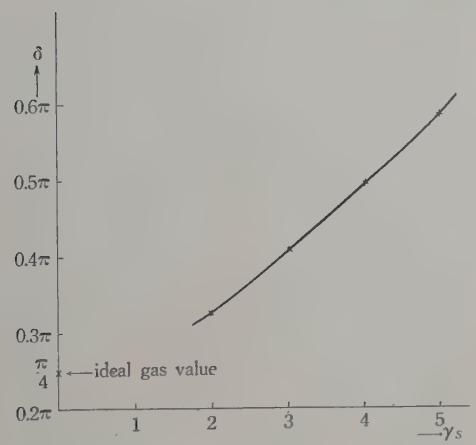


Fig. 4.

$\nu = 5 \times 10^{-2}$. These results, we think, are not inconsistent with the experimental facts though they are very few.[4] The details of our investigations will be published in the near future.

1) T. Matsubara, Prog. Theor. Phys. **14** (1955), 351.
2) E. H. Sondheimer and A. H. Wilson, Proc. Roy. Soc. **210A** (1951), 173.
3) J. Hubbard, Proc. Roy. Soc. **240A** (1957), 539.
4) T. G. Berlincourt and M. C. Steel, Phys. Rev. **95** (1954), 1421.
T. G. Berlincourt, Phys. Rev. **99** (1955), 1716.
5) H. Kanazawa and N. Matsudaira, Prog. Theor. Phys. **23** (1960), 433.
6) G. Dresselhaus, Phys. Rev. **114** (1959), 736.

S-Wave Pion-Nucleon Phase Shifts

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May 2, 1960

Previously, Chew, Goldberger, Low and Nambu (C-G-L-N)[1] have derived a formula of S-wave pion-nucleon phase shifts α_1 and α_3 from the non-subtracted dispersion relation in which the momentum transfer between the incident and scattered pions is held fixed, assuming that the imaginary part appearing in the dispersion integral is to be dominated by the 3-3 state. And they have shown that thus estimated $\alpha_1 - \alpha_3$ agrees with the experiments, and further $\alpha_1 + 2\alpha_3$ also is not inconsistent with the experiments if one cuts off the dispersion integral at about the energy of the threshold of the pion production. At first sight, however, it seems to be curious that the behaviour of S-wave is affected by 3-3 state having the different parity and angularmomentum in their dispersion relations. In this note, we shall show how this can be interpreted within the static meson theory.

We suppose their result is due to the change of the S-wave effective coupling constant by the P-wave cloud surrounding the nucleon core. Then we first assume simply that, if P-wave cloud is absent, the S-wave interaction could be represented by the Hamiltonian obtained by the Foldy transformation[2] from *Ps-Ps* theory, that is,

$$H_1=4\pi\left(\frac{g_0^2}{2M}\boldsymbol{\phi}^2+\frac{g_0^2}{(2M)^2}\boldsymbol{\tau}\cdot[\boldsymbol{\phi}\times\boldsymbol{\pi}]\right), \tag{1}$$

where M is nucleon mass, $\boldsymbol{\phi}$ is the field operator of pion, $\boldsymbol{\pi}\equiv\partial\boldsymbol{\phi}/\partial t$ and g_0 is the unrenormalized and unrationalized *Ps-Ps* coupling constant which is related by $f_0=g_0/2M$ with the *Ps-Pv* coupling constant, where meson mass is taken to be unity. H_1 will be changed, however, by the *P*-wave cloud surrounding the nucleon core. This effect can be estimated using the Chew-Low static theory[3] by taking the expectation value of H_1 with respect to the one-nucleon state followed by the *P*-wave cloud. Now, the expectation value of the first term in the right-hand side of (1), say H_c, which contributes to $\alpha_1+2\alpha_3$, is

$$\langle\Psi_0 H_c\Psi_0\rangle=4\pi g_0^2/2M\cdot\boldsymbol{\phi}^2. \tag{2}$$

Thus, as is expected, H_c is not changed by the *P*-wave cloud. On the other hand, the expectation value of the second term, say H_π, which contributes to $\alpha_1-\alpha_3$, is

$$\langle\Psi_0 H_\pi\Psi_0\rangle=4\pi\langle\Psi_0\boldsymbol{\tau}\Psi_0\rangle\cdot g_0^2/(2M)^2\cdot[\boldsymbol{\phi}\times\boldsymbol{\pi}], \tag{3}$$

where $\langle\Psi_0\boldsymbol{\tau}\Psi_0\rangle$ is not $\boldsymbol{\tau}$ and may be calculated explicitly by using Miyazawa's technique.[4] Thus the result is as follows:

$$\langle\Psi_0 H_\pi\Psi_0\rangle=4\pi\frac{1}{2M}\left[\frac{g_r^2}{2M}+\frac{4M}{\pi}\times\int_1^\infty\frac{d\omega}{q^2}\mathrm{Im}f_3^{(-)}(\omega)\right]\boldsymbol{\tau}\cdot[\boldsymbol{\phi}\times\boldsymbol{\pi}], \tag{4}$$

where g_r^2 is the unrationalized and renormalized coupling constant and $f_3^{(-)}(\omega)$ is nearly equal to $-f_{33}(\omega)/3$, if one assumes the 3-3 state is dominant. This expression completely agrees with C-G-L-N's result.[1] As has been stated previously, however, H_c is not changed by the *P*-wave cloud, further g_0^2 is always larger than g_r^2 in the static theory. Thus, the calculated $\alpha_1+2\alpha_3$ is larger than that calculated by H_c with the renormalized coupling constant, and of course does not agree with the experiments nor with C-G-L-N's result. In this respect, we can not conclude anything by such a simple interpretation, but we may suppose the C-G-L-N's expression of $\alpha_1+2\alpha_3$ is not accurate because their dispersion relation for the symmetrical part might not hold without one subtraction. Thus, it may not be excluded from our interpretation that the observed small $\alpha_1+2\alpha_3$ might be due to other origins such as the pion-pion interaction as has been pointed out by Miyazawa and Kawarabayashi.[5]

The author would like to thank Prof. K. Nakabayasi, I. Sato and the other members of Tohoku University for their constant interest and for helpful discussions.

1) G. F. Chew, M. L. Goldberger, F. E. Low and Y. Nambu, Phys. Rev. **106** (1957), 1337.
2) L. L. Foldy, Phys. Rev. **87** (1952), 973.
3) G. F. Chew and F. E. Low, Phys. Rev. **101** (1956), 1570.
4) H. Miyazawa, Phys. Rev. **101** (1956), 1564.
5) H. Miyazawa and K. Kawarabayashi, Prog. Theor. Phys. **23** (1960), 490.

An Example of a Scattering System in Jauch's Sense

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In a previous note[1] we considered a mathematical model of a quantum mechanical system which consists of two particles moving in the one-dimensional space and is described by the Hamiltonian $H=-\partial^2/\partial x_1^2-\partial^2/\partial x_2^2+V(x_2)+K$. Here $V(x_2)$ is a potential such that the Hamiltonian of the second particle $-d^2/dx_2^2+V(x_2)$ has one and only one eigenvalue λ and K is an integral operator of rank 1. For this system we showed that the existence of the *wave operators* $W_\pm=s\text{-}\lim_{t\to\pm\infty}[\exp(itH)\cdot\exp(-itH_0)]$, $H_0=-\partial^2/\partial x_1^2-\partial^2/\partial x_2^2$, does not in general imply the unitarity of the "scattering operator" $S=W_+^*W_-$. As has been pointed out,[1] however, this S cannot properly be called the scattering operator of the system, because this system should be considered a *multi-channel system* in which H_0 is merely one of the channel Hamiltonians.

In such a multi-channel problem, Jauch[2] defined the notion of a *scattering system*, generalizing the *simple scattering system*[3] in the case of a single-channel problem. Roughly speaking, a scattering system is a system in which the scattering operator exists and describes the scattering process completely. Jauch developed a general theory of scattering systems, assuming their existence. It seems, however, that so far there has been given no concrete example of the scattering system in this strict sense. The object of this note is to point out that the system mentioned above actually furnishes an example of the scattering system with the choice of channels stated below. Thus there is nothing strange, from the physical point of view, in the mathematical result given in Ref. 1). (As for the uniqueness of the choice of channels, see Zinnes.[4])

This system has two channels. One of them is characterized by the channel Hamiltonian H_0 and the other by $H_1=-\partial^2/\partial x_1^2+\lambda$. We denote by D_k, $k=0, 1$, the set of all square integrable functions $u=u(x_1, x_2)$ such that the limits $\lim_{t\to\pm\infty}\exp(itH)\exp(-itH_k)u=u_\pm^{(k)}$ exist and by $R_\pm^{(k)}$ the set of all such $u_\pm^{(k)}$. Then D_0 coincides[1] with the Hilbert space $L^2(E_2)$ of all square integrable functions $u(x_1, x_2)$; it can further be shown that D_1 consists of all functions of the type $u_1(x_1)\varphi(x_2)$, where $\varphi(x_2)$ is the eigenfunction of the second particle corresponding to the eigenvalue λ and $u_1(x_1)$ is an arbitrary function of $L^2(E_1)$. Physically speaking, D_0 is the aggregate of all the asymptotic states in which both particles are free and D_1 is that in which the first particle is free and the second lies in the bound state with the eigenfunction $\varphi(x_2)$.

Now, the essential requirement imposed on the scattering system is that (in a two-channel system as in our case) the relation

$$R_+^{(0)}\oplus R_+^{(1)}=N=R_-^{(0)}\oplus R_-^{(1)} \quad (1)$$

holds.[2] Here N is the subspace of all continuum states of H (strictly speaking, N is the *absolutely continuous subspace*[5] of $L^2(E_2)$ with respect to H). But these relations can be immediately verified using the result given in Ref. 1), showing that our system is indeed a scattering system in Jauch's sense.

Incidentally, we note further that a similar result holds for a system of finite number of particles, if the Hamiltonian of each individual particle has at most a finite number of bound states and K is, for example, a self-adjoint operator belonging to the *trace class*.[5]

In these examples the interaction K is effective only when all constituent particles gather near the scattering center, the origin of the configuration space. The method used above, which is based on the possibility of the separation of variables in $H-K$, cannot be applied to the physically more interesting problem in which K includes the mutual interactions depending only on the relative coordinates of the constituent particles. This problem is treated by Hack[6] to some extent. It seems, however, that so far the fundamental relations corresponding to (1) has not been proved.

1) T. Kato and S. T. Kuroda, Nuovo Cimento, **14** (1959), 1102.
2) J. M. Jauch, Helv. Phys. Acta, **31** (1958), 661.
3) J. M. Jauch, Helv. Phys. Acta, **31** (1958), 127.
4) I. I. Zinnes, Nuovo Cimento, **12** Suppl. No. 1 (1959), 87.
5) S. T. Kuroda, Nuovo Cimento, **12** (1959), 431.
6) M. N. Hack, Nuovo Cimento, **13** (1959), 231.

On the Non-Axial Deformation of the Medium and Heavy Even-Even Nuclei

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In this paper, we calculate systematically the asymmetry of the deformation of medium and heavy even-even nuclei with $N \gtrsim 50$ (N: neutron number) from their energy ratio of low-lying states on the basis of the asymmetric rotator model,[1] and point out that these even-even nuclei are divided into two groups each of which having the nearly equal asymmetry.

In the asymmetric rotator model the energy levels are classified into two types: the normal rotational levels and the anomalous rotational levels. The normal rotational levels correspond to the ground state rotational bound of the axially symmetric rotator and the dependence to the asymmetry, γ, of these levels are rather weak but the anomalous rotational levels have very strong γ-dependence. The energy ratio between these rotational levels are determined by *one* parameter γ_0. Therefore, we can estimate the value of γ_0 from the experimental data of the ratio.

In this model, there are two spin 2 states, of which the lower one is the first excited state and the normal rotational level and the higher one is the anomalous rotational level. If we write

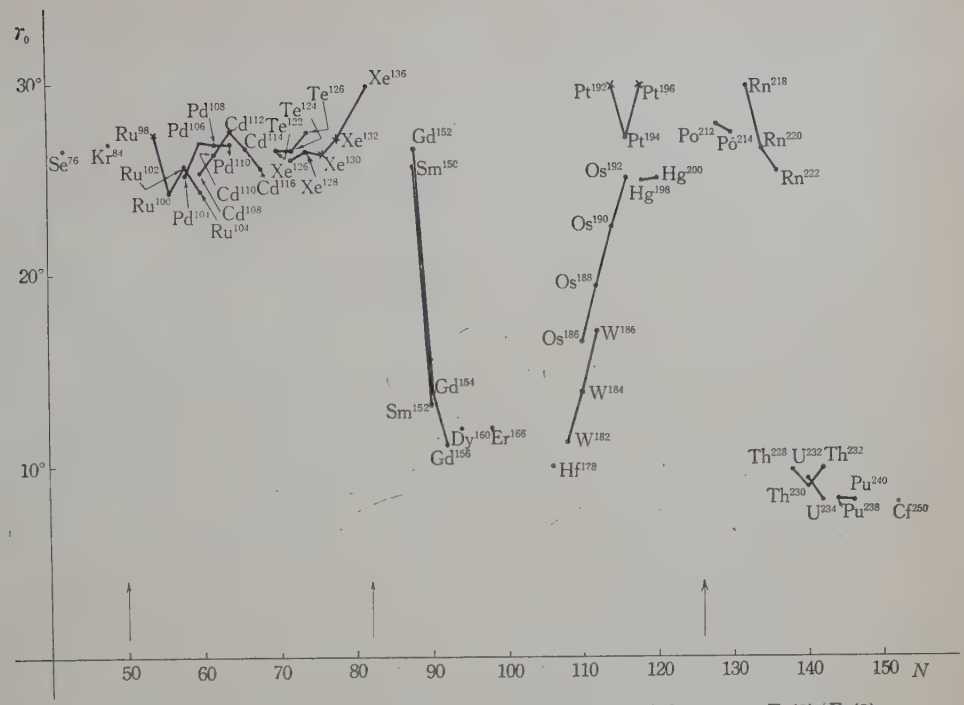

Fig. 1. N dependence of the asymmetry γ_0 calculated from $p_1=E_2(2)/E_1(2)$.

the energy ratio of these two levels as $p_1=E_2(2)/E_1(2)$, then the asymmetry $\gamma_0(p_1)$ is given by the following formula,

$$\sin^2(3\gamma_0(p_1))=(9/2)(p_1/(p_1+1)^2).$$

$\gamma_0(p_1)$ of about 50 nuclei with $N\gtrsim 50$ are shown in Fig. 1. Here, we restricted γ_0 to $0\leq\gamma_0\leq\pi/6$. There is only one spin 3 state and this state is also the anomalous rotational level. If we introduce $p_2=E(3)/E_1(2)$, we find $\gamma_0(p_2)$ by

$$\sin^2(3\gamma_0(p_2))=(9/2)(p_2-1)/(p_2^2),$$

In Davidov-Filippov's asymmetric rotator model, the minimum value of p_1 is equal to 2, therefore when the experimental value of p_1 is smaller than 2, we suppose that in such nucleus the energy levels are pushed down by some perturbation such as the rotation-vibration interaction, and assume that $\gamma_0=30°$. From Fig. 1, it is found that except for the nuclei of the near closed-shell regions, almost all nuclei with $N\gtrsim 50$ are grouped into $\gamma_0\sim 26°$ and $\gamma_0\sim 10°$. The transitions between these two groups occur at $N=88$—90, $N=110$—116 and $N=136$—138. Moreover, this grouping of the values of γ_0 corresponds to the so-called vibrational region and rotational region respectively. We can see that in the $N=110$—116 region the level structure changes rapidly but continuously with γ_0.

The nuclei which have the $E(3)$ state of this model are very few, being only about 16 in $N\gtrsim 50$. In these nuclei, $\gamma_0(p_2)$ agrees well with $\gamma_0(p_1)$ in good

approximation. This fact means not only that the relation $E_1(2)+E_2(2)=E(3)$ is satisfied in good approximation, but also that these levels which are interpreted usually as the γ-vibrational band in Bohr-Mottelson's collective model[2] can be interpreted as the anomalous rotational levels of the non-axially symmetric nucleus.

If we use the curves of the γ-dependence of energy ratios calculated by Davidov and Filippov,[1] we can determine the approximate value of γ_0 for the normal rotational levels. These γ_0 are somewhat larger than γ_0 determined from the anomalous rotational levels. However, if we take into account experimental errors and inaccuracy in the assignment of γ_0 from normal rotational levels, we may say that γ_0 determined from the anomalous and normal rotational level is rather in good agreement.

From the above systematics, even the nuclei which have been considered to have axially symmetric equilibrium deformation have finite asymmetry, $\gamma_0\sim 10°$, more strictly speaking, $\gamma_0\sim 12°$ to rare earth region and $\gamma_0\sim 9°$ to actinide region.

In Bohr's collective model,[2] the equilibrium value for axial-asymmetry is zero. Gottfried[3] calculated the asymmetry from the minimum total energy for the particles system moving in the anisotropic oscillator potential

$$V(r)=\frac{1}{2}m\omega^2[(x^2\xi^2+y^2\xi^{-2})\lambda+z^2\lambda^{-2}],$$

$$0\leq\lambda\leq\infty,\ 0\leq\xi\leq\infty.$$

He found the maximum asymmetry $\xi_0=1.045$ to the particle number $A=40\sim 70$ and the change of the electric-quadrupole moment due to this asymmetry $dQ=0.59\%$. From these results he concluded that he could neglect the effects of the asymmetry of the deformation. The value for our parameter to the asymmetry corresponding to this $\xi_0=1.045$, however, amounts to $\gamma_0\sim 18°$. Therefore, this fact shows that we can not necessarily neglect the effects of the asymmetry.

We found that, from our systematics, the medium and heavy even-even nuclei are grouped into $\gamma_0\sim 26°$ and $\gamma_0\sim 10°$ and, at the special numbers of neutrons, the rapid transitions between these two groups take place. This agrees qualitatively with the results of Gursky.[4]

1) A. S. Davidov and G. F. Filippov, Nuclear Phys. **8** (1957), 237.
A. S. Davidov and V. S. Rostovsky, Nuclear Phys. **12** (1959), 58.
2) K. Alder, A. Bohr, T. Huus, B. Mottelson and A. Winther Rev. Mod. Phys. **28** (1956), 432.
A. Bohr, Dan. Mat. Fys. Medd. **26** (1952), No. 14.
3) K. Gottfried, Phys. Rev. **103** (1956), 1017.
4) M. Gursky, Phys. Rev. **98** (1955), 1205 (A). See also L. Wilets and M. Jean, Phys. Rev. **102** (1956), 788.

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On the Quantization of Physical Space-time Operators

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In order to find a clue to extending the conventional field theory in a consistent way, a physical interpretation of Dirac-Bergmann's field theory is proposed from the point of view of the uncertainty principle. The physical space-time operators introduced by them are connected with the measurement of the position. Further a new formulation of the auxiliary condition, which is the direct consequence of the quantization of the physical space-time operators, is constructed, and then the consistency and the detailed structure of the formalism are made clear.

§ 1. Introduction

In the conventional field theory, space-time coordinates are treated as c-number parameters, and field operators (consequently energy-momentum density, current density and such quantities are expressed as the products of field operators also) are represented as their functions. On the other hand, many authors[1] investigated the possibility to regard space-time coordinates as operators, or to quantize them, in order to get rid of the so-called divergence difficulties. However, no promising guiding principle to quantize space-time coordinates has been found. Among many attempts, a series of investigations[2,-4] which was begun by the famous Dirac's work[5] has an exceptional attraction for the present author, although these works were developed regardless of the divergence difficulties. For their quantization of the physical coordinate operators seems to express the uncertainty principle in the measurements, as will be seen in the following discussions. The purpose of this article is to make the physical meaning of the quantization clear and to analyse the nature of the extended Hilbert space.

The meaning of the uncertainty principle in quantum mechanics is as follows. As long as the particle, for which the uncertainty relation between its position and momentum holds, is used in the observation of the motion of some particle, the position and the momentum of the object also cannot be measured simultaneously and exactly. If any particle which is free from the uncertainty principle does exist, we would be able to use this particle to make the simultaneous measurement of the position and the momentum of the object, whether they are elementary or not. It may in fact be possible to take the standpoint that the uncertainty relation is derived from such a theory that introduces some hidden variable free from the uncertainty principle, as Bohm[6] did. However, our experience seems to require

that the uncertainty principle holds for all particles. In the conventional field theory, elementary particles are represented by the quantized feld operator $\varphi(x)$. But the space-time coordinates x^μ themselves should not be considered as the objects of the uncertainty relation. Therefore we must introduce new variables X^μ's as the objects, which will be called physical space-time operators in the following.

In order to find a clue to introducing the physical space-time operator X^μ in the field theory, let us begin with the consideration of the case of quantum mechanics. It is the time parameter t which corresponds to the parameter x^μ in field theory. Now any dynamical variable $F(t)$ at an arbitrary time t can be expressed by the matrix elements between the energy eigenstates

$$\langle E'|F(t)|E''\rangle. \tag{1·1}$$

However, such matrix elements themselves has no direct physical meaning, because the simultaneous measurements of the time t and the energy E are not allowed unless the uncertainty relation

$$\Delta t \Delta E \gtrsim h, \tag{1·2}$$

is discarded. Further, the direct comparison of such quantities with experiment is not necessary. Does it mean the theory contains the superfluous quantities which have no correspondence to the actual experiments? Should we try to construct a theory in which such superfluous quantities are eliminated? In fact, we can eliminate the parameter t from the theory, if we can introduce the time operator T which satisfies the commutation relation

$$[T, H]=i\hbar \tag{1·3}$$

as will be exhibited in the Appendix.* Here H is the Hamiltonian operator of the system. The operator T can be interpreted to represent the measurable physical time, and the commutation relation (1·3) is nothing but the operator expression of the uncertainty relation (1·2).

In the conventional field theory, there is a similar situation. Any dynamical operator $F(x)$, which is the product of field operators, can be expressed by the matrix elements between the eigenstates of the total energy-momentum operator P_μ,

$$\langle P_\mu'|F(x)|P_\mu''\rangle. \tag{1·4}$$

However, it is physically meaningless to refer to the matrix elements of any variables at the definite space-time point between the energy-momentum eigenstates, on account of the uncertainty relation of the position and the energy-momentum of the test-body. At first sight, one may imagine that it is possible to extend the aforementioned method in quantum mechanics to field theory, keeping in mind the correspondence between the time parameter t in the former and the space-time

* A similar account was given independently by F. Engelmann and E. Fick.[7]

parameter x^μ in the latter. But in field theory, we must consider infinitely many space points at one time (the spacelike surface), and generally it is not possible to use only one set of the space-time coordinates X^μ's.* Therefore we must preserve the space-time parameters, which we shall write as $u^s (s=0, 1, 2, 3)$ in the following, and impose the commutation relation

$$[X^\mu(u),\ P_\nu(u')]=i\hbar\,\delta_\nu{}^\mu\,\delta_3(u-u') \qquad (1\cdot5)$$

$$(\text{for } u^0=u^{0\prime}, \text{ and } \mu, \nu=0, 1, 2, 3),$$

on the physical space-time operators $X^\mu(u)$'s. $X^\mu(u)$'s are the functions of the parameter u^s and implicitly depend on the energy-momentum density operator $p_\nu(u)$ through u^s. Then we cannot refer to the physical space-time independently of the behaviour of the energy momentum of the system. (This will be important in the extremely high energy phenomena or the phenomena at an extremely small distance.)

This may be regarded as the natural extension of the commutation relation (1·3). For the relation (1·5) is the differential form of the relation

$$[X^\mu(u),\ P_\nu]=i\hbar\,\delta_\nu{}^\mu, \qquad (1\cdot6)$$

where P_μ is the total energy momentum operator

$$P_\mu=\int p_\mu(u)\,d^3u. \qquad (1\cdot7)$$

The relation (1·5) just means that (1·6) holds even if the energy-momentum of the system is confined in a very small region as the case of Finkelstein's particle model.[8]

The parameter u^s has no physical meaning other than a kind of index specifying each point in the four-dimensional manifold. Therefore the theory must be invariant under the arbitrary (continuous ?) coordinate transformations by u^s. Such quantities as distance and metric, which characterize the physical space, are defined in terms of the fields $X^\mu(u)$'s. For example, the square of line element is given by

$$ds^2=g_{\mu\nu}\,dX^\mu\,dX^\nu=g_{\mu\nu}(u)\,\frac{\partial X^\mu}{\partial u^r}\,\frac{\partial X^\nu}{\partial u^s}\,du^r\,du^s. \qquad (1\cdot8)$$

Such point of view is similar to the one in the theory of general relativity where the metric of the space is determined by the distribution of matter in its neighborhood.

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* For instance, Finkelstein[8] considered in his unified field theory that an elementary particle was represented by the field confined in a small space-region, and defined its position operator X^i and kinematical energy-momentum of each particle P_j by the center of gravity and the total energy-momentum of the field, respectively. Then they satisfy the commutation relation

$$[X^i, P_j]=i\hbar\,\delta_j{}^i \quad (i, j=1, 2, 3).$$

Of course, such a definition of X^i may be reasonable only in the case of one particle problem, but it may be considered insufficient in this sense.

Here we must make a comment about the structure of the Hilbert space extended by quantizing the parameter t or x^μ. The operator T satisfying the commutation relation (1·3) cannot be defined in the ordinary Hilbert space, because usually the dynamical systems are considered to have the lowest energy, namely the spectrum of the Hamiltonian H cannot distribute from $-\infty$ to $+\infty$.[9] Similarly in the field theory the eigenvalues of the total energy-momentum operator of the physical system, P_μ', are restricted by the conditions $\sum_i P_i'^2 - P_0'^2 \leq 0$ and $P_0' \geq 0$. Therefore the operator X^μ, for which the covariant commutation relation (1·6) or (1·5) holds, cannot be defined in the Hilbert space only with the positive definite norm, even if the bases are increased so as to include the new dynamical variable X^μ. Therefore if we want to introduce the operator X^μ, it is expected that we shall be forced to consider the Hilbert space with the indefinite norm, in other words the negative energy states.

Now the commutation relation (1·5) can be obtained in a sense by simplifying and rearranging the formulation of the quantization of the physical space-time variables developed by Dirac,[5] Bergmann and the others,[2)-4)] so as to make it fit for the present purpose (§ 2). In § 3, by analysing the properties of some constraints, it will also be shown that the quantization is nothing but a way to formulate the uncertainty principle of the physical space-time. Therefore our speculation which leads us to the commutation relation (1·5) may be considered as partly justified. § 4 will be offered to reformulate the theory in the form which may be called "altered theory" after Belinfante.[10] In this form, the structure of the extended Hilbert space can be seen more easily, and there we shall encounter with the indefinite Hilbert space, as was already expected in the aforementioned speculation.

§ 2. Quantization of physical space-time operators

The physical space-time operator was introduced by Bergmann and many other authors[2)-4)] in the course of the quantization of the gravitational field. But for the introduction of the space-time operator, the gravitational field does not play any role. The essential points are the separation of the physical space-time variables x^μ from the back ground of the parameter space u^ν and Dirac's generalized variational method[5] for the system of the dynamical variables which are not necessarily independent of one another. Further, the existence of the gravitational field does not only make the theory very complicated, but also is of no use for investigating the physical meaning of the quantization of the operator x^μ. Therefore in this section, we begin with the rearrangement of their formalism for the system of ordinary fields such as electromagnetic field, spinor field, or scalar field, to make it suitable for obtaining the commutation relation which is physically equivalent to (1·5).

We start from the action integral*

$$S\equiv\int L(y_A,\ y_{A,\sigma})d^4x\equiv\int JLd^4u. \tag{2·1}$$

Here L is the ordinary Lagrangian density and the function of the field variable of the kind A, y_A, and of its derivative, $y_{A,\sigma}\equiv\partial y_A/\partial x^\sigma(\sigma=0,\ 1,\ 2,\ 3)$, which can be expressed $y_{A,\sigma}\equiv y_{A|s}\cdot\partial u^s/\partial x^\sigma$ in terms of the derivative with respect to the parameter u^s, $y_{A|s}\equiv\partial y_A/\partial u^s(s=0,\ 1,\ 2,\ 3)$.** Further J is the Jacobian $J\equiv|\partial(x)/\partial(u)|$, and JL is regarded throughout as a function of y_A, $y_{A|i}$, $x^\rho_{|i}$ $(i=1,\ 2,\ 3)$,*** and the so-called velocities $y_{A|0}\equiv\dot{y}_A$ and $x^\rho_{|0}\equiv\dot{x}^\rho$. The canonical momenta conjugate to x^ρ and y_A are defined by

$$\pi^A=\frac{\partial(JL)}{\partial\dot{y}_A}\equiv\frac{\partial L}{\partial y_{A,\sigma}}l_\sigma, \tag{2·2}$$

and

$$\lambda_\rho=\frac{\partial(JL)}{\partial\dot{x}^\rho}\equiv-T_\rho{}^\sigma l_\sigma, \tag{2·3}$$

respectively. Here

$$T_\rho{}^\sigma\equiv\frac{\partial L}{\partial y_{A,\sigma}}y_{A,\rho}-\delta_\rho{}^\sigma L, \tag{2·4}$$

and $l_\sigma\equiv J\cdot\partial u^0/\partial x^\sigma$ which is independent of the velocities. The geometrical meaning of l_ρ is nothing but the normal vector of the super-surface $u^0=$const. in the physical space-time world $\{x^\sigma\}$, since

$$l_\rho x^\rho_{|i}\equiv J\frac{\partial u^0}{\partial x^\rho}\frac{\partial x^\rho}{\partial u^i}\equiv J\frac{\partial u^0}{\partial u^i}\equiv 0.$$

By eliminating the velocities, $\dot{y}_A$ and $\dot{x}^\rho$, several independent weak equations, the so-called " constraints ",

$$\phi_\alpha(y_A,\ y_{A|i},\ x^\rho_{|i},\ \pi^A,\ \lambda_\rho)=0\quad(\alpha=1,\ 2,\ \cdots M), \tag{2·5}$$

are obtained from the equations (2·2) and (2·3). The variational principle $\delta S=0$ gives the field equations

* We distinguish between the strong equation and the weak equation after Dirac's definition.[5] The former is interpreted as the operator equation and the latter as the condition imposed on the state vector in the quantized theory.

** In the quantized theory, the order of the factors in the products of non-commutative quantities must be defined. It seems suitable to define the order of the factors in $y_{A,\sigma}$ implicitly as

$$\frac{\partial y_A}{\partial u^s}\equiv\frac{1}{2}\left(\frac{\partial x^\rho}{\partial u^s}\frac{\partial y_A}{\partial x^\rho}+\frac{\partial y_A}{\partial x^\rho}\frac{\partial x^\rho}{\partial u^s}\right),$$

but in this paper we shall not enter into the detailed discussion about this point.

*** The indices i, j, k refer to 1, 2, 3, and r, s, t, μ, ν, ρ, and σ run from 0 to 3. In this article, we follow Pirani-Schild's notations.[3]

$$\frac{\partial(JL)}{\partial y_A}-\left(\frac{\partial(JL)}{\partial y_{A|s}}\right)_{|s}\equiv J\left[\frac{\partial L}{\partial y_A}-\left(\frac{\partial L}{\partial y_{A,\sigma}}\right)_{,\sigma}\right]=0, \tag{2·6}$$

and

$$\frac{\partial(JL)}{\partial x^\sigma}-\left(\frac{\partial(JL)}{\partial x^\sigma_{|s}}\right)_{|s}\equiv J\frac{\partial T_\rho{}^\sigma}{\partial x^\sigma}=0. \tag{2·7}$$

The latter equations are nothing but the conservation law of energy-mementum of the field, and should be derived from the former. Therefore (2·5) ought to contain at least four constraints which depend on λ_ρ. In such simple problems as cited below, $T_\rho{}^\sigma l_\sigma$ in Eq. (2·3) has the form which does not contain the velocities,

$$T_\rho{}^\sigma l_\sigma=p_\rho(y_A,\ y_{A|s},\ x^\rho_{|s},\ \pi^A). \tag{2·8}$$

Therefore the concerned four constraints are

$$\varphi_\rho\equiv\lambda_\rho+p_\rho=0. \tag{2·9}$$

Here we may write down the expressions of p_ρ for the simple case as examples;

(i) electromagnetic field (A_μ)[3]

$$p_\rho\equiv l^{-2}A_{\alpha\beta\rho}(l^\alpha\pi^\beta-l^\beta\pi^\alpha)-l_\rho l^{-2}\frac{1}{2}g_{\alpha\beta}\pi^\alpha\pi^\beta$$

$$-\frac{1}{4}l_\rho l^{-2}g^{\lambda\sigma}(2g^{\alpha\nu}g^{\beta\mu}-g^{\alpha\mu}g^{\beta\nu})A_{\alpha\beta\lambda}A_{\mu\nu\sigma}, \tag{2·10}$$

(ii) fermion field (ψ)[4]

$$p_\rho\equiv\frac{\hbar}{2}(\bar\psi_{\rho\mu}\gamma^\mu\psi-\bar\psi\gamma^\mu\psi_{\rho\mu})+\hbar m l_\rho\bar\psi\psi, \tag{2·10'}$$

(iii) neutral scalar field (ϕ)

$$p_\rho\equiv l^{-2}l^\beta\phi_{\rho\beta}\pi+\frac{1}{2}l_\rho[l^{-4}l^\alpha l^\beta g^{\mu\sigma}\phi_{\mu\alpha}\phi_{\sigma\beta}-l^{-2}\pi^2+\mu^2\phi^2]. \tag{2·10''}$$

There, $y_{A\alpha\beta}\equiv y_{A,\alpha}l_\beta-y_{A,\beta}l_\alpha\equiv y_{A|i}l^i_{\alpha\beta}$, $l^i_{\alpha\beta}\equiv\dfrac{\partial l_\beta}{\partial x^\alpha_{|i}}\equiv J\left(\dfrac{\partial u^i}{\partial x^\alpha}\dfrac{\partial u^0}{\partial x^\beta}-\dfrac{\partial u^i}{\partial x^\beta}\dfrac{\partial u^0}{\partial x^\alpha}\right)$ and $l^2=l_\rho l^\rho$. They are also independent of velocities.

Hamiltonian density defined by

$$JH\equiv\pi^A\dot y_A+\lambda_\rho\dot x^\rho-JL \tag{2·11}$$

can be generally rewritten as

$$JH\equiv\dot x^\rho\varphi_\rho+v_{\alpha'}\phi_{\alpha'}=0, \tag{2·12}$$

as proved by Dirac,[5] where $\phi_{\alpha'}$ is the other constraint different from φ_ρ and $v_{\alpha'}$ is some quantity which depends on the velocities. It is well known that the number of $\phi_{\alpha'}$ is one, two and four for the electromagnetic field, the fermion field and the gravitational field respectively.[10]

The Poisson bracket can be reinterpreted as the commutation relation, except for the case of fermion field, and then

$$[y_A(u),\ \pi^B(u')\ \ \ \

total Hilbert space is the direct sum of the sub-spaces with respect to $y_A(u)$ and the one of $x^\rho(u)$. Here we must take care of the fact that for a set of u's there are infinitely many eigenvalues $x'^\rho(u)$ which extend from $-\infty$ to $+\infty$, and further that the operators $x^\rho(u)$ for different u^i's (on the same u^0) mutually commute. Expanding the arbitrary state vector $\Psi[u^0]$ which is defined on $[u^0]$ only with respect to the subspace of $x^\rho(u)$, it can be written as

$$\Psi[u^0]=\int_{-\infty}^{\infty} \prod_{[u^0]} d^4x'^\sigma(u)\cdot\varphi(\{x'(u)\})\cdot\prod_{[u^0]}\xi(x'^\sigma(u)). \tag{3·1}$$

Here $\xi(x'^\sigma(u))$ is the eigenvector corresponding to the eigenvalue $x'^\sigma(u)$, and $\prod_{[u^0]}$ means the product taken over all points on the surface $[u^0]$. $\varphi(\{x'(u)\})$ is still a state vector in the subspace concerning the field variables.

The ortho-normal relation of $\xi(x'^\sigma(u))$ is

$$\xi^*(x'^\sigma(u))\xi(x''^\sigma(u'))=\delta_3(u,\ u')\delta_4(x'^\sigma-x''^\sigma). \tag{3·2}$$

Further, in our representation the operator λ_ρ can be expressed as

$$\lambda_\rho(u)=-i\hbar\frac{\delta}{\delta x'^\rho(u)}, \tag{3·3}$$

by the commutation relation (2·14). Therefore, the constraints

$$\varphi_\rho\Psi=(\lambda_\rho+p_\rho)\Psi=0 \tag{3·4}$$

can be written as

$$i\hbar\frac{\delta\varphi(\{x'(u)\})}{\delta x'^\rho(u)}=p_\rho'(u)\varphi(\{x'(u)\}), \tag{3·5}$$

where $p_\rho'(u)$ is such expression that is obtained by substituting the eigenvalues $x_{|i}'^\rho(u)$ in the place of the operator $x_{|i}^\rho(u)$ in $p_\rho(u)$ which is the function of y_A, $y_{A|i}$, $x_{|i}^\rho$, π^A. The consistency between the equations (3·5),

$$i\hbar\left[\frac{\delta p_\rho'(u')}{\delta x'^\sigma(u)}-\frac{\delta p_\sigma'(u)}{\delta x'^\rho(u')}\right]=[p_\sigma'(u),\ p_\rho'(u')], \tag{3·6}$$

is guaranteed by the first group of Eq. (2·15).* In the expansion of $\Psi[u^0]$ in (3·1), each coefficient $\varphi(\{x'(u)\})$ has the similar properties to the state vector in the conventional field theory. The essential difference between our formalism and the conventional one is in the character of the set of $x'^\rho(u)$, $\{x'(u)\}$, which is an assembly of the values of $x'^\rho(u)$ for all the points on $[u^0]$. If there were not

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* For any operator F which is the function of y_A, $y_{A|i}$, x'^ρ, $x_{|i}'^\rho$ and π^A, we have

$$\frac{\delta F(u')}{\delta x'^\rho(u)}=\delta_3(u-u')\frac{\partial F(u')}{\delta x'^\rho(u')}-\sum_i\frac{\partial\delta_3(u-u')}{\partial u^i}\frac{\partial F(u')}{\partial x_{|i}'^\rho(u')}.$$

For the simple examples referred to in § 2, Eqs. (2·10), etc., Eq. (3·6) can be verified by the direct calculations.

any constraint such as (3·4), there would be no restriction on the nature of the assembly $\{x'(u)\}$, and then $\{x'(u)\}$ might not construct even a continuous surface in the physical space-time world. For $x'^{\rho}(u)$ can have any value between $-\infty$ and $+\infty$, and $x'^{\rho}(u)$ does not depend on the values at u'^{i} different from u^{i}, as emphasized at the beginning of this section. It is desirable that the constraint (3·5) restricts the assembly $\{x'(u)\}$ to be spacelike surface in the physical space-time world. Of course, it will be impossible to see what conditions are generally imposed on the assembly. But in the simple examples, we can see a feature of the restriction through the explicit forms of $p_{\rho}(u)$ which are given in Eqs. (2·10), (2·10′) and (2·10″). In these examples, each term of $p_{\rho}(u)$ always contains the factor l^{-2}.* Therefore $p_{\rho}'(u)$ becomes infinity when the assembly $\{x'(u)\}$ constitutes a continuous surface and touches the light cone, because the normal vector of the surface l_{ρ} there satisfies $l^{2}=0$. The equation (3·5) tells us that at these boundaries $\varphi(\{x'(u)\})$ change discretely.

Now, if the constraint (3·5) and/or any other situation such as the existence of the solution of the equations of motion (2·6) select the spacelike surface out of $\{x'(u)\}$, the coefficient $\varphi(\{x'(u)\})$ may correspond to the state vector in the conventional field theory. Then Eq. (3·5) is nothing but the four-dimensional generalization of the Schroedinger equation.[11] The superposition of $\varphi(\{x'(u)\})$ in the expansion (3·1) may be regarded as an expression of the uncertainty principle $\Delta x^{\sigma}\Delta p_{\sigma}\gtrsim h$. Further the matrix elements of any operator $F(x(u))$, which correspond to $\langle P_{\mu}'|F(x)|P_{\mu}''\rangle$ in the conventional theory, are represented as

$$(\Psi_{\alpha},\ F(x(u))\Psi_{\beta})=\int_{[u^{0}]} II\, d^{4}x'^{\sigma}(u)\cdot\varphi_{\alpha}^{*}(\{x'(u)\})F(x'(u))\varphi_{\beta}(\{x'(u)\}). \quad (3\cdot7)$$

The objection argued in the Introduction about the uncertainty principle with respect to x^{μ} and P_{ν} in the conventional field theory will be removed, even if we take the eigenstate of the energy-momentum density of the system as $\varphi_{\alpha}(\{x'(u)\})$'s. For the expression is superposed over the eigenvalues $x'^{\rho}(u)$ on the right-hand side of Eq. (3·7).

If the correspondence holds literally, our theory will give the same result as the conventional one, and will be nothing but its reformulation. However, in the author's opinion, our formalism would be beyond the conventional one, although it will be very difficult to see how the former deviates from the latter. In order to explore this problem we shall reformulate our theory in the fashion of the altered theory in the next section.

Concluding this section, we may call our attention to the fact that the operator x^{ρ} is not the "true observable". Usually such variables that commute with all the constraints are called "true observables", and regarded as the clearly defined

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* In Eq. (2·10′), at first sight, it does not seem to be the case, but, for example, the last term becomes $ml_{\rho}\bar{\psi}\psi\equiv ml^{-2}l_{\rho}\pi\gamma^{\lambda}l_{\lambda}\psi$ as $\pi\equiv\bar{\psi}\gamma^{\mu}l_{\mu}$.

physical quantities.[12),13)] The operator $x^\rho(u)$ do not commute with the constraints $\varphi_\rho=0$, and their physical meaning is not necessarily clear.

§ 4. New formulation of the auxiliary condition

The existence of the constraints in our formalism is due to taking the much more variables than the really independent ones in the variational procedure. As the result the Hilbert space involves the superfluous bases which have no really physical meaning. As recipes for such situations we have two methods, the "true theory" and the "altered theory".[10)] In the former treatment, only the true observables, which is defined as the quantities commutable with all constraints, are quantized as the independent variables. Therefore the commutation relations become different from the formal results such as Eqs. (2·13) and (2·14). In the altered theory, the suitable additional terms are added to the Lagrangian density, so as to make all the equations strong ones. The state vector, consequently the Hilbert space, is restricted by the auxiliary conditions which make the additional terms of Lagrangian and the constraints appearing in the original formalism of the preceding sections vanish simultaneously. A well-known example of such a problem is the quantum-electrodynamics. There the method of Coulomb gauge, where only the transverse waves are quantized and the longitudinal waves are rearranged to Coulomb force, is the true theory. The method corresponding to the altered theory is Fermi's one which uses the Lorentz gauge and the Lorentz condition is regarded as the auxiliary condition on the state vector.[14)] For the gravitational field, the altered theory is formulated by Gupta[15)] in the linear approximation, but the true theory has not been constructed.

Now for our problem it will be very difficult to make the theory in the form of the true theory. Bergmann et al.[12)] regarded $x^\rho(u)$ as c-number and the theory reduced to the conventional one if the surface $[u^0]$ is restricted to be spacelike. But this is not the true theory, because the field variables y_A and π^A do not commute with the constraints (2·9), that is, they are not the true variables. In this section we shall reformulate our theory in the form of the altered theory, and try to make the first step to understand the physical meaning of our scheme and to extend the conventional field theory in a consistent way.

In this section the original Lagrangian density which was used in § 2 is denoted by L_0 and the additional terms by L_{add}. Further, we assume that the other constraints than $\varphi_\rho=0$ are removed by the already known method of the altered theory, and that the corresponding additional Lagrangian density is already involved in L_0. Therefore the modified field equations are all strong equations, and becomes

$$\frac{\partial(JL_0)}{\partial y_A}-\left(\frac{\partial(JL_0)}{\partial y_{A|s}}\right)_{|s}=J\left[\frac{\partial L_0}{\partial y_A}-\left(\frac{\partial L_0}{\partial y_{A,\sigma}}\right)_{,\sigma}\right]\equiv\xi^A \qquad (4\cdot1)$$

and

$$\frac{\partial(JL_0)}{\partial x^\sigma}-\left(\frac{\partial(JL_0)}{\partial x^\sigma_{|s}}\right)_{|s}\equiv J\frac{\partial T_\sigma{}^\rho}{\partial x^\rho}\equiv\zeta_\sigma. \tag{4·2}$$

Here

$$\zeta^A\equiv\left(\frac{\partial(JL_{add})}{\partial y_{A|s}}\right)_{|s}-\frac{\partial(JL_{add})}{\partial y_A} \tag{4·3}$$

and

$$\zeta_\sigma\equiv\left(\frac{\partial(JL_{add})}{\partial x^\sigma_{|s}}\right)_{|s}-\frac{\partial(JL_{add})}{\partial x^\sigma}, \tag{4·4}$$

and $T_\sigma{}^\rho$ is the same expression as Eq. (2·4) with L_0 instead of L. Canonical conjugate variables are

$$\pi^A\equiv\frac{\partial(JL_0)}{\partial\dot{y}_A}+\eta^A,\quad \eta^A\equiv\frac{\partial(JL_{add})}{\partial\dot{y}_A} \tag{4·5}$$

and

$$\lambda_\rho\equiv\frac{\partial(JL_0)}{\partial\dot{x}^\rho}+\eta_\rho,\quad \eta_\rho\equiv\frac{\partial(JL_{add})}{\partial\dot{x}_\rho}. \tag{4·6}$$

Now the form of L_{add} must be determined. First L_{add} must contain $x^\sigma_{|s}$ explicitly. The left-hand side of Eqs. (4·1) and (4·2) is not independent as in the case of (2·6) and (2·7), and we have

$$\zeta_\sigma\equiv-\zeta^A y_{A,\sigma}. \tag{4·7}$$

It results from the fact that $x^\sigma_{|s}$ are involved in L_0 only through the form $y_{A,\sigma}\equiv y_{|s}\cdot\partial u^s/\partial x^\sigma$. If L_{add} also does not involve $x^\sigma_{|s}$ explicitly, the right-hand side of Eqs. (4·3) and (4·4) satisfy Eq. (4·7) identically. Then the field equations are not independent, and cannot be strong equations. Secondly we may suppose that L_{add} is a function of $x^\sigma_{|s}$ alone. Then $\zeta^A\equiv0$ and $\eta^A\equiv0$, and the equations of motion and the expressions of canonical conjugate variables by y_A have the same forms as (2·6) and (2·2), where L must be replaced by L_0. But the difference is that the new equations are strong equations, in contrast with the weak equations (2·6) and (2·2). The equations of motion of x^σ become

$$\left(\frac{\partial(JL_{add})}{\partial x^\sigma_{|s}}\right)_{|s}-\frac{\partial(JL_{add})}{\partial x^\sigma}\equiv0, \tag{4·8}$$

owing to Eqs. (4·1)–(4·4) and (4·7). On the other hand, Eq. (4·6) can be rewritten as

$$\lambda_\rho\equiv-p_\rho+\eta_\rho,\quad p_\rho\equiv T_\rho{}^\sigma l_\sigma, \tag{4·9}$$

by the definitions (2·3) and (2·8). For the old constraints $\varphi_\rho\equiv\lambda_\rho+p_\rho=0$ to hold effectively, we must regard $\eta_\rho=0$ as the auxiliary conditions on state vectors. As we have seen in § 3, the old constraints may be able to restrict the configuration of the $x'^\sigma(u)$ to be spacelike. Therefore it is desirable to determine the form

of L_{add} in such a way that $\eta_\rho=0$ selects such a configuration out of the state vectors.

For simplicity we assume the bilinear form of η_ρ as JL_{add},

$$JL_{add}\equiv cg^{\rho\sigma}\eta_\rho\eta_\sigma, \tag{4·10}$$

as in cases of the quantum-electrodynamics and of the gravitational field. Then JL_{add} vanishes in the sense of weak equation, owing to $\eta_\rho=0$. In order that the second equation of (4·6) holds, η_ρ must be the linear function of $\partial x^\rho/\partial u^0$, and we can put*

$$\eta_\rho\equiv\kappa^3\left(g_{\rho\sigma}\frac{\partial x^\sigma}{\partial u^0}+\kappa^2 l_\rho\right)=0. \tag{4·11}$$

Here $g_{\rho\sigma}=\begin{pmatrix}1&&&\\&1&&\\&&1&\\&&&-1\end{pmatrix}$ and κ is a constant of the dimension of $(\text{length})^{-1}$. Then the constant c in Eq. (4·10) is $(2\kappa^2)^{-1}$. The geometrical restriction imposed by the definition becomes clearer, if we construct the form

$$\eta_\rho l^\rho\equiv\kappa^2(J+\kappa^2 l^2)=0,$$

by taking account of $l_\rho\cdot\partial x^\rho/\partial u^0\equiv J$. The surface $[u^0]$ is restricted to be spacelike ($l^2<0$) or timelike ($l^2>0$), in accordance with the sign of J. On the other hand, there is no continuous coordinate transformation connecting the system of x^σ with $J>0$ to the one with $J<0$, when the u^s coordinate is fixed. Therefore the spacelike surface $[u^0]$ is separated discretely from the timelike one, and the situation seems to correspond just to the fact that Eq. (3·5) picked out the spacelike configuration of the physical space-time as discussed in § 3. The expression (4·11) may be insufficient to our purpose, as it does not exclude the timelike surface $[u^0]$. But in the following we shall be contented with Eq. (4·11).

By this definition of JL_{add}, the field equations of $x^\rho(u)$, Eqs. (4·8), becomes

$$g_{\sigma\rho}\frac{\partial^2 x^\rho}{\partial u^{02}}-\kappa^4 l^i_{\sigma\mu}g^{\mu\nu}l^j_{\nu\rho}\frac{\partial^2 x^\rho}{\partial u^i\partial u^j}\equiv 0. \tag{4·12}$$

The auxiliary condition (4·11) holds for all $[u^0]$, that is, $\partial\eta_\rho/\partial u^0=0$, as easily shown by Eqs. (4·12). Further, the consistency of the conditions

$$[\eta_\rho(u),\ \eta_\sigma(u')]\equiv 0\ \ (\text{for } u^0=u^{0\prime}), \tag{4·13}$$

can be verified, on account of the relations

$$\frac{\partial x^\sigma}{\partial u^0}\equiv g^{\sigma\rho}\{\kappa^{-2}(\lambda_\rho+p_\rho)-\kappa^2 l_\rho\} \tag{4·14}$$

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* Equation (4·11) means that u^0-line is orthogonal to the surface $[u^0]$. For the vector $\partial x^\sigma/\partial u^0$ is tangential to the u^0-line and l_ρ is the normal vector of the surface $[u^0]$.

and

$$\left[\frac{\partial x^\sigma(u)}{\partial u^0},\ x^\rho(u')\right] \equiv -i\hbar\kappa^{-2} g^{\sigma\rho}\delta_3(u-u'). \qquad (4\cdot15)$$

(for $u^0=u^{0\prime}$)

The former relation is the result of Eqs. (4·9) and (4·11), and the latter is derived from it by Eqs. (2·13) and (2·14).

Eq. (4·15) is the most remarkable character of this altered theory. According to this non-commutability of x^ρ operator this formalism will have many different nature from the conventional field theory. Here the unit κ^{-1} introduced by L_{add} plays the role of the universal length which characterizes the new theory, and it reduces to the ordinary local theory when κ^{-2} is neglected.

Hamiltonian density is written as

$$JH \equiv JH_0 + JH_{add} \equiv \dot{x}^\rho(\lambda_\rho + p_\rho) - JL_{add} \equiv \dot{x}^\rho \eta_\rho - \frac{1}{2\kappa^2} g^{\rho\sigma}\eta_\rho\eta_\sigma$$

$$\equiv \frac{1}{2}\kappa^2\left(g_{\rho\sigma}\frac{\partial x^\rho}{\partial u^0}\frac{\partial x^\sigma}{\partial u^0} - \kappa^4 l^2\right). \qquad (4\cdot16)$$

By the use of $\mathfrak{H} \equiv \int JH d^3u$ we obtain the equation of motion for any operator F

$$\frac{\partial F}{\partial u^0} \equiv \frac{1}{i\hbar}[F, \mathfrak{H}], \qquad (4\cdot17)$$

as usually done. Further, from (4·14) we obtain

$$\left[\frac{\partial x^\sigma}{du^0},\ y_A\right] \equiv \kappa^{-2} g^{\sigma\rho}[p_\rho,\ y_A],$$
$$\left[\frac{\partial x^\sigma}{\partial u^0},\ \pi^A\right] \equiv \kappa^{-2} g^{\sigma\rho}[p_\rho,\ \pi^A]. \qquad (4\cdot18)$$

If we write as

$$\frac{\partial x^\rho}{\partial u^0} \equiv \frac{\partial \bar{x}^\rho}{\partial u^0} + \kappa^{-2} g^{\rho\sigma} p_\sigma, \qquad (4\cdot19)$$

$\partial\bar{x}^\rho/\partial u^0$ commutes with y_A and π^A. Then Eq. (4·16) becomes

$$JH \equiv \frac{\kappa^2}{2}\left(g_{\rho\sigma}\frac{\partial \bar{x}^\rho}{\partial u^0}\frac{\partial \bar{x}^\sigma}{du^0} - \kappa^2 l^2\right) + \frac{1}{2}\left\{\frac{\partial \bar{x}^\rho}{\partial u^0},\ p_\rho\right\} + \frac{1}{2}\kappa^{-2} g^{\rho\sigma} p_\rho p_\sigma. \qquad (4\cdot20)$$

In the Schroedinger representation all the operators in (4·20) are constant. By canonical transformation that eliminates the first term, the operators $x^\rho(u)$'s are transformed into the same form as Eqs. (4·12) and the operators y_A and π^A remain unchanged. In this representation the Schroedinger equation for the state vector $\Psi'(u^0)$ is given by

$$i\hbar\frac{d\Psi'(u^0)}{du^0} = \mathfrak{H}'(u^0)\Psi'(u^0), \qquad (4\cdot21)$$

where

$$\mathfrak{H}'(u^0)\equiv\int d^3u\left[\frac{1}{2}\left\{\frac{\partial\bar{x}^\rho}{\partial u^0},\ p_\rho\right\}+\frac{1}{2}\kappa^{-2}g^{\rho\sigma}p_\rho p_\sigma\right]. \tag{4·22}$$

$\partial\bar{x}^\rho/\partial u^0$ in (4·19) means the derivative of $x^\rho(u)$ with respect to u^0 in this representation, in which the x^ρ operator for any u^0 commutes with y_A and π^A. However, if we take the complete Heisenberg representation for all variables, we shall face to such non-commutabilities as Eqs. (4·18) and the behavior of each field will become very complicated.

In the remaining part of this section we shall show how the auxiliary conditions $\eta_\rho=0$ restrict our Hilbert space. Here we shall leave out the other fields except for $x^\rho(u)$ from our considerations. Owing to the non-linearity of the field equations (4·12) the exact treatment seems almost impossible. Therefore we write as

$$x^\rho(u)\equiv\kappa^{-1}(\delta_s{}^\rho u^s+A^\rho(u)), \tag{4·23}$$

and assume that the higher powers of A^ρ_ι's can be neglected in comparison with one. This is the same treatment as Gupta's approximation used for the gravitational field.[15] Then, we may approximate l_ρ's and their derivatives $l^\iota_{\rho\sigma}$'s as

$$\left.\begin{aligned} &l_0\equiv\kappa^{-3}\left(1+\sum_\iota\frac{\partial A^\iota}{\partial u^\iota}\right),\quad l_\iota\equiv-\kappa^{-3}\frac{\partial A^0}{\partial u^\iota},\\ &l^\iota_{\iota 0}\equiv\kappa^{-2}\left(1+\sum_{\iota\neq\jmath}\frac{\partial A^\iota}{\partial u^\jmath}\right),\quad l^\jmath_{\iota 0}\equiv-\kappa^{-2}\frac{\partial A^\jmath}{\partial u^\iota},\quad l^\jmath_{\iota\jmath}\equiv-\kappa^{-2}\frac{\partial A^0}{\partial u^\jmath}, \end{aligned}\right\} \tag{4·24}$$

and $l^k_{\iota\jmath}\equiv 0$.

Further, the auxiliary conditions (4·11) and field equations (4·12) becomes

$$\frac{\partial A^0}{\partial u^0}-\sum_\iota\frac{\partial A^\iota}{\partial u^\iota}=0,\quad \frac{\partial A^\iota}{\partial u^0}-\frac{\partial A^0}{\partial u^\iota}=0, \tag{4·25}$$

$$\frac{\partial^2 A^0}{\partial u^{02}}-\sum_\iota\frac{\partial^2 A^0}{\partial u^{\iota 2}}\equiv 0,\quad \frac{\partial^2 A^\iota}{\partial u^{02}}-\sum_\jmath\frac{\partial^2 A^\jmath}{\partial u^\iota\partial u^\jmath}\equiv 0, \tag{4·26}$$

respectively.

In terms of the Fourier components

$$a^\rho(h)\equiv\int d^4u\,A^\rho(u)\,e^{-i\hbar u+ih_0u^0},$$

Eq. (4·26) can be expressed as

$$(-h_0{}^2+h^2)a^0\equiv 0,\quad -h_0{}^2a^\iota+\sum_\jmath h_\iota h_\jmath a^\jmath\equiv 0.\quad (h^2=\sum_\iota h_\iota{}^2).$$

As will be seen easily by multiplying h_ι on the second equations,

$$-\sum_i h_i h_0{}^2 a^i + \sum_{i,j} h_i{}^2 h_j a^j \equiv -(h_0{}^2 - h^2)\sum_j h_j a^j \equiv 0,$$

the transversed components of A^i defined by $\sum_j h_j a^j \equiv 0$ satisfy the equations $\partial^2 A^i / \partial u^{0 2} \equiv 0$, that is, they depend on u^0 linearly. Further, A^0 and the longitudinal components of A^i satisfy the wave equations $\Box_u A^\sigma \equiv 0$ which means they propagate like photon in u^s space. Therefore, we may write A^0 as

$$A^0(u) \equiv \sum_{\boldsymbol{h}} (2h_0 \Omega)^{-1/2} [a^0(\boldsymbol{h})\, e^{i\boldsymbol{h}u - ih_0 u^0} + a^{+0}(\boldsymbol{h})\, e^{-i\boldsymbol{h}u + ih_0 u^0}] \tag{4·27}$$

and A^i as the summation of the longitudinal components

$$\sum_{\boldsymbol{h}} (2h_0 \Omega)^{-1/2} h_i h^{-1} [a_3(\boldsymbol{h})\, e^{i\boldsymbol{h}u - ih_0 u^0} + a_3{}^+(\boldsymbol{h})\, e^{-i\boldsymbol{h}u + ih_0 u^0}] \tag{4·27′}$$

and the transversed components

$$\sum_{\boldsymbol{h},\lambda} (2h_0 \Omega)^{-1/2} e_\lambda{}^i [(1 - ih_0 u^0) a_\lambda(\boldsymbol{h})\, e^{i\boldsymbol{h}u} + (1 + ih_0 u^0) a_\lambda{}^+(\boldsymbol{h})\, e^{-i\boldsymbol{h}u}], \tag{4·27″}$$

where $h_0 = \sqrt{h^2}$ and $\boldsymbol{e}_\lambda^2 = 1$, $\boldsymbol{e}_\lambda \boldsymbol{h} = \boldsymbol{e}_1 \boldsymbol{e}_2 = 0$ for $\lambda = 1, 2$. Owing to Eqs. (4·15) each component must satisfy the commutation relations

$$\begin{aligned} &[a^0(\boldsymbol{h}),\ a^{+0}(\boldsymbol{h}')] \equiv -\delta_{\boldsymbol{h}\boldsymbol{h}'} \\ \text{and} \quad &[a_i(\boldsymbol{h}),\ a_j{}^+(\boldsymbol{h}')] \equiv \delta_{ij}\delta_{\boldsymbol{h}\boldsymbol{h}'}. \end{aligned} \tag{4·28}$$

Due to the first equation the A^0 operator requires the indefinite metric in Hilbert space. This situation was already expected as the result of such quantization of x^ρ as (1·5), although, strictly speaking, the formal origin may be considered as different. If we formally define the vacuum for x^ρ operator as

$$A^{\rho(+)}(u)\, \Psi_0 = 0, \tag{4·29}$$

(where we used the sign (+) to take only the annihilation operator up), we must modify the auxiliary condition as

$$\eta_\rho^{(+)}\, \Psi = 0, \tag{4·30}$$

in conformity with (4·29). Then by (4·25) we have

$$h_0 a^0 + h a_3 = 0, \quad h_i a_3 + h_i a^0 = 0, \quad h_0 a_\lambda = 0 \tag{4·31}$$

or

$$a^0 + a_3 = 0, \quad a_\lambda = 0. \tag{4·31′}$$

Namely, the state restricted by the auxiliary conditions will contain only the components of A^0 and the longitudinal components of A^i, and it will be the sum of the vacuum state and the zero norm states in the similar way to the case of the quantum-electrodynamics.[14] Hamiltonian (4·16) becomes

$$\begin{aligned} \mathfrak{H} \equiv \frac{1}{2} \sum_{\boldsymbol{h}} h_0 [&(a_3{}^+ a_3 + a_3 a_3{}^+) - (a^{0+} a^0 + a^0 a^{0+}) \\ &+ \frac{1}{2} \sum_\lambda (a_\lambda^+ a_\lambda + a_\lambda a_\lambda^+)] + \sum_{\boldsymbol{h}} [a^0 + a_3 + a^{0+} + a_3{}^+], \end{aligned} \tag{4·32}$$

and owing to Eqs. (4·31′) there remains only zero point energies in the expectation value of $\mathfrak{H}$. Thus the freedom of x^ρ operator does not appear in the physical states. This is the reasonable result from the viewpoint that the quantization of x^ρ should give no new dynamical variables. However, the existence of x^ρ operator might affect the structure of this system implicitly, in the similar manner to that the longitudinal components of the electromagnetic field give the Coulomb force.

Perhaps in the higher approximation of A_{fs}^{σ} these situations will remain without substantial modifications.

§ 5. Concluding Remarks

In the preceding sections we tried to formulate the conjecture which was exhibited in § 1, and to some extent our purpose was achieved. However, it leaves much to be considered. First, strictly speaking, the commutation relation (1·5) is not established yet. The relation is replaced by Eqs. (2·14) and (2·9) in our Dirac-Bergmann's theory, and the latter equation is only a weak equation. The relation should be obtained as a strong equation, but it will be impossible without any new guiding principle. Secondly, it was made clear that our auxiliary condition (4·11) plays the role to separate the spacelike surface $[u^0]$ from the timelike one, but the relation does not pick up only the spacelike surface $[u^0]$. The timelike surface u^0 might be rejected, if we solve the equations of motion explicitly. However, such wishful thinking will be too optimistic. On the other hand the assumed auxiliary condition (4·11) may be considered as only a tentative scheme. It may be asked whether the altered theory is determined uniquely or not, or whether we should seek some further principle to set up the altered theory or not.

Our theory is not only very complicated owing to the non-commutability between x^ρ and y_A and the non-linear character of the equations of motion of x^ρ, but also it is not clear how the introduction of the universal length κ^{-1} in L_{add} modifies the original theory exhibited in § 2. But it is shown that the theory can be constructed consistently without any new mathematical difficulty, and that the altered theory gives no modification of the field equations of y_A fields. Further it may be thought that the quantization of the operator x^ρ plays some role to weaken the singularity near the light cone,* and that the non-commutability of y_A and x^ρ makes the microcausality in the conventional field theory more complicated.

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* We have two ideas as to weakening the singularity near the light cone in the propagators by introducing the non-linear equations. One is the method of Heisenberg's non-linear spinor equation[16] and the other is Deser's theory which make use of the gravitational field.[17]

Appendix

In this appendix we shall show that it is possible to eliminate the parameter t from the ordinary quantum mechanics, if we tentatively assume the existence of the operator T which satisfy the commutation relation

$$[T,\ H]=i\hbar, \tag{A·1}$$

with the Hamiltonian operator H of the system. For simplicity, we consider the one-dimensional problem. Its Hamiltonian is the function of the variables x and p, $H(x, p)$, and these variables satisfy the commutation relation

$$[x,\ p]=i\hbar. \tag{A·2}$$

In the classical dynamics x and p are regarded as the functions of time parameter t and the equations of motion can be written in the canonical form

$$\frac{dx(t)}{dt}=\frac{\partial H}{\partial p},\qquad \frac{dp(t)}{dt}=-\frac{\partial H}{\partial x}. \tag{A·3}$$

On the other hand, the relations

$$[x,\ f]=-\frac{\hbar}{i}\frac{\partial f}{\partial p},\quad [p,\ f]=\frac{\hbar}{i}\frac{\partial f}{\partial x}, \tag{A·4}$$

hold for the arbitrary function $f(x, p)$, owing to the commutation relation (A·2). Especially we have

$$[x,\ H]=-\frac{\hbar}{i}\frac{\partial H}{\partial p},\quad [p,\ H]=\frac{\hbar}{i}\frac{\partial H}{\partial x}. \tag{A·5}$$

Therefore we may regard Eqs. (A·3) as the operator equations in the quantum mechanics, and rewrite them as

$$\frac{\hbar}{i}\frac{dx}{dt}=[H,\ x],\qquad \frac{\hbar}{i}\frac{dp}{dt}=[H,\ p]. \tag{A·6}$$

Now, if we can introduce the operator T satisfying Eq. (A·1), we shall have the relation

$$\frac{\hbar}{i}\frac{\partial x}{\partial T}=[H,\ x],\qquad \frac{\hbar}{i}\frac{\partial p}{\partial T}=[H,\ p], \tag{A·7}$$

in the same way as the relations (A·4) result from the commutation relation (A·2). Here x and p must be regarded as the function of the operator T and H, and does not contain the parameter t. Comparing these equations with Eqs. (A·5), the relations

$$\frac{\partial x}{\partial T}=\frac{\partial H}{\partial p},\qquad \frac{\partial p}{\partial T}=-\frac{\partial H}{\partial x} \tag{A·8}$$

result.

Eqs. (A·7) and (A·8) are very similar to Eqs. (A·5) and (A·3). The difference lies only in the meaning of the time derivatives of their left-hand side. But both formalisms have the same contents, because the latter can be obtained from the former by introducing the parameter t as the tool. Namely we can introduce the new operator $F(t)$ corresponding to any operator F in such a way as

$$\langle E'|F(t)|E''\rangle=\langle E'|F|E''\rangle e^{i/\hbar(E'-E'')t} \tag{A·9}$$

in the representation in which H is diagonal. Then we have the relation

$$\frac{d\langle E'|F(t)|E''\rangle}{dt}=\frac{i}{\hbar}(E'-E'')\langle E'|F(t)|E''\rangle$$

$$=\left\langle E'\left|\frac{i}{\hbar}[H,\ F]\right|E''\right\rangle e^{i/\hbar(E'-E'')t}=\left\langle E'\left|\frac{\partial F}{\partial T}\right|E''\right\rangle e^{i/\hbar(E'-E'')t}, \tag{A·10}$$

and consequently a correspondence between dF/dt and $\partial F/\partial T$ is obtained. Conversely, if we express the new theory in terms of such operators $F(t)$'s as defined above, the relations (A·5) and (A·3) are reduced from Eqs. (A·7) and (A·8). It means that, if we can adopt the commutation relation (A·1) instead of the canonical equations (A·3), we shall gain the operator equations (A·8), and the operator T may be considered to represent the measurable physical time which concerns with the uncertainty relation (1·2). Thus we could construct a formalism equivalent to the ordinary quantum mechanics without the superfluous parameter t.

We can consider that the commutation relation (A·1) plays the role of Schroedinger equation in the conventional theory, because the latter equation is the counterpart in the Schroedinger representation of the equations of motion (A·3) in the Heisenberg representation, and the relation (A·1) takes the place of Eqs. (A·3). Unfortunately, the operator T satisfying (A·1) cannot be defined owing to the existence of the lowest energy as mentioned in § 1. Therefore we cannot use the commutation relation (A·1) for ordinary theories, and must be contented with the Schroedinger equation in order to express the uncertainty principle between time and energy.

References

1) M. Markow, Journ. of Phys. U.S.S.R. **2** (1940), 453.
H. S. Snyder, Phys. Rev. **71** (1947), 38.
H. Yukawa, Phys. Rev. **77** (1950), 219.
2) P. G. Bergmann and J. H. Brunning, Rev. Mod. Phys. **21** (1949), 480.
3) F. A. E. Pirani and A. Schild, Phys. Rev. **79** (1950), 986.
4) B. S. de Witt and C. M. De Witt, Phys. Rev. **87** (1952), 116.
5) P. A. M. Dirac, Canadian Jaurnal of Math. **2** (1950), 129.
6) D. Bohm, Phys. Rev. **85** (1952), 166, 180.
7) F. Engelmann and E. Fick, Nuovo Cimento Supplement **12** (1959), 63.
8) R. J. Finkelstein, Phys. Rev. **75** (1949), 1079.

9) J. von Neumann, *Die Mathematishe Grundlagen der Quantenmechanik* (Springer Verlag, Berlin, 1932).
10) F. J. Belinfante, D. I. Caplan and W. L. Kennedy, Rev. Mod. Phys. **29** (1957), 518.
11) P. A. M. Dirac, Phys. Rev. **73** (1948), 1092.
12) P. G. Bergmann and R. Schiller, Phys. Rev. **89** (1953), 4.
13) J. L. Anderson, Phys. Rev. **110** (1958), 1197.
14) S. N. Gupta, Proc. Phys. Soc. **A63** (1950), 681.
K. Bleuler, Helv. Phys. Acta **23** (1950), 567.
or W. Heitler, *The Quantum Theory of Radiation* (Clarendon Press, Oxford, 1954), 3rd ed., pp. 87–103.
15) S. N. Gupta, Proc. Phys. Soc. **A65** (1952), 161, 608.
16) W. Heisenberg, Z. Naturforsh. **9a** (1954), 292; Rev. Mod. Phys. **29** (1957), 269.
17) S. Deser, Rev. Mod. Phys. **29** (1957), 417.

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Scattering of Electrons by the Thermal Motion of Impurity Ions

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The scattering of electrons by the thermal motion of the excess electronic potential of impurity ions is discussed. The excess potential is assumed as a screened Coulomb interaction of electrons with excess charges of impurity ions. The change in the electrical resistance of non-magnetic and substitutional dilute alloys due to this type of scattering is calculated at high temperature. It is assumed that the thermal motion of solute ions is quite similar to that of the perfect lattice. Calculated results are compared with the experimental deviations of the electrical resistance from Matthiessen's rule in cases of Cu-, Ag- and Au-based alloys. It is shown that the agreement between them is fairly good in most cases.

§ 1. Introduction

The scattering of electrons in impure metals is in general caused by the excess impurity potential as well as by the lattice vibration. The impurity scattering which is caused by the excess potential is usually treated under the assumption that impurity ions are rigid at their positions. In other words, only the elastic part of the impurity scattering is usually taken into account.

On the other hand, deviations of the electrical resistance of dilute alloys from Matthiessen's rule have been observed at high temperature[1] as well as at low temperature.[2] It is therefore needed to estimate second order effects such as the thermal motion of the excess impurity potential, the temperature dependence of the screening constant, the change in the number of electrons and in the phonon spectrum, etc.

The effect of the change in the Debye temperature and the effect of the localized mode of lattice vibration on the electrical resistance have been estimated by Jones[3] and by Fukuda,[4] respectively, and their results are much smaller than the experimental results. The second order correction to the solution of the Bloch equation which is obtained by a variation method is also too small to account for the experimental deviation according to Sondheimer.[5]

In the present paper we shall discuss the effect of the thermal motion of the excess impurity potential on the electrical resistance of dilute alloys in case that the valence of the impurity atom is different from that of the solvent atom. The potential of an electron in an impure crystal may be devided into the periodic and the excess part. The latter is the difference of potential in the impure crystal

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from that in the perfect crystal. Impurity ions in the crystal vibrate thermally in general and the excess potential caused by them also vibrates. Electrons in the crystal are therefore scattered by impurity ions inelastically as well as elastically. The inelastic scattering consists of two parts, one of which is the part of the phonon scattering, the scattering by the thermal distortion of the periodic potential, and the other is the scattering by the thermal motion of the excess impurity potential, which will be discussed in the present paper. Since the thermal motion of the impurity ion is a part of the thermal motion of the lattice, the scattering of electrons by the thermal motion of the excess potential may be considered as the scattering by phonons through the excess impurity potential.

In the next section we shall first calculate matrix elements of the excess potential caused electrostatically by an impurity ion. There we shall assume that the thermal motion of impurity atoms is quite similar to that of solvent atoms in the pure lattice. As mentioned already, the thermal motion of an impurity atom, as well as the lattice vibration, is different from that of a solvent atom in the pure crystal. It is expected, however, that the thermal motion of the impurity atom is not significantly different from that of the pure lattice when the mass and the valence of the impurity atom are not much different from those of the solvent atom. Furthermore it is noted that the scattering of electrons by the thermal motion of the excess potential is a phenomenon which would occur even if the presence of the impurity atoms did not disturb the lattice vibration.

Our assumption that the thermal motion of impurity atoms is quite similar to that of the pure solvent lattice will be useful as the first order approximation at least when the mass and the valence of the impurity atoms are not much different from those of the solvent atoms. The correction to our result has to be added after detailed information on the thermal motion of impure crystals is obtained, which is, however, a future problem. In the same section the excess potential will be assumed to be a screened Coulomb potential.

In § 3, the electrical resistance of monovalent metal-based dilute alloys in which solute atoms are non-magnetic and substitutional will be calculated with the use of the Born approximation and the high temperature approximation. Numerical values of our results are obtained and compared with the experimental results in § 4. In the last section we shall discuss the effect of the temperature dependence of the screening constant and also the increase in the number of electrons on the electrical resistance.

Interference occurs between the scattering by the thermal distortion of the periodic potential and by the thermal motion of the excess impurity potential. The correction to the electrical resistance due to this interference will be calculated in Appendix 1.

§ 2. Matrix element

The excess impurity potential may be given by the difference of the electronic

potential of the impurity ion from that of the solvent ion substituted by the impurity ion in case of a substitutional impurity. When the impurity atom has the valence which is different from the valence of the solvent atom, the greatest part of the excess impurity potential may be considered to be caused by the difference of the valence between the solute and the solvent atom. It may be treated with the Thomas-Fermi method, and it takes, in the first order approximation, the form of the screened Coulomb interaction, if the position of the impurity ion is rigid. From the viewpoint of the collective description, the long-range part of Coulomb interaction contributes to the collective mode and modifies it. The short-range part may be represented, in the first order approximation, by the screened type of Coulomb interaction.

An impurity ion in a crystal vibrates thermally. As discussed in the previous section, we assume that the motion of the impurity atom is the same as that of the solvent atom in the pure crystal. Our assumption will be useful as the first order approximation when the mass and the valence of the impurity atom have the values similar to those of the solvent atom. The displacement of the impurity atom, $\boldsymbol{u}(\boldsymbol{R})$, can then be expanded with the coordinate of unperturbed phonons as follows,

$$\boldsymbol{u}(\boldsymbol{R})=\sum_{q}\sum_{j}\left(\frac{\hbar}{2d\omega}\right)^{1/2}\boldsymbol{e}\{a\exp(i\boldsymbol{q}\cdot\boldsymbol{R}+i\omega t)+a^{*}\exp(-i\boldsymbol{q}\cdot\boldsymbol{R}-i\omega t)\}, \quad (1)$$

where $\boldsymbol{R}$ is the equilibrium position vector of the impurity atom, d the density of the solvent metal, $\hbar$ Planck's constant divided by 2π, and $\boldsymbol{e}$, a and ω are abbreviations of $\boldsymbol{e}_{qj}$, the polarization, a_{qj}, the annihilation operator, and ω_{qj}, the angular frequency of the phonon which is specified by the wave number vector $\boldsymbol{q}$ and the direction of polarization j, respectively. Matrix elements of the annihilation operator, a, and the creation operator, a^{*}, are given by

$$\langle n-1|a|n\rangle=n^{1/2},\ \langle n+1|a^{*}|n\rangle=(n+1)^{1/2}, \quad (2)$$

where n is the abbreviation of $n(\boldsymbol{q},j)$ which is the number of phonons specified by $\boldsymbol{q}$ and j.

The excess impurity potential should be influenced by the thermal motion of the impurity ion. The interaction of electrons with a moving impurity ion may be divided into two parts, one of which is the interaction through the distortion of the periodic potential and the other through the moving excess impurity potential. The excess impurity potential may be described with the screened Coulomb potential as in the case of the rigid impurity in the usual temperature range. It is then given by

$$V(\boldsymbol{r}-\boldsymbol{R}-\boldsymbol{u})=\frac{Ze^{2}}{|\boldsymbol{r}-\boldsymbol{R}-\boldsymbol{u}|}\exp\{-p|\boldsymbol{r}-\boldsymbol{R}-\boldsymbol{u}|\}, \quad (3)$$

where $\boldsymbol{r}$ is the position vector of the electron, Z the excess charge of the impurity

ion, e the electronic charge and p the screening constant which is, in the first order approximation, given by[6]

$$p=\{(3N/\pi)^{1/3}\,4me^2/\hbar^2\}^{1/2}. \tag{4}$$

Here N is the total number of electrons in the metal and m is the effective mass of an electron.

Let us calculate matrix elements of the excess potential. The initial and the final states of an electron are assumed to be plane wave

$$\varphi_k(\boldsymbol{r})=\exp\,(i\boldsymbol{k}\cdot\boldsymbol{r}+iE_kt/\hbar), \tag{5}$$

for the mathematical convenience. Here $\boldsymbol{k}$ is the wave number vector and E_k is the energy of the electron which is given by

$$E_k=\frac{\hbar^2}{2m}k^2. \tag{6}$$

A matrix element, without the time factor, is given by

$$\begin{aligned}&\langle \boldsymbol{k}'\boldsymbol{n}'|V|\boldsymbol{k}\boldsymbol{n}\rangle\\&=\langle \boldsymbol{n}'|e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot(\boldsymbol{R}+\boldsymbol{u})}|\boldsymbol{n}\rangle\int\frac{Ze^2}{r}e^{-pr+i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}}d\boldsymbol{r}\\&=-\frac{4\pi Ze^2e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{R}}}{p^2+(\boldsymbol{k}-\boldsymbol{k}')^2}\langle \boldsymbol{n}'|e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{u}}|\boldsymbol{n}\rangle,\end{aligned} \tag{7}$$

where $\boldsymbol{n}$ and $\boldsymbol{n}'$ are vectors whose components are $n(\boldsymbol{q},j)$'s and $n'(\boldsymbol{q},j)$'s, respectively. The operator $\exp(i\boldsymbol{k}\cdot\boldsymbol{u})$ in the right-hand side of Eq. (7) may be expanded in power series with respect to u, then the last factor in Eq. (7) is reduced to

$$\begin{aligned}&\langle \boldsymbol{n}'|\,\exp\,\{i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{u}\}\,|\boldsymbol{n}\rangle\\&=\delta(\boldsymbol{n}',\boldsymbol{n})\\&+i\sum_{\boldsymbol{q},j}(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{e}_j\left(\frac{\hbar}{2d\omega}\right)^{1/2}\{e^{-i\boldsymbol{q}\cdot\boldsymbol{R}}n(\boldsymbol{q}j)^{1/2}\delta(n'(\boldsymbol{q}j),n(\boldsymbol{q}j)-1)\\&+e^{i\boldsymbol{q}\cdot\boldsymbol{R}}(n(\boldsymbol{q}j)+1)^{1/2}\delta(n'(\boldsymbol{q}j),\,n(\boldsymbol{q}j)+1)\}\,\delta(\boldsymbol{n}_1',\boldsymbol{n}_1)+\cdots,\end{aligned} \tag{8}$$

where $\boldsymbol{n}_1$ is a reduced vector in which the component $n(\boldsymbol{q},j)$ is excluded.

The first term in the right-hand side of Eq. (8) corresponds to the elastic part and the second term to the one-phonon process. From Eqs. (7) and (8), the first order term with respect to the phonon coordinate in the expansion series of the matrix element is reduced to

$$\begin{aligned}\langle \boldsymbol{k}'n-1|V|\boldsymbol{k}n\rangle=-i\left(\frac{\hbar}{2d\omega}\right)^{1/2}&\frac{4\pi Ze^2}{p^2+(\boldsymbol{k}-\boldsymbol{k}')^2}(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{e}n^{1/2}\\&\cdot\exp\,\{i(\boldsymbol{k}-\boldsymbol{k}'+\boldsymbol{q})\cdot\boldsymbol{R}\},\end{aligned} \tag{9}$$

$$\langle \boldsymbol{k}'n+1|V|\boldsymbol{k}n\rangle = -i\left(\frac{\hbar}{2d\omega}\right)^{1/2} \frac{4\pi Ze^2}{p^2+(\boldsymbol{k}-\boldsymbol{k}')^2}(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{e}(n+1)^{1/2}$$
$$\cdot\exp\{i(\boldsymbol{k}-\boldsymbol{k}'-\boldsymbol{q})\cdot\boldsymbol{R}\}.$$

It can be easily shown that the second order term, the two phonon process, is usually much smaller than the first order term.

The total excess potential is given by the sum of all individual ones, namely

$$\bar{V}(\boldsymbol{r}) = \sum_m V_m(\boldsymbol{r}),$$

where $V_m(\boldsymbol{r})$ is the excess potential due to the m-th impurity ion. The matrix element of the total excess potential is therefore given by

$$\begin{aligned}\langle \boldsymbol{k}'n-1|\bar{V}|\boldsymbol{k}n\rangle &= \sum_n \langle \boldsymbol{k}'n-1|V_m|\boldsymbol{k}n\rangle \\ &= \sum_m \exp\{i(\boldsymbol{k}-\boldsymbol{k}'+\boldsymbol{q})\cdot\boldsymbol{R}_m\}\langle \boldsymbol{k}'n-1|V_0|\boldsymbol{k}n\rangle, \end{aligned} \tag{10}$$

where in the last expression (9) has been used. In order to obtain the transition probability, the square of matrix element is needed. From (10) we obtain

$$\begin{aligned}|\bar{V}|^2 &= |\sum_m V_m|^2 = \sum_m |V_m|^2 + \sum_{m\neq n}\sum V_m^* V_n \\ &= Nx|V_0|^2 + \sum_{m\neq n}\sum |V_0|^2 \exp\{i(\boldsymbol{k}-\boldsymbol{k}'+\boldsymbol{q})\cdot(\boldsymbol{R}_n-\boldsymbol{R}_m)\}, \end{aligned} \tag{11}$$

where Nx is the number of impurity ions in which N is the number of solvent atoms in the perfect lattice. Provided the distribution of impurity ions is quite random, we may make use of the random-phase assumption and (11) is reduced to

$$|\bar{V}|^2 = Nx\{1+(Nx-1)\delta_{\boldsymbol{k}',\boldsymbol{k}+\boldsymbol{q}}\}|V_0|^2. \tag{12}$$

The second term in the right-hand side of Eq. (12) will be estimated in Appendix 2, and it will be shown that the effect of second term on the electrical resistance is much smaller than that of the first one.

§ 3. Electrical resistance

With the use of matrix elements (9) and squares (12) the probability of the transition of an electron from a state $\boldsymbol{k}$ to another state $\boldsymbol{k}'$ is, in the first order approximation, given by

$$\begin{aligned}P(\boldsymbol{k}',\boldsymbol{k}) = Nx\sum_{qj}\frac{16\pi^3Z^2e^4}{d\omega}\frac{\{(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{e}_j^2\}}{\{p^2+(\boldsymbol{k}-\boldsymbol{k}')^2\}^2} \\ \times\{n\delta(E'-E-\hbar\omega)+(n+1)\delta(E'-E+\hbar\omega)\}. \end{aligned} \tag{13}$$

Now we assume that the frequency of phonons is independent of the direction of polarization. Furthermore the Debye model is used for the phonon spectrum, so that $\omega_q = vq$, where v is the velocity of sound which is assumed to be independent of the direction of polarization. Because of these assumptions all quantities other than $\boldsymbol{e}_j$ in Eq. (9) are independent of j, and therefore the summation

with respect to j can be easily performed to yield

$$\sum_j \{(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{e}_j\}^2=(\boldsymbol{k}-\boldsymbol{k}')^2. \tag{14}$$

When the temperature is sufficiently higher than the Debye temperature, the energy of the phonon may be neglected in the argument of δ-function in Eq. (13), and the relaxation time may be calculated simply in the following way.[3]

$$1/\tau=(1/8\pi^3)\int P(\boldsymbol{k}', \boldsymbol{k})(1-\cos\theta)d\boldsymbol{k}', \tag{15}$$

where θ is the angle between k and k'. Inserting Eq. (13) into Eq. (15), neglecting $\hbar\omega$ in the argument of δ-function, assuming $n+1\sim n$ and using the relation (14), we obtain

$$\frac{1}{\tau}=\frac{2\sqrt{2}Z^2e^4m^{1/2}k^2\theta NTxF(\kappa)}{\pi\hbar^3v^3dE^{1/2}}, \tag{16}$$

where k is Boltzmann's constant, θ the Debye temperature and

$$F(\kappa)=1-(p^2/2\kappa^2)\ \log\ (1+4\kappa^2/p^2)+p^2/(p^2+4\kappa^2). \tag{17}$$

And we have used the relation $n=kT/\hbar\omega$ which is valid at high temperature. From (16) the electrical resistance of a monovalent metal containing a small concentration of solute atoms is obtained as

$$\rho=\frac{2\sqrt{2}Z^2e^2m^{3/2}k^2\theta TxF(\kappa')}{\pi\hbar^3v^3d\zeta^{1/2}}, \tag{18}$$

where ζ is the Fermi energy and κ' is the wave number of an electron on the Fermi surface. Eq. (18) is rewritten as

$$\rho=92.6\frac{Z^2\theta TxF(\kappa')}{v^3d\zeta^{1/2}}\quad\text{ohm-cm}, \tag{19}$$

provided that the mass of electron is equal to the true mass.

§ 4. Numerical values

It may be concluded from the result (18) or (19) obtained in the preceding section that the correction to the electrical resistance due to the thermal motion of impurity ions depends linearly on the temperature and on the impurity concentration and it is proportional to the square of the excess charge of impurity. These dependences are consistent approximately with experimental results. Let us estimate numerical values. Values of quantities which are necessary for the numerical estimation of Eq. (19) are listed in Table I. With the use of values N listed in Table I the screening constant p which is given by Eq. (4) is obtained as $p=1.77\times10^8\,\mathrm{cm}^{-1}$ for Cu and $1.73\times10^8\,\mathrm{cm}^{-1}$ for Ag and Au. From (17) the value of $F(\kappa')$ becomes 0.267 for Cu and 0.233 for Ag and Au. With the use of these values the change of electrical resistance ρ is finally reduced to

$$\rho=\begin{cases}5.25\times10^{-9}\ Z^2Tx & \text{ohm-cm for Cu,}\\ 1.18\times10^{-8}\ Z^2Tx & \text{ohm-cm for Ag,}\\ 1.11\times10^{-8}\ Z^2Tx & \text{ohm-cm for Au.}\end{cases} \tag{20}$$

They are comparable to Linde's experimental values.[1]

Table I.

	Cu	Ag	Au
d(in gr-cm^{-3})	8.93	10.5	19.3
v(in m-sec^{-1})	366	265	208
Θ(in ^{0}K)	310	220	185
ζ(in 10^{-12} erg)	11.1	8.25	8.85
C(in 10^{-11} erg)	1.24	1.00	1.00
N(in 10^{22} cm^{-3})	8.5	5.9	5.9
κ'(in 10^8cm^{-1})	1.37	1.21	1.21

§ 5. Discussions

We have estimated the effect of the thermal motion of impurity ions on the high temperature electrical resistance of dilute alloys under several assumptions. First we have assumed that impurity ions move in the same manner as the perfect solvent lattice. For the calculation of the normal part of electron-phonon interaction the effect of the change in the phonon spectrum will be small. It will, on the other hand, appreciably influence the interaction of electrons with impurity ions when the mass and the valence of impurity atoms are much different from those of solvent ions, since the lattice vibration is considered to be most different from the normal motion near positions of impurity ions. Our assumption is, however, expected to be useful as the first order approximation at least when the mass and the valence of the impurity atom are not much different from those of the solvent atom.

Second, the screening constant given by (4) is only the first order approximation. The deviation of our screening constant from the true form of the excess potential will become larger as the excess charge Z becomes larger. Furthermore we have made use of the Born approximation in the calculation of the electrical resistance. The Born approximation can usually be applied in our case as well as in the case of the normal scattering by phonons since both types of scattering are essentially the same order phenomena.

There are many other mechanisms which give rise to the deviation of electrical resistance from Matthiessen's rule. We shall briefly discuss the temperature dependent part of screening constant and the increase of the number of electrons in the following.

a) The temperature dependent part of the screening constant

The screening constant is given by[6]

$$p^2=\frac{4\pi ne^2}{kT}\frac{F_{-1/2}}{F_{1/2}}, \tag{21}$$

where

$$F_l=\frac{1}{l!}\int_0^\infty \frac{x^l dx}{\exp(x-\eta)+1}, \tag{22}$$

in which $\eta=\zeta/kT$ (ζ is the Fermi energy). When $\eta\gg 1$, the integral (22) may be expanded in a power series with respect to η^{-1}, and we obtain

$$F_l=\frac{\eta^{l+1}}{(l+1)!}\left[1+\frac{\pi^2}{6}l(l+1)\eta^{-2}\right]. \tag{23}$$

Introducing (23) into (21), we obtain

$$p=p_0\left[1-\frac{\pi^2}{12}\eta^{-2}\right]. \tag{24}$$

According to Mott,[7] the residual resistance is given by

$$\rho_{res}=\frac{Z^2e^2m^{1/2}xG(\kappa')}{2\sqrt{2}\zeta^{3/2}}, \tag{25}$$

where

$$G(\kappa)=\log(1+4\kappa^2/p^2)-4\kappa^2/(p^2+4\kappa^2)$$

and κ' is the wave number at the Fermi surface. From (24) and (25) we obtain the correction to the residual resistance due to the correction to the screening constant

$$\frac{\Delta\rho}{\rho_{res}}=\frac{\pi^2}{6}\frac{(kT/\zeta)^2}{(1+p^2/4\kappa'^2)G}\lesssim(kT/\zeta)^2 \quad \text{for monovalent metals,}$$

which is much smaller than (20) at usual temperature.

(b) The increase in the number of electrons

The increase in the number of electrons gives rise to changes in values of N, m, ζ and C. These changes yield the correction to the phonon-scattering resistance ρ_{ph}. ρ_{ph} is given by[8]

$$\rho_{ph}\propto m^{1/2}N^{-1}(C/\zeta)^2\zeta^{1/2}. \tag{26}$$

Since C is essentially the kinetic energy, C/ζ is considered, in the first order approximation, to be independent of the increase of electrons. The effective mass of electron will change its value appreciably in cases of noble metals in which the Fermi level touches almost an edge of the first Brillouin zone. When Nx of solute atoms are added, N is replaced by $N(1+Zx)$. The change of phonon-scattering resistance $\Delta\rho$ is therefore approximately given by

$$\Delta\rho/\rho_{ph}=(1/2)(\Delta m/m+\Delta\zeta/\zeta)-Zx. \tag{27}$$

Since the correct estimation of Δm and $\Delta\zeta$ is impossible at the present stage,

we can not estimate $\Delta\rho$ here. If $\Delta m/m$ and $\Delta\zeta/\zeta$ were negligible compared with Zx, the right-hand side of Eq. (27) would be reduced to $-\rho_{ph}Zx$ the absolute value of which is comparable to (19), and therefore the effect of the thermal motion of impurity ions would be canceled by the effect of the increase of electrons. In practical cases, however, Δm will not be negligible. In fact, for example, in a one-dimensional lattice the effective mass changes its sign as well as its value when the number of electrons goes over a half of the number of atoms, and therefore the hole picture must be adopted in this case. With the use of the hole picture it is shown that the change of electrical resistance is positive.

Though conditions in three-dimensional cases are much different from those in one-dimensional cases, it is expected that the correction to the electrical resistance due to the increase of electrons is not given by $-\rho_{ph}Zx$ only. Anyway more careful discussions on the effect of the increase in the number of electrons are needed in order to obtain the change of electrical resistance and to understand fully the deviation of electrical resistance from Matthiessen's rule.

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Appendix 1

Interference effect

The elastic scattering by impurity ions does not interfere with any of the inelastic scattering by impurity ions and by phonons. On the other hand the inelastic scattering by impurity ions does interfere with that by phonons, since the matrix element of the former is an imaginary quantity as well as that of the latter, as shown in the text.

The matrix element of the perturbation which is due to one phonon specified by the wave number $\boldsymbol{q}$ and the polarization j is well known[8]

$$\langle \boldsymbol{k}'n-1|V_p|\boldsymbol{k}n\rangle=-i\frac{2}{3}C\left(\frac{\hbar}{2d\omega}\right)^{1/2}n^{1/2}(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{e}, \qquad (A\cdot1)$$

where

$$\boldsymbol{k}'-\boldsymbol{k}-\boldsymbol{q}=0, \qquad (A\cdot2)$$

and also

$$\langle \boldsymbol{k}'n+1|V_p|\boldsymbol{k}n\rangle=-i\frac{2}{3}C\left(\frac{\hbar}{2d\omega}\right)^{1/2}(n+1)^{1/2}(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{e}, \qquad (A\cdot3)$$

where

$$\boldsymbol{k}'-\boldsymbol{k}+\boldsymbol{q}=0, \qquad (A\cdot4)$$

provided that umklapp-processes are neglected. In Eqs. (A·1) and (A·3) C is

the coupling constant which is defined by

$$C=\frac{\hbar^2}{2m}\int|\mathrm{grad}\,\chi|^2 d\tau \qquad (A\cdot 5)$$

where χ is the periodic part of the Bloch function. The transition probability can be obtained from Eqs. (9), (A·1) and (A·3). The scattering may be treated as if elastic and also Eq. (15) may be used as is usual in the calculation of electrical conductivity at high temperature. After an easy calculation the electrical resistance ρ' due to the interference effect is obtained

$$\rho'=-\frac{Zm^{1/2}k^3C\theta^3GTx}{3\sqrt{2}\hbar^2v^4d\zeta^{3/2}}$$

$$=-1.68\times10^{-8}\frac{ZC\theta^2GTx}{dv^4\zeta^{3/2}}\ \text{ohm-cm}, \qquad (A\cdot 6)$$

where

$$G=1-(\hbar vp/k\theta)^2\log\{1+(k\theta/\hbar vp)^2\}.$$

With the use of the values of quantities listed in § 4 of the text the numerical value of (A·6) is obtained

$$\rho'=\begin{cases}-8.45\times10^{-10}\,ZTx & \text{ohm-cm for Cu,}\\ -2.20\times10^{-9}\,ZTx & \text{ohm-cm for Ag,}\\ -2.24\times10^{-9}\,ZTx & \text{ohm-cm for Au}\end{cases} \qquad (A\cdot 7)$$

Eqs. (A·7) show that the correction due to the interference is always smaller than the correction due to the thermal motion of impurity ions at least in noble metals.

Appendix 2

Transition conserving the total wave number

The transition probability which comes from the second term in the right-hand side of Eq. (12) is

$$P(\boldsymbol{k}',\boldsymbol{k})=(Nx)^2\sum_q\frac{16\pi^3Z^2e^4}{d\omega}\frac{q^2}{\{p^2+q^2\}^2}$$

$$\times\{n\delta(E'-E-\hbar\omega)+(n+1)\delta(E'-E+\hbar\omega)\}, \qquad (A\cdot 8)$$

where 1 is neglected compared with Nx. With the use of Eq. (15) we obtain the relaxation time, from which we obtain the electrical resistance

$$\rho''=\frac{\sqrt{2}\pi Z^2e^2m^{1/2}kTNfx^2}{v^2d\zeta^{3/2}}, \qquad (A\cdot 9)$$

where

$$f=\log(1+q_0^2/p^2)-q_0^2/(p^2+q_0^2),$$

in which q_0 is the maximum wave number of lattice vibration. The ratio of (A·9) to the normal resistance due to phonon[8] is

$$\frac{\rho''}{\rho_{ph}}=\frac{64\pi^2Z^2e^4}{q_0^4C^2}fN^2x^2 \qquad (A\cdot10)$$

$$\simeq 6.5\times10^{-2}Z^2x^2 \quad \text{for Cu-based alloys,} \qquad (A\cdot11)$$

which is much smaller than (19) of the text.

References

1) J. O. Linde, Thesis, Lund, 1939.
2) E. Krautz & H. Schultz, Z. Naturforsch. **129** (1957), 710.
   P. Alley & B. Serin, Phys. Rev. **116** (1959), 334.
3) See, for example, D. K. C. MacDonald, *Handbuch der Physik* (Springer, Berlin, 1956), Vol. 14.
4) H. Jones, *Handbuch der Physik* (Springer, Berlin, 1956), Vol. 19.
5) Y. Fukuda, private communication.
6) E. H. Sondheimer, Proc. Roy. Soc. **A203** (1950), 75.
7) R. B. Dingle, Phil. Mag. **46** (1955), 831.
8) N. F. Mott, Proc. Camb. Phil. Soc. **32** (1936), 281.
9) A. Sommerfeld & H. Bethe, *Handbuch der Physik* (Springer, Berlin, 1933), Vol. 24/2.

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Collective Correlation between Vacuum Nucleons and the S-Wave Pion-Nucleon Scattering

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Scattering of S-wave pion by nucleon was calculated taking into account the collective correlation between vacuum nucleons. The point is that there can exist a strong collective correlation between vacuum nucleons just as between electrons in metal at the absolute zero of temperature, since interaction between vacuum nucleons due to pion is dominantly attractive. The effect of this correlation was calculated using Bogoliubov's method which was very successful in the theory of superconductivity. The phase shift δ_3 was calculated using Tamm-Dancoff approximation including up to two pions and one nucleon pair.

It was shown that by taking into account this effect, the result of Dyson et al. using the same approximation was improved greatly, and an overall agreement with experiment was obtained. δ_1 was not calculated in this paper.

§ 1. Introduction

It is well known that the *ps-ps* meson theory does not reproduce the S-wave pion-nucleon scattering experiment or the anomalous magnetic moment of the nucleon. This is in sharp contrast to brilliant successes of quantum electrodynamics or of the P-wave meson theory. Several attempts have been made for this.[1)–14)] None of them, however, seems to have gained a final success.

In this paper we want to discuss the effect of collective correlation between vacuum nucleons on this problem. The outline is as follows. We assume, according to Dirac, that the vacuum is a state in which all negative energy states of the nucleon are occupied. Between these nucleons there exists interaction due to pion. Now it should be noted that there exists a remarkable similarity between these vacuum nucleons and electrons in metal at the absolute zero of temperature. That is, if we assume a free electron gas model, electrons in metal fill levels up to E_F(E_F: Fermi energy), and they interact through phonon with one an other. If we replace electron by nucleon, E_F by $-mc^2$ (m: rest mass of the nucleon) and phonon by pion, this is nothing but the vacuum we want to discuss. The problem of electrons in metal has been investigated in detail in connection with the theory of superconductivity,[15),16)] and it has been found that, if the interaction is dominantly attractive, a kind of condensation occurs as the result of collective correlation between electrons, and this becomes the origin of superconductivity. Here it should be noted that this is a very general feature of the fermion system whose in-

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teraction is dominantly attractive. Therefore, we must expect that exactly the same situation would occur in the case of vacuum since internucleon force due to pion is dominantly attractive. If this is the case, it is easily supposed that the effect of condensation would suppress the S-wave scattering amplitude since in ps-ps theory S-wave scattering occurs only through the process of nucleon pair formation. Thus, we can expect that the situation would be greatly improved if we can take into account these effects correctly.

The actual calculation is made as follows. We take as the Hamiltonian of our system

$$H=H_0+H', \tag{1·1}$$

where H_0 is the free Hamiltonian, and H' is the usual γ_5-interaction. We rewrite (1·1) as

$$H=H_0+H_2+(H'-H_2), \tag{1·2}$$

where H_2 is the two-body interaction term due to H'.* H_0+H_2 can be diagonalized by a method similar to that of B.C.S. or Bogoliubov. We regard H_0+H_2 as unperturbed Hamiltonian, and calculate the phase shift due to $(H'-H_2)$. In this way we may expect that the desired effect is taken into account at least in the sense of Hartree approximation.

Our system differs from electrons in metal in that their energy is negative, and that there exists a large energy gap $2mc^2$ above its highest level. It can be shown, however, that Bogoliubov's method can still be applied, and a vacuum is obtained which is essentially different from the one obtained by the perturbation method. This is because terms expressing the effect of collective correlation are essentially singular at the origin when considered as a function of the coupling constant. Therefore, by this way of approach, a method of approximation is obtained which is essentially different from the ones used thus far in field theories.

We applied this method to the scattering of S-wave pion by nucleon. There exist two phase shifts δ_1 and δ_3, corresponding to $T=1/2$ and $3/2$. (T: magnitude of the total isotopic spin). As there exists an additional complication connected with the renormalization in the case of $T=1/2$, however, we shall calculate only δ_3 in this paper.** The calculation was made using Tamm-Dancoff approximation including up to two pions and one nucleon pair. It was shown that by including this effect, the result of Dyson et al.[4] which was made in the

* Strictly speaking, H_2 should be determined self-consistently so that two-body correlation due to $(H'-H_2)$ vanishes when we regard H_0+H_2 as unperturbed system. In actual calculations, however, we shall take as H_2 simply nuclear force calculated in the second order perturbation. These two agree in the lowest order approximation.

** There exists another doubt in applying the Tamm-Dancoff approximation in the case $T=1/2$, since in this case crossing symmetry is violated. We know, however, that the zero energy limit of the slope of $(\delta_1-\delta_3)$ can be calculated unambiguously by dispersion theory,[17] and if zero energy limit of the slopes of δ_1 and δ_3 are known, their energy dependences can also be calculated by dispersion theory.[11] So, we shall be contented with calculating only δ_3 in this paper.

same approximation was greatly improved, and the result excellently agreed with experiment.

§ 2. Structure of the vacuum

As the first step, we shall re-examine the structure of vacuum taking into account the collective correlation between vacuum nucleons.

According to § 1, we shall take as H_2 the two-body interaction due to nuclear force calculated in the lowest order perturbation. Though this has complicated spin and isotopic spin dependence, we shall replace it by a simple Yukawa potential. Thus,

$$H_2=-\frac{1}{2}g^2\int\psi^*(\boldsymbol{x})\psi^*(\boldsymbol{x}')\frac{e^{-\kappa(|x-x'|)}}{|x-x'|}\psi(\boldsymbol{x}')\psi(\boldsymbol{x})d\boldsymbol{x}\ d\boldsymbol{x}', \qquad (2\cdot1)$$

where κ is reciprocal of the Compton wave length of the pion. Other notations will be obvious.

Therefore, in momentum representation, H_0+H_2 is given by*

$$\begin{aligned} H_0+H_2=&-\sum_{k,\sigma}E_k\ a^*_{k\sigma}\ a_{k\sigma} \\ &-\frac{1}{2V}\sum_{\substack{k_1,k_1'\\ k_2,k_2'\\ k_1-k_2=k_1'+k_2'\\ \sigma_1,\sigma_2}} J(\boldsymbol{k}_2-\boldsymbol{k}'_2)\,a^*_{k_2'\sigma_2}\ a_{k_2\sigma_2}\ a^*_{k_1'\sigma_1}\ a_{k_1\sigma_1}, \end{aligned} \qquad (2\cdot2)$$

where

$$\left.\begin{aligned} &E_k=\hbar^2k^2/2m, \\ &J(\boldsymbol{k}_2-\boldsymbol{k}_2')=4\pi g^2/(\kappa^2+|\boldsymbol{k}_2-\boldsymbol{k}_2'|^2), \end{aligned}\right\} \qquad (2\cdot3)$$

and $a^*_{k\sigma}$ and $a_{k\sigma}$ are creation and annihilation operator for vacuum nucleon with momentum $\boldsymbol{k}$ and spin σ. Hereafter, we shall proceed exactly in parallel with B. C. S. or Bogoliubov. Rewriting the interaction term as

$$-\frac{1}{2V}\sum J(\boldsymbol{k}-\boldsymbol{k}')\,a^*_{k'\sigma_1}\ a^*_{-k'+p,\sigma_2}\ a_{-k+p,\sigma_2}\ a_{k\sigma},$$

we shall take only one component corresponding to $\boldsymbol{p}=0$ and $\sigma_1=-\sigma_2$. Then, expressing spin up and down by $\uparrow$ and $\downarrow$, (2·2) becomes

$$H=H_0+H_{red},$$

with

$$H_0=-\sum E_k(a^*_{k\uparrow}\ a_{k\uparrow}+a^*_{-k\downarrow}\ a_{-k\downarrow}),$$

---

* We shall take $-mc^2$ as zero point of the energy. Non-relativistic approximation is used for nucleon.

and

$$H_{rel}=-\frac{1}{V}\sum_{k,k'}J(\boldsymbol{k}-\boldsymbol{k}')a^*_{k'\uparrow}\ a^*_{-k'\downarrow}\ a_{-k\downarrow}\ a_{k\uparrow}. \tag{2·4}$$

Our problem is to calculate the eigenvalue and eigenfunction of the system described by (2·4). Since (2·4) is not bilinear in a and a^*, it is impossible to diagonalize (2·4) by a canonical transformation. Therefore, we shall take Hartree approximation. The $(\boldsymbol{k}\uparrow,\ -\boldsymbol{k}\downarrow)$ component of (2·4) is given by

$$-\frac{1}{V}\left\{\sum_{k'}J(\boldsymbol{k}-\boldsymbol{k}')a^*_{k'\uparrow}a^*_{-k'\downarrow}\right\}a_{-k\downarrow}\ a_{k\uparrow}-\frac{1}{V}\left\{\sum_{k'}J(\boldsymbol{k}-\boldsymbol{k}')a_{-k'\downarrow}\ a_{k'\uparrow}\right\}a^*_{k\uparrow}\ a^*_{-k\downarrow}. \tag{2·5}$$

We shall replace the expression in { } by its expectation value*

$$\frac{1}{V}\sum_{k'}J(\boldsymbol{k}-\boldsymbol{k}')\langle a^*_{k'\uparrow}a^*_{-k'\downarrow}\rangle_{Av}=\frac{1}{V}\sum_{k'}J(\boldsymbol{k}-\boldsymbol{k}')\langle a_{-k'\downarrow}\ a_{k'\uparrow}\rangle_{Av}=I_k, \tag{2·6}$$

and shall try to determine I_k self-consistently afterwards. Thus, $\boldsymbol{k}$-th component of H becomes

$$H_k=-E_k(a^*_{k\uparrow}\ a_{k\uparrow}+a^*_{-k\downarrow}\ a_{-k\downarrow})-I_k(a^*_{k\uparrow}\ a^*_{-k\downarrow}+a_{-k\downarrow}\ a_{k\uparrow}). \tag{2·7}$$

(2·7) can be diagonalized by the Bogoliubov transformation:

$$\begin{aligned}\alpha_{k0}&=u_k\ a_{k\uparrow}-v_k\ a^*_{-k\downarrow}\\ \alpha_{k1}&=u_k\ a_{-k\downarrow}+v_k\ a^*_{k\uparrow},\end{aligned} \tag{2·8}$$

where u_k and v_k are subjected to the condition

$$u_k^2+v_k^2=1. \tag{2·9}$$

(2·8) is a canonical transformation if (2·9) is satisfied. Inserting (2·8) into (2·7),

$$\begin{aligned}H_k=&-E_k\{(u_k^2-v_k^2)(\alpha^*_{k0}\ \alpha_{k0}+\alpha^*_{k1}\ \alpha_{k1})\\ &+2u_k\ v_k(\alpha^*_{k0}\ \alpha^*_{k1}+\alpha_{k1}\ \alpha_{k0})+2v_k^2\}\\ &-I_k\{-2u_k\ v_k(\alpha^*_{k1}\ \alpha_{k1}+\alpha^*_{k0}\ \alpha_{k0})\\ &+(u_k^2-v_k^2)(\alpha_{k1}\ \alpha_{k0}+\alpha^*_{k0}\ \alpha^*_{k1})+2u_k\ v_k\}.\end{aligned} \tag{2·10}$$

By equating the coefficient of off-diagonal terms to zero, we get

$$-2u_k\ v_k\ E_k=I_k(u^2{}_k-v_k^2) \tag{2·11}$$

from which we have, assuming that $v_k^2>u_k^2$,**

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* It is easily shown that $(1/V)\sum J(\boldsymbol{k}-\boldsymbol{k}')\langle a^*{}_{k'\uparrow}\ a^*{}_{-k'\downarrow}\rangle_{Av}$ and $(1/V)\sum J(\boldsymbol{k}-\boldsymbol{k}')\langle a_{-k'\downarrow}\ a_{k'\uparrow}\rangle_{Av}$ must be equal and real to preserve the hermiticity of the total Hamiltonian.

** We also assume that $u_k,\ v_k>0$.

$$u_k^2=\frac{1}{2}\left(1-\frac{E_k}{\sqrt{E_k^2+I_k^2}}\right),$$

and (2·12)

$$v_k^2=\frac{1}{2}\left(1+\frac{E_k}{\sqrt{E_k^2+I_k^2}}\right).$$

Substituting (2·12) in (2·10), H becomes

$$H=\sum_k\sqrt{E_k^2+I_k^2}\,(\alpha_{k0}^*\,\alpha_{k0}+\alpha_{k1}^*\,\alpha_{k1})-2\sum_k(E_k\,v_k^2+I_k\,u_k\,v_k). \quad (2\cdot13)$$

(2·13) defines the ground state as

$$\begin{aligned}\alpha_{k0}\,\psi&=0,\\ \alpha_{k1}\,\psi_\nu&=0,\end{aligned} \quad (2\cdot14)$$

with the ground energy

$$E_g=-\sum_k(2E_k\,v_k^2+I_k\,u_k\,v_k).^* \quad (2\cdot15)$$

ψ_ν thus defined includes the effect of two-body correlation in the Hartree approximation, and we shall assume that this is the true vacuum for meson-nucleon system.

To determine I_k, it is necessary to express a_k by α_k. Thus,

$$\begin{aligned}a_{k\uparrow}^*\,a_{-k\downarrow}^*+a_{-k\downarrow}\,a_{k\uparrow}&=(u_k^2-v_k^2)\,(\alpha_{k1}\,\alpha_{k0}+\alpha_{k0}^*\,\alpha_{k1}^*)\\ &\quad+2u_k\,v_k(\alpha_{k1}\,\alpha_{k1}^*-\alpha_{k0}^*\,\alpha_{k0}),\end{aligned}$$

from which we get

$$\langle a_{k\uparrow}^*\,a_{-k\downarrow}^*+a_{-k\downarrow}\,a_{k\uparrow}\rangle_{Av}=2u_kv_k. \quad (2\cdot16)$$

Inserting (2·16) into (2·6),

$$\frac{1}{V}\sum J(\boldsymbol{k}-\boldsymbol{k}')\,u_{k'}\,v_{k'}=I_k. \quad (2\cdot17)$$

If we approximate $J(\boldsymbol{k}-\boldsymbol{k}')$ by

$$J(\boldsymbol{k}-\boldsymbol{k}')=\begin{cases}\text{const.}=J & \text{for } k,\ k'\lesssim\bar{\bar{k}},\\ 0 & \text{otherwise.}\end{cases}$$

then I_k is also constant for $k<\bar{\bar{k}}$, and zero otherwise. Therefore,

$$\frac{J}{2V}\sum_k^{k\lesssim\bar{\bar{k}}}\frac{I}{\sqrt{E_k^2+I^2}}=I,$$

---

* In the second term in the right-hand side of (2·13), the contribution from the two-body interaction is calculated doubly as the result of the Hartree approximation. Therefore we must divide it by 2 to get correct result.

where we have omitted sufix k in I_k. Thus, introducing state density per unit volume $N(E)$, we get, assuming that $I \neq 0$,*

$$\frac{J}{2}\int_0^{\hbar^2\bar{k}^2/2m}\frac{N(E)}{\sqrt{E^2+I^2}}dE=I, \tag{2·18}$$

which can be written as

$$\frac{1}{2}JN(\bar{E})\int_0^{\hbar^2\bar{k}^2/2m}\frac{dE}{\sqrt{E^2+I^2}}=1, \tag{2·19}$$

where $\bar{E}$ is some mean value of E. From this we get

$$I=\frac{\hbar^2\bar{k}^2}{2m}\Big/\sinh\left(\frac{2}{JN(\bar{E})}\right). \tag{2·20}$$

If we assume that the state density is not so much altered from the free case, $N(E)$ is given by

$$N(E)\simeq N_0(E)=\frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{E},** \tag{2·21}$$

where $N_0(E)$ is the state density for free case. J may be approximated by

$$J\simeq 4\pi g^2/\bar{k}^2. \tag{2·22}$$

Inserting these expressions, we get

$$JN(\bar{E})\simeq\frac{g^2}{\hbar c}\cdot\frac{mc}{\hbar}\cdot\frac{1}{\bar{k}},$$

where $\bar{k}$ is the momentum corresponding to $\bar{E}$. Thus, assuming that $\bar{k}\simeq\bar{\bar{k}}\simeq\xi\kappa$ and $g^2/\hbar c\simeq 0.1$, we get

---

* $I=0$ gives vacuum which is identical with the free case. We can show, however, that this state is energetically higher than the one with $I\neq 0$. Our system differs from electrons in metal, in addition to that the energy is negative, in the point that there is no quantity which corresponds to the Fermi energy. In the latter case, superconducting state is realized by some of the electrons going above the Fermi surface. This state is energetically more stable, because although kinetic energy increases by some electrons going above the Fermi surface, potential energy decreases due to the effect of attractive potential which was of no effect in the normal state due to the Pauli principle. In the case of vacuum nucleons, kinetic energy of the total vacuum nucleons increases by evacuating some of the negative energy nucleons. Nevertheless, total energy can be lowered by doing so by exactly the same reason why superconducting state was energetically more stable than the normal one. The aim of our calculation is to find the lowest energy state thus produced in the Hartree approximation. The answer is just given by (2·14). Therefore, if we want to define vacuum as eigenstate of the total Hamiltonian with the lowest energy eigenvalue, we must consider this state as the true vacuum. We shall not discuss where evacuated nucleons go, since all that is needed to define vacuum is to find the eigenstate of the total Hamiltonian with the lowest energy eigenvalue.

** See note added in proof.

$$I \simeq (\mu/m) \cdot \bar{\xi}^2 / \sinh(\bar{\xi}) \cdot \mu c^{2*}. \tag{2·23}$$

Therefore, if $\bar{\xi}$ is of order unity, I would be of order $(\mu/m) \cdot \mu c^2 \simeq 20\text{Mev}$. But this result is very sensitive to $\bar{\xi}$, and we can not trust it more than order of magnitude. Therefore, we shall leave I as an undetermined parameter, and shall try to determine it afterwards so that best agreement with experiment is obtained.

To see the structure of this vacuum more concretely, it is instructive to express it in terms of a. Transformation function for the Bogoliubov transformation is given by[18]

$$\begin{aligned} &U = e^{-iS}, \\ &S = i\sum \theta_k (a^*_{k\uparrow}\, a^*_{-k\downarrow} - a_{-k\downarrow}\, a_{k\uparrow}), \\ &(\theta_k = \cos^{-1} u_k). \end{aligned} \tag{2·24}$$

Using this, we can express ψ_ν as

$$\psi_\nu = \prod_k (u_k + v_k\, a^*_{k\uparrow}\, a^*_{-k\downarrow})\psi_v \tag{2·25}$$

where ψ_v is the vacuum for a particle.

Let us consider first the case $I=0$. In this case,

$$\begin{aligned} &u_k = 0, \\ &v_k = 1, \quad \text{for} \quad E_k > 0. \end{aligned} \tag{2·26}$$

Therefore, (2·25) represents a state in which all negative energy states are occupied, which is nothing but the usual free vacuum. When $I=0$, however, u_k and v_k deviate from (2·26) near $E_k=0$. Therefore, in this case, level density is decreased from the free case near $E_k=0$. From (2·25), the probability of finding a pair with specified momentum $\boldsymbol{k}$ is given by $v_k{}^2$. Therefore, level density is given by

$$N(E_k) = v_k{}^2 N_0(E_k). \tag{2·27}$$

In the case of quantum electrodynamics, interparticle force is the repulsive Coulomb force. Therefore, in this case no peculiar situation occurs.

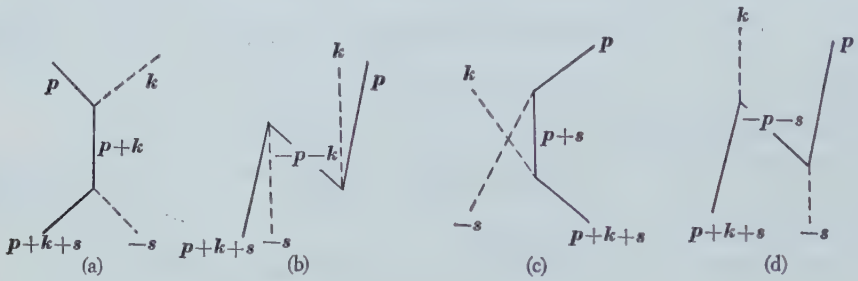
There are several defects in this formalism since our calculation is based on a crude approximation which does not satisfy normal symmetry or invariance properties. The most serious point is that (2·7) we took as our starting point does not satisfy the conservation law of nucleon number. Therefore, the result is in general not the eigenstate of nucleon number, and especially ψ_ν is not the eigenstate of nucleon number. This makes obscure the concept of antinucleon as hole of nucleon in negative energy state. This defect is, however, inevitable so long as we use Bogoliubov's method, and we shall not touch this point further

* μ is the rest mass of the pion.

in this paper.*

§ 3. S-wave pion-nucleon scattering

In this section, we shall calculate phase shift for S-wave scattering taking into account the effect of collective correlation. As noted in § 1, we shall calculate only δ_3. The calculation is made according to Dyson et al.[4]** based on the Tamm-Dancoff approximation including two mesons. As is well known, the following four graphs contribute to the scattering in this approximation if we neglect self-energy terms.



First let us consider what modification is needed when we modify vacuum according to § 2. For this purpose it is necessary to write down the total Hamiltonian including positive energy nucleons. For this we shall write

$$
\begin{aligned}
H=&\sum(E_k+2mc^2)\,(b^*_{k\uparrow}\,b_{k\uparrow}+b^*_{-k\downarrow}\,b_{-k\downarrow})\\
&-\frac{1}{V}\sum J(\boldsymbol{k}-\boldsymbol{k}')\,b^*_{k'\uparrow}\,b^*_{-k'\downarrow}\,b_{-k\downarrow}\,b_{k\uparrow} \qquad (3\cdot1)\\
&-\sum E_k(a^*_{k\uparrow}\,a_{k\uparrow}+a^*_{-k\downarrow}\,a_{-k\downarrow})\\
&-\frac{1}{V}\sum J(\boldsymbol{k}-\boldsymbol{k}')\,a^*_{k'\uparrow}\,a^*_{-k'\downarrow}\,a_{-k\downarrow}\,a_{k\uparrow},
\end{aligned}
$$

where $b^*_{k\uparrow,\downarrow}$ and $b_{k\uparrow,\downarrow}$ are creation and annihilation operators for positive energy nucleons. There are many other interaction terms. However, we neglect them all as unessential. We shall first consider transition between positive energy states. Since (3·1) does not contain terms connecting positive and negative energy states, nucleons in positive energy state suffer interaction from nucleon in

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* Our vacuum is not stable when nucleon exists in positive energy state, since this nucleon can annihilate through $n\to\alpha+\pi$. It can be easily shown, however, that lifetime τ due to this process is approximately given by $\tau_0\,u_k{}^2$, where τ_0 is the reaction time for the strong interaction, and $\boldsymbol{k}$ is the wave number of the emitted α particle. $\boldsymbol{k}$ must be large, of order of $(mc/\hbar)$ to satisfy the conservation law of energy. Therefore $u_k{}^2$ is very small, of order of $(\mu/m)^2\simeq10^{-2}$. So, τ can be long enough as compared with the reaction time of the π-n scattering. So, we shall neglect this possibility in this paper. We expect that τ will be infinite in the exact theory in which the conservation law of nucleon number is strictly satisfied.

** This work will be referred to as (I) in the following.

positive energy state only. But the effect is negligible if the number of nucleons in positive state is small, since I contains N and approaches zero very quickly as N goes to zero.* Therefore, even when a few particles is added in the positive energy state, vacuum is not changed, and the wave function of the total system is given by the product of (2·25) and that of free nucleons in positive energy state, that is,

$$\psi=\prod_{k}(u_k+v_k\, a^*_{k\uparrow}\, a^*_{-k\downarrow})\cdot b^*_{k'\uparrow,\downarrow}\cdots\psi_v. \tag{3·2}$$

Therefore, in transitions between positive energy states, no modification is necessary both for matrix element and energy denominator.

Next we consider transitions between positive and negative energy states. In this case, it is necessary to know first the wave function for the excited state of the vacuum. The state vector in which α_{k0} or α_{k1} is excited is given by[15),16),18)]

$$\prod_{k'\neq k}(u_{k'}+v_{k'}\, a^*_{k'\uparrow}\, a^*_{-k'\downarrow})\cdot a^*_{k\uparrow}\,\psi_v, \tag{3·3}$$

$$\prod_{k'\neq k}(u_{k'}+v_{k'}\, a^*_{k\uparrow}\, a^*_{-k\downarrow})\cdot a^*_{-k\downarrow}\,\psi_v. \tag{3·3'}$$

It can be shown that (3·3) and (3·3′), together with vacuum and more highly excited states, form a complete orthonormal set. The state in which a few positive energy nucleons are added to these excited states is given by

$$\prod_{k'\neq k}(u_{k'}+v_{k'}\, a^*_{k'\uparrow}\, a^*_{-k'\downarrow})\cdot a^*_{k\uparrow}\cdot b^*_{k''\uparrow,\downarrow}\cdots\psi_v, \tag{3·4}$$

$$\prod_{k'\neq k}(u_{k'}+v_{k'}\, a^*_{k'\uparrow}\, a^*_{-k'\downarrow})\cdot a^*_{-k\downarrow}\cdot b^*_{k''\uparrow,\downarrow}\cdots\psi_v. \tag{3·4'}$$

According to the spirit of our approach, we must assume that negative energy particles which appear in the course of the calculation are not a, but α particles. Therefore, from (3·3) ((3·3′)) and (3·4) ((3·4′)), it is clear that we must multiply an extra factor v_k for matrix element corresponding to transitions between positive and negative energy states, where k is the momentum of the negative energy nucleon to or from which transition occurs.

The energy denominator for this process is given by

$$\sqrt{E_k^2+I^2}+E_{k'}+2mc^2. \tag{3·5}$$

I is of order μc^2. Therefore, if $E_k\lesssim I$, $\sqrt{E_k^2+I^2}$ can be neglected as compared with $E_{k'}+2mc^2$. On the other hand, if $E_k\gg I$, (3·5) can be written as

$$E_k+E_{k'}+2mc^2.$$

Therefore, in both cases we can use the usual expression for energy denominator.

Taking these points into account, the Tamm-Dancoff equation in which the

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* Bogoliubov transformation treats positive and negative energy states unsymmetrically. This violates the invariance under charge conjugation. Therefore, we shall not consider charge conjugation in this paper.

effect of the collective correlation is taken into account is given by multiplying v^2_{p+k} and v^2_{p+s} to kernels corresponding to (b) and (d). The calculation can be made entirely analogous to (I) if we take into account only this point. Thus the basic equation is given by*

$$[E-E(p)-\omega(k)]g_{u\alpha}(\boldsymbol{p};\boldsymbol{k})$$
$$=-\frac{G^2}{16\pi^3}\int d^3s\left[\frac{M^4}{E(p)E(\boldsymbol{p}+\boldsymbol{k}+\boldsymbol{s})\omega(k)\omega(s)}\right]^{1/2}\bar{u}(\boldsymbol{p})\gamma_5\times$$
$$\left[\frac{1}{E(\boldsymbol{p}+\boldsymbol{k})}\left\{\frac{\tau_\alpha\Lambda_+(\boldsymbol{p}+\boldsymbol{k})\tau_\delta}{E-E(\boldsymbol{p}+\boldsymbol{k})}+\frac{v^2_{p+k}\tau_\alpha\Lambda_-(-\boldsymbol{p}-\boldsymbol{k})\tau_\delta}{E-E(p)-E(\boldsymbol{p}+\boldsymbol{k})-E(\boldsymbol{p}+\boldsymbol{k}+\boldsymbol{s})-\omega(k)-\omega(s)}\right\}\right.$$
$$\left.+\frac{1}{E(\boldsymbol{p}+\boldsymbol{s})}\left\{\frac{\tau_\delta\Lambda_+(\boldsymbol{p}+\boldsymbol{s})\tau_\alpha}{E-E(\boldsymbol{p}+\boldsymbol{s})-\omega(k)-\omega(s)}+\frac{v^2_{p+s}\tau_\delta\Lambda_-(-\boldsymbol{p}-\boldsymbol{s})\tau_\alpha}{E-E(\boldsymbol{p}+\boldsymbol{s})-E(\boldsymbol{p}+\boldsymbol{k}+\boldsymbol{s})-E(p)}\right\}\right]$$
$$\times\gamma_5 v(\boldsymbol{p}+\boldsymbol{k}+\boldsymbol{s})g_{u\delta}(\boldsymbol{p}+\boldsymbol{k}+\boldsymbol{s};-\boldsymbol{s}), \tag{3·6}$$

where $g_{u\alpha}(\boldsymbol{p};\boldsymbol{k})$ is the amplitude with nucleon specified by momentum $\boldsymbol{p}$ and spin index u and pion specified by momentum $\boldsymbol{k}$ and charge index α, $u(\boldsymbol{p})$ and $v(\boldsymbol{p}')$ are Dirac spinor, and $\Lambda_\pm$ are projection operators for positive and negative energy state.

Going to the center of mass system, we shall put

$$g_{u\alpha}(\boldsymbol{p};\boldsymbol{k})\to g_{u\alpha}(\boldsymbol{p};-\boldsymbol{p})\equiv g_{u\alpha}(\boldsymbol{p}),$$

and put $\sum_{u=1,2}u(\boldsymbol{p})g_{u\alpha}(\boldsymbol{p})=g_\alpha(\boldsymbol{p})$. Then, introducing large and small component of $g_\alpha(\boldsymbol{p})$ by

$$g_\alpha(\boldsymbol{p})=\begin{pmatrix} g_\alpha{}^+(\boldsymbol{p}) \\ g_\alpha{}^-(\boldsymbol{p}) \end{pmatrix},$$

separating the angular part by exactly the same method as in (I) and leaving only the terms that contribute to δ_3, we get

$$[E-E(p)-\omega(p)]g(\boldsymbol{p})=\frac{G^2}{8\pi^2}\int_0^\infty\frac{s^2ds}{[E(p)\omega(p)E(s)\omega(s)]^{1/2}}$$
$$\times\sum_n\left[(Y_n'-Z_n)\left(S_n[E(p)+M]+\frac{psR_n}{E(s)+M}\right)\right.$$
$$\left.+\left(\frac{1}{C}X_n+\frac{1}{B}X'_n-\frac{1}{C}Z_n-\frac{1}{B}Y_n'\right)\right. \tag{3·7}$$
$$\left.\times\left\{\frac{psR_n(E(p)+E(s)+M)}{E(s)+M}+(E(p)+M)(E(p)+E(s)-M)S_n\right\}\right]g(s),$$

---

* Hereafter we shall use the relativistic formalism and shift the zero point of the energy from $-mc^2$ to zero to proceed in parallel with (I) as closely as possible.

as the integral equation to determine the scattering amplitude for $T=(3/2)$, where we introduced X_n' and Y_n' in addition to Dyson's X_n, Y_n and Z_n:*

$$X_n'=\left(n+\frac{1}{2}\right)(r\bar{E})^{-1}\int_{1-r}^{1+r}v^2(z)\,dz\,P_n\left(\frac{z^2-1-r^2}{2r}\right),$$

$$Y_n'=\left(n+\frac{1}{2}\right)(r\bar{E})^{-1}\int_{1-r}^{1+r}v^2(z)\,dz\frac{z}{b+z}P_n\left(\frac{z^2-1-r^2}{2r}\right), \tag{3·8}$$

where P_n's are Legendre polynoms, and

$$\begin{aligned} A&=E(p)+\omega(p)+E(s)+\omega(s)-E,\\ B&=E(p)+E(s)-E,\\ C&=\omega(p)+\omega(s)-E,\\ \bar{E}&=(E(p+s)+E(p-s))/2,\\ r&=ps/\bar{E}^2, \end{aligned} \tag{3·9}$$

and

$$z=E(\boldsymbol{p}+\boldsymbol{s})/\bar{E}.$$

R_n and S_n are defined by

$$R_n\,g(\boldsymbol{s})=\frac{1}{4\pi}\int d\omega_s\frac{(\boldsymbol{\sigma p})}{p}\cdot\frac{(\boldsymbol{\sigma s})}{s}\cdot P_n(\theta_s-\theta_p)g(\boldsymbol{s}), \tag{3·10}$$

and

$$S_n\,g\,(\boldsymbol{s})=\frac{1}{4\pi}\int d\omega_s\;P_n(\theta_s-\theta_p)g(\boldsymbol{s}).$$

Rewriting (3·7),

$$[E-E(p)-\omega(p)]g(p)=\frac{G^2}{8\pi^2}\int_0^\infty\frac{s^2ds}{[E(p)\omega(p)E(s)\omega(s)]^{1/2}}L(p,\,s)g(s), \tag{3·11}$$

where

$$L(p,\,s)=\sum_n\Big[(E(p)+M)[(Y_n'-Z_n)+\left\{\frac{1}{C}(X_n-Z_n)+\frac{1}{B}(X_n'-Y_n')\right\}\times$$

---

* X_n, Y_n, Z_n are defined by

$$X_n=\left(n+\frac{1}{2}\right)(r\bar{E})^{-1}\int_{1-r}^{1+r}dz\,P_n\left(\frac{z^2-1-r^2}{2r}\right),$$

$$Y_n=\left(n+\frac{1}{2}\right)(r\bar{E})^{-1}\int_{1-r}^{1+r}dz\,\frac{z}{b+z}P_n\left(\frac{z^2-1-r^2}{2r}\right),$$

$$Z_n=\left(n+\frac{1}{2}\right)(r\bar{E})^{-1}\int_{1-r}^{1+r}dz\,\frac{z}{c+z}P_n\left(\frac{z^2-1-r^2}{2r}\right).$$

$$\times (E(p)+E(s)-M)]S_n+\frac{ps}{E(s)+M}[(Y_n'-Z_n)+\left\{\frac{1}{C}(X_n-Z_n)+\frac{1}{B}(X_n'-Y_n')\right\}$$

$$\times (E(p)+E(s)+M)]R_n]. \tag{3·12}$$

Among terms in (3·12) only those proportional to S_0 and R_1 contribute to the S-wave scattering. Thus, for S-wave scattering,

$$L(p, s)=(E(p)+M)[(Y_0'-Z_0)+\left\{\frac{1}{C}(X_0-Z_0)+\frac{1}{B}(X_0'-Y_0')\right\}$$

$$\times (E(p)+E(s)-M)]+\frac{1}{3}\frac{ps}{E(s)+M}[(Y_1'-Z_1)+\left\{\frac{1}{C}(X_1-Z_1)+\frac{1}{B}(X_1'-Y_1')\right\}$$

$$\times (E(p)+E(s)+M)],^{*}$$

which is simplified, neglecting terms of order (μ/m), to

$$L(p, s)=(E(p)+M)\{Y_0'+\frac{1}{B}(X_0'-Y_0')(E(p)+E(s)-M)\}. \tag{3·13}$$

From definition (3·8) we get

$$X_0'=\frac{1}{2}\cdot\frac{1}{r\bar{E}}\int_{1-r}^{1+r}v^2(z)dz.$$

We can approximate this by

$$X_0'=\frac{1}{\bar{E}}v^2(1)=\frac{1}{\bar{E}}\times\frac{1}{2}\left(1+\frac{\bar{E}-mc^2}{\sqrt{(\bar{E}-mc^2)^2+I^2}}\right), \tag{3·14}$$

since r is of order $(\mu/m)^2$. By exactly the same way, we get

$$Y_0'=\frac{1}{B+\bar{E}}\times\frac{1}{2}\left(1+\frac{\bar{E}-mc^2}{\sqrt{(\bar{E}-mc^2)^2+I^2}}\right). \tag{3·15}$$

Substituting (3·14) and (3·15) in (3·13),

$$L(p, s)=\frac{E(p)+M}{\bar{E}}\left(1+\frac{\bar{E}-M}{\bar{E}+B}\right)\times\frac{1}{2}\left(1+\frac{\bar{E}-mc^2}{\sqrt{(\bar{E}-mc^2)^2+I^2}}\right).^{**} \tag{3·16}$$

$g(p)$ has singularity at $p=k$. To separate this singularity, we put

$$g(p)=\delta(E-E(p)-\omega(p))+\frac{f(p)}{E-E(p)-\omega(p)}. \tag{3·17}$$

Inserting (3·17) into (3·11) we get

---

* Factor (1/3) in front of the second term is the eigenvalue of R_1.

** (3.16) reduces to the result of Dyson et al. in the limit $I\to 0$, as it should.

$$f(p)=\frac{G^2}{8\pi^2}\int_0^\infty \frac{s^2}{[E(s)\omega(s)E(p)\omega(p)]^{1/2}}L(p,s)g(s)ds. \tag{3·18}$$

Born approximation to (3·18) is obtained by replacing $g(s)$ by $\delta(E-E(s)-\omega(s))$. Thus,

$$f_B(p)=\frac{G^2}{8\pi^2}\cdot\left[\frac{E(k)\omega(k)}{E(p)\omega(p)}\right]^{1/2}\cdot\frac{k}{E}\cdot L(p,k). \tag{3·19}$$

Using $f_B(p)$, (3·18) can be written as

$$f(p)=f_B(p)+\frac{G^2}{8\pi^2}\mathcal{P}\int_0^\infty \frac{s^2ds}{[E(s)\omega(s)E(p)\omega(p)]^{1/2}} \times L(p,s)\frac{1}{E-E(s)-\omega(s)}\cdot f(s). \tag{3·20}$$

If (3·20) is solved, phase shift can be calculated by

$$\tan\delta_3=-\pi f(k). \tag{3·21}$$

§ 4. Approximate solution of the integral equation

It is difficult to solve (3·18) or (3·20) exactly. Therefore, we shall try to obtain an approximate solution using Chew's variational method.[19]

To apply this method, it is necessary to cut off the integral in the right-hand side at some appropriate value K. Then using static approximation for the nucleon, we get

$$f(p)=f_B(p)+\mathcal{P}\int_0^K s^2K(p,s)\frac{f(s)}{\omega_0-\omega}ds, \tag{4·1}$$

where

$$K(p,s)=\frac{\lambda}{7[\omega(p)\omega(s)]^{1/2}}L(p,s), \tag{4·2}$$

$$L(p,s)=1+\frac{(p^2+s^2)/14}{\sqrt{(p^2+s^2)^2/14^2+I^2}}, \tag{4·3}$$

and

$$f_B(p)=\frac{\lambda k}{7}\left[\frac{\omega(k)}{\omega(p)}\right]^{1/2}L(p,k), \tag{4·4}$$

and μ was used as unit for momentum and energy. $\lambda=G^2/8\pi^2$ and ω_0 is the energy of the incident pion.

Chew's approximate solution to (4·1) is given by

$$f(k)=\frac{f_B(k)}{1+\Delta}, \tag{4·5}$$

with

$$\Delta=\frac{1}{K(k,k)}\mathcal{P}\int_0^K\frac{K^2(p,k)p^2}{\omega(p)-\omega(k)}dp. \tag{4·6}$$

First let us consider the case $I=0$. In this case, $L(p,s)=2$. Thus neglecting terms of order $(1/K)^2$ against unity,

$$\mathcal{P}\int_0^K\frac{K^2(p,k)p^2}{\omega(p)-\omega(k)}dp=\frac{4\lambda^2}{49\omega(k)}\left[K+\sqrt{1+k^2}\log 2K-k\log\left(k+\sqrt{1+k^2}\right)-\frac{k^2}{K}\right]. \tag{4·7}$$

Therefore,

$$f^{(0)}(k)=\frac{f_B{}^{(0)}(k)}{1+\Delta^{(0)}} \tag{4·8}$$

with

$$\Delta^{(0)}=\frac{2\lambda}{7}\left[K+\sqrt{1+k^2}\log 2K-k\log\left(k+\sqrt{1+k^2}\right)-\frac{k^2}{K}\right], \tag{4·9}$$

where we have introduced suffix 0 to specify the case corresponding to $I=0$. (4·8) reproduces the result of relativistic calculation of (I) fairly well if we take $K=3\sim7$.* (See Fig. 1.) Therefore, we shall assume that Chew's method gives a

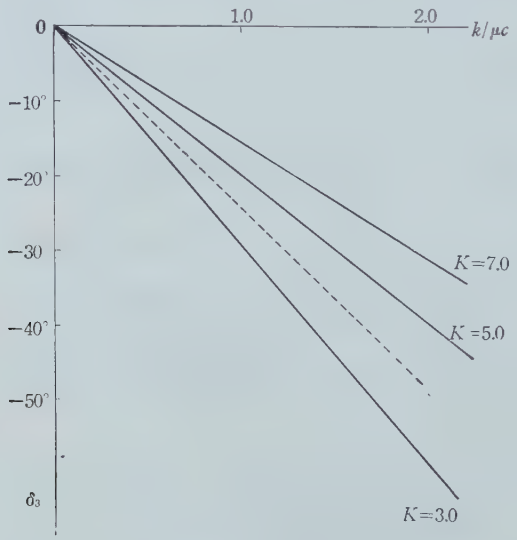

Fig. 1. Comparison of various calculations.
- - -: relativistic calculation by Dyson et al.
——: Chew's variational method.

* We assume $G^2/4\pi=15$. We use rationalized coupling constant in § 3 and § 4. Natural unit is used also.

good result in the case $I \neq 0$ also. When $I \neq 0$, $L(p, k) \neq 2$. The difference is, however, restricted to the region with small p, since I_k is of order 1 and vanishes for $k > \xi$. More precisely, $L(p, k)$ is approximated by

$$L(p, k) = \begin{cases} 2 & \text{for } k \gtrsim \xi, \\ 2 & \text{for } k \lesssim \xi,\ p \gtrsim \xi, \\ 1 & \text{for } k \lesssim \xi,\ p \lesssim \xi. \end{cases} \tag{4·10}$$

So, in approximation neglecting terms of order $(1/K)$ against unity, (4·7) remains valid in this case also. Therefore, noticing that $L(k, k) = v^2_{\sqrt{2}k}$, we get

$$f(k) = \frac{v^2_{\sqrt{2}k} f_B{}^{(0)}(k)}{1 + \Delta^{(0)}/v^2_{\sqrt{2}k}} . \tag{4·11}$$

which reduces, neglecting terms of order (μ/m), to

$$f(k) \simeq v^4_{\sqrt{2}k} f^{(0)}(k). \tag{4·12}$$

Thus, $f(k)$, and consequently δ_3, is multiplied by approximately $v^4_{\sqrt{2}k}$ as compared with the case $I=0$. As $v^2_{\sqrt{2}k}$ varies from 1/2 to 1 as k varies from 0 to infinity, this means that δ_3 is reduced to about 1/4 at low energies by the effect of collective correlation. (4·8) gives

$$\delta_3 \simeq -(20° \sim 30°)k, \tag{4·13}$$

for $G^2/4\pi = 15$ and $K = 3 \simeq 7$. The introduction of the effect of collective correlation modifies this to

$$\delta_3 \simeq -(5° \sim 7°)k \tag{4·14}$$

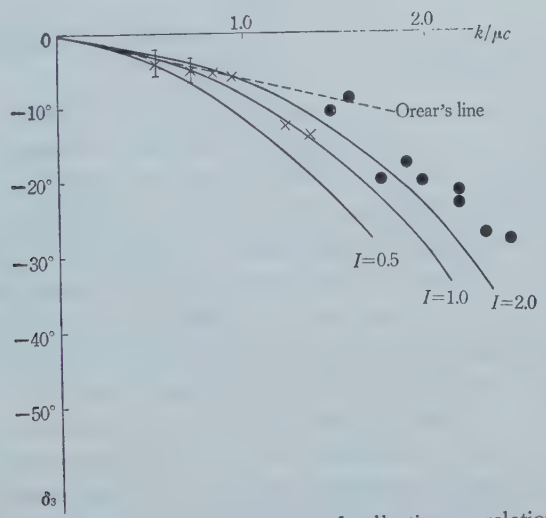

Fig. 2. δ_3 corrected by the effect of collective correlation.* K was taken to be 5.0.
Bethe-Hoffmann[22]
Pontecorvo.[21]

* We assume that I is constant in energy region given in Fig. 2, assuming that $\xi \geqq 2$.

at low energies. At high energies, (4·13) remains unchanged. Therefore, energy dependence of δ_3 deviates from linear line in contrast to the case $I=0$. It should be noted that the slopes of both low and high energy limits are independent of the newly introduced parameter I. Only the position of the turning point depends upon it. (4·14) is in good agreement with Orear's phase shift ($\delta_3=-6\cdot3°k$),[20] while the predicted energy dependence seems to be in agreement with recent high energy data.[21] Thus, we see that both the slope at zero energy limit and the energy dependence of δ_3 can be satisfactorily accounted for by introducing the effect of collective correlation between vacuum nucleons. As examples, we give phase shift for $K=5.0$ and $I=0.5$, 1.0 and 2.0 in Fig. 2. As will be seen, the agreement with experiment is surprisingly good for $I=1.0$ and 2.0. It is satisfactory that the agreement is obtained by the value of I near 1.0, since our calculation has shown that I is just of this order of magnitude.

§ 5. Concluding remarks

In § 4, we had to introduce cutoff momentum K to apply Chew's method. If we can solve (3.18) or (3.20) exactly, it is unnecessary to introduce it, and the result of § 4 strongly suggests that in this case we can reproduce Orear's phase shift without introducing any ambiguous parameter such as K or I, since the relativistic Tamm-Dancoff approximation gives

$$\delta_3=-24°k \qquad (5\cdot1)$$

unambiguously.[4] We are now trying to solve (3·18) numerically, with use of an electronic computor.

In this paper, we did not discuss the effect of pion-pion interaction. It is very probable that this interaction affects pion-nucleon scattering since the nucleon is always surrounded by the pion cloud. This effect on the S-wave scattering was discussed by Kawarabayashi and Miyazawa.[14] They showed that with a reasonable sign and the strength of the pion-pion interaction, it is possible to reproduce the qualitative features of the S-wave scattering if we assume extension of order κ^{-1} for source function of the nucleon. If, however, such an extended source is assumed, phase shift will drop off to zero for momentum larger than about μc. This seems to be in contradiction with high energy data.[21]

We have seen that at least in S-wave problem, the situation is greatly improved by introducing the effect of collective correlation between vacuum nucleons. It is expected that this effect also affects other phenomena which include nucleon antinucleon pair formation such as the iso-scalar part of the anomalous magnetic moment of the nucleon or 2γ decay of the neutral pion. We have not yet made detailed calculation in these cases, but the tendency will not be bad since this effect suppresses the effect of virtual pair in general.

Thus far, field theory has been formulated in a completely relativistic invariant

form and in this form many successes have been made. Although this is on one hand a great merit of the theory, it seems to have had on the other hand the danger that it was sometimes treated too formally. In this point, concepts or techniques developed in low temperature physics or plasma physics are very instructive. The object of quantum field theory is of course an interacting system of many particles, and in this sense the object is entirely the same. Therefore, peculiar situations arising in the latter should always have their counter parts in the former, and this suggests that methods of approximation usually used in quantum field theory are not always adequate. To evade this point, it would be of interest, as a complementary approach to the usual approximations, to try to introduce these techniques into the field theory. It is not yet certain whether the S-wave problem is the example of such situation, but our calculation would at least remove the impression that *ps-ps* meson theory is inherently in contradiction with experiment.

References

1) S. D. Drell and E. M. Henley, Phys. Rev. **88** (1952), 1053.
2) N. Fukuda, S. Goto, S. Okubo and K. Sawada, Prog. Theor. Phys. **12** (1954), 79.
3) M. M. Levy and R. E. Marshak, Nuovo Cimento **XI** (1954), 366.
4) F. J. Dyson, M. Ross, E. E. Salpeter, S. S. Schweber, M. K. Sundaresan, W. M. Visscher and H. A. Bethe, Phys. Rev. **95** (1954), 1644.
5) M. M. Levy, Phys. Rev. **98** (1955), 1470.
6) M. H. Kalos and R. H. Dalitz, Phys. Rev. **100** (1950), 1515.
7) E. M. Lomon, Nuovo Cimento **IV** (1956), 106.
8) A. Martin, Nuovo Cimento **IV** (1956), 369.
9) C. J. Goebel, Phys. Rev. **101**(1956), 468.
10) S. D. Drell, M. H. Friedman and F. Zachariasen, Phys. Rev. **104** (1956), 236.
11) K. Tanaka, Phys. Rev. **108** (1957), 1629.
12) W. Gilbert, Phys. Rev. **108** (1957), 1078.
13) R. Sugano, Prog. Theor. Phys. **22** (1959), 381.
14) K. Kawarabayashi and H. Miyazawa, Prog. Theor. Phys. **23** (1960), 490.
15) J. Bardeen, L. N. Cooper, and J. R. Schriefer, Phys. Rev. **106** (1957), 162; **107** (1957), 1175.
16) N. N. Bogoliubov, J. E. T. P. **34** (1958), 58.
17) M. L. Goldhaber, H. Miyazawa and R. Oehme, Phys. Rev. **99** (1955), 986.
18) K. Yoshida, Phys. Rev. **111** (1958), 1255.
19) G. F. Chew, Phys. Rev. **93** (1954), 341.
20) J. Orear, Phys. Rev. **96** (1954), 176.
21) B. Pontecorvo, *Report at Ninth Annual Conference on High Energy Nuclear Physics* (Kiev).
22) H. A. Bethe and de Hoffmann, *Mesons and Fields*, Vol. II (Interscience Publishers, 1956).

Note added in proof : We assume that the total number of the vacuum nucleon is given by the condition that it gives minimum value for E_g. More specifically, we add λN(N is the number operator for vacuum nucleons) to the original Hamiltonian (2·4) according to the standard technique of Bogoliubov, and determine λ by the condition $\langle N\rangle_{Av}=N$, where N is given by

$$\frac{\partial E_g}{\partial N}=0, \quad \text{and} \quad \frac{\partial^2 E_g}{\partial N^2}<0.$$

Obviously, $\lambda=0$ at the weak coupling limit, and we neglect in this paper its possible dependence upon J for simplicity.

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Field Theoretical Interpretation of "*B-Matter*". I

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The property of "*B-matter*" which was introduced in the Nagoya model is studied on the standpoint of the current field theory. First, a reasonable interaction Lagrangian is assumed between the "*B-particle*" and leptons. Then, the following conditions are required: Through this interaction, the proton P, the neutron N and the lamda particle Λ should be constructed as bound states of the "*B-particle*" and leptons, and further more the β-decay should occur through the same interaction. Results are summarized as follows: the "*B-particle*" must have a mass about $10^5 \sim 10^6$ times electron mass in order that the above requirements may be fulfilled. It might be possible to describe the strong and weak interactions (excluding the μ-decay) by means of the above interaction in a unified way.

§ 1. Whereabouts of the problem

One of the main objects of the Nagoya model[1] recently presented by the Nagoya group is to construct all elementary particles except photon by introducing a new material (called "*B-matter*") and to examine the applicability of the current field theory on the basis of this model. At present, our knowledges about the B-matter are as follows: (1) B-matter obeys bose-statistics and (2) composes P, N and Λ combined with ν, e^- and μ^-, respectively. (3) It has, therefore, a positive charge and (4) a mass larger than those of P, N and Λ, and (5) does not exist in free states. (6) The number of B-matters (which is counted by baryon number) is conserved by itself.

Many people believe that the B-matter should not be quantized and is not described by the current field theory since it has never been found by experiments. The belief, however, has not been justified at all. It is, therefore, indispensable to study whether the current field theory is applicable to the B-matter or not. We examine this point in this article. Our standpoint is as follows: If the B-matter has the property neither to be quantized nor to be decribed by the current field theory, inconsistency would appear in the theory in which the B-matter is treated as a particle by the current field theory. We should like to make clear this inconsistency and to see to what extent the current field theory is applicable to the B-matter.

The B-matter is assumed to have a particle-like property so as to be described by the current field theory. We shall call this the "*B-particle*" hereafter. Then we assume a suitable interaction between the B-particle and leptons and require

that the strong and weak interactions should be deduced from the same interaction.

Let us begin with the determination of the interaction Lagrangian. Since the number of B-particles and of leptons are conserved separately, the simplest one is

$$L' = \{g_1(\bar{\nu}\,\nu) + g_2(\bar{e}\,e) + g_3(\bar{\mu}\,\mu)\}B^*B.$$

More complicated cases may be essentially the same as the above. We further assume a symmetry with respect to ν, e and μ, that is, $g_1 = g_2 = g_3 \equiv g$, which will not change the essence of our task.

$$L' = g(\bar{\psi}_i\ \psi_i)B^*B \qquad (i = \nu, e, \mu). \tag{1}$$

We require that the strong and weak interactions are to be described by this interaction Lagrangian.

Now, why is B-particle in free state not found by experiments? We consider the reason as follows: (a) The mass of B-particle is so large that it cannot be produced at low energies. (b) B-particle would easily make a stable bound state combined with a lepton. (c) B-particle in a free state is very unstable and decays into a proton and an anti-neutrino at once (i.e. $B \to p + \bar{\nu}$) (Fig. 1).

Fig. 1. Decay of B-particle.

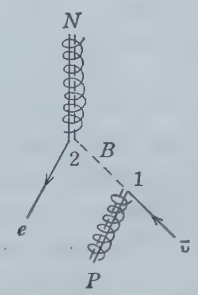

Fig. 2. β-decay.

An interesting point is to know whether the above interaction is able to construct P, N and Λ and to describe the weak interactions like β-decay, and what properties the B-particle and the coupling constant g must have in order to satisfy the above requirements. Since, in the case of β-decay, for example, a bare B-particle appears in the virtual state, the matrix element of the decay is considered to be small on account of the B-particle propagator, provided the mass of B-particle is large. This is, however, not necessarily the case, because the matrix element depends upon the value of the coupling constant and the amplitudes of the two-body wave function ($P = B + \nu$, $N = B + e$) which are closely related to the mass of B-particle.

§ 2. Calculation and summary

i) *Wave function and mass of nucleon*

In this subsection, we construct P, N and Λ as bound states of B-particle and leptons and examine the relation between the mass of B-particle M_B and the coupling constant g which must be satisfied in order that the masses of P, N and Λ may be given as the energies of the bound states.

The wave functions of P, N and Λ defined by

$$\Psi_i(x_1,\ x_2)\equiv\langle 0|T[B(x_1)\psi_i(x_2)]|s_i\rangle \quad (s_1=P,\ s_2=N,\ s_3=\Lambda) \tag{2}$$

satisfy

$$\Psi_i(x_1,\ x_2)=ig\int\Delta_F(x_1-x')S_F^i(x_2-x')\Psi_i(x',\ x')\,(dx') \tag{3}$$

where we take a chain approximation. $\Delta_F(x)$ and $S_F^i(x)$ $(i=1,\ 2,\ 3)$ are propagators of B-particle and leptons, respectively. The solution of the equation are easily obtained as follows:[2)]

$$\left.\begin{aligned}&\Psi(x_1,\ x_2)\equiv\Psi(x)e^{iPX},\\&\Psi(x)=\frac{ig}{(2\pi)^4}\int(dp)\frac{i\gamma(bP-p)-m}{[(aP+p)^2+M_B^2-i\varepsilon][(bP-p)^2+m^2-i\varepsilon]}e^{ipx}\Psi(0),\\&x\equiv x_1-x_2,\ X\equiv ax_1+bx_2,\ a\equiv M_B/(M_B+m),\ b\equiv m/(M_B+m),\end{aligned}\right\} \tag{4}$$

where we drop the lepton suffix i. P and p denote the total and relative energy-momentum of the system and M_B and m are the masses of B-particle and leptons, respectively. $\Psi(0)$ is a constant spinor to be determined by the normalization of the wave function.

If we put $x_1=x_2$, Eq. (4) is an eigenvalue equation. We determine the coupling constant g so that the masses of P, N and Λ are given as the energy eigenvalues of the equation. As the equation, however, contains a divergent integral, we make use of the cutoff method so as to make this convergent, by taking the mass of B-particle as the cutoff energy. It may be reasonable for us to consider that the B-particle has the largest mass among elementary particles in the world. It would thus be meaningless to argue about energy larger than the mass of B-particle. Both straight cut and Feynman cut may be suitable. Though the equation (4) contains an infrared divergence in the case where the lepton is neutrino $(m_\nu=0)$, we do not worry about it and avoid this divergence by assuming that neutrino has non-zero mass $(\neq 0)$, since it is beside our interest.

As the mass of B-particle is very large, masses of other particles are negligible compared with it. The equation therefore becomes very simple and leads to the following relation which must be fulfilled between M_B and g in order that the equation has a constant energy eigenvalue (i.e. masses of P, N and Λ).

$$gM_B\sim 1 \text{ or } g\sim 1/M_B. \tag{5}$$

This shows that the larger the mass of B-particle, the smaller the coupling constant becomes. This situation is reasonable since the cutoff energy depends upon the mass of B-particle.

ii) *Normalization of nucleon wave function*

Now let us consider the normalization of the above two-body wave function Ψ. If we notice the conservation of B-particle, the normalization in lowest order is written in the following form as is seen in the Appendix.

$$g\int(dx_1)\left[\Psi^{\dagger}(x_0,\ x_1)\frac{\partial\Delta_F(x_0-x_1)}{\partial x_0^0}-\frac{\partial\Psi^{\dagger}(x_0,\ x_1)}{\partial x_0^0}\Delta_F(x_0-x_1)\right]\Psi(x_1,\ x_1)=1/V, \tag{6}$$

where V is the normalization volume. Substituting the solution (4) in the above equation and looking at $\Psi(0)$, g and M_B only, we see the left-hand side of Eq. (6) becomes $\sim g^2|\Psi(0)|^2$. Therefore,

$$|\Psi(0)|^2\sim 1/g^2V. \tag{7}$$

iii) *β-decay*

Finally, we calculate the matrix element of β-decay which we feel most interesting. The expression of the matrix element is

$$S_{if}=g^2\int\Psi_N^{\dagger}(2,\ 2)\Delta_F(1-2)\Psi_P(1,1)\Psi_{\bar{\nu}}(1)\Psi_e(2)\,(dx_1)\,(dx_2), \tag{8}$$

where $\Psi_{\bar{\nu}}(x)\equiv\langle 0|\bar{\psi}_{\nu}(x)|\bar{\nu}\rangle$ and $\psi_e(x)\equiv\langle 0|\psi_e(x)|e\rangle$. Substituting the solution (4) and using Eqs. (5) and (7), we see

$$S_{if}\sim\delta^4(P^N-P^P-p^e-p^{\nu})\cdot 1/M_B^2 \tag{9}$$

where the unit volume is taken for the normalization volumes and the δ-function expresses the conservation of the total energy-momentum of the system. This means that the larger the mass of B-particle, the smaller the matrix element S_{if} becomes, that is, the more scarcely the β-decay occurs. If f is the coupling constant of the usual four fermi coupling of β-decay, we have a relation between f and M_B as

$$M_B^2\sim 1/f. \tag{10}$$

Since f is about $10^{-7}/m_{\pi}^2$ (m_{π}=meson mass), we see $M_B^2\sim 10^7 m_{\pi}^2$ or $M_B\sim(10^3\sim 10^4)m_{\pi}$.

iv) *Summary and discussion*

In the subsection iii) we have seen that if we describe the B-matter as a particle by making use of the current field theory, the B-particle must have a mass about $10^5\sim 10^6 m_e$ (=electron mass) in order to explain the β-decay. It is consistent with the assumption (a) in section 1.

If we assume that the matrix element of a reaction in which a bare B particle appears as a virtual state is small, then decay interactions which conserve respective numbers of leptons (the neutrino number n_{ν}, the electron number n_e and the μ-meson number n_{μ}) are generally weak. The examples are β-decay (i.e. $N\rightarrow P+e^-+\bar{\nu}$) and $K_{\mu 2}$-decay (i.e. $K^{\pm}\rightarrow\mu^{\pm}+\nu(\bar{\nu})$). The interaction (1), however,

cannot explain the decays which do not conserve respective lepton numbers. The simplest and typical example is μ-decay (i.e. $\mu^{\pm}\to e^{\pm}+\nu+\bar{\nu}$). This means that another interaction

$$L''=g'(\bar{\mu},\ e)(\bar{\nu},\ \nu) \tag{11}$$

is indispensable. Then all decay modes seem to be explained by both interactions (1) and (11).

The remaining problems to be explained in future are as follows:*

a) The B-particle makes bound states with the leptons and the anti-leptons in the same manner. The situation turns out that two kinds of the protons and doubly charged baryons are existent at the same time.

b) The parity non-conservation in the weak interactions is not explained in this article.

c) The strong interaction in the usual sense is not discussed at all.

d) The decay of the K^0-meson ($K^0\to\mu^{\mp}+e^{\pm}$) occurs in the same order as the weak interaction.

Among the above problems a)$\sim$c) will be solved in the next article.

The author would like to express his gratitude to Dr. S. Tanaka and Dr. T. Murota for their interest shown in this work and for critical discussions.

Appendix**

Derivation of the normalization (6)

The normalization of nucleon wave function is performed according to the method of Nishijima,[3] if we notice the conservation of the number of B-particles. First we hypothetically introduce an external field q_μ which interacts only with B-particle.

$$L_{int}=g(\bar{\psi}\ \psi)B^*B-j_\mu\ q_\mu,$$

$$j_\mu(x)=i\left(B(x)\frac{\partial B^*(x)}{\partial x_\mu}-\frac{\partial B(x)}{\partial x_\mu}B^*(x)\right).$$

If we define a transformation function

$$U=T^*[\exp(i\int_{-\infty}^{+\infty}(dx)\,L_{int}(x))],$$

we see

$$i\frac{\delta}{\delta q_\mu(0)}T[UABC\cdots Z]=T[Uj_\mu(0)ABC\cdots Z].$$

On the other hand, the two-body Green function defined by

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* The author is indebted to Dr. Y. Ohnuki for pointing out these problems.

** The notation of reference 3) is used throughout.

$$K(1,2;3,4)\equiv\langle 0|B(1)\psi(2)B^*(3)\bar{\psi}(4)|0\rangle$$

satisfies

$$K(1,2;3,4)=\Delta_F(1-3)S_F(2-4)$$
$$+ig\int(dx_5)(dx_6)\Delta_F(1-5)S_F(2-6)\delta(5-6)K(5,6;3,4).$$

Differentiating the above equation with respect to q_μ, putting $q_\mu=0$ and substituting $(3,4)^{in}$ for (3,4), then we have

$$\Psi_\mu(1,2;0)=ig\int(dx_5)\Big[i\frac{\delta}{\delta q_\mu(0)}\{\Delta_F(1-5)S_F(2-5)\}_{q=0}\Psi(5,5)$$
$$+\Delta_F(1-5)S_F(2-5)\Psi_\mu(5,5;0)\Big],$$

where

$$\Psi_\mu(1,2;0)\equiv\langle 0|T[j_\mu(0)B(1)\psi(2)]|s\rangle.$$

In the lowest order we see

$$i\frac{\delta}{\delta q_\mu(0)}\{\Delta_F(1-5)S_F(2-5)\}_{q=0}\simeq i\frac{\delta}{\delta q_\mu(0)}\Delta_F(1-5)\cdot S_F(2-5)$$
$$\simeq i\int(dx_3)(dx_4)\Delta_F(1-3)D_\mu(3,4;0)\Delta_F(4-5)S_F(2-5),$$

where

$$D_\mu(3,4;0)=\partial\delta(3-0)/\partial x_\mu{}^0\cdot\delta(4-0)-\delta(3-4)\cdot\partial\delta(4-0)/\partial x_\mu{}^0.$$

Then we have the equation for $\Psi_\mu(1,2;0)$

$$\Psi_\mu(1,2;0)=ig\int(dx_5)\Delta_F(1-5)S_F(2-5)\Psi_\mu(5,5;0)$$
$$-g\int(dx_3)(dx_4)(dx_5)\Delta_F(1-3)S_F(2-5)D_\mu(3,4;0)\Delta_F(4-5)\Psi(5,5).$$

The solution of the above equation is easily obtained if we consider the equation for the two-body Green function $K(1,2;3,4)$.

$$\Psi_\mu(1,2;0)=-g\int(dx_3)(dx_4)(dx_5)K(1,2;3,5)D_\mu(3,4;0)\Delta_F(4-5)\Psi(5,5).$$

If we substitute $(1,2)^{out}$ for $(1,2)$, $\Psi_\mu(1,2;0)$ is rewritten as follows:

$$\Psi_\mu(1,2;0)\to\sum_{s'}\langle 0|T[B(1)\psi(2)]|s'\rangle\langle s'|j_\mu(0)|s\rangle,$$

where the summation s' is taken over all bound states. Therefore we see

$$\langle s|j_\mu(0)|s\rangle=-g\int(dx_3)(dx_4)(dx_5)\Psi^\dagger(3,5)D_\mu(3,4;0)\Delta_F(4-5)\Psi(5,5)$$

$$=g\int(dx_5)\left[\Psi^{\dagger}(0,5)\frac{\partial\Delta_F(0-5)}{\partial x_\mu^{0}}-\frac{\partial\Psi^{\dagger}(0,5)}{\partial x_\mu^{0}}\Delta_F(0-5)\right]\Psi(5,5).$$

As the right-hand side of the above equation is independent of x_μ^{0}, it is equal to the inverse of the normalization volume V.

References

1) Z. Maki, M. Nakagawa, Y. Ohnuki and S. Sakata, Prog. Theor. Phys. **23**(1960), 1174.
2) H. Yamamoto, Prog. Theor. Phys. **23**(1960), 1100.
3) K. Nishijima, Prog. Theor. Phys. **10**(1953), 549; **12** (1954), 279.

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The Electric Polarizability of the Neutron

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Electric polarizability of the neutron is calculated with the relativistic meson theory in the lowest order of perturbation method. A cutoff calculation is also examined. There are remarkable cancellations among the large contributions from the pion current so that those from the nucleon currents and from the K-meson interactions become important. The electric polarizability obtained amounts to a several times as large as 10^{-43}cm^3 and is expected not to exceed 10^{-42}cm^3. Further discussions are made in connection with the Foldy term, static theory and the problem of mass difference.

§ 1. Introduction

Recently, interest in the subject of the electric polarizability of the neutron has developed, and several experimental methods have been reported for determining it.[1] The electromagnetic property of a particle is characterized by its electric charge and then by its electric dipole moment, magnetic dipole moment, and so forth. The electric charge of the neutron is confirmed to be zero with strict accuracy. The electric dipole moment of the elementary particle has been discussed in connection with the invariance of the theory under the time reversal, i.e. the presence of the intrinsic electric dipole moment implies the non-invariance of the theory under this transformation. Experimentally, only an upper limit is known for this quantity:[2] $d \leq e(5\times10^{-20})\text{cm}$.** Since, however, the electron-nucleon scatterings in several hundred Mev region have revealed a finite extent of the nucleon of order of a few times its Compton wavelength—the root mean square radius $r_{rms}=0.8\times10^{-13}\text{cm}$—, it appears that the nucleon has some structure. Accordingly, an electric dipole moment may be induced, for example, in the neutron when it is placed in an external electric field. When the induced electric dipole moment is denoted by

$$+\alpha E,$$

one obtains a perturbing Hamiltonian of non-linear type such as

$$H=-(1/2)\cdot\alpha E^2. \tag{1}$$

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** This value of the intrinsic moment predicts an electric polarizability of $\sim10^{-55}\,\text{cm}^3$ for the neutron, a value which is extremely smaller than that of our present concern.

The constant α is called the electric polarizability of the system. If the characteristic dimension of the neutron structure is assumed to be r_{rms}, the electric polarizability of the neutron may be crudely estimated as

$$\sim e^2(r_{rms})^3 = 3.7\times 10^{-42}\ \mathrm{cm}^3.$$

We should also note that some of experiments have presented values of comparative order of magnitude to this for the electric polarizability of neutron, though most of them scatter over a wide range in magnitude.[1)]

Complying with these present situations we calculate the electric polarizability of the neutron in the perturbation method with current relativistic meson theory. A cutoff calculation is further applied to this problem, because the reality might be reflected rather in cutoff calculation than in non-cutoff one. Of course these straightforward calculations cannot always be expected to give reliable answers. It is however necessary to know what results are derived from these routine methods, in order to have an outlook on the future development.

§ 2. Contributions from the pion-nucleon coupling

We describe the interaction among three fields—the nucleon, the pion and the electromagnetic fields—by the following Hamiltonian:

$$H=\int (H_1+H_2+H_3)\,d\sigma, \tag{2}$$

where

$$H_1=ig\,\bar{\psi}\,\gamma_5\,\tau_\lambda\,\phi_\lambda\,\psi, \tag{2·1}$$

$$H_2=-ie\bar{\psi}\cdot(1+\tau_3)/2\cdot\gamma_\mu\,\psi A_\mu-(1/2)\mu\bar{\psi}\sigma_{\lambda\nu}\,F_{\lambda\nu}\,\psi, \tag{2·2}$$

$$H_3=ie(\phi^*\partial_\nu\,\phi-\partial_\nu\,\phi^*\cdot\phi)\,A_\nu+e^2\,\phi^*\,\phi\,A_\nu\,A_\nu, \tag{2·3}$$

in the conventional notation.

With (2) the lowest order diagrams which contribute to the electric polarizability are given in Fig. 1. It includes, besides g^2 terms, terms of g^4 and higher order in the perturbational language, because the anomalous magnetic moment is phenomenologically included in (2). This may probably be due to the effect of the meson-nucleon interactions.

After conventional mathematical calculations there results an effective Hamiltonian for the one-nucleon system modified in an external electric field, arising from the graphs in Fig. 1:

$$M=M_A+M_B+\cdots+M_{Dm}\,; \tag{3}$$

$$M_A\simeq 8e^2a_4b_4(i\pi^2/m^3)\cdot N$$

$$\times\Big[\bar{u}u(-1/2)I_{211}$$

$$+\bar{u}(\gamma_4R_0/m)u(-1/2)I_{211}$$

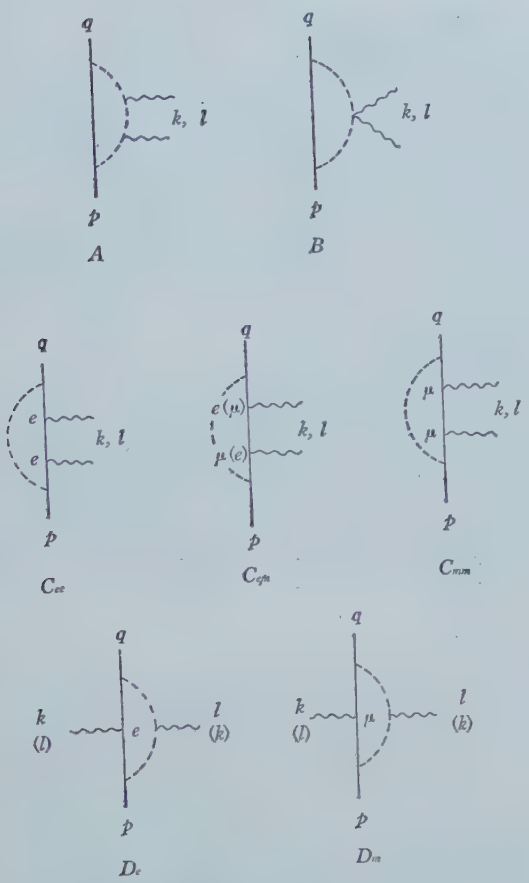

Fig. 1. Feynman diagrams considered for the electric polarizability of the neutron

$$+\bar{u}(R_0/m)^2u(1/4)I_{232}$$
$$+\bar{u}(Q/m)^2u\{-(3/4)I_{322}$$
$$+(1/2)I_{312}-(3/8)I_{412}+I_{343}$$
$$-(2/3)I_{333}+(1/2)I_{433}\}], \quad (3\cdot1)$$

$$M_B\simeq -2e^2a_4b_4(i\pi^2/m^3)\cdot N$$
$$\times[\bar{u}u\{-I_{111}\}$$
$$+\bar{u}(Q/m)^2u\{+(1/6)I_{312}\}], \quad (3\cdot2)$$

$$M_{Cee}\simeq -2e^2a_4b_4(i\pi^2/m^3)\cdot N$$
$$\times[\bar{u}u\{+I_{121}-2I_{042}+I_{252}\}$$
$$+\bar{u}(R_0/m)^2u\{I_{142}\}$$
$$+\bar{u}(\gamma_4R_0/m)u\ \{I_{121}-I_{142}\}$$
$$+\bar{u}(Q/m)^2u\{(1/2)I_{032}$$
$$-(29/12)I_{042}+(3/2)I_{052}$$
$$-(1/3)I_{063}+(1/2)I_{073}\}], \quad (3\cdot3$$

$$M_{Cem}\simeq -2e\mu a_4b_4(i\pi^2/m^2)\cdot N$$
$$\times[\bar{u}(\gamma_4R_0/m)u\{-2I_{142}\}$$
$$+\bar{u}(R_0/m)^2u\{I_{142}\}$$
$$+\bar{u}(Q/m)^2u\{I_{021}-(5/6)I_{042}$$
$$-(3/2)I_{142}+(1/6)I_{052}\}], \quad (3\cdot4)$$

$$M_{Cmm}\simeq -2\mu^2a_4b_4(i\pi^2/m)\cdot N$$
$$\times\bar{u}(Q/m)^2u\{(3/2)I_{021}-(1/2)I_{031}-(1/2)I_{052}-I_{121}-I_{142}\}, \quad (3\cdot5)$$

$$M_{De}\simeq -4e^2a_4b_4(i\pi^2/m^3)\cdot N$$
$$\times[\bar{u}(\gamma_4R_0/m)u\{-I_{121}-I_{232}\}$$
$$+\bar{u}(R_0/m)^2u\{(1/2)I_{132}-I_{142}\}$$
$$+\bar{u}(Q/m)^2u\{-I_{232}-I_{253}\}], \quad (3\cdot6)$$

$$M_{Dm}\simeq 4e\mu a_4b_4(i\pi^2/m^2)\cdot N$$
$$\times\bar{u}(Q/m)^2u\{-(1/2)I_{211}+I_{232}\}, \quad (3\cdot7)$$

where

$$I_{rst} \equiv \int_0^1 dx \frac{(1-x)^r x^s}{\{x^2+(m_\pi/m)^2(1-x)\}^t},$$

$$a_\mu \equiv A_\mu(k), \qquad b_\mu \equiv A_\mu(l),$$

$$N \equiv g^2 \sqrt{\frac{m}{p_0}} \sqrt{\frac{m}{q_0}} \frac{1}{(2\pi)^3} \delta(p+k+l-q),$$

$$R \equiv p+q, \qquad Q \equiv k+l.$$

The notation used in the above expressions is:

m and m_π ···masses of nucleon and pion respectively,

k and l ···4-momenta of incident electromagnetic fields,

p and q ···4-momenta of initial and final nucleon respectively.

$\simeq$ means that both sides are equal up to terms of order $(k\cdot l)$. The other terms which have been dropped in (3) do not contribute to the electric polarizability. In (3) we have written down all the expressions in their diagonal form or with even operators, by making use of the relations:

$$(i\gamma p+m)u=0,$$
$$\bar{u}(i\gamma q+m)=0. \tag{4}$$

(3) contains several types of terms—ones which are proportional to $(k\cdot l)$ explicitly and the others which are not. In order to estimate the contributions to the electric polarizability from these terms one should proceed with the interpretation in the form of (1), in terms of the non-relativistic spinor wave functions. The reduction into this form is unambiguously made in the case of the terms proportional to $(k\cdot l)$. The terms which do not involve $(k\cdot l)$ explicitly give different contributions according to the methods of reduction. The first way of reduction is the use of Foldy-Tani-Wouthuysen transformation. The second is the simple omission of the small spinor components by making $m\to\infty$ therein. The third consists in rewriting and transforming (3) into the "2×2" matrix form without neglecting the small components which are now expressed with the large components. In any case, however, the results are similar and the difference in the method of reduction is not of significance. Thus among alternatives the result obtained through the Foldy-Tani-Wouthuysen transformation alone is presented in Table I, in the 2nd and the 4th columns. In the 3rd and the 5th columns the contributions from $(k\cdot l)$ terms are presented. All the numerical values are obtained by taking

$$\frac{g^2}{4\pi} = 15. \tag{5}$$

§ 3. Cutoff calculations

The higher order contributions as well as some unknown internal effects might be considered to be embodied into the cutoff calculations. Though its exact form is not known, some measure of it may be tangible.

In the process of leading to (3) of the last section we meet integrals over the 4-momentum space of the virtual particles, t_μ. On introducing an auxiliary scalar integration variable, x in I_{rst} of (3), and averaging over the direction of the 4-vector t_μ, they take the form of

$$\int \frac{(t^2)^{m-2}}{(t^2+c)^n} dt.$$

Since all the integrals appearing in the present calculations are convergent, it is convenient to cut them off in the following way rather than to introduce a new form factor. Thus we proceed as follows.

$$\int \frac{(t^2)^{m-2}}{(t^2+c)^n} dt = \lim_{\Lambda\to\infty} \int_0^{\Lambda^2} \frac{i\pi^2 \sigma^{m-1}}{(\sigma+c)^n} d\sigma$$

$$\to i\pi^2 \int_0^{\Lambda^2} \frac{\sigma^{m-1}}{(\sigma+c)^n} d\sigma. \qquad (6)^*$$

The results are presented in brackets in Table I, on taking

$$\Lambda = m. \qquad (7)$$

Effects of cutoff differ from one diagram to another. Large terms (α_A, α_B, α_{De}) are not drastically modified, while the small terms are strongly affected. Owing to the remarkable cancellations between large terms which remain unchanged in the cutoff calculations too, the total result considerably changes with cutoff used.

§ 4. *K*-meson contributions

The neutron can dissociate into K-meson and hyperons in virtual state, which gives further contributions to the neutron electric polarizability. Contrary to the pion-nucleon interaction, however, few are known about the K-meson-baryon interactions.

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* The cutoff method (6) is taken here for the sake of simplicity. That the behavior of this cutoff is similar to what is called straight cutoff[8] would be shown by taking an example of the nucleon magnetic moment calculations. Applications of (6) and (7), and the straight cutoff method to it lead to:

	non-cutoff	straight cutoff at m	(6) & (7)	experimental
N	−3.9	−1.9	−2.17	−1.9128
P	0.52	0.61	0.59	1.7962

Table I. Pion contributions to the electric polarizability of the nucleon
$K=(1/\pi^2)g^2e^2(1/m^3)$
$=1.3\times10^{-42}\text{cm}^3$; $g^2/4\pi=15$.
The figures in brackets are the results of cutoff calculation.

	Neutron		Proton	
	Foldy transf.	*kl*-term	Foldy transf.	*kl*-term
α_A	$+0.793K$ $(+0.754K)$	$+0.674K$ $(+0.649K)$	$+0.793K$ $(+0.754K)$	$+0.674K$ $(+0.649K)$
α_B	$-1.283K$ $(-1.274K)$	$-1.208K$ $(-1.214K)$	$-1.283K$ $(-1.274K)$	$-1.208K$ $(-1.214K)$
α_{Cee}	$+0.030K$ $(+0.082K)$	$+0.089K$ $(+0.135K)$	$+0.015K$ $(+0.041K)$	$+0.045K$ $(+0.068K)$
α_{Cem}	$+0.005K$ $(+0.063K)$	$+0.038K$ $(+0.092K)$	$+0.003K$ $(+0.032K)$	$+0.019K$ $(+0.046K)$
α_{Cmm}	$-0.039K$ $(+0.050K)$	$-0.039K$ $(+0.050K)$	$-0.041K$ $(+0.052K)$	$-0.041K$ $(+0.052K)$
α_{D_e}	$+0.477K$ $(+0.436K)$	$+0.462K$ $(+0.440K)$	0 (0)	0 (0)
α_{D_m}	$+0.046K$ $(+0.075K)$	$+0.046K$ $(+0.075K)$	$+0.052K$ $(+0.085K)$	$+0.052K$ $(+0.085K)$
Total	$+0.029K$ $(+0.150K)$	$+0.062K$ $(+0.227K)$	$-0.461K$ $(-0.310K)$	$-0.459K$ $(-0.314K)$
Foldy term α_M	-6.2×10^{-44} cm^3		-13×10^{-44} cm^3	

$$H_{KA}=ig_{ANK}\{\bar{p}\gamma_5\Lambda K^+ + \bar{n}\gamma_5\Lambda K^0 + \bar{\Lambda}\gamma_5 pK^- + \bar{\Lambda}\gamma_5 n\bar{K}^0\},$$
$$H_{K\Sigma}=ig_{\Sigma NK}\{\bar{p}\gamma_5\Sigma^0K^+ - \bar{n}\gamma_5\Sigma^0K^0 + \sqrt{2}\,\bar{n}\gamma_5\Sigma^-K^+ + \sqrt{2}\bar{p}\gamma_5\Sigma^+K^0$$
$$+\bar{\Sigma}^0\gamma_5 pK^- - \bar{\Sigma}^0\gamma_5 n\bar{K}^0 + \sqrt{2}\bar{\Sigma}^-\gamma_5 nK^- + \sqrt{2}\bar{\Sigma}^+\gamma_5 p\bar{K}^0\}. \quad (8)$$

The choice of the pseudoscalar interactions is for the present matter of convention.*,3) The structure of H_{KY} is the same as (2) of the pion-nucleon interaction, so that the resulting contributions are estimated from (3) by substituting new parameters for g, μ, m's. Numerical values are listed in Table II.

* Sugano and Komatsuzawa[3] conclude, from their analysis of the K–N scatterings with use of the dispersion relations, that the possible sets of K–N–Hyperon interactions are such that

$$\begin{cases} \Lambda NK \text{ interaction}.....\text{pseudoscalar} \\ \Sigma NK \text{ interaction}.....\text{either scalar or pseudoscalar} \end{cases}$$

with the coupling strength:

$$g^2_{\Lambda NK}/4\pi \sim 5, \qquad g^2_{\Sigma NK}/4\pi \sim 0.$$

Table II. K-meson contributions to the electric polarizability of the nucleon
$K' \equiv (1/\pi^2) g^2_{\Sigma NK} e^2 (1/m m_\Sigma{}^2) = (g^2_{\Sigma NK}/g^2) \times 1.0 \times 10^{-42} \mathrm{cm}^3$,
$K'' \equiv (1/\pi^2) g^2_{\Lambda NK} e^2 (1/m m_\Lambda{}^2) = (g_{\Lambda NK}/g^2) \times 1.1 \times 10^{-42} \mathrm{cm}^3$;
μ_i: anomalous magnetic moment of particle i, in unit of $|e|/2m_i$.

	Neutron		Proton	
	ΛNK	ΣNK	ΛNK	ΣNK
A	0	$+0.049K'$	0	$+0.025K'$
B	0	$-0.071K'$	0	$-0.036K'$
C_{ee}	0	$+0.024K'$	0	$+0.024K'$
C_{em}	0	$-0.026\mu_- K'$	0	$+0.026\mu_+ K'$
C_{mm}	$-0.007\mu_\Lambda{}^2 K''$	$\begin{pmatrix} -0.013\mu_-{}^2 K' \\ -0.007\mu_0{}^2 K' \end{pmatrix}$	$-0.007\mu_\Lambda{}^2 K''$	$-0.007\mu_0{}^2 K'$
D_e	0	$+0.159K'$	0	0
D_m	0	$+0.005\mu_- K'$	0	$+0.003\mu_0{}^2 K'$

§ 5. Summary and discussions

In Table I, all the contributions calculated are presented separately. Large terms are A, B, and D_e. Others are an order of magnitude smaller than these.

It is remarkable that the strong cancellations are shown among the three large terms of α_A, α_B and α_{De}, so that the other contributions from the smaller terms become important each of which is less than $10^{-43}\,\mathrm{cm}^3$. Accordingly, the K-baryon effects, for example, will become notable. Since the K-baryon interactions as well as the electromagnetic properties of K and Hyperons, such as the magnetic moments, are not established, their effects cannot be estimated exactly. In Table II these are listed with movable parameters. Then it appears reasonable to state that the K-contributions are of $10^{-43}\mathrm{cm}^3$ at most.

Turning to the cutoff calculations, the large terms are affected only slightly by the cutoff in contrast with the smaller terms which suffer drastic modifications. The situation of the strong cancellation among the large terms remains so unchanged that the total contribution is quite dependent on the cutoff. The results are tabulated in brackets in Table I. They give the figure of $10^{-43}\,\mathrm{cm}^3$.

Summarizing, we may expect that the straightforward calculation with the present meson theory will give an electric polarizability of the neutron which amounts to the order of $10^{-43}\mathrm{cm}^3$ but does not exceed $10^{-42}\mathrm{cm}^3$.

In relation to the foregoing results we shall make some comments on the previous studies. The first comment is concerned with the magnetic moment of the neutron. The mere existence of the magnetic moment μ already predicts electric polarizability $-(\mu^2/m)$ which we denote as α_M.*,[4] This is small and nega-

* α_M is called the Foldy term in Table 1.

tive, and was first derived by Foldy from the one-particle Hamiltonian of the neutron in an external electric field in a manner analogous to the case of interpreting the electron-neutron interaction in terms of the Foldy term. Comparing this with ours we notice that generally the larger contributions are expected from the proper meson field effects than that which are predicted from the mere existence of the anomalous magnetic moment interaction,

$$|\alpha| > |\alpha_M|.$$

The second comment is to be made on the relation between our results and those of the static meson theory.[5),6)] Static cutoff meson theory with the coupling constant $(f^2/4\pi)=0.08$ predicts, by taking account of graphs A and B,*[5)]

$$\alpha=1.00\times10^{-42}\,\mathrm{cm}^3 \text{ for cutoff function } 1/\{1+(t/5.6)^2\} \tag{9}$$

or

$$\alpha=1.13\times10^{-42}\,\mathrm{cm}^3 \text{ for cutoff function } \exp\{-t^2/2(5.6)^2\}.$$

These are to be compared with $-0.5\times10^{-42}\,\mathrm{cm}^3$ of the present relativistic calculations.

It is shown that if we separately consider the contributions from the intermediate states of positive energy and those from its negative energy states, and if we denote them with suffices $(+)$ and $(-)$ respectively, the contributions $\alpha_A+\alpha_B$ are rewritten as

$$\begin{aligned}
&\alpha_A=\alpha_A^{(+)}+\alpha_A^{(-)}=0.8K(0.8K),\\
&\alpha_B=\alpha_B^{(+)}+\alpha_B^{(-)}=-1.3K(-1.3K),\\
&\alpha_A^{(+)}=-1.3K(-1.1K),\quad \alpha_B^{(+)}=+2.7K(+2.5K),\\
&\alpha_A^{(-)}=+2.1K(+1.9K),\quad \alpha_B^{(-)}=-4.0K(-3.8K),
\end{aligned} \tag{10}$$

and

$$\alpha_A^{(+)}+\alpha_B^{(+)}=1.4K(1.4K). \tag{11}$$

(Figures in brackets are the cutoff results. $K=1.3\times10^{-42}\,\mathrm{cm}^3$) This figure is comparable to the static theoretical results, and it is several times as large as ours of the relativistic calculation with cutoff.

Finally, we mention about the electric polarizability of the proton and about the mass correction due to it. It is obtained by evaluating graphs of similar kinds

---

* Barashenkov and Barvashov[6)] give

$$\alpha=1.6\times10^{-42}\mathrm{cm}^3 \text{ for cutoff function } 1/\{1+(t/5.6)^2\},$$

and

$$\alpha=1.8\times10^{-42}\mathrm{cm}^3 \text{ for cutoff function } \exp\{-t^2/2(5.6)^2\},$$

instead of $1.00\times10^{-42}\,\mathrm{cm}^3$ and $1.13\times10^{-42}\mathrm{cm}^3$ of 5).

to those of Fig. 1 and is presented in the Tables (the 4th and the 5th columns).

The value we get is

$$\alpha_P = -0.461K \text{ for non-cutoff calculation,}$$
$$\alpha_P = -0.310K \text{ for cutoff at } m. \qquad (12)$$

The difference between electric polarizabilities of the proton and the neutron is

$$\alpha_N - \alpha_P = 9.36(e^2/m^3) \text{ for non-cutoff calculation,}$$
$$= 8.79(e^2/m^3) \text{ for cutoff calculation,}$$

and

$$e^2/m^3 = 6.8 \times 10^{-44} \text{ cm}^3.$$

Then one can consider that this difference between the electromagnetic interactions of the proton and the neutron might cause a further mass difference between them. If we temporarily assume as the effective interaction through the electric polarizability

$$H = \frac{\alpha}{4}\int F_{\mu\nu} F_{\mu\nu} \bar{\psi}\psi \, d\sigma, \qquad (13)^*$$

notwithstanding the fact that there are no *a priori* implications that one may extend the non-relativistic Hamiltonian (1) to the relativistic form (13), then some estimates of its effects on the problem in question would become attainable. The result will be shown to be highly divergent:

$$\Delta M \equiv \Delta M_N - \Delta M_P = (1/15.7)\lambda m_e (\Lambda/m)^4,$$

where Λ is a cutoff parameter, m_e is electron mass and λ is the difference between the polarizability of the proton and the neutron in units of e^2/m^3.

If we may cut the integral at the nucleon mass m, then we get

$$\Delta M = +(1/15.7)\lambda m_e.$$

If Λ should be twice the nucleon mass,

$$\Delta M = +(16/15.7)\lambda m_e.$$

This is large compared with the observed mass difference $2.53\, m_e$, so if this were the case, the usual considerations on the small mass difference should be re-examined and some of the results should be altered considerably.**,***,[7]

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* This predicts the same magnitude for the electric and the magnetic polarizabilities, which does not necessarily seem to be the case. For the present experimental status see ref. 1).

** If we recall that the expansion in k and l was used to obtain α, the cut at m would be considered to be practically the maximum. In case of the polarizability of the electron, it is at least

$$\sim (e^2/\Delta m_e^3); \quad (e^2 = 1/137),$$

which corresponds to the term α_M in the text. Due to this, the electron mass may be shifted by

$$\Delta m_e = (1/15.7) \times \lambda_e \, m_e (\Lambda/m_e)^4,$$

where λ_e is the electric polarizability of the electron in units of e^2/m_e^3. If we take $\Lambda \sim m_e$, then this is very small in comparison with m_e. If Λ were taken as m, for instance, then mass shift would become large and this cut would not be justifiable.

*** A connection of α with the small mass difference among the elementary particles might be interesting. It is, however, not clear that this is reflected, for example, in the fact that

$$m_{K^+} - m_{K^0} = -3.9 \text{ Mev} < 0,$$
$$m^+ - m^0 = +4.59 \text{ Mev} > 0.$$

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References

1) A. S. Baldin, *Proceedings of the Conference in Padua-Venice* (September 1957).
$4\times10^{-43}\text{cm}^3<\alpha<1.4\times10^{-42}\text{cm}^3$.
R. M. Thaler, Phys. Rev. **114** (1959), 827.
$0<\alpha<2\times10^{-41}\ \text{cm}^3$.
I. Aleksandrov, J.E.T.P. **33** (1957), 294.
$\alpha\sim10^{-40}\ \text{cm}^3$.
L. L. Foldy, unpublished.
$\alpha\simeq4\times10^{-42}\ \text{cm}^3$.
G. Breit and N.L. Rustgi, Phys. Rev. **114** (1959), 830.
$\alpha<2\times10^{-42}\text{cm}^3$.
At the Kiev Conference on High Energy Physics the electric and the magnetic polarizabilities of the proton were reported:
$\alpha_E{}^P=(9\pm2)\times10^{-43}\ \text{cm}^3$: electric polarizability,
$\alpha_M{}^P=(2\pm2)\times10^{-43}\text{cm}^3$: magnetic polarizability.
2) E. M. Pursell and N.F. Ramsey, Phys. Rev. **78**(1950), 807.
3) R. Sugano and A. Komatsuzawa, Prog. Theor. Phys. **23** (1960), 287.
4) L. L. Foldy, Phys. Rev. Letters **3** (1959), 105.
5) A. Kanazawa, Soryushi-ron Kenkyu (mimeographed circular in Japanese) **19** (1959), 469.
6) V.S. Barashenkov and B.M. Barbashov, Nuclear Physics **9** (1958/1959), 426.
7) For example, K. Hiida and M. Sawamura, Prog. Theor. Phys. **18** (1957), 451.
M. Kato and G. Takeda, Prog. Theor. Phys. Supplement **7** (1958), 35.
H. Katsumori, Prog. Theor. Phys. **19** (1958), 342.
S. Gasiorowicz and A. Petermann, Phys. Rev. Letters **1** (1958), 457.
P.T. Matthews and J.L. Uretsky, Phys. Rev. Letters **3** (1959), 297.
8) S. Goto, Prog. Theor. Phys. **12** (1954), 699.

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The Theory of the Nucleon Level Structure in Terms of the Pion-Pion Resonance. I

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The existence of a sharp pion-pion resonance is assumed for the $T=j=1$ state at about 600 Mev. This resonance state is approximately replaced by a real $T=j=1$ particle $\rho^{\pm,0}$ and its interactions with the pion field and the electro-magnetic field are determined. The calculation of pion-nucleon scatterings through ρN intermediate states shows in the Born approximation that very strong attractive forces seem to exist for both $T=1/2$, $d_{3/2}$ and $T=1/2$, $f_{5/2}$ pion-nucleon states at about the ρ-production threshold energy. The former is identified as the second and the latter as the third resonance observed in the pion-nucleon scatterings. A similar calculation predicts the possible appearance of the second resonance and the suppression of the third one in the photo-pion productions. A tentative model for explaining the fourth maximum observed in the pion-nucleon scattering is proposed along the similar line.

§ 1. Introduction

Since the measurements of the total cross-section for the pion-nucleon scattering were performed by Cool et al.[1] in the 1 Bev region, it has become known that the nucleon has several excited levels besides the well-known (3/2, 3/2) resonance. Recently more detailed experiments on the pion-nucleon scattering and the photo-pion production by nucleons have been made[2] and some information about the resonance energies, the spin, the isotopic spin, the parity and some other properties of the excited levels are obtained (Table I).

Table I. Table of the excitation energy W, the isotopic spin T, the spin j, the orbital pion-nucleon angular momentum l, and the parity of the excited levels.

W(Mev)	T	j	l	Parity	
0	1/2	1/2		+	(N)
~300	3/2	3/2	1	+	(N_1)
~580	1/2	3/2	2 or 1	±(?)	(N_2)
~740	1/2	$\geq 5/2$	≥ 2	±(?)	(N_3)
~940	3/2	?	?	±(?)	(N_4)

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One of the authors[3] assumed the existence of the pion-pion resonance in the $T=j=1$ state and explained the large bump of the total pion-nucleon scattering cross-section in the $T=1/2$ state at about one Bev (At that time the second and third resonances were not separately observed). The predicted bump was too broad, because the impulse approximation was used for treating the collision between the incoming pion and the pion-cloud of the target nucleon.

In this paper we follow along the line proposed then and try to explain the sharp second and third resonances. The main difference from the previous paper[3] is to avoid the impulse approximation. We shall proceed under the following assumption:

(A) The assumed pion-pion resonance state can be approximately treated as an actual particle $\rho^{\pm,0}$. This is, of course, an isovector and ordinary vector particle $(T=j=1)$ and the mass is taken as about 600 Mev.

After the phenomenological model of the pion-pion resonance in the $T=j=1$ state had been proposed by Takeda, Tamm et al.[4] predicted the resonance by calculating the pion-pion scattering in the Tamm-Dancoff approximation, and recently Chew and Mandelstam[5] have also pointed out the possible existence of the resonance by a dispersion-like treatment of the pi-meson theory. Thus the sharp pion-pion resonance might be able to be justified based on the present field theory. Frazer and Fulco[6] analyzed the electro-magnetic structure of the nucleons introducing the sharp pion-pion resonance and brought us a better agreement between the observed nucleon form factor and the theoretical one. Since the width of the assumed resonance is rather small $(\Gamma'\approx 0.3\sim 0.5)$, we may be allowed to consider this state approximately as a particle.

The effective interaction Hamiltonian between the pion field and the ρ field is as follows:

$$H_{\rho\pi\pi}=F\sum_{\alpha,\beta,\gamma}\chi_i^{\gamma}\partial_i\varphi^{\alpha}\varphi^{\beta}\epsilon_{\alpha\beta\gamma}. \qquad \text{(I)}$$

The Greek suffices α, β, and γ are the suffices of the isotopic spin space and can take the value either 1, 2, or 3. The Latin suffix i is a suffix in the ordinary space and is either 1, 2, 3, or 4. $\epsilon_{\alpha\beta\gamma}$ is the three-dimensional anti-symmetric unit tensor. χ and φ are the ρ and π field operators, respectively.

Similarly, the effective interaction Hamiltonian among the ρ field, the pion field, and the electro-magnetic field is given by

$$H_{\rho\pi\gamma}=iE\sum_{\substack{\alpha\\ i,j,k,l}}\epsilon_{ijkl}(\partial_i A_j-\partial_j A_i)\partial_k\varphi^{\alpha}\chi_l^{\alpha}, \qquad \text{(II)}$$

where A_i represents the electro-magnetic field operator and ϵ_{ijkl} is the four-dimensional anti-symmetric unit tensor.

The magnitude of the coupling constant, F or E, introduced above is estimated in § 2 and the large value of $F(F^2/4\pi\approx 5)$ is responsible for the second and third resonances (N_2 and N_3).

The $\rho\pi\pi$ interaction (I) together with the ordinary $NN\pi$ interaction brings a pion-nucleon state into a ρ-nucleon state through one pion exchange (Fig. 6) and the iteration of this gives the pion-nucleon scattering in the Born approximation (Fig. 5a). In § 3 we evaluate the pion-nucleon potential in this Born approximation and find strong attractive potentials both for the $T=1/2$, $d_{3/2}$ and $T=1/2$, $f_{5/2}$ pion-nucleon states. Based on this calculation plausible arguments are given for assigning the 2nd and 3rd resonances as those in the $T=1/2$, $d_{3/2}$ and $T=1/2$, $f_{5/2}$ states, respectively.

The observed second resonance in the photo-pion production is due to excitation of the resonant ρ-N state through a ρ particle production by the incoming γ ray in the pion field of the target nucleon. In § 4 a calculation in the Born approximation is performed using the $\rho\pi\gamma$ interaction (II) and the $NN\pi$ interaction (Fig. 11). The results show that the second resonance is indeed excited, but not the third one. This is consistent with the experimental result for not finding the prominent third resonance in the photo-pion production in contrast with its finding in the pion-nucleon scattering.

A tentative argument is given in § 5 for interpreting the broad fourth maximum observed in the π^+-p scattering cross-section as due to the effect of the pion-pion resonance.

§ 2. The ρ field and the effective $\rho\pi\pi$ and $\rho\pi\gamma$ interactions

Following our assumption (A), we regard the pion-pion resonant state with $T=j=1$ as a vector and iso-vector particle $\rho^{\pm,0}$ whose mass m_ρ is about 600 Mev. The free particle wave equation for ρ is taken as

$$(\Box-m_\rho{}^2)\ \chi_i{}^\alpha=0, \tag{1}$$

and

$$\partial_i\chi_i{}^\alpha=0. \tag{2}$$

Here $\alpha(=1, 2, \text{or } 3)$ is an isotopic spin component and $i(=1, 2, 3, \text{or } 4)$ is an ordinary spin component of the ρ field.

The interactions of the ρ field with the pion and the electro-magnetic fields are given by $H_{\rho\pi\pi}(I)$ and $H_{\rho\pi\gamma}(II)$. These are to be regarded as effective Hamiltonians and should be defined as follows:

The pion-pion scattering amplitude $f(\alpha' p', \beta' q'; \alpha p, \beta q)$ for two pions $(\alpha p, \beta q)$ going into two pions $(\alpha' p', \beta' q')$ is[7]

$$f(\alpha' p', \beta' q'; \alpha p, \beta q)=\frac{2\pi^2}{i}\int dx \exp\left[i\ \frac{q+q'}{2}x\right]<\alpha' p'|j_{\beta'}\left(-\frac{x}{2}\right)j_\beta\left(\frac{x}{2}\right)|\alpha p\rangle$$

$$=\frac{2\pi^2}{i}\int dx \exp\left[i\frac{q+q'}{2}x\right]\sum_n\langle\alpha' p'|j_{\beta'}\left(-\frac{x}{2}\right)|n\rangle\langle n|j_\beta\left(\frac{x}{2}\right)|\alpha p\rangle, \tag{3}$$

where $j_\beta(x)$ is the pion source current defined by

$$j_\beta(x) = i\frac{\delta S}{\delta \varphi_\beta(x)} S^\dagger, \tag{4}$$

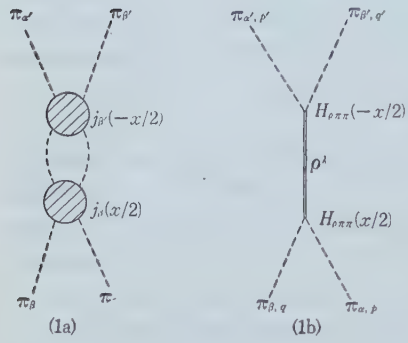


Fig. 1. The diagrams for pion-pion scatterings.

and n designates an intermediate state which is an eigen-state of the total Hamiltonian of the system. We assume that the pion-pion scattering amplitude is appreciable because of the $T=j=1$ resonant state and replace the summation over n by the intermediate single ρ particle state. Furthermore, the amplitude $\langle n|j_\beta(x)|\alpha p\rangle$ where n is a ρ particle state is approximated by the corresponding matrix element of the effective Hamiltonian (I) (Figs. 1). A similar assumption is involved in defining the Hamiltonian (II) as effective 3π–γ interaction.

Our assumption contains the following fact: If the pion-pion collision cross section can be measured in future, the cross section should be predominantly contributed by that in the $T=j=1$ state at least in the low energy region ($\lesssim 1$ Bev).

The forms of the effective Hamiltonians are chosen by the Lorentz invariance, the charge independence, the gauge invariance and the following condition: The energy-momentum four-vectors of ingoing or outgoing π and γ particles into or from the vertices are small compared with the characteristic energy-momentum of the 4π and $3\pi\gamma$ vertices. The latter is the reciprocals of the space and time-like extension of the 4π and $3\pi\gamma$ vertices and are equal to about 2 Mc when we use the perturbation calculation for the vertices. In fact the form of the Hamiltonians can be obtained by calculating the lowest order Feynman graphs for the vertices (Figs. (2a)$\sim$(2c)), replacing the two outgoing pions in the $T=j=1$ state by ρ, and taking low energy-momentum limit of the outgoing and incoming particles.

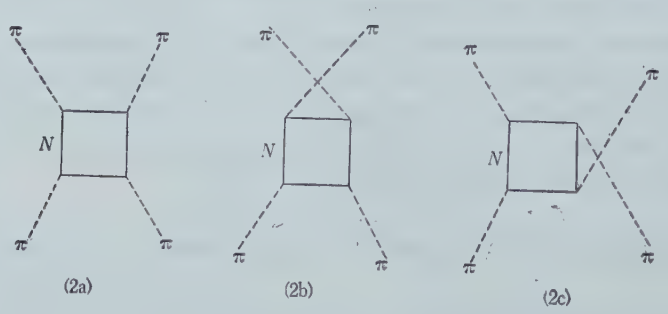


Fig. 2. The 4π vertices in the lowest order perturbation.
The dashed lines represent pions and the full lines nucleons.

The magnitudes of the $\rho\pi\pi$ and $\rho\pi\gamma$ couplings, F and E, are approximately determined by the energy W_r and the width Γ of the assumed pion-pion resonance. We calculate the pion-pion scattering amplitude $f(\alpha' p', \beta' q'; \alpha p, \beta q)$ (3) first by employing $H_{\rho\pi\pi}$ in perturbation and then compare it with the one-level formula of Frazer and Fulco.[6] The imaginary part of the p-wave scattering amplitude $f_{11}(p)$ in the centre of mass system is given by (see Appendix I)

$$p \mathrm{Im} f_{11}(p) = \frac{F^2}{6} \frac{p^3}{W} \delta(m_\rho^2 - W^2), \tag{5a}$$

where p is the pion momentum and W the total energy in the centre of mass system. The one level formula employed by Frazer and Fulco gives (see Appendix I)

$$p \mathrm{Im} f_{11}(p) = \frac{(\Gamma/2)^2}{(W - W_r)^2 + (\Gamma/2)^2}, \tag{5b}$$

where

$$\Gamma \approx \frac{8\Gamma' p^3}{W_r^2}. \tag{5c}$$

Since we assume the width Γ is very small and the resonance state is approximated by an actual particle ρ with a definite mass, Eq. (5b) can be approximated by

$$\begin{aligned} p \mathrm{Im} f_{11}(p) &\approx \pi \Gamma W_r \delta(W_r^2 - W^2) \\ &\approx \frac{8\pi \Gamma' p^3}{W_r} \delta(W_r^2 - W^2). \end{aligned} \tag{5b'}$$

Comparison between Eq. (5a) and Eq. (5b') gives

$$\frac{F^2}{4\pi} \approx 12\Gamma' \text{ and } W_r \approx m_\rho. \tag{6}$$

The first equation of (6) can also be obtained by calculating the decay life of ρ going into 2π. According to Frazer and Fulco we take here[8]

$$\Gamma' \approx 0.3 \sim 0.5 \text{ and } W_r \approx 600 \text{Mev}, \tag{7}$$

which gives

$$F^2/4\pi \approx 5, \tag{8}$$

and

$$m_\rho \approx 600 \text{ Mev}. \tag{9}$$

Next we shall proceed to determination of the $\rho\pi\gamma$ coupling constant E (See Appendix I). Let us evaluate first the $\pi\gamma \to 2\pi$ vertex in the $T=j=1$ state corresponding to the lowest order Feynman graph (Figs. 3) and compare it with the similar lowest order $2\pi \to 2\pi$ vertex. A straightforward calculation gives us

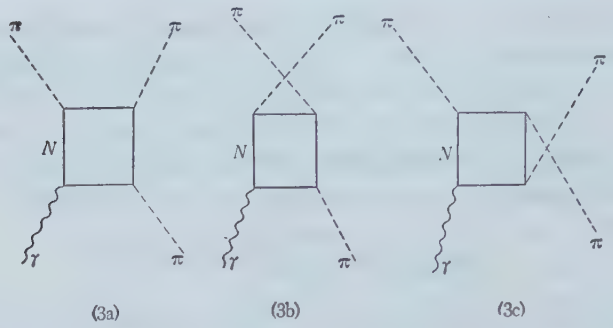


Fig. 3. The 3π-γ vertices in the lowest order perturbation. The dashed lines, the full lines and the wave lines are pions, nucleons and γ-rays, respectively.

$$H_{3\pi,\gamma}=\frac{i}{24\pi^2}\frac{eg^3}{M^3}\sum_{\alpha\beta\gamma}\sum_{ijkl}\epsilon_{\alpha\beta\gamma}\epsilon_{ijkl}A_i\partial_j\varphi^\alpha\partial_k\varphi^\beta\partial_l\varphi^\gamma$$

$$=\frac{i}{48\pi^2}\frac{eg^3}{M^3}\sum_{\alpha\beta\gamma}\sum_{ijkl}\epsilon_{\alpha\beta\gamma}\epsilon_{ijkl}F_{ij}\partial_k\varphi^\alpha\partial_l\varphi^\beta\varphi^\gamma, \tag{10}$$

and

$$H_{4\pi}=-\frac{1}{6\pi^2}\frac{g^4}{M^2}\sum_{\alpha\beta\alpha'\beta'}\sum_{t}(\delta_{\alpha\alpha'}\delta_{\beta\beta'}-\delta_{\alpha\beta'}\delta_{\beta\alpha'})\partial_t\varphi^{\alpha'}\partial_t\varphi^\alpha\varphi^{\beta'}\varphi^\beta, \tag{11}$$

where g is the pion-nucleon coupling constant ($g^2/4\pi\approx 15$) The latter expression was calculated by Igi and Kawarabayashi[9] before.

The 2π resonant state with $T=j=1$ which should be replaced by a ρ particle state can be composed of the pion field φ except its normalization factor A:

$$\chi_i^\gamma=A\frac{1}{\sqrt{2}}\sum_{\alpha,\beta}\epsilon_{\alpha\beta\gamma}\partial_i\varphi^\alpha\varphi^\beta. \tag{12}$$

$H_{\rho\pi\pi}$ is obtained from $H_{4\pi}$ (Eq. (11)) through substituting Eq. (12) and taking

$$A=-\frac{1}{3\sqrt{2}\,\pi^2}\frac{g^4}{FM^2}. \tag{13}$$

Analogously we can obtain $H_{\rho\pi\gamma}$ from $H_{3\pi\gamma}$ (Eq. (10)) by employing Eqs. (12) and (13). The result gives the $\rho\pi\gamma$ coupling constant E as

$$E=+\frac{eg^3}{8\sqrt{2}\,\pi^2M^3A}=-\frac{3}{8}\frac{eF}{gM}. \tag{14}$$

Since $g^2/4\pi\approx 15$ and $F^2/4\pi\approx 5$, we obtain

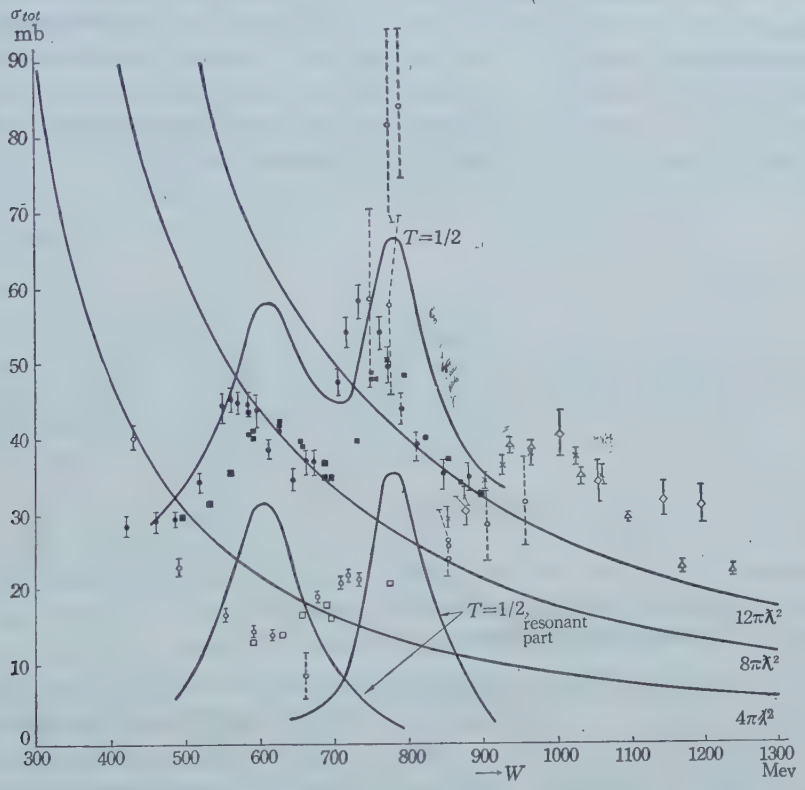
$$\left|\frac{E}{e}\right|\approx\frac{1}{5}\frac{1}{M}. \tag{14'}$$

The magnitude of the $\rho\pi\gamma$ coupling constant E evaluated above is less reliable than that of F. The enhancement of the resonance state excitation by $N\gamma$ ($N\gamma\to N\rho$) over that of the lowest perturbation calculation is taken as equal to the en-

hancing factor for the $\pi N \to \rho N$ excitation. This has no real justification and we shall replace the nucleon mass M in Eqs. (14) and (14′) by the effective mass M^*, hoping M^* to be evaluated more reliably in future.

§ 3. The second and third resonances in pion-nucleon scatterings

In this section we discuss the high energy pion-nucleon scattering by using



$\pi^- + p$ J. C. Brisson et al., Phys. Rev. Letters **3** (1959), 561.

H. C. Burrowes et al., Phys. Rev. Letters. **2** (1959), 119.

$\pi^+ + p$ J. C. Brisson et al., Phys. Rev. Letters. **3** (1959), 561.

H. C. Burrowes et al., Phys. Rev. Letters **2** (1959), 119.

M. J. Longo et al., Phys. Rev. Letters **3** (1959), 568.

T. J. Devlin et al., preliminary data

R. Cool et al., Phys. Rev. **103** (1956), 1082.

$\pi^- + p \to \Lambda + K^\circ$ 1958 Annual International Conference on High Energy Physics at CERN, p. 148. The unit of the cross section is $10\mu b$.

The curves for the total and resonant $T=\frac{1}{2}$ cross sections are taken from H. C. Burrowes et al., Phys. Rev. Letters **2** (1959) 119.

Fig. 4. The total $\pi^{\pm}$-p cross sections and $\pi^- + p \to \Lambda^0 + K^0$ cross section.

the effective $\rho\pi\pi$ interaction as well as the ordinary $NN\pi$ interaction. The experimental curves of the pion-nucleon scattering cross sections are shown in Fig. 4. It will be shown that the strong $\rho\pi\pi$ interaction $H_{\rho\pi\pi}$ gives rather strong attractive potential of the pion-nucleon system in the $T=1/2$ state for $W\approx m_\rho$ and the observed second and third resonances are identified as those in the $T=1/2$, $d_{3/2}$ and $T=1/2$, $f_{5/2}$ states, respectively.

Since we are interested here in pion-nucleon scatterings of rather high energy region, the large observed cross sections can be expected only through a long-range pion-nucleon interaction. The longest range potential between a pion and a nucleon is 2-pion exchange potential and the corresponding lowest order Feynman graphs are shown in Figs. 5. Graphs containing nucleon loops are disregarded because such are assumed to be effectively represented by $H_{\rho\pi\pi}$.

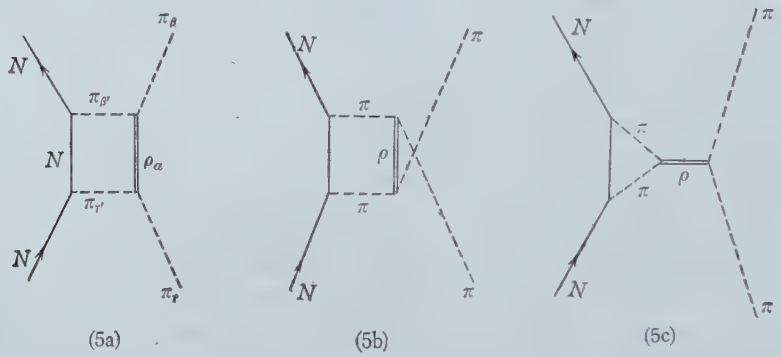


Fig. 5. The lowest order Feynman diagrams for the π-N scatterings through two pion exchange.

The scattering amplitude corresponding to Fig. 5a becomes very large for $W\approx m_\rho$, because energies of intermediate ρN states can be very near or identical with the initial energy W of the colliding system. Furthermore the nucleon-pion interaction becomes attractive, since most intermediate ρN states have an energy larger than W (Appendix II). On the other hand, for Feynman graphs 5b and 5c the energy differences between the intermediate states and the initial state are rather large for $W\approx m_\rho$ and at least of an order of W, resulting in rather small contributions to the π-N force from these graphs compared to that of 5a. So we disregard the graphs 5b and 5c hereafter, although they too give a 2-pion exchange attractive force between N and π.

Since we expect a large attractive force at $W\approx m_\rho$ to come out from the graph 5a and particularly from that having those intermediate ρN states with small relative momentum q, we can confine ourselves to evaluate such contributions of Fig. 5a. The isotopic spin dependence of the π-N potential is then given by

$$\langle\pi_\beta|V|\pi_\gamma\rangle\sim\sum_{\alpha,\beta',\gamma'}\epsilon_{\alpha\beta\beta'}\tau_{\beta'}\epsilon_{\alpha\gamma\gamma'}\tau_{\gamma'}=2\delta_{\beta\gamma}+1/2[\tau_\beta,\tau_\gamma]=4P_{1/2}+P_{3/2}, \quad (15)$$

where $P_T(T=1/2 \text{ or } 3/2)$ is the projection operator on the isotopic spin T state. Equation (15) shows that the attractive π-N potential is 4 times larger in the $T=1/2$ state than in the $T=3/2$ state according to our lowest order perturbation calculation and the factor 4 is expected to become much larger in the scattering amplitudes obtained by a better approximation. Thus our result predicts resonance scatterings only for the $T=1/2$ state, being consistent with the observed second and third resonances.

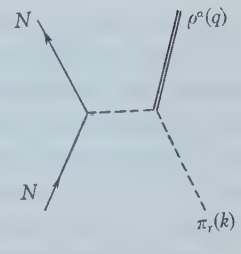


Fig. 6. The lowest order Feynman diagram for the $\pi N \to \rho N$ transition.

Next we discuss in which states of the angular momentum and parity we obtain large attractive forces. This can be examined by evaluating the $\pi N \to \rho N$ transition matrix element for ρN states of small $\boldsymbol{q}$ (Fig. 6). Again we evaluate the matrix element in lowest order perturbation:

$$\langle \rho^{\alpha}, q|T|\gamma, k\rangle = \sum_{\gamma'} \epsilon_{\alpha\gamma'\gamma}\tau_{\gamma'} Ff \times \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{k}-\boldsymbol{q})[\boldsymbol{\rho}\cdot(2\boldsymbol{k}-\boldsymbol{q})-\rho_0(2k_0-q_0)]}{\mu[(\boldsymbol{k}-\boldsymbol{q})^2-(k_0-q_0)^2+\mu^2]}, \quad (16)$$

where $(\boldsymbol{\rho}, \rho_0)$ is the unit polarization vector of ρ particle and $(\boldsymbol{k}, k_0)$ and $(\boldsymbol{q}, q_0)$ are the energy-momentum 4 vectors of the pion and ρ particle, respectively. The pion-nucleon coupling is taken as the pseudo-vector one

$$H_{NN\pi} = \frac{f}{\mu}\sum_{\alpha}\tau_{\alpha}(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\varphi^{\alpha} \quad (f^2/4\pi \approx 0.08), \qquad \text{(III)}$$

because the recoil momenta of the target nucleon are still small compared with M and furthermore we are interested in the intermediate states with a positive energy nucleon. The part of the amplitude (16) independent of $\boldsymbol{q}$ corresponds to the $\pi N \to \rho N$ amplitude with the ρ orbital angular momentum $l_{\rho}=0$ and is given except its isotopic spin dependence by

$$2Ff\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{k})(\boldsymbol{\rho}\cdot\boldsymbol{k})}{m_{\rho}^2\mu} \approx -\frac{4\pi}{\sqrt{3}}\frac{2Ff}{\mu}(P(s_{1/2}\to S_{1/2})+P(d_{3/2}\to S_{3/2})). \quad (17)$$

In deriving Eq. (17) we have made approximations $\rho_0=0$ and $k_0 \approx m_{\rho}$, the former being the result of Eq. (2) and $\boldsymbol{q}=0$. The projection operators $P(s_{1/2}\to S_{1/2})$ and $P(d_{3/2}\to S_{3/2})$ are the unit amplitudes for πN in $s_{1/2}$ and $d_{3/2}$ states going into ρN in $S_{1/2}$ and $S_{3/2}$ state, respectively.

Similarly, the amplitude (16) linear in $\boldsymbol{q}$ and leading to the state with $l_{\rho}=1$ is given by

$$-\frac{2Ff}{\mu}\left[\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{q})(\boldsymbol{\rho}\cdot\boldsymbol{k})}{m_{\rho}^2}+\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{k})(\boldsymbol{\rho}\cdot\boldsymbol{q})}{m_{\rho}^2}-2\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{k})(\boldsymbol{\rho}\cdot\boldsymbol{k})(\boldsymbol{k}\cdot\boldsymbol{q})}{m_{\rho}^4}\right]$$

$$=-\frac{16\pi}{15}\frac{Ffq}{m_{\rho}\mu}[\sqrt{10}P(f_{5/2}\to P_{5/2}^{2})+\frac{1}{2}\sqrt{\frac{5}{2}}P(p_{3/2}\to P_{3/2}^{2})$$

$$+\frac{5}{2\sqrt{2}}P(p_{3/2}\to P_{3/2}^{1})+5P(p_{1/2}\to P_{1/2}^{0})-\frac{5}{\sqrt{2}}P(p_{1/2}\to P_{1/2}^{1})], \qquad (18)$$

thereby $\rho_0\approx 1/m_\rho\cdot(\boldsymbol{q}\cdot\boldsymbol{\rho})$ and $k_0\approx m_\rho$ are employed. $P(f_{5/2}\to P_{5/2}^{2})$ is the unit amplitude for πN in the $f_{5/2}$ state going into ρN in the $l_\rho=1$, $j=5/2$ state. The small upper index 2 designates the magnitude of the sum of $\boldsymbol{l}_\rho$ and the polarization vector $\boldsymbol{\rho}$. Other P's are also defined in the same way. The explicit expressions for those projection operators P are given in Appendix III.

Since strong attraction can be obtained by transitions through ρN intermediate states with small $\boldsymbol{q}$, the possible angular momenta for resonance states near $W=m_\rho$ should be either in the expression (17) or in (18). The strength of the attractive π-N potential in our perturbation computation is same for $s_{1/2}$ and $d_{3/2}$ states and 10: $(5/8+25/8)$: $(25+25/2)=8/3:1:10$ for $f_{5/2}$, $p_{3/2}$ and $p_{1/2}$ states.

The attractive potential $\langle\pi_\beta|V|\pi_\gamma\rangle$ for $l_\rho=0$ would become largest for W being nearly equal to but a little smaller than m_ρ, because then all the intermediate states give positive contributions to the attractive potential. On the other hand, the $\langle\pi_\beta|V|\pi_\gamma\rangle$ for $l_\rho=1$ should be largest at a W larger than that for $l_\rho=0$, because the $\pi N\to\rho N$ matrix element (18) is proportional to q and it requires the ρN intermediate states with appreciable q to be effective. Therefore, if any resonance should occur by the attractive potential, a resonance through ρN states with $l_\rho=0$ should be at $W\lesssim m_\rho$ and one through ρN states with $l_\rho=1$ is likely to be at $W\gtrsim m_\rho$.

Experimentally the second resonance (N_2) at $W_2\approx$ 580 Mev ($\lesssim m_\rho$) requires $j=3/2$ and the third one (N_3) at $W_3\approx 740$ Mev ($\gtrsim m_\rho$) requires $j\geqq 5/2$. Therefore it is quite consistent from our results to identify N_2 as the $d_{3/2}$ resonance and N_3 as the $f_{5/2}$ one.[10] Our simple perturbation calculation has also predicted equally strong or stronger attractive force for $s_{1/2}$ and $p_{1/2}$ states compared to those in $d_{3/2}$ and $f_{5/2}$ states. Here we can only make plausible arguments for not actually having those as resonance states: Besides the 2-pion exchange potential (Figs. 5) there are many other πN potentials with a shorter range and they are of course not negligible for scatterings with a small πN angular momentum and so particularly for $l_\pi=0$ and 1. Furthermore, the πN scattering graphs (Fig. 7) beyond the lowest perturbation ones contain $3\pi N$ and $2\pi\rho N$ loops and the $2\pi\rho N$ loop gives us a divergent expression for $l_\pi=0$ (not for $l_\pi\geqq 1$). Therefore it would be of little meaning to discuss strength of the πN potential for s and p states based on our effective Hamiltonian $H_{\rho\pi\pi}$.

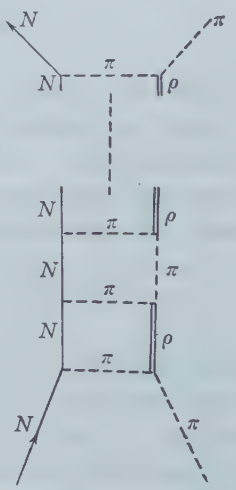


Fig. 7. A Feynman diagram for the π-N scattering beyond Born approximation.

We shall make several remarks here. The first remark is on a possible presence of a $T=1/2$ and $p_{1/2}$ resonance. Since the attractive potential (see Eq.

(18)) in the $p_{1/2}$ state is exceedingly large, there would be a chance for having a $p_{1/2}$ resonance at a nearly same energy with the $f_{5/2}$ resonance in spite of the above arguments against it. A slightly favourable but rather weak evidence for this resonance is as follows: If the third resonance is an $f_{5/2}$ resonance, we can compute the elasticity of the pion-nucleon scattering in that particlular state $f_{5/2}$ by[11]

$$\frac{\sigma_{el}^{j}}{\sigma_{tot}^{j}}=\frac{\sigma_{tot}^{j}}{2(2j+1)\pi\lambda\!\!\!^{-}_{r}{}^{2}} \quad (j=5/2). \tag{19}$$

Here σ_{tot}^{j} and σ_{el}^{j} are the total and elastic scattering cross sections for the $f_{5/2}$ state at the resonance energy W_r. The $\lambda\!\!\!^{-}_{r}$ is the pion wavelength at $W=W_r$. A rough estimation for the experimental value of σ_{tot}^{j} is about 36 mb and this together with $12\pi\lambda\!\!\!^{-}_{r}{}^{2}\approx 42$ mb gives $\sigma_{el}^{j}/\sigma_{tot}^{j}\approx 6/7$. This leads to a small inelastic cross section $\sigma_{inel}^{j}(\approx(1/7)\sigma_{tot}^{j})$ for this resonant state which should be compared with $\sigma_{inel}/\sigma_{tot}\approx 1/2$ measured at about $W\approx W_3$.[12] If we assume the coincidence of the two resonating states $f_{5/2}$ and $p_{1/2}$, then we obtain a more satisfactory value of $\sigma_{el}/\sigma_{tot}\approx 36mb/16\pi\lambda\!\!\!^{-}_{r}{}^{2}\approx 9/14$ leading to $\sigma_{inel}/\sigma_{tot}\approx 5/14$.

The second remark is about the location of the resonance energies. We have argued for W_2 being smaller than m_ρ and W_3 being likely to be larger than m_ρ. A different way to lead to this prediction may be along the similar line with Baz[13] although he interpreted the third resonance as due to a strong Λ-K attractive potential. According to our model the second and third resonances are due to the π-N scatterings through the ρ-N intermediate states with $l_\rho=0$ and $l_\rho=1$, respectively. This might be interpreted as follows: The ρ particle is trapped in a strong attractive potential due to the nucleon for a time period of an order of $\hbar/\Gamma$, Γ being the width of the π-N resonance. Since there is no centrifugal barrier for a ρ in the $l_\rho=0$ state, the energy of this metastable intermediate state should be less than m_ρ and we should expect $W_2\lesssim m_\rho$. On the other hand, there is a centrifugal barrier for a ρ in $l_\rho=1$ state, which is rather strong for a low momentum ρ particle. Therefore W_3 could be and likely to be $W_3\gtrsim m_\rho$ (see Figs. 8). The width of these resonant state comes from $\rho N\to\pi N$, $2\pi N\cdots$decays as well as from emission of a free ρ particle subsequently decaying into 2 pions for the $l_\rho=1$ case. We show in Fig. 9 the relation between the resonance energies and the free ρ particle production thresholds. Our model predicts W_2 and W_3 being rather near to the ρN threshold energy. The close correlation between W_4 and the ρN_1 threshold energy will be discussed in § 5.

The final remark is about the observed resonance in ΛK^0 production by pion-nucleon collisions. As shown in Fig. 4 this resonance occurs at just the same energy with W_3 and furthermore ΛK^0 production requires $T=1/2$ state.[14] Therefore it would be natural to identify them with the same resonance in the $f_{5/2}$ state. Although the centrifugal barrier for ΛK^0 in the f state is large, the centre of mass momentum of K^0 is already appreciable at $W\approx W_3(k\approx 1.7\mu)$. Therefore the

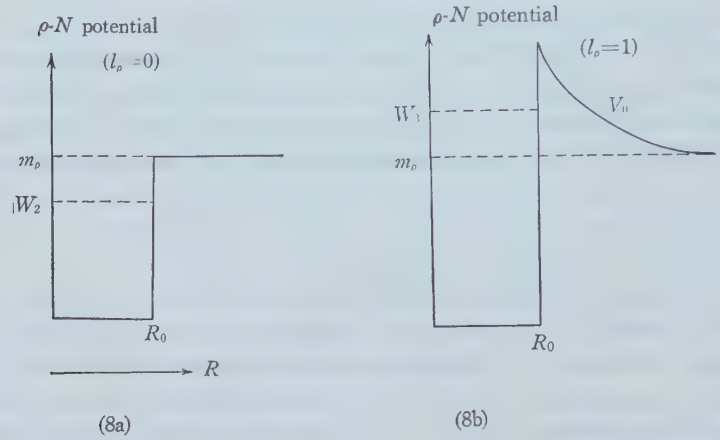


Fig. 8. Metastable ρ-N states corresponding to the second (Fig. 8a) and third (Fig. 8b) resonances.

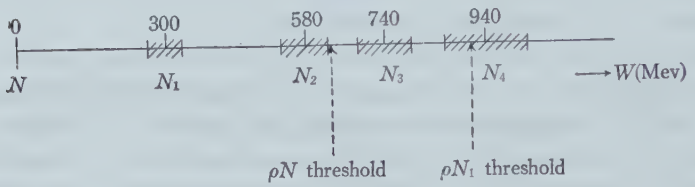
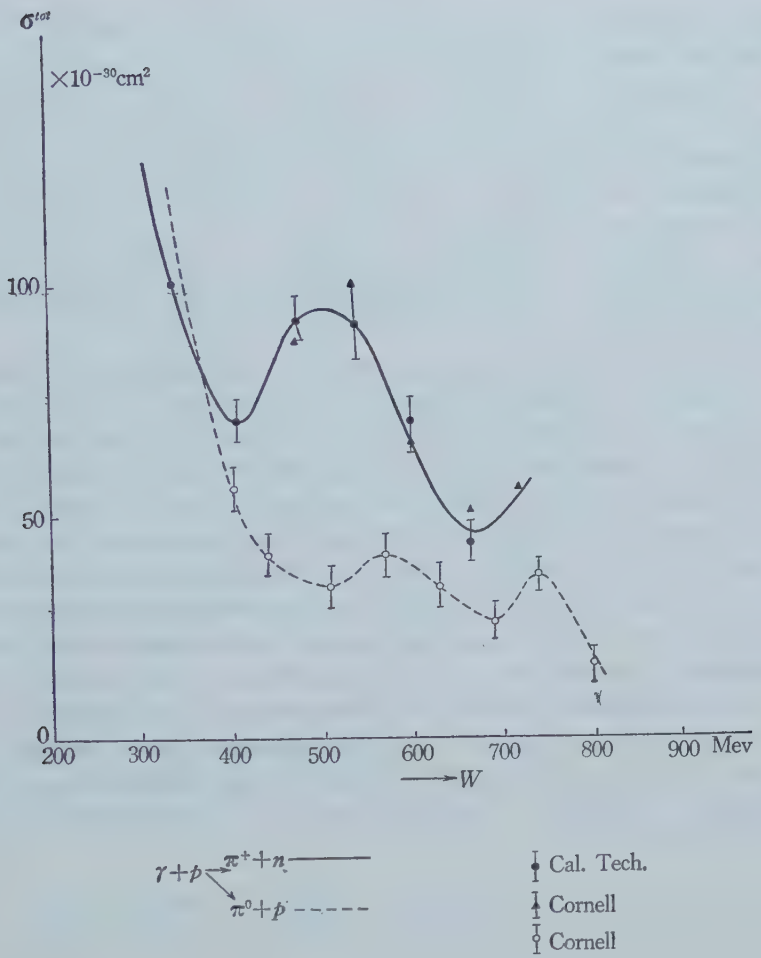


Fig. 9. The excitation energies of the various levels and the threshold energies for ρN and ρN_1 productions.

experimental ratio of the partial widths $\Gamma_{\rho N\to\Lambda K^0}/\Gamma_{\rho N\to\pi\pi N}$, which is about equal to $\sigma_{\pi^-p\to\Lambda K^0}/\sigma^{tot}_{\pi^-p}\approx 1/40$, is not unreasonably large for the $f_{5/2}$ assignment. The observed angular distribution of produced Λ and particularly its backward peak are consistent with the $f_{5/2}$ resonance together with some ΛK^0 production in the $s_{1/2}$ state. The observed polarization of Λ seems to require further mixture of either $p_{1/2}$ and (or) $p_{3/2}$ states. In any case the $f_{5/2}$ resonance together with small s and p wave productions of ΛK^0 is quite consistent with the existing data of ΛK^0 production by π-p collision near $W\approx W_3$.

§ 4. The second and third resonances in photo-pion productions

The measured cross sections of the photo-pion production by nucleons show also the resonant structures at $W\approx W_1$ and $W\approx W_2$ which are identified with the same resonance levels observed in the pion-nucleon scatterings (Fig. 10). Experimental evidence for the presence of the 3rd resonance in photo-pion productions is not conclusive at present and even if it exists the present experimental data exclude the possibility of it being a very pronounced maximum such as observed in the pion-nucleon scatterings.



(Reported by G. Bernaldini at The 9th Conf. on High Energy Nuclear Physics)

Fig. 10. The total cross sections for $\gamma+p\rightarrow\pi^+ + n$ and $\gamma+p\rightarrow\pi^0+p$.

The excitation of the ρN intermediate states by one pion exchange between the incident γ and the target nucleon can be done by $H_{\rho\pi\gamma}$ and the corresponding Feynman graph is shown in Fig. 11. The transition amplitude corresponding to this graph is

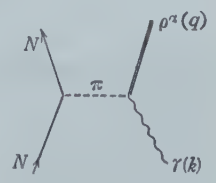


Fig. 11. The lowest order Feynman graph for $\gamma N\rightarrow\rho N$.

$$\langle \rho_i{}^\alpha, q|T|e_j, k\rangle = \frac{Ef}{\mu}\ \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{q}-\boldsymbol{k})}{(\boldsymbol{q}-\boldsymbol{k})^2+\mu^2-(q_0-k)^2}$$
$$\times\tau_\alpha\sum_{k,l}\epsilon_{ijkl}\rho_i{}^\alpha e_j k_k(q-k)_l, \qquad (20)$$

where ρ_i and e_j are the unit polarization vectors of the ρ and γ, respectively.

The isotopic spin dependence of T clearly shows that only the $T=1/2$ ρN

states can be excited by this one pion exchange force, since T is proportional to τ_α.

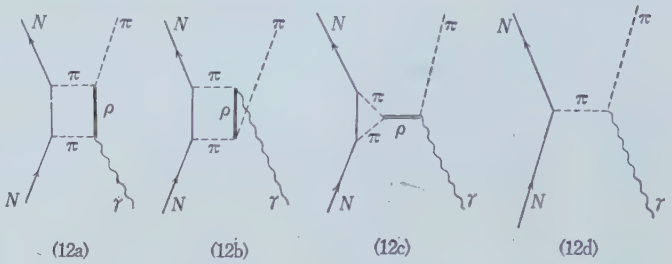


Fig. 12. The lowest order Feynman diagrams for photo-pion productions by one or two pion exchange.

The lowest order Feynman graphs of the photo-pion productions by a long range force up to two pion exchange are shown in Figs. 12. Fig. 12a corresponds to the photo-pion production through the ρN excitation by one pion exchange described above and, therefore, occurs only in the $T=1/2$ state. Figs. 12b and 12c give us also the photo-pion productions only in the $T=1/2$ state, but production amplitudes of them are small compared with that of Fig. 12a by similar reasons as those for neglecting Figs. (5b) and (5c) of the pion-nucleon scatterings. Fig. 12d is of one pion exchange type through the ordinary $\pi\pi\gamma$ interaction and as important as Fig. 12a. Since the photo-pion production amplitude due to Fig. 12d is not sensitive on photon energy near $W\approx m_\rho$, it gives one of the main contributions for the non-resonant part of the cross sections but not very important for the resonance (The isotopic spin dependence of this amplitude is $1:\sqrt{2}$ in favour of the $T=3/2$ state.).

Next we shall examine whether the same resonance states identified as $T=1/2$, $d_{3/2}$ and $T=1/2$, $f_{5/2}$ in § 3 can be excited with a relatively large width by the photo-nucleon collisions. The parts of the $\gamma N\to\rho N$ amplitude (17) independent of q and linear in q are

$$\frac{Ef}{\mu}\frac{2k-m_\rho}{m_\rho(2k-m_\rho)+\mu^2}(\boldsymbol{\sigma},\boldsymbol{k})(\boldsymbol{e}\cdot\boldsymbol{k}\times\boldsymbol{\rho})\tau_\alpha\approx\frac{Efm_\rho}{\mu}\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{k})(\boldsymbol{e}\cdot\boldsymbol{k}\times\boldsymbol{\rho})}{k^2}\tau_\alpha, \tag{21}$$

and

$$Efq\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{q})(\boldsymbol{e}\cdot\boldsymbol{k}\times\boldsymbol{\rho})+(\boldsymbol{\sigma}\cdot\boldsymbol{k})(\boldsymbol{\rho}\times\boldsymbol{q}\cdot\boldsymbol{e})}{kq\mu}, \tag{22}$$

where similar approximations as those used in obtaining Eqs. (15) and (16) are employed.

If we use the projection operator $E_L(l_j^n)$ or $M_L(l_j^n)$ for bringing a nucleon into a $\rho+N$ state with the total angular momentum j, orbital angular momentum l, and the sum of $\boldsymbol{l}$ and $\boldsymbol{\rho}$ equal to n by electric or magnetic 2^L-pole radiation, Eqs. (21) and (22) become as follows:

$$2i\frac{Efm_\rho}{\mu}\left[\sqrt{2}M_2(S_{3/2}^1)+\sqrt{\frac{2}{3}}E_1(S_{3/2}^1)-2\sqrt{\frac{2}{3}}E_1(S_{1/2}^1)\right], \tag{21'}$$

and

$$\frac{4\pi}{3}\frac{Efq}{\mu}[\sqrt{2}\,M_1(P_{1/2}^{0})-\sqrt{5}\,M_1(P_{3/2}^{2})+\sqrt{3}\,E_2(P_{3/2}^{1})]. \qquad (22')$$

Explicit expressions for $E_L(l_j^n)$ and $M_L(l_j^n)$ are given in Appendix III.

The second $d_{3/2}$ resonance which corresponds to the $S_{3/2}^{1}$ ρN state is indeed excited both by the electric dipole and magnetic quadrupole radiation (Eq. (21')). On the other hand, the third $f_{5/2}$ resonance corresponding to the $P_{5/2}^{2}$ ρN state is unexcited contrary to its excitation by the pion-nucleon scatterings. Thus our model explains the presence of observed ($T=1/2$, $d_{3/2}$) second resonance and also the suppression of the third ($T=1/2$, $f_{5/2}$) resonance in the photo-pion productions.

§ 5. The fourth maximum of the pion-nucleon scattering cross sections

The broad bump of the total π^+-p collision cross section observed at $W\approx 940$ Mev has not so far been experimentally examined in datail. In § 3 we have mentioned the coincidence of the location of this bump with the ρ-N_1 threshold energy (Fig. 9), which suggests us a model to interpret the bump as a maximum of the inelastic scattering cross sections $\pi N\to\rho N_1\to 3\pi N$ due to final state interactions among the produced pions and the nucleon.

The $\pi N\to\rho N_1$ transition by one pion exchange potential is illustrated in the Feynman graph of Fig. 13, where an appropriate effective Hamiltonian $H_{NN_1\pi}$ should be used for the $NN_1\pi$ vertex. The isotopic spin dependence of $\pi N\to\rho N_1$ transition amplitudes is independent of any details of the choice of $H_{NN_1\pi}$ and the relative magnitudes of various $\pi N\to\rho N_1$ amplitudes are

$$f(\pi^+ p\to\rho^+ N_1^+) : f(\pi^+ p\to\rho^0 N_1^{++}) : f(\pi^- p\to\rho^- N_1^+) : f(\pi^- p\to\rho^0 N_1^0)$$
$$=\sqrt{2/3} : 1 : \sqrt{2/3} : \sqrt{1/3}. \qquad (23)$$

From this the inelastic scattering cross section for $\pi N\to\rho N_1$ in the $T=3/2$ state is larger than that in the $T=1/2$ state by factor 5/2, thus indicating that the maximum shows up only in the π^+-p collisions.

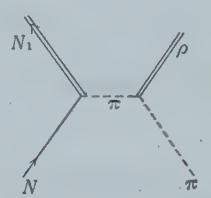


Fig. 13. The lowest order Feynman diagram for $\pi N\to\rho N_1$ by one pion exchange.

Since the observed bump is broad and the ρ and N_1 are both unstable resonance states, it is unlikely to ascribe the bump to a resonance in a particular ρ-N_1 state. Instead it would be more natural to assume it being contributed by inelastic cross sections in many different states. In fact, even if we limit the relative ρ-N_1 orbital angular momentum l_ρ to 0 or 1, many π-N states with different angular momentum up to $f_{7/2}$ can contribute.

We calculate the energy dependence of the inelastic cross section $\pi N\to\rho N_1\to 3\pi N$ based on the statistical model, thereby taking into account final state interactions. These are interactions between any two of the three pions in the $T=l=1$

state and those between the remaining pion and the nucleon in the $T=j=3/2$ state. According to Belenkii[15] a statistical weight $S(W, 3\pi)$ for producing 3π through ρN_1 intermediate state is

$$S(W, 3\pi)\propto\frac{2l'+1}{\pi}\frac{2l''+1}{\pi}\int dp'\,dp''\,dq\,\frac{\partial\eta'}{\partial p'}\frac{\partial\eta''}{\partial p''}\delta(W-W'-W''-W_q), \tag{24}$$

where $l'=l''=1$.

W' and p' are the centre of mass total energy and momentum of the two pions decaying from the virtual ρ particle. W'' and p'' are those of the third pion and nucleon resulting from decay of the N_1 in the virtual state. The η' and η'' are the scattering phase shifts between the two pions and between the pion and nucleon, respectively. W_q is the relative kinetic energy between the two pion system and the pion-nucleon system. Eq. (24) can be considered as the statistical weight for producing the ρ and N_1 particles whose masses are variable.

If we use one level formula for the phase shifts, then we obtain

$$\frac{\partial\eta'}{\partial W'}=\frac{\Gamma'}{\Gamma'^2+(W'-W_\rho)^2},$$

and

$$\frac{\partial\eta''}{\partial W''}=\frac{\Gamma''}{\Gamma''^2+(W''-W_1)^2}. \tag{25}$$

Here Γ' and Γ'' are the width of ρ and N_1, respectively. The resonance energies W_ρ and W_1 are given by $W_\rho\approx 600$ Mev and $W_1\approx 300$ Mev.

Since important contributions come from $W'\approx W_\rho$ and $W''\approx W_1$, W_q is the relative kinetic energy between relatively heavy $\rho(\approx m_\rho)$ and $N_1(\approx M+W_1)$ particles and may be neglected against W' and W''.

Under these simplifications and making approximations $W'\approx p'$ and $W''\approx p''$, the energy dependence of $S(W, 3\pi)$ near $W\approx W_4$ is given by

$$S(W, 3\pi)\propto\int_0^W dW''\int_0^{W-W''} dW'\,W'^2W''^2\delta(W-W'-W'')\frac{\partial\eta'}{\partial W'}\frac{\partial\eta''}{\partial W''}$$

$$=\int_0^W dW''(W-W'')^2W''^2\frac{\Gamma''}{\Gamma''^2+(W''-W_1)^2}\frac{\Gamma'}{\Gamma'^2+(W-W''-W_\rho)^2}. \tag{26}$$

Although the details of the shape of $S(W, 3\pi)$ depend on values of Γ' and Γ'', it has a maximum around $W\approx W_1+W_\rho$ with a broad width $\gtrsim\Gamma'+\Gamma''$. Let us illustrate the actual shape of $S(W, 3\pi)$ for an extreme case, where one of Γ' and Γ'' is negligibly small. If $\Gamma'\ll\Gamma''$, $\Gamma'/\Gamma'^2+(W-W''-W_\rho)^2$ can be approximately replaced by $\pi\delta(W-W''-W_\rho)$ and we obtain

$$S\propto(W-W_\rho)^2W_\rho^2\frac{\Gamma''}{\Gamma''^2+(W-W-W_1)^2}, \tag{27}$$

confirming the above statement. The $S(W, 3\pi)$ (Eq. (24)) is only a part of the inelastic cross sections and there are many other processes contributing. But what we are looking for is a special reason for giving a maximum at $W \approx W_4$ and other processes would be supposed to give energy insensitive contributions to the cross section. The main reason for this is due to the fact that the measured total cross sections both for π^+-p and π^--p are nearly constant around and beyond $W = W_4$ except the fourth bump in π^+-p collisions.

Our tentative model for the fourth maximum requires that the $\pi N \to 3\pi N$ inelastic process is mainly responsible for the maximum. This would be subject to direct experimental test in future.

§ 6. Concluding remarks

We have thus proposed a model to interpret the second and third resonances of the pion-nucleon scatterings in terms of the assumed $T=l=1$ pion-pion resonance. The ρ particle is only introduced as an effective substitute for this resonance and should not be taken as introduced arbitrarily. Since the pion-pion p-wave resonance has a possibility to be justified by more thorough treatment of the pion-pion scattering in the pi-meson theory, our model is more or less faithful to the present meson theory.

The mass and width of ρ particle are taken from those values obtained by the analysis of nucleon electro-magnetic structure.[6] Also the type of the effective Hamiltonian $H_{\rho\pi\pi}$ and $H_{\rho\pi\gamma}$ are based on the perturbation treatment of the meson theory and the coupling constants F and E are determined by the pion-nucleon coupling constant f and the width of the pion-pion resonance. Therefore, there would be little ambiguity in our choice of the properties of ρ particle and its interaction with other particles.

According to our model, very strong attractive potentials are obtained for the $T=1/2$, $d_{3/2}$ and $T=1/2$, $f_{5/2}$ π-N states at $W \approx m_\rho$. And the former is identified as the second resonance state and the latter as the third one. The same $T=1/2$, $d_{3/2}$ state is shown to be excited by the photo-pion production at $W \approx m_\rho$. On the other hand, the $T=1/2$, $f_{5/2}$ state is shown to be unexcited in conformity with the experimental results.

We have neglected the effects of strange particle interactions, assuming them rather unimportant as a cause of the resonances. Although third resonance is very near to the ΛK^0 threshold energy, strong ΛK^0 interactions would ordinarily imply an s or p wave resonance contrary to $f_{5/2}$ resonance. The experiments seem to require $j \geqq 5/2$ and are against the possibility of the ΛK^0 attractions being responsible for the resonance.

Our model for the fourth maximum is also based on the presence of the pion-pion resonance, but it should be regarded as one of various different possibilities. No detailed information on this maximum is yet given by experiments.

In our next paper we shall discuss the resonant pion-nucleon scatterings beyond the Born approximation. Thereby the competition between the pion production and the elastic scattering and the effect of the finite width of the pion-pion resonance will be taken into account.

Acknowledgement

We are very much obliged to the financial aid by Nippon Gakujutsu Shinkokai Foundation. Also we are indebted to Miss F. Nunokawa for help in preparing the manuscript.

Appendix I

Determination of the effective coupling constants F and E

(1) Effective coupling constant F for $\rho\pi\pi$ interaction

Let us consider the π-π scattering $(\alpha p,\ \beta q)\to(\alpha' p',\ \beta' q')$, where p's and q's are the four-momenta of the pions, and α's and β's designate their charge states. S-matrix element for this process can be written as

$$S=1-i(2\pi)^4\delta^4(p+q-p'-q')\,(2\pi)^{-6}(16\,p_0q_0p_0'q_0')^{-1/2}T(\alpha' p',\ \beta' q';\ \alpha p,\ \beta q). \tag{A·1}$$

In the center of mass system, the invariant amplitude T is decomposed into the sum of partial amplitudes:

$$T(p'\alpha'\beta';\ p\alpha\beta)=-16\pi W(4\pi)\sum_{l,T}f_{lT}(p)\sum_m Y_l^m(\boldsymbol{p}')\,Y_l^m(\boldsymbol{p})^*P_T, \tag{A·2}$$

where $\boldsymbol{p}$, $\boldsymbol{p}'$ and W are respectively the initial and final momenta of single pions and the total energy of the system. P_T and $\sum_m Y_l^m(\boldsymbol{p}')\,Y_l^m(\boldsymbol{p})^*$ are the isotopic spin (T) and angular momentum (l) projection operators, respectively. For example,

$$P(T=1)=\frac{1}{2}(\delta_{\alpha\alpha'}\delta_{\beta\beta'}-\delta_{\alpha\beta'}\delta_{\beta\alpha'}),$$

$$P(l=1)=\sum_m Y_1^m(\boldsymbol{p}')\,Y_1^m(\boldsymbol{p})^*=\frac{3}{4\pi}\frac{(\boldsymbol{p}\cdot\boldsymbol{p}')}{p^2}. \tag{A·3}$$

The unitarity condition for the S-matrix implies the partial amplitude for the state (l, T) to be of the form

$$f_{lT}(p)=\frac{1}{p}\exp[i\delta_{lT}(p)]\sin[\delta_{lT}(p)], \tag{A·4}$$

provided that the inelastic processes can be disregarded.

According to the analysis of Frazer and Fulco,[6] the π-π scattering resonance formula is approximately as follows:

$$\left(\frac{\nu+\mu^2}{\nu^3}\right)^{1/2}\exp(i\delta_{11})\sin\delta_{11}\approx\frac{\Gamma'}{\nu_r-\nu-i\Gamma'(\nu^3/\nu+\mu^2)^{1/2}},\qquad(\text{A}\cdot5)$$

where, in our notation, $\nu=p^2$ and ν_r is the value of ν at the resonance peak. They have also shown that the electromagnetic structure of the nucleon can be explained by introducing such a resonance at $\nu_r\sim3.5\mu^2$ with the width $\Gamma'\approx0.3\sim0.5$. Returning to our original notation, (A·5) reduces to

$$pf_{11}(p)=\frac{\Gamma/2}{W_r-W-i\Gamma/2},\qquad(\text{A}\cdot6)$$

with

$$W_r=2(\nu_r+\mu^2)^{1/2}\approx(18\mu^2)^{1/2}\approx600\ \text{Mev}$$

and

$$\Gamma=16p^3\Gamma'/W(W_r+W).$$

W is the centre of mass total energy and thus W_r corresponds to the rest mass of our ρ-particle (§ 2. Eq. (6)). It is natural for evaluation of Γ' to take its value at resonance peak, and thus we obtain Eq. (5c) of the text:

$$\Gamma=8p^3\Gamma'/W_r^2.$$

Next, we consider the π-π scattering via the intermediate ρ-particle with effective interaction $H_{\rho\pi\pi}$. The S-matrix element for this process can be obtained by standard perturbation method (§ 2. Fig. (1b)) and resultant amplitude T is

$$T(\alpha'p',\beta'q';\alpha p,\beta q)=F^2\sum_\lambda\epsilon_{\alpha'\beta'\lambda}\epsilon_{\alpha\beta\lambda}\frac{(p'-q')(p-q)}{(p+q)^2+m_\rho^2+i\varepsilon},\qquad(\text{A}\cdot7)$$

and accordingly, in the center of mass system,

$$T(p'\alpha'\beta';p\alpha\beta)=4F^2\frac{\boldsymbol{p}'\cdot\boldsymbol{p}}{m_\rho^2-W^2+i\varepsilon}\sum_\lambda\epsilon_{\alpha'\beta'\lambda}\epsilon_{\alpha\beta\lambda}.\qquad(\text{A}\cdot8)$$

Combining (A·3), (A·8) and

$$\sum_\lambda\epsilon_{\alpha'\beta'\lambda}\epsilon_{\alpha\beta\lambda}=\delta_{\alpha\alpha'}\delta_{\beta\beta'}-\delta_{\alpha\beta'}\delta_{\beta\alpha'},$$

we obtain

$$T(p'\alpha'\beta',p\alpha\beta)=\frac{8\pi}{3}\frac{4F^2p^2}{m_\rho^2-W^2+i\varepsilon}\sum_m Y_1^m(\boldsymbol{p}')\,Y_1^m(\boldsymbol{p})^*P_{T=1}.\qquad(\text{A}\cdot9)$$

This gives, by (A·2),

$$f_{11}(p)=-\frac{F^2}{6\pi W}\frac{p^2}{m_\rho^2-W^2+i\varepsilon}.\qquad(\text{A}\cdot10)$$

Eq. (5a) of the text is only a direct consequence of (A. 10).

With these results ((A·6) and (A·10) or (5a) and (5b) of the text), the determination of the effective coupling constant F is straightforward, and by the

method explained in the text, we finally obtain,

$$F^2/4\pi \approx 5 \quad \text{and} \quad m_\rho \approx 600 \text{ Mev.} \tag{A·11}$$

(2) Effective coupling constant E for $\rho\pi\gamma$ interaction

Unfortunately we have not any direct information about the 3π-γ interaction, in contrast with the case of π-π scattering. So, we assume tentatively that the rate of enhancement of the $T=l=1$ 2π state produced by γ-π is equal to that of the $T=l=1$ resonance state of π-π collision, as being illustrated by Fig. (A·1). Under this tentative assumption, we proceed as follows.

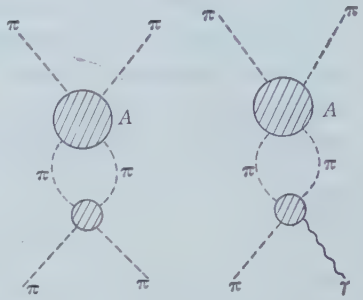


Fig. A·1. The diagrams used for evaluation of E. The lowest order perturbation calculation is used for evaluation of the small vertices.

First, consider the correspondence between ρ-particle and the two-pion system in the state $T=l=1$. The normalized isotopic-spin eigenfunctions $\eta_{\pm,0}$ of the two-pion system in the state $T=1$ are given by

$$\eta_\pm = \frac{\pm 1}{\sqrt{2}}(-\alpha_\pm\beta_0+\alpha_0\beta_\pm), \quad \eta_0 = \frac{1}{\sqrt{2}}(-\alpha_+\beta_-+\alpha_-\beta_+), \tag{A·12}$$

where $\alpha_{\pm,0}$ and $\beta_{\pm,0}$ are the normalized isotopic-spin eigenfunctions of the single pions, the suffix $(+, -, 0)$ representing their charge values. Transformation to the usual representation $(j=1, 2, 3)$ by

$$\alpha_+ = -\frac{1}{\sqrt{2}}(\alpha_1+i\alpha_2), \quad \alpha_0=\alpha_3, \quad \alpha_- = \frac{1}{\sqrt{2}}(\alpha_1-i\alpha_2), \text{ etc.,} \tag{A·13}$$

reduces (A·12) to a compact form

$$\eta_j = -\frac{i}{\sqrt{2}}(\boldsymbol{\alpha}\times\boldsymbol{\beta})_j = -\frac{i}{\sqrt{2}}\sum_{k,l}\epsilon_{klj}\alpha_k\beta_l, \tag{A·14}$$

where $\boldsymbol{\alpha}$ is a vector in the isotopic-spin space whose components are $(\alpha_1, \alpha_2, \alpha_3)$, and ϵ_{klj} is a three-dimensional antisymmetric unit tensor. Similarly, the normalized angular-momentum eigenfunctions $\xi_{\pm,0}$ of the two-pion system in the state $l=1$ are given by

$$\begin{cases} \xi_\pm \equiv Y_1^\pm(\boldsymbol{q}) = \mp\sqrt{\dfrac{3}{8\pi}}(q_x \pm iq_y)/q, \\ \xi_0 \equiv Y_1^0(\boldsymbol{q}) = \sqrt{\dfrac{3}{4\pi}}q_z/q, \end{cases} \tag{A·15}$$

where $\boldsymbol{q}$ is the relative momentum between two pions and z denotes the direction of the quantization axis. Thus the transformation analogous to (A·13) brings (A·15) into

$$\xi=\sqrt{\frac{3}{4\pi}}\frac{\boldsymbol{q}}{q}. \qquad \text{(A·16)}$$

Referring to (A·14) and (A·16), we obtain the following correspondence between ρ-particle and the two-pion system:

$$\chi_i^{\gamma}=\pm\frac{i}{\sqrt{2}}A(q_1-q_2)_i\sum_{\alpha,\beta}\epsilon_{\alpha\beta\gamma}\varphi^{\alpha}(q_1)\varphi^{\beta}(q_2)=\frac{1}{\sqrt{2}}A\sum_{\alpha,\beta}\epsilon_{\alpha\beta\gamma}\partial_i\varphi^{\alpha}\varphi^{\beta}. \qquad \text{(A·17)}$$

(+sign when ρ is annihilated and $-$sign when ρ is created.)

In (A·17), each of α, β and γ runs over 1, 2, 3, and represents the charge state of the relevant particle, while i runs from 1 to 4, representing a four-vector character in the ordinary space-time. A in Eq. (A·17) is the unknown normalization factor and contains the effect of the enhancement of the $T=l=1$ state by π-π resonance.

Now, in the center of mass system of the reaction $2\pi\to\rho$, our effective Hamiltonian $H_{\rho\pi\pi}$ reduces to

$$H_{2\pi\to\rho}=2iF\sum_{\alpha,\beta,\gamma}\epsilon_{\alpha\beta\gamma}(\chi^{\gamma}\boldsymbol{p})\varphi^{\alpha}(\boldsymbol{p})\varphi^{\beta}(-\boldsymbol{p}),$$

where $\boldsymbol{p}$ is the momentum of one of the pions in the center of mass system. Thus, by (A. 17), we obtain

$$\begin{aligned}H_{4\pi}&=2\sqrt{2}\,AF\sum_{\alpha,\beta}\sum_{\alpha',\beta'}\sum_{\gamma}\epsilon_{\alpha\beta\gamma}\epsilon_{\alpha'\beta'\gamma}(\boldsymbol{q},\ \boldsymbol{p})\varphi^{\alpha'}(\boldsymbol{q})\varphi^{\beta'}(-\boldsymbol{q})\varphi^{\alpha}(\boldsymbol{p})\varphi^{\beta}(-\boldsymbol{p})\\&=4\sqrt{2}\,AF\sum_{\alpha,\beta}\sum_{\alpha',\beta'}P_{T=1}(\alpha'\beta';\ \alpha\beta)(\boldsymbol{q}\cdot\boldsymbol{p})\varphi^{\alpha'}(\boldsymbol{q})\varphi^{\beta'}(-\boldsymbol{q})\varphi^{\alpha}(\boldsymbol{p})\varphi^{\beta}(-\boldsymbol{p}).\end{aligned} \qquad \text{(A·18)}$$

On the other hand, Igi and Kawarabayashi[9] have computed the effective 2π-2π Hamiltonian based on the diagrams shown in Fig. (2), and obtained

$$V=-\left(\frac{g^2}{4\pi}\right)^2\frac{64}{3M^2}(\boldsymbol{q}\cdot\boldsymbol{p})\sum_{\alpha,\beta}\sum_{\alpha',\beta'}P_{T=1}(\alpha'\beta';\ \alpha\beta)\varphi^{\alpha'}(\boldsymbol{q})\varphi^{\beta'}(-\boldsymbol{q})\varphi^{\alpha}(\boldsymbol{p})\varphi^{\beta}(-\boldsymbol{p}) \qquad \text{(A·19)}$$

for the $T=l=1$ state. Comparing (A. 19) with (A. 18), we obtain Eq. (13) of the text, as the correspondence between F and $PS(PS)$ coupling constant g:

$$A=-\frac{1}{3\sqrt{2}\,\pi^2}\frac{g^4}{M^2}\frac{1}{F}. \qquad \text{(A·20)}$$

The perturbation calculation of the $\gamma\pi\to2\pi$ reaction, which is based on the diagrams shown in Fig. (3) (analogously to the calculation of $2\pi\to2\pi$ by Igi and Kawarabayashi), gives as a matrix element of an effective Hamiltonian

$$V=\frac{1}{(2\pi)^2}\frac{eg^3}{M^3}\sum_{\alpha,\beta,\gamma}\epsilon_{\alpha\beta\gamma}\sum_{i,j,k,l}\epsilon_{ijkl}e_i p_j q_{1k}q_{2l}\varphi^{\alpha}\varphi_1{}^{\beta}\varphi_2{}^{\gamma}, \qquad \text{(A·21)}$$

where $\varphi=\varphi(p)$ is an initial pion field operator and φ_1 and φ_2 are those of the

final two pions (momenta q_1 and q_2). e_i is a polarization vector of the incident photon. On the other hand, our effective Hamiltonian can be written as

$$H_{\gamma\pi\to p}=2iE\sum \epsilon_{ijkl}e_i k_j p_k \chi_l{}^{\alpha}\varphi^{\alpha},$$

which, owing to the replacement (A. 17), reduces to

$$\begin{aligned} H&=+\sqrt{2}\,AE\sum \epsilon_{\alpha\beta\gamma}\epsilon_{ijkl}e_i k_j p_k(q_{1l}-q_{2l})\varphi^{\alpha}\varphi_1{}^{\beta}\varphi_2{}^{\gamma} \\ &=+2\sqrt{2}\,AE\sum \epsilon_{\alpha\beta\gamma}\epsilon_{ijkl}e_i p_j q_{1k}q_{2l}\varphi^{\alpha}\varphi_1{}^{\beta}\varphi_2{}^{\gamma}. \end{aligned} \tag{A·22}$$

In order to obtain the last version of (A. 22), we employ the conservation law of energy momentum.

Comparison between (A. 21) and (A. 22) yields that

$$E=+\frac{eg^3}{8\sqrt{2}\,\pi^2 M^3 A}. \tag{A·23}$$

Thus, under our tentative assumption explained in the beginning of this subsection (which means that we take A in (A. 23) as to be equal to (A. 20)),

$$\frac{E}{e}=-\frac{3}{8M}\,\frac{F}{g}. \tag{A·24}$$

With the values $g^2/4\pi\approx 15$ and $F^2/4\pi\approx 5$, we finally obtain.

$$\left|\frac{E}{e}\right|\approx\frac{1}{5}\cdot\frac{1}{M}. \tag{A·25}$$

Appendix II

Sign of the pion-nucleon potential in perturbation calculation

The pion-nucleon potential V in the lowest order perturbation is given by

$$V=\sum_n \frac{\langle f|H|n\rangle\langle n|H|i\rangle}{W_i-W_n}, \tag{A·26}$$

where i, n, and f are the initial, intermediate, and final states and $W_i(W_i=W_f)$ and W_n are the initial and intermediate energies. The $\langle n|H|i\rangle$ is $\langle n|H_{NN\pi}|i\rangle$ in the low energy pion-nucleon scatterings and $\langle \rho N|T|\pi N\rangle$ in the case of our text (Fig. 5a).

Since we can take $i=f$ for discussing the sign of V of the pion-nucleon elastic scatterings, Eq. (A. 26) can be written as

$$V=\sum_n \frac{|\langle n|H|i\rangle|^2}{W_i-W_n}, \tag{A·27}$$

and the sign of W_i-W_n determines the sign of V.

In the p-wave pion-nucleon scatterings (Fig. A2) $W_i<W_n$ for π^+-p scatterings and $W_i>W_n$ for π^--p scatterings. Therefore we get an attractive potential for the former and a repulsive one for the latter.

Fig. A·2. The diagrams of the p-wave pion-nucleon scatterings.

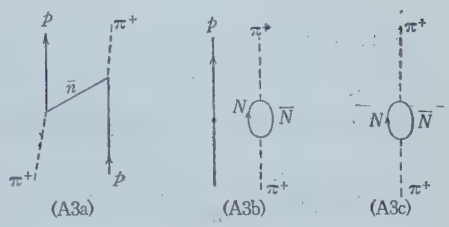


Fig. A·3. The diagrams of the s-wave π^+-nucleon scatterings.

Similarly, the pion-nucleon force through ρ-N intermediate states (Fig. 5a of the text) is attractive for $W_i < m_\rho$, since always $W_n \geq m_\rho > W_i$. Even when $W_i > m_\rho$ as in our case of the third resonance, the potential is attractive since most of intermediate ρ-N states have W_n larger than W_i when W_i is near m_ρ. This is the case with our third resonance.

An exceptional case appears in scatterings through formation of a virtual nucleon pair such as in the lowest order s-wave pion-nucleon scatterings (Fig. A3a). Here the two nucleons are in the identical state when $i=f$ and the transition is prohibited by the Pauli principle. The non-vanishing contribution to the potential comes from the process of the initial pion going into itself through a virtual nucleon pair ($N\bar{N}$) production in the presence of the initial nucleon (Fig. A3b). This transition should be measured relative to the transition of the pion (π_i) going into itself in absence of the nucleon (Fig. A3c) and the resulting con tribution is entirely due to such process of Fig. A3b having the virtual nucleon identical with the initial nucleon (p_i). Therefore we obtain

$$-\frac{\langle \pi_i|H|p_i\bar{N}\rangle\langle p_i\bar{N}|H|\pi_i\rangle}{E_\pi - E_{N\bar{N}}} = -\frac{|\langle \pi_i|H|p_i\bar{N}\rangle|^2}{W_i - W_n} > 0, \qquad (A\cdot 27)$$

leading to a repulsive potential in spite of $W_n > W_i$.

Appendix III

Tables of projection operators

(a) Projection operators $P(l_J \to L_J^j)$ for $\pi N \to \rho N$

Notation

J: Total angular momentum

l and L: π-N and ρ-N relative orbital angular momenta

j: sum of the intrinsic ρ-spin and the ρ-N relative orbital angular momentum

$\boldsymbol{k}$ and $\boldsymbol{q}$: momenta of π and ρ in the center of mass system

$\boldsymbol{\rho}$: polarization vector of ρ

$\boldsymbol{\sigma}$: nucleon Pauli spin matrix

By $P(l_J \to L_J^j)$, we denote the projection operator for production of ρ in the state (J, L, j) by absorption of π in the state (J, l).

$$P(s_{1/2}\to S_{1/2}^{1})=-\frac{1}{4\pi}\frac{1}{\sqrt{3}}(\boldsymbol{\sigma}\cdot\boldsymbol{\rho})$$

$$P(d_{3/2}\to S_{3/2}^{1})=-\frac{1}{4\pi}\frac{1}{\sqrt{3}}[3(\boldsymbol{\sigma}\cdot\boldsymbol{k})(\boldsymbol{\rho}\cdot\boldsymbol{k})/k^2-(\boldsymbol{\sigma}\cdot\boldsymbol{\rho})]$$

$$P(p_{1/2}\to P_{1/2}^{0})=\frac{1}{4\pi}(\boldsymbol{\sigma}\cdot\boldsymbol{k})(\boldsymbol{\rho}\cdot\boldsymbol{q})/kq$$

$$P(p_{1/2}\to P_{1/2}^{1})=\frac{1}{4\pi}\frac{i}{\sqrt{2}}(\boldsymbol{\sigma}\cdot[\boldsymbol{q}\times\boldsymbol{\rho}])(\boldsymbol{\sigma}\cdot\boldsymbol{k})/kq$$

$$=-\frac{1}{4\pi}\frac{1}{\sqrt{2}}[(\boldsymbol{\sigma}\cdot\boldsymbol{\rho})(\boldsymbol{q}\cdot\boldsymbol{k})-(\boldsymbol{\sigma}\cdot\boldsymbol{q})(\boldsymbol{\rho}\cdot\boldsymbol{k})-i(\boldsymbol{k}\cdot[\boldsymbol{q}\times\boldsymbol{\rho}])]/kq$$

$$P(p_{3/2}\to P_{3/2}^{1})=\frac{1}{4\pi}\frac{1}{\sqrt{2}}[(\boldsymbol{\sigma}\cdot\boldsymbol{\rho})(\boldsymbol{q}\cdot\boldsymbol{k})-(\boldsymbol{\sigma}\cdot\boldsymbol{q})(\boldsymbol{\rho}\cdot\boldsymbol{k})+2i(\boldsymbol{k}\cdot[\boldsymbol{q}\times\boldsymbol{\rho}])]/kq$$

$$P(p_{3/2}\to P_{3/2}^{2})=-\frac{1}{8\pi}\sqrt{\frac{2}{5}}[3\{(\boldsymbol{\sigma}\cdot\boldsymbol{q})(\boldsymbol{\rho}\cdot\boldsymbol{k})+(\boldsymbol{\sigma}\cdot\boldsymbol{\rho})(\boldsymbol{q}\cdot\boldsymbol{k})\}$$
$$-2(\boldsymbol{\sigma}\cdot\boldsymbol{k})(\boldsymbol{q}\cdot\boldsymbol{\rho})]/kq$$

$$P(f_{5/2}\to P_{5/2}^{2})=\frac{3}{8\pi}\sqrt{\frac{2}{5}}[\{(\boldsymbol{\sigma}\cdot\boldsymbol{\rho})(\boldsymbol{q}\cdot\boldsymbol{k})+(\boldsymbol{\sigma}\cdot\boldsymbol{k})(\boldsymbol{q}\cdot\boldsymbol{\rho})+(\boldsymbol{\sigma}\cdot\boldsymbol{q})(\boldsymbol{\rho}\cdot\boldsymbol{k})\}/kq$$
$$-5(\boldsymbol{\sigma}\cdot\boldsymbol{k})(\boldsymbol{\rho}\cdot\boldsymbol{k})(\boldsymbol{q}\cdot\boldsymbol{k})/k^3q]. \qquad (A\cdot28)$$

These projection operators satisfy the orthonormality condition

$$\mathrm{Tr}\int d\Omega_q P(l_J\to L_J^{j})^{\dagger}\ P(l'_{J'}\to L'^{j'}_{J'})=\frac{1}{4\pi}(2J+1)\delta_{JJ'}\delta_{ll'}\delta_{LL'}\delta_{jj'}, \qquad (A\cdot29)$$

where the symbol Tr means the sum over the three independent orientations of the ρ-polarization vector $\boldsymbol{\rho}$ and the trace with respect to the two-dimensional Pauli spin space.

(b) Projection operators for $\gamma N\to\rho N$

Notation

$\boldsymbol{k}$ and $\boldsymbol{e}$: momentum and polarization vector of the photon

$\boldsymbol{q}$ and $\boldsymbol{\rho}$: momentum and polarization vector of ρ

$E_l(L_J^{j})$ and $M_l(L_J^{j})$ are the projection operators for production of ρ in the state (J, L, j) by absorption of electric and magnetic $2l$-pole radiation, respectively. The meaning of J, L and j is same as in Subsection (a).

$$E_1(S_{1/2}^{1})=\frac{1}{4\pi}\frac{1}{\sqrt{6}}[(\boldsymbol{e}\cdot\boldsymbol{\rho})-i(\boldsymbol{\sigma}\cdot[\boldsymbol{e}\times\boldsymbol{\rho}])]$$

$$E_1(S_{3/2}^{1})=\frac{1}{4\pi}\frac{1}{\sqrt{6}}[2(\boldsymbol{e}\cdot\boldsymbol{\rho})+i(\boldsymbol{\sigma}\cdot[\boldsymbol{e}\times\boldsymbol{\rho}])]$$

$$M_2(S_{3/2}^{1})=-\frac{1}{4\pi}\frac{i}{\sqrt{2}}[2(\boldsymbol{\sigma}\cdot\boldsymbol{k})(\boldsymbol{\rho}\cdot[\boldsymbol{e}\times\boldsymbol{k}])/k^2+(\boldsymbol{\sigma}\cdot[\boldsymbol{e}\times\boldsymbol{\rho}])]$$

$$M_1(P_{1/2}^{0}) = \frac{1}{4\pi}\frac{i}{\sqrt{2}}(\boldsymbol{\sigma}\cdot[\boldsymbol{e}\times\boldsymbol{k}])\,(\boldsymbol{q}\cdot\boldsymbol{\rho})/kq$$

$$M_1(P_{1/2}^{1}) = \frac{1}{4\pi}\frac{1}{2}[\{(\boldsymbol{\rho}\cdot\boldsymbol{e})\,(\boldsymbol{q}\cdot\boldsymbol{k}) - (\boldsymbol{\rho}\cdot\boldsymbol{k})\,(\boldsymbol{e}\cdot\boldsymbol{q})\} + i\{(\boldsymbol{\sigma}\cdot\boldsymbol{q})\,(\boldsymbol{\rho}\cdot[\boldsymbol{e}\times\boldsymbol{k}]) - (\boldsymbol{\sigma}\cdot\boldsymbol{\rho})\,(\boldsymbol{q}\cdot[\boldsymbol{e}\times\boldsymbol{k}])\}]/kq$$

$$M_1(P_{3/2}^{1}) = \frac{1}{4\pi}\frac{1}{2}[2\{(\boldsymbol{\rho}\cdot\boldsymbol{e})\,(\boldsymbol{q}\cdot\boldsymbol{k}) - (\boldsymbol{\rho}\cdot\boldsymbol{k})\,(\boldsymbol{e}\cdot\boldsymbol{q})\} - i\{(\boldsymbol{\sigma}\cdot\boldsymbol{q})\,(\boldsymbol{\rho}\cdot[\boldsymbol{e}\times\boldsymbol{k}]) - (\boldsymbol{\sigma}\cdot\boldsymbol{\rho})\,(\boldsymbol{q}\cdot[\boldsymbol{e}\times\boldsymbol{k}])\}]/kq$$

$$E_2(P_{3/2}^{1}) = \frac{i}{4\pi}\frac{\sqrt{3}}{2}[(\boldsymbol{\sigma}\cdot\boldsymbol{k})\,(\boldsymbol{e}\cdot[\boldsymbol{\rho}\times\boldsymbol{q}]) + (\boldsymbol{\sigma}\cdot\boldsymbol{e})\,(\boldsymbol{k}\cdot[\boldsymbol{\rho}\times\boldsymbol{q}])]/kq$$

$$M_1(P_{3/2}^{2}) = \frac{1}{4\pi}\frac{i}{2\sqrt{5}}[2(\boldsymbol{q}\cdot\boldsymbol{\rho})\,(\boldsymbol{\sigma}\cdot[\boldsymbol{e}\times\boldsymbol{k}]) + 3(\boldsymbol{\sigma}\cdot\boldsymbol{q})\,(\boldsymbol{e}\cdot[\boldsymbol{\rho}\times\boldsymbol{k}]) + 3(\boldsymbol{\sigma}\cdot\boldsymbol{\rho})\,(\boldsymbol{e}\cdot[\boldsymbol{q}\times\boldsymbol{k}])]/kq$$

$$E_2(P_{3/2}^{2}) = \frac{1}{8\pi}\sqrt{\frac{3}{5}}[2\{(\boldsymbol{\rho}\cdot\boldsymbol{e})\,(\boldsymbol{q}\cdot\boldsymbol{k}) + (\boldsymbol{\rho}\cdot\boldsymbol{k})\,(\boldsymbol{e}\cdot\boldsymbol{q})\} - i\{(\boldsymbol{q}\cdot\boldsymbol{k})\,(\boldsymbol{\sigma}\cdot[\boldsymbol{e}\times\boldsymbol{\rho}]) + (\boldsymbol{\rho}\cdot\boldsymbol{e})\,(\boldsymbol{\sigma}[\boldsymbol{k}\times\boldsymbol{q}]) + (\boldsymbol{\rho}\cdot\boldsymbol{k})\,(\boldsymbol{\sigma}\cdot[\boldsymbol{e}\times\boldsymbol{q}]) + (\boldsymbol{e}\cdot\boldsymbol{q})\,(\boldsymbol{\sigma}\cdot[\boldsymbol{k}\times\boldsymbol{\rho}])\}]/kq$$

$$E_2(P_{5/2}^{2}) = \frac{1}{8\pi}\sqrt{\frac{3}{5}}[3\{(\boldsymbol{\rho}\cdot\boldsymbol{e})\,(\boldsymbol{q}\cdot\boldsymbol{k}) + (\boldsymbol{\rho}\cdot\boldsymbol{k})\,(\boldsymbol{e}\cdot\boldsymbol{q})\} + i\{(\boldsymbol{q}\cdot\boldsymbol{k})\,(\boldsymbol{\sigma}\cdot[\boldsymbol{e}\times\boldsymbol{\rho}]) + (\boldsymbol{\rho}\cdot\boldsymbol{e})\,(\boldsymbol{\sigma}\cdot[\boldsymbol{k}\times\boldsymbol{q}]) + (\boldsymbol{\rho}\cdot\boldsymbol{k})\,(\boldsymbol{\sigma}\cdot[\boldsymbol{e}\times\boldsymbol{q}]) + (\boldsymbol{e}\cdot\boldsymbol{q})\,(\boldsymbol{\sigma}[\boldsymbol{k}\times\boldsymbol{\rho}])\}]/kq$$

$$M_3(P_{5/2}^{2}) = \frac{i}{8\pi}\sqrt{\frac{3}{10}}[5\{(\boldsymbol{\sigma}\cdot\boldsymbol{k})\,(\boldsymbol{q}\cdot\boldsymbol{k})\,(\boldsymbol{\rho}\cdot[\boldsymbol{k}\times\boldsymbol{e}]) + (\boldsymbol{\sigma}\cdot\boldsymbol{k})\,(\boldsymbol{\rho}\cdot\boldsymbol{k})\,(\boldsymbol{q}\cdot[\boldsymbol{k}\times\boldsymbol{e}]) + (\boldsymbol{q}\cdot\boldsymbol{k})\,(\boldsymbol{\rho}\cdot\boldsymbol{k})\,(\boldsymbol{\sigma}\cdot[\boldsymbol{k}\times\boldsymbol{e}])\}/k^3q - \{(\boldsymbol{\sigma}\cdot\boldsymbol{q})\,(\boldsymbol{\rho}\cdot[\boldsymbol{k}\times\boldsymbol{e}]) + (\boldsymbol{\sigma}\cdot\boldsymbol{\rho})\,(\boldsymbol{q}\cdot[\boldsymbol{k}\times\boldsymbol{e}]) + (\boldsymbol{q}\cdot\boldsymbol{\rho})\,(\boldsymbol{\sigma}[\boldsymbol{k}\times\boldsymbol{e}])\}/kq]. \tag{A·30}$$

These projection operators satisfy the orthonormality condition

$$\int d\Omega_q \,\mathrm{Tr}\, E_l(L_J^{j})^{\dagger}\, E_{l'}(L_{J'}^{\prime j'}) = \frac{1}{4\pi}(2J+1)\delta_{JJ'}\delta_{ll'}\delta_{LL'}\delta_{jj'}, \tag{A·31}$$

$$\int d\Omega_q \,\mathrm{Tr}\, M_l(L_J^{j})^{\dagger}\, M_{l'}(L_{J'}^{\prime j'}) = \frac{1}{4\pi}(2J+1)\delta_{JJ'}\delta_{ll'}\delta_{LL'}\delta_{jj'}, \tag{A·32}$$

$$\int d\Omega_q \,\mathrm{Tr}\, M_l(L_J^{j})^{\dagger}\, E_{l'}(L_{J'}^{\prime j'}) = \int d\Omega_q \,\mathrm{Tr}\, E_l(L_J^{j})^{\dagger} M_{l'}(L_{J'}^{\prime j'}) = 0. \tag{A·33}$$

In the above equations, the symbol Tr means the summation over the independent directions of the polarization vectors $\boldsymbol{e}$ and $\boldsymbol{\rho}$ of the photon and ρ, and the trace with respect to the Pauli spin space.

References

1) R. Cool, O. Piccioni, and D. Clark, Phys. Rev. **103** (1956), 1082.
2) *Proceedings of the 9th Conference on High Energy Nuclear Physics (1959)*, Reports by G. Bernaldini and B. Pontecorvo.
J. C. Brisson, J. Detoef, P. Falk-Vairant, L. van Rossum, G. Valladas, and Luke C. L. Yuan, Phys. Rev. Letters **3** (1959), 561.
H. C. Burrowes, D. O. Caldwell, D. H. Frisch, D. A. Hill, D. M. Ritson, R. A. Schluter, and M. A. Wahlig, Phys. Rev. Letters **2** (1959), 119.
M. J. Longo, J. A. Helland, W. N. Hess, B. J. Moyer, and Victor Perez-Mendez, Phys. Rev. Letters **3** (1959), 568.
3) G. Takeda, Phys. Rev. **100** (1955), 440.
4) I. E. Tamm, private communication. (See *Proceedings of the 9th Conference on High Energy Nuclear Physics (1959)*, Report by L. I. Shiff).
5) G. F. Chew and S. Mandelstam, UCRL-8728.
6) W. R. Frazer and J. R. Fulco, Phys. Rev. Letters 2 (1959), 365.
7) See, e. g. N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields*, pp. 597-602.
8) The values for Γ' and W_r taken by Frazer and Fulco (ref. 6) are $\Gamma'=0.3\sim0.5$ and $W_r=2\sqrt{1+3.5}\mu\approx600$ Mev.
9) K. Igi and K. Kawarabayashi, Prog. Theor. Phys. **20** (1958), 576.
10) R. R. Wilson, Phys. Rev. **110** (1958), 1212.
R. F. Peierls, Phys. Rev. Letters **1** (1958), 174.
11) C. N. Yang, *Proceedings of the 5th Conference on High Energy Nuclear Physics (1955)*, 37.
12) The ratio $\sigma_{inel}/\sigma_{tot}$ was measured only for π^--P collisions and experimental accuracy is rather poor. Furthermore, tne measured ratio is one averaged over various different j states and does not mean the same ratio for the resonant state. See R. R. Crittenden, J. H. Scandrett, W. D. Shephard, and W. D. Walker, Phys. Rev. Letters **2** (1959), 121.
13) A. I. Baz, Phil. Mag. **4** (1959), 1035.
14) *Proceedings of the 8th Conference on High Enegy Nuclear Physics (1958)*, Reports by J. Steinberger, pp. 148-150, and B. T. Feld, pp. 158-159.
15) S. Z. Belen'kii, Soviet Phys. JETP **5** (1957), 952; Nuclear Phys. **2** (1957), 259.

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Mathematical Foundation of the Quantum Theory of Free Fields

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We assume that there is a quantum mechanical system at each point of the space-time, and on this assumption we can precisely construct the quantum theory of free fields. This theory is mathematically meaningful, and contains a theory which is completely equivalent to the usual one from the physical standpoint. Especially the physical meaning and the mathematical contents of field operators are clarified.

§1. Introduction

Though it is believed in general that the present quantum theory of free fields is a closed one, we cannot agree to such an opinion, because the meanings and roles of field operators are not yet completely clarified. Of course, we can treat the problem consistently in outline, considering the field operators to be purely formal.[1] But, we think that the role of field operators is very important physically and mathematically, and that a closer study of this problem is necessary. In this paper we shall show how we can treat the field operators consistently not only from a physical point of view but also from a mathematical point of view.

We begin our study by considering the physical import of field operators. Any field operator $\boldsymbol{q}(x)$ should be regarded as an "idealized" operator. For example, if it is a creation operator, it creates a particle which is at only one point. Of course, there will be no such particles in real nature, but it is very useful to utilize these idealized particles. In fact, if we do not use such idealized notions in quantum mechanics, it will be very difficult to understand the theory physically. On the other hand, if $\boldsymbol{q}(x)$ represents an observable quantity, it represents the density of a physical quantity. The density is of an "idealized" character, too, but it is obvious that, if we have no such notion as density, it is difficult to understand the character of the field quantity [ref. 2), II, § 4].

In order that we can treat the idealized operators consistently, it is necessary to introduce an "idealized" system. Namely it is necessary to assume that there is an idealized system at each point x of the space-time, and that the whole system can be constructed by these idealized ones. In the following, such an idealized system is called a "primary system," and we assume that each primary system is composed of the "primary states" and the "primary operators." On the other hand, the system which is composed of the states and the operators in

the proper sense is called a "proper system." The proper system has the physical reality in contrast to the primary system. As each primary system should be an idealized system of the usual quantum mechanical one, it will have the well-known quantum mechanical character. In these respects we may say that our theory is nothing but a faithful and mathematically meaningful realization of the second quantization following the program of Heisenberg and Pauli.[3]

Though our theory is mathematically meaningful, our argument here is not necessarily mathematically rigorous in order to avoid physically meaningless confusions.

§ 2. Primary system

As stated in § 1, we assume that at each point $x=(\boldsymbol{x}, t)$ of the space-time there is a quantum mechanical system which is called a "primary system." Namely we assume that there is such a separable Hilbert space $\mathfrak{h}(x)$ that any "primary state" is a normalized vector of $\mathfrak{h}(x)$ and any "primary operator" is an operator in $\mathfrak{h}(x)$. Furthermore we assume that the set of these primary operators forms a *-algebra $\mathfrak{a}(x)$.[2] The expectation value of a primary operator $q(x)$ for a primary state $\omega(x)\in\mathfrak{h}(x)$ is given by the inner product

$$(\omega(x), q(x)\omega(x))_{\mathfrak{h}}. \tag{2·1}$$

As the space-time is homogeneous, all primary systems should be equivalent to each other.

It is well known that the quantum theory of fields is nothing but the theory of a system of particles, and we assume that this feature is preserved also in the primary system. That is, we assume that there is a set of primary operators $\{q_\alpha; \alpha=1, 2, \cdots, N\}$ which satisfies the following conditions:

i) Every primary operator can be generated by these q_α's.

ii) Under a Lorentz transformation L, q_α is transformed according to the well-known equation

$$'q_\alpha=L_{\alpha\beta}q_\beta,^{\dagger} \tag{2·2}$$

where $L_{\alpha\beta}$ is determined by the spin of the field under consideration.

iii) The commutation relations are

$$[q_\alpha, q_\beta^*]_{\mp}=\delta_{\alpha\beta}, \tag{2·3}$$

where subscripts $-$ and $+$ correspond to the integer and the half-integer spins respectively. Eq. (2·3) shows that q_α and q_α^* are the annihilation and the creation operators, and there is a vacuum $\omega_0(x)\in\mathfrak{h}(x)$ such that

$$q_\alpha(x)\omega_0(x)=0, \quad \alpha=1, 2, \cdots, N. \tag{2·4}$$

It should be noted that Eq. (2·3) is not necessarily Lorentz invariant. This means that, though the theory as a whole is Lorentz invariant, the set $\{q_\alpha\}$ which

† In the case of the electromagnetic field, q_i, $i=1, 2, 3$, and q_0^* are mixed under some Lorentz transformations, but there is no need to enter into this problem in this paper.[4]

satisfies the above conditions is not Lorentz invariant, namely, the set $\{'q_\alpha\}$ is not necessarily such a set after a Lorentz transformation is performed. The situation will be clarified in the example c) in § 5.

Next we consider the system on a flat space-like surface σ, namely at a time t in some Lorentz frame S. In order to describe the system on σ, we construct a direct product of all primary systems on σ [ref. 2), II]. Let

$$\mathfrak{h}^{\omega_0}(\sigma)=\bigotimes_{x\in\sigma}^{\omega_0}\mathfrak{h}(x), \tag{2·5}$$

where the right-hand side is the incomplete direct product[5] of $\{\mathfrak{h}(x)\,;\,x\in\sigma\}$ which contains the element

$$\Omega_0(\sigma)=\times_{x\in\sigma}\omega_0(x). \tag{2·6}$$

It is obvious that $\mathfrak{h}^{\omega_0}(\sigma)$ is not separable. The physical meaning of the space $\mathfrak{h}^{\omega_0}(\sigma)$ is the following. $\Omega_0(\sigma)$ is a vacuum, that is, a state of no particle on σ. If there is a particle at a point $a\in\sigma$, then the state is given by a vector

$$\Omega(\sigma)=\times_{x\in\sigma}\omega(x), \tag{2·7}$$

where

$$\omega(x)=\omega_1(a),\quad x=a,$$

$$\omega(x)=\omega_0(x),\quad x\neq a,$$

and $\omega_1(a)$ is a state with one particle. When there is two or more particles, the state can be represented in a way analogous to Eq. (2·7). $\mathfrak{h}^{\omega_0}(\sigma)$ is nothing but a Hilbert space which is spanned by these states $\Omega(\sigma)$. In some cases, there may be another set of primary operators $\{q'_\alpha\}$ and a vacuum ω'_0 which is not equal to ω_0. In these cases $\mathfrak{h}^{\omega_0}(\sigma)$ and $\mathfrak{h}^{\omega_0'}(\sigma)$ are not equal to each other. The most typical example is given by the two kinds of vacuums for the Dirac electron field.

Let $\mathfrak{a}(\sigma)$ be the direct product of $\mathfrak{a}(x)$ [ref. 2), II],

$$\mathfrak{a}(\sigma)=\bigotimes_{x\in\sigma}\mathfrak{a}(x). \tag{2·8}$$

That is, $\mathfrak{a}(\sigma)$ is an algebra generated by such elements as

$$\times_{x\in\sigma}q(x),\quad q(x)\in\mathfrak{a}(x),$$

where $q(x)$ is the unit element $e(x)$ except for only a finite number of $q(x)$. For each element of $\mathfrak{a}(\sigma)$, an operator is defined in $\mathfrak{h}^{\omega_0}(\sigma)$, but it is to be remarked that $\mathfrak{a}(\sigma)$ does not depend on ω_0 contrary to $\mathfrak{h}^{\omega_0}(\sigma)$. This fact seems to us to show that $\mathfrak{a}(\sigma)$ is more essential than $\mathfrak{h}^{\omega_0}(\sigma)$ in the quantum theory of fields. Corresponding to a primary operator $q(a)$, an element of $\mathfrak{a}(\sigma)$ is defined

$$\times_{x\in\sigma}\dot{q}(x),$$

where

$$q(x)=q(a),\quad x=a,$$

$$q(x)=e(x),\quad x\neq a.$$

We denote it by the same symbol $q(a)$, too, there being no danger of confusion. In the following the operators $q_\alpha(x)\in\mathfrak{a}(\sigma)$, $\alpha=1, 2, \cdots, N$, corresponding to the primary operator $q_\alpha(x)\in\mathfrak{a}(x)$ are especially important. In the case of the half-integer spin, it is well known that a slight modification is necessary in order to preserve the anti-commutation relation

$$[q_\alpha(x),\ q_\beta(x')]_+=0,\quad x\neq x',$$

and we assume that this modification is already performed for these $q_\alpha(x)$.

The vector $\Omega(\sigma)$ correctly represents an idealized state, in which each particle is at exactly one point, but does not represent a proper physical state. Therefore we may say that the system composed of $\mathfrak{h}^{\infty 0}(\sigma)$ and $\mathfrak{a}(\sigma)$ is also a kind of primary system, and we call it a primary σ-system.

§ 3. Proper system

We are now in a position to consider the "proper system." In quantum mechanics a state which is represented by a wave function $\psi'(\boldsymbol{x})$ can be considered as a superposition with coefficients $\psi'(\boldsymbol{x})$ of idealized states, in which the particle is at $\boldsymbol{x}$. The relation between the proper state vector $\Psi(\sigma)$ and the state vectors $\Omega(\sigma)$ of the primary σ-system is the same as that between $\psi'(\boldsymbol{x})$ and idealized states. Such a vector $\Psi(\sigma)$ can be defined precisely as follows. Let $\Psi(\sigma)$ be an antilinear functional on $\mathfrak{h}^{\infty 0}(\sigma)$, and denote the value at $\Omega(\sigma)\in\mathfrak{h}^{\infty 0}(\sigma)$ by

$$(\Omega(\sigma),\ \Psi(\sigma)).$$

Put

$$\left.\begin{aligned}
&\Psi(\sigma\,;0)=(\Omega_0(\sigma),\ \Psi(\sigma)),\\
&\Psi(\sigma\,;\alpha,\ \boldsymbol{x})=(1/\sqrt{1\,!})(q_\alpha^*(\boldsymbol{x})\Omega_0(\sigma),\ \Psi(\sigma)),\\
&\Psi(\sigma\,;\alpha,\ \boldsymbol{x}\,;\beta,\ \boldsymbol{y})=(1/\sqrt{2\,!})(q_\beta^*(\boldsymbol{y})q_\alpha^*(\boldsymbol{x})\Omega_0(\sigma),\ \Psi(\sigma)),\\
&\cdots\cdots\cdots.
\end{aligned}\right\}\quad(3\cdot1)$$

Then it is obvious that each of them corresponds to the vacuum, the state with one particle, etc., and we call them the components of $\Psi(\sigma)$. It is easy to show that the components of $\Psi(\sigma)$ are symmetric or antisymmetric functions according to whether the field is of the integer spin or the half-integer.

At first, we assume that all components of $\Psi(\sigma)$ belong to $\mathfrak{S}$, and that only a finite number of them are not equal to zero.† Put the set of all these $\Psi(\sigma)$ be $\mathfrak{H}_{\mathfrak{S}}(\sigma)$, and $\mathfrak{H}_{\mathfrak{S}}(\sigma)$ is a vector space. If $\theta(\boldsymbol{x})$ is a positive definite distribution, then we can define an inner product in $\mathfrak{H}_{\mathfrak{S}}(\sigma)$ as follows:

$$\left.\begin{aligned}
&(\Psi_2(\sigma),\ \Psi_1(\sigma))_{\mathfrak{H}}\\
&\quad=\overline{\Psi}_2(\sigma\,;0)\Psi_1(\sigma\,;0)+\sum_\alpha\breve{\theta}_x(\widetilde{\Psi}_2(\sigma\,;\alpha,\ \boldsymbol{x})*\Psi_1(\sigma\,;\alpha,\ \boldsymbol{x}))\\
&\quad+\sum_{\alpha,\beta}(\breve{\theta}_x\times\breve{\theta}_y)(\widetilde{\Psi}_2(\sigma\,;\alpha,\ \boldsymbol{x}\,;\beta,\ \boldsymbol{y})*\Psi_1(\sigma\,;\alpha,\ \boldsymbol{x}\,;\beta,\ \boldsymbol{y}))+\cdots,
\end{aligned}\right\}\quad(3\cdot2)$$

† Mathematical symbols and terms are explained in the Appendix.

or symbolically

$$(\Psi_2(\sigma), \Psi_1(\sigma))_{\mathfrak{H}} = \breve{\theta}(\widetilde{\Psi}_2(\sigma) * \Psi_1(\sigma)), \tag{3·2′}$$

and the norm of $\Psi(\sigma)$ is defined by

$$|\Psi(\sigma)|^2 = (\Psi(\sigma), \Psi(\sigma))_{\mathfrak{H}}.$$

Let the completion of $\mathfrak{H}_{\mathfrak{S}}(\sigma)$ be $\mathfrak{H}(\sigma)$, and $\mathfrak{H}(\sigma)$ is the Hilbert space which is spanned by the proper state vectors. We can show that $\mathfrak{H}(\sigma)$ is separable in all physically meaningful cases [see the Appendix]. A vector $\Psi(\sigma)$, whose components are all zero except for the component

$$\Psi(\sigma\,;0)=1,$$

is the vacuum corresponding to $\Omega_0(\sigma)$, and we denote it by $\Psi_0(\sigma)$.

If $F(\boldsymbol{x})$ is a square integrable function in the sense of Eq. (A·4), then we can define an operator $\boldsymbol{q}_\alpha^*(F)$ in $\mathfrak{H}(\sigma)$ as follows:

$\boldsymbol{q}_\alpha^*(F)$ is such a creation operator that

$$\left.\begin{aligned} &(\boldsymbol{q}_\beta^*(\boldsymbol{x})\Omega_0(\sigma),\ \boldsymbol{q}_\alpha^*(F)\Psi_0(\sigma)) = \delta_{\alpha\beta}F(\boldsymbol{x}), \\ &(\boldsymbol{q}_\delta^*(\boldsymbol{y})q_\gamma^*(\boldsymbol{x})\Omega_0(\sigma),\ \boldsymbol{q}_\beta^*(F_2)\boldsymbol{q}_\alpha^*(F_1)\Psi_0(\sigma)) \\ &\quad = \delta_{\delta\beta}\delta_{\gamma\alpha}F_2(\boldsymbol{y})F_1(\boldsymbol{x}) \pm \delta_{\delta\alpha}\delta_{\gamma\beta}F_2(\boldsymbol{x})F_1(\boldsymbol{y}), \\ &\cdots, \end{aligned}\right\} \tag{3·3}$$

where + and − correspond to the integer and half-integer spins respectively. It is obvious that every vector in $\mathfrak{H}(\sigma)$ can be constructed from $\Psi_0(\sigma)$ by operating an appropriate number of operators $\boldsymbol{q}_\alpha^*(F)$, and that

$$\left.\begin{aligned} &(\boldsymbol{q}_\beta^*(F_2)\Psi_0(\sigma),\ \boldsymbol{q}_\alpha^*(F_1)\Psi_0(\sigma))_{\mathfrak{H}} = \delta_{\beta\alpha}\breve{\theta}(\widetilde{F}_2 * F_1), \\ &(\boldsymbol{q}_\delta^*(F_4)\boldsymbol{q}_\gamma^*(F_3)\Psi_0(\sigma),\ \boldsymbol{q}_\beta^*(F_2)\boldsymbol{q}_\alpha^*(F_1)\Psi_0(\sigma))_{\mathfrak{H}} \\ &\quad = \delta_{\delta\beta}\delta_{\gamma\alpha}\breve{\theta}(\widetilde{F}_4 * F_2)\cdot\breve{\theta}(\widetilde{F}_3 * F_1) \pm \delta_{\delta\alpha}\delta_{\gamma\beta}\breve{\theta}(\widetilde{F}_4 * F_1)\cdot\breve{\theta}(\widetilde{F}_3 * F_2), \\ &\cdots. \end{aligned}\right\} \tag{3·4}$$

$\boldsymbol{q}_\alpha(F)$, defined as the Hermitian conjugate of $\boldsymbol{q}_\alpha^*(F)$, is an annihilation operator, and

$$\left.\begin{aligned} &\boldsymbol{q}_\beta(F_2)\boldsymbol{q}_\alpha^*(F_1)\Psi_0(\sigma) = \delta_{\beta\alpha}\breve{\theta}(\widetilde{F}_2 * F_1)\Psi_0(\sigma), \\ &\boldsymbol{q}_\gamma(F_3)\boldsymbol{q}_\beta^*(F_2)\boldsymbol{q}_\alpha^*(F_1)\Psi_0(\sigma) \\ &\quad = \delta_{\gamma\beta}\breve{\theta}(\widetilde{F}_3 * F_2)\boldsymbol{q}_\alpha^*(F_1)\Psi_0(\sigma) \pm \delta_{\gamma\alpha}\breve{\theta}(\widetilde{F}_3 * F_1)\boldsymbol{q}_\beta^*(F_2)\Psi_0(\sigma), \\ &\cdots. \end{aligned}\right\} \tag{3·5}$$

The commutation relations are given by

$$\left.[\boldsymbol{q}_\alpha^*(F_1),\ \boldsymbol{q}_\beta^*(F_2)]_{\mp} = 0,\right.$$

$$\left.\begin{aligned}&[\boldsymbol{q}_\alpha(F_1), \boldsymbol{q}_\beta(F_2)]_\mp=0,\\&[\boldsymbol{q}_\alpha(F_1), \boldsymbol{q}^*_\beta(F_2)]_\mp=\delta_{\alpha\beta}\check{\theta}(\widetilde{F}_1*F_2).\end{aligned}\right\} \quad (3\cdot6)$$

§ 4. Field operator

In order to define various field operators, which are not operators in $\mathfrak{H}(\sigma)$, it is necessary to introduce a dual space $\mathfrak{H}^*_{\mathfrak{S}}(\sigma)$ to $\mathfrak{H}_{\mathfrak{S}}(\sigma)$. It is obvious that, if $\Psi_2(\sigma)\in\mathfrak{H}_{\mathfrak{S}}(\sigma)$, then the inner product defined by Eq. (3·2) has a precise meaning on condition that every component of $\Psi_1(\sigma)$ belongs to $\mathfrak{D}'_c$. Therefore, we define an extension of $\mathfrak{H}_{\mathfrak{S}}(\sigma)$ as follows. Let $\Psi^*(\sigma)$ be an antilinear functional on $\mathfrak{h}^{\infty 0}(\sigma)$ whose components belong to $\mathfrak{D}'_c$, and the set of these $\Psi^*(\sigma)$ be $\mathfrak{H}^*_{\mathfrak{S}}(\sigma)$. For any elements $\Psi(\sigma)\in\mathfrak{H}_{\mathfrak{S}}(\sigma)$ and $\Psi^*(\sigma)\in\mathfrak{H}^*_{\mathfrak{S}}(\sigma)$, a complex value is defined by Eq. (3·2),

$$(\Psi(\sigma), \Psi^*(\sigma))_{\mathfrak{H}}.$$

It is easy to show that, for any $F\in\mathfrak{D}'_c$, we can define an operator $\boldsymbol{q}_\alpha(F)$ in $\mathfrak{H}_{\mathfrak{S}}(\sigma)$ and $\boldsymbol{q}^*_\alpha(F)$ in $\mathfrak{H}^*_{\mathfrak{S}}(\sigma)$ in the same way as in § 3, and that they satisfy the same equations as (3·3) and (3·4). Furthermore, these operators are adjoint to each other, that is, for any $\Psi(\sigma)\in\mathfrak{H}_{\mathfrak{S}}(\sigma)$, $\Psi^*(\sigma)\in\mathfrak{H}^*_{\mathfrak{S}}(\sigma)$,

$$(\Psi(\sigma), \boldsymbol{q}^*_\alpha(F)\Psi^*(\sigma))_{\mathfrak{H}}=(\boldsymbol{q}_\alpha(F)\Psi(\sigma), \Psi^*(\sigma))_{\mathfrak{H}}. \quad (4\cdot1)$$

$\boldsymbol{q}_\alpha(F)$ is an operator in $\mathfrak{H}_{\mathfrak{S}}(\sigma)$, and $\boldsymbol{q}^*_\alpha(F)$ is in $\mathfrak{H}^*_{\mathfrak{S}}(\sigma)$, therefore we cannot define the product of these operators in general, and the commutation relations of them have no mathematically precise meaning.

The most typical examples of field operators are given by

$$\boldsymbol{q}_\alpha(\boldsymbol{a})=\boldsymbol{q}_\alpha(\delta(\boldsymbol{x}-\boldsymbol{a})),$$
$$\boldsymbol{q}^*_\alpha(\boldsymbol{a})=\boldsymbol{q}^*_\alpha(\delta(\boldsymbol{x}-\boldsymbol{a}))$$

and, if we formally calculate the commutator, we get

$$[\boldsymbol{q}_\alpha(\boldsymbol{a}), \boldsymbol{q}^*_\beta(\boldsymbol{b})]_\mp=\delta_{\alpha\beta}\theta(\boldsymbol{a}-\boldsymbol{b}). \quad (4\cdot2)$$

This shows that $\boldsymbol{q}_\alpha(\boldsymbol{a})$ and $\boldsymbol{q}^*_\alpha(\boldsymbol{a})$ are nothing but the field operators in the usual theory, and we have succeeded to give them precise mathematical meanings. It is remarkable that, though the commutation relation (4·2) has no precise meaning, our theory contains in Eqs. (2·3) and (3·2) mathematically meaningful contents which are completely equivalent to Eq. (4·2). Eq. (4·2) is very useful in practice, because it can be considered as a formal representative of Eq. (3·6).

Let $F(\boldsymbol{x}, a)\in\mathfrak{D}'_c$ contain a real parameter a, then we can define the differentiation and the integration by

$$\left.\begin{aligned}&(\partial/\partial a)\boldsymbol{q}_\alpha(F(\boldsymbol{x}, a))=\boldsymbol{q}_\alpha((\partial/\partial a)F(\boldsymbol{x}, a)),\\&(\partial/\partial a)\boldsymbol{q}^*_\alpha(F(\boldsymbol{x}, a))=\boldsymbol{q}^*_\alpha((\partial/\partial a)F(\boldsymbol{x}, a)),\end{aligned}\right.$$

$$\left.\begin{aligned}&\int \boldsymbol{q}_\alpha(F(\boldsymbol{x}, a))\bar{g}(a)\,da=\boldsymbol{q}_\alpha\Big(\int F(\boldsymbol{x}, a)g(a)da\Big),\\&\int \boldsymbol{q}^*_\alpha(F(\boldsymbol{x}, a))g(a)\,da=\boldsymbol{q}^*_\alpha\Big(\int F(\boldsymbol{x}, a)g(a)da\Big).\end{aligned}\right\}\quad(4\cdot3)$$

Of course, it is necessary that the differentiation and the integration in the right-hand side are well-defined in $\mathfrak{D}'_c$. It is obvious that $\boldsymbol{q}_\alpha(\boldsymbol{a})$ and $\boldsymbol{q}^*_\alpha(\boldsymbol{a})$ are indefinitely differentiable with respect to a_i, $i=1, 2, 3$, and if $F\in\mathfrak{D}'_c$,

$$\left.\begin{aligned}&\boldsymbol{q}_\alpha(F)=\int \boldsymbol{q}_\alpha(\boldsymbol{a})\,\overline{F}(\boldsymbol{a})\,d\boldsymbol{a},\\&\boldsymbol{q}^*_\alpha(F)=\int \boldsymbol{q}^*_\alpha(\boldsymbol{a})F(\boldsymbol{a})\,d\boldsymbol{a}\end{aligned}\right\}\quad(4\cdot4)$$

in the precise sense.

It is worthy of notice that, though the Fourier transformation has a precise meaning in $\mathfrak{S}'$, the field operators in the so-called momentum representation lose their mathematical meanings in some cases. Formally we can define the Fourier transformation of field operators by the following equations:

$$\left.\begin{aligned}&(1/(2\pi)^3)\int \boldsymbol{q}_\alpha(F(\boldsymbol{x}-\boldsymbol{a}))\exp(-i\boldsymbol{k}\cdot\boldsymbol{a})\,d\boldsymbol{a}=\boldsymbol{q}_\alpha(F(\boldsymbol{k})\exp(i\boldsymbol{k}\cdot\boldsymbol{x})),\\&(1/(2\pi)^3)\int \boldsymbol{q}^*_\alpha(F(\boldsymbol{x}-\boldsymbol{a}))\exp(i\boldsymbol{k}\cdot\boldsymbol{a})\,d\boldsymbol{a}=\boldsymbol{q}^*_\alpha(F(\boldsymbol{k})\exp(i\boldsymbol{k}\cdot\boldsymbol{x})),\end{aligned}\right\}\quad(4\cdot5)$$

where

$$F(\boldsymbol{x})=\int F(\boldsymbol{k})\exp(i\boldsymbol{k}\cdot\boldsymbol{x})\,d\boldsymbol{k}.$$

If $\theta(\boldsymbol{x})$ belongs to $\mathfrak{D}'_c$, we can easily show that Eq. (4·5) determines mathematically meaningful operators, and they are the field operators in the momentum representation. On the contrary, if $\theta(\boldsymbol{x})$ does not belong to $\mathfrak{D}'_c$, Eq. (4·5) loses its mathematical meaning in general. For example, the distribution $\theta(\boldsymbol{x})$ of the electromagnetic field does not belong to $\mathfrak{D}'$, so, in this case, the field operators in the momentum representation have no precise meaning.

§ 5. Equation of motion

In this section we consider the relations between the systems at different times. From the principle of causality, a state $\Psi(\sigma_0)$ on a surface σ_0 should completely determine the other states $\Psi(\sigma)$ on any surface σ. This means that there should be a one-to-one correspondence between $\mathfrak{H}(\sigma_0)$ and $\mathfrak{H}(\sigma)$, and it is very likely that, by this correspondence, $\mathfrak{H}_{\mathfrak{S}}(\sigma_0)$ is transformed onto $\mathfrak{H}_{\mathfrak{S}}(\sigma)$ and $\mathfrak{H}^*_{\mathfrak{S}}(\sigma_0)$ onto $\mathfrak{H}^*_{\mathfrak{S}}(\sigma)$ simultaneously. If these conditions are satisfied, we can consider that all spaces $\mathfrak{H}_{\mathfrak{S}}(\sigma)$, $\mathfrak{H}(\sigma)$, and $\mathfrak{H}^*_{\mathfrak{S}}(\sigma)$ do not vary with σ, and we can denote them by $\mathfrak{H}_{\mathfrak{S}}$, $\mathfrak{H}$, and $\mathfrak{H}^*_{\mathfrak{S}}$ respectively. Namely, we can adopt the Heisenberg representation

As the differentiation of field operators is mathematically meaningful, the well-known equation of motion has a mathematical meaning, too, and there will be no need to discuss it in detail. For our purpose, it is more convenient to introduce a Green function. It is obvious that all field operators must be represented by those at the time t_0. Furthermore, as we are dealing with free fields, the number of particles must be conserved. All these conditions are satisfied if and only if there is a kernel function $G_{\alpha\beta}(\boldsymbol{x}, t) \in \mathcal{O}_c'$ such that

$$\left.\begin{aligned} \boldsymbol{q}_\alpha(\boldsymbol{x}, t) &= \bar{G}_{\alpha\beta}(\boldsymbol{x}, t-t_0) * \boldsymbol{q}_\beta(\boldsymbol{x}, t_0), \\ \boldsymbol{q}_\alpha^*(\boldsymbol{x}, t) &= G_{\alpha\beta}(\boldsymbol{x}, t-t_0) * \boldsymbol{q}_\beta^*(\boldsymbol{x}, t_0). \end{aligned}\right\} \quad (5\cdot1)$$

It is easy to show that

$$\left.\begin{aligned} \boldsymbol{q}_\alpha(F, t) &= \boldsymbol{q}_\beta(\breve{G}_{\alpha\beta}(t-t_0) * F, t_0), \\ \boldsymbol{q}_\alpha^*(F, t) &= \boldsymbol{q}_\beta^*(\breve{G}_{\alpha\beta}(t-t_0) * F, t_0). \end{aligned}\right\} \quad (5\cdot2)$$

As $\breve{G}(t) \in \mathcal{O}_c'$, Eq. (5·2) secures the above-mentioned correspondences between $\mathfrak{H}_{\mathfrak{S}}(\sigma_0)$ and $\mathfrak{H}_{\mathfrak{S}}(\sigma)$, and between $\mathfrak{H}_{\mathfrak{S}}^*(\sigma_0)$ and $\mathfrak{H}_{\mathfrak{S}}^*(\sigma)$. From Eq. (5·1), it can be shown that $G_{\alpha\beta}(\boldsymbol{x}, t)$ should satisfy the equation

$$\sum_\gamma \left\{ G_{\alpha\gamma}(\boldsymbol{x}, t_1) * G_{\gamma\beta}(\boldsymbol{x}, t_2) \right\} = G_{\alpha\beta}(\boldsymbol{x}, t_1+t_2). \quad (5\cdot3)$$

Furthermore, in every physically meaningful case, we can show that

$$\bar{G}_{\alpha\beta}(-t) = \breve{G}_{\beta\alpha}(t), \quad (5\cdot4)$$

and this secures the invariance of Eq. (3·6). In fact, from Eqs. (4·2), (5·1), (5·3), and (5·4),

$$[\boldsymbol{q}_\alpha(\boldsymbol{x}, t),\ \boldsymbol{q}_\beta^*(\boldsymbol{y}, t)]_\mp = \sum_\gamma \{\bar{G}_{\alpha\gamma}(t) * \breve{G}_{\beta\gamma}(t) * \theta\}(\boldsymbol{x}-\boldsymbol{y}) = \delta_{\alpha\beta}\theta(\boldsymbol{x}-\boldsymbol{y}),$$

that is, the formal equation (4·2) is valid at any time t, and this means the invariance of Eq. (3·6). Hence, Eq. (5·1) determines a unitary transformation in $\mathfrak{H}$. Put

$$\Theta_{\alpha\beta}(\boldsymbol{x}, t) = \bar{G}_{\alpha\beta}(\boldsymbol{x}, t) * \theta(\boldsymbol{x}), \quad (5\cdot5)$$

and $\Theta_{\alpha\beta}(\boldsymbol{x}, t)$ belongs to $\mathcal{S}'$. We can rewrite the theory symmetrically with respect to $\boldsymbol{x}$ and t. For example,

$$(\boldsymbol{q}_\beta^*(F_2, t_2)\Phi_0,\ \boldsymbol{q}_\alpha^*(F_1, t_1)\Phi_0)_{\mathfrak{H}} = \breve{\Theta}_{\beta\alpha}(t_2-t_1)(\tilde{F}_2 * F_1), \quad (5\cdot6)$$

$$[\boldsymbol{q}_\alpha(\boldsymbol{x}, t_1),\ \boldsymbol{q}_\beta^*(\boldsymbol{y}, t_2)]_\mp = \Theta_{\alpha\beta}(\boldsymbol{x}-\boldsymbol{y}, t_1-t_2), \quad (5\cdot7)$$

and the others will be obvious.

Thus we have succeeded in giving a precise mathematical meaning to the quantum theory of free fields. In order to justify our theory, it is only necessary to confirm the following in each case:

There are two distributions, $\theta(\boldsymbol{x})$ and $G_{\alpha\beta}(\boldsymbol{x}, t)$, and they satisfy the following

conditions.

i) $\theta(\boldsymbol{x})$ is positive definite and belongs to $\mathcal{S}'$.

ii) $G_{\alpha\beta}(\boldsymbol{x}, t)$ belongs to $\mathfrak{D}_c'$, and satisfies the usual equation of motion, Eq. (5·3), and Eq. (5·4).

iii) Defining $\theta_{\alpha\beta}(\boldsymbol{x}, t)$ by Eq. (5·5), we can show that Eq. (5·7) coincides with the usual commutation relation.

Here we add a brief sketch of some examples.

a) *Scalar field* $U(x)$

Divide $U(x)$ into positive and negative frequency parts $U^+(x)$ and $U^-(x)$, and

$$[U^+(x),\ U^{+*}(x')]=i\Delta^+(x-x'),$$
$$[U^{-*}(x),\ U^-(x')]=-i\Delta^-(x'-x).$$

The two distributions $i\Delta^+(\boldsymbol{x}, 0)$ and $-i\Delta^-(\boldsymbol{x}, 0)$ are positive definite, because [see the Appendix]

$$i\Delta^+(\boldsymbol{x}, 0)=-i\Delta^-(\boldsymbol{x}, 0)$$
$$=(1/(2\pi)^3)\int(1/2K_0)\ \exp\ (i\boldsymbol{k}\cdot\boldsymbol{x})\,d\boldsymbol{k},$$

where

$$K_0=\sqrt{\boldsymbol{k}^2+\kappa^2}.$$

If we put

$$G^{\pm}(\boldsymbol{x}, t)=(1/(2\pi)^3)\int\exp\ (i\boldsymbol{k}\cdot\boldsymbol{x})\ \exp\ (\mp iK_0t)\,d\boldsymbol{k},$$

then

$$U^{\pm}(\boldsymbol{x}, t)=\int G^{\pm}(\boldsymbol{x}-\boldsymbol{y}, t)\,U^{\pm}(\boldsymbol{y}, 0)\,d\boldsymbol{y}.$$

Thus, putting $U^+=q_1$, $U^{-*}=q_2$, we get

$$\theta(\boldsymbol{x})=i\Delta^+(\boldsymbol{x}, 0)=-i\Delta^-(\boldsymbol{x}, 0)\in\mathcal{S}',$$
$$G_{11}(\boldsymbol{x}, t)=G_{22}(\boldsymbol{x}, t)=\bar{G}^+(\boldsymbol{x}, t)=G^-(\boldsymbol{x}, t)\in\mathfrak{D}_c',$$
$$\theta_{11}(\boldsymbol{x}, t)=\theta_{22}(\boldsymbol{x}, t)=i\Delta^+(\boldsymbol{x}, t)=-i\Delta^-(-\boldsymbol{x}, -t).$$

b) *Electromagnetic field* $A_\mu(x)$

When we put

$$A_\mu(x)=A_\mu{}^+(x)+A_\mu{}^-(x),$$

then

$$[A_\mu{}^+(x),\ A_\nu{}^-(x')]=i\delta_{\mu\nu}D^+(x-x'),$$

and

$$A_\mu{}^{\pm}(\boldsymbol{x}, t)=\int G_0^{\pm}(\boldsymbol{x}-\boldsymbol{y}, t)\,A_\mu{}^{\pm}(\boldsymbol{y}, 0)\,d\boldsymbol{y},$$

where

$$G_0^{\pm}(\boldsymbol{x}, t) = G^{\pm}(\boldsymbol{x}, t)|_{\kappa=0}.$$

Putting

$$A_0^- = q_0, \quad A_i^+ = q_i, \quad i=1, 2, 3,$$

we get

$$\theta(\boldsymbol{x}) = iD^+(\boldsymbol{x}, 0) = -iD^-(\boldsymbol{x}, 0) \in \mathfrak{S}',$$
$$G_{00}(\boldsymbol{x}, t) = G_0^+(\boldsymbol{x}, t) \in \mathfrak{D}_c',$$
$$G_{ii}(\boldsymbol{x}, t) = G_0^-(\boldsymbol{x}, t) \in \mathfrak{D}_c',$$
$$\Theta_{00}(\boldsymbol{x}, t) = -iD^-(\boldsymbol{x}, t),$$
$$\Theta_{ii}(\boldsymbol{x}, t) = iD^+(\boldsymbol{x}, t).$$

c) *Spinor field* $\psi_\alpha(x)$ $(\kappa \neq 0)$

It is well known that

$$[\psi_\alpha(\boldsymbol{x}, t), \psi_\beta^*(\boldsymbol{x}', t)]_+ = \delta_{\alpha\beta}\delta(\boldsymbol{x}-\boldsymbol{x}'),$$

and

$$\psi(\boldsymbol{x}, t) = -\int iS(\boldsymbol{x}-\boldsymbol{y}, t)\gamma_4\psi(\boldsymbol{y}, 0)\, d\boldsymbol{y},$$

where

$$S(x) = (\gamma_\nu\partial_\nu - \kappa)\Delta(x).$$

It is obvious that

$$\theta(\boldsymbol{x}) = \delta(\boldsymbol{x}) \in \mathfrak{S}',$$
$$G(\boldsymbol{x}, t) = i\bar{S}(\boldsymbol{x}, t)\bar{\gamma}_4 \in \mathfrak{D}_c',^{\dagger}$$
$$\Theta(\boldsymbol{x}, t) = -iS(\boldsymbol{x}, t)\gamma_4.$$

Eq. (5·4) is satisfied on condition that γ_i, $i=1, 2, 3$, and γ_4 are Hermitian matrices.

For the primary operators, a commutation relation

$$[\psi_\alpha, \psi_\beta^*]_+ = \delta_{\alpha\beta}$$

is not Lorentz invariant. A covariant commutation relation is given by an equation

$$[\psi, \bar{\psi}]_+ = -i\gamma_\mu n_\mu,$$

where

$$\bar{\psi} = \psi^*\gamma_4$$

and n_μ is a time-like unit vector. Physical meaning of this commutation relation is the following.

† If $\kappa=0$, $G(\boldsymbol{x}, t)$ does not belong to $\mathfrak{D}_c'$, and our theory is not applicable without any modification.

In order to specify the physical meaning of a primary operator ψ_α, it is necessary to appoint not only the Lorentz frame S_0 but also the forms of matrices γ_μ. After a Lorentz transformation L is performed, we get a new Lorentz frame,

$$S=LS_0,$$

and new primary operators,

$$'\psi_\alpha=L_{\alpha\beta}\psi_\beta.$$

But the physical meaning of $'\psi_\alpha$ cannot be specified without the knowledge of γ_μ and S_0. Thus the appearance of a specified vector in the commutation relation can be understood, and n_μ is nothing but the unit vector in the direction of the time axis of S_0.

§ 6. Concluding remarks

We have stated the theory using a space-like surface $t=$const., but it will be obvious that we can use any space-like surface σ on condition that there is a uniquely defined vacuum $\omega_0\in\mathfrak{h}$.† For physicists, it is customary to use any space-like surface σ, and our theory affords a mathematical foundation to it. This fact is very important from the viewpoint of general relativity, because, in this case, there is no flat surface in general. Therefore most physicists will be able to accept our theory intuitively also in this respect.

Now, we summarize the chief contents of this paper. At first, we have introduced a primary system at each point, and have constructed a direct product of them on a space-like surface σ, that is, we have constructed a primary σ-system. As the meanings of Eqs. (2·2) and (2·3) are well known, there is no room for ambiguity as to these systems. Next, we have introduced the space $\mathfrak{H}_\mathfrak{F}(\sigma)$, whose elements are well-defined functionals on $\mathfrak{h}^{\omega_0}(\sigma)$, and have explicitly defined an inner product in $\mathfrak{H}_\mathfrak{F}(\sigma)$. Thus we have defined the Hilbert space $\mathfrak{H}(\sigma)$ concretely, and this enables us to define and to treat various field operators correctly.

It is, however, worthy of notice that our theory can contain essentially new contents. It is easy to see that, in the above theory, such a state vector as $q_\alpha^*(\boldsymbol{x})q_\beta^*(\boldsymbol{x})\Omega_0$ makes no contribution to $\mathfrak{H}$. But, if we add such terms as

$$\sum_{\alpha,\beta}\breve{\theta}_\alpha(\widetilde{\Psi}_2(\sigma\,;\,\alpha,\,\boldsymbol{x}\,;\,\beta,\,\boldsymbol{x})*\Psi_1(\sigma\,;\,\alpha,\,\boldsymbol{x}\,;\,\beta,\,\boldsymbol{x}))$$

to the right-hand side of Eq. (3·2), the above vector becomes meaningful, and there appears a completely new feature. In this case $\mathfrak{H}$ consists of two parts $\mathfrak{H}_1$, $\mathfrak{H}_2$, where $\mathfrak{H}_1$ is the original one and is orthogonal to $\mathfrak{H}_2$. $\mathfrak{H}_2$ is a space corresponding to a new kind of particles, and it is an open question what physical meaning $\mathfrak{H}_2$ has.

If there are two or more fields, the primary system of the composite system

† It may happen that there is no vacuum which belongs to $\mathfrak{h}$ and is invariant under the Lorentz transformation,[4] but we do not consider this problem in this paper.

is given by the direct product of primary systems of the original fields, and the theory is unchanged except for the interactions of these fields. As to the interactions, we can introduce an essentially new type of them which is characterized by primary operators such as $q_\alpha q_\beta$. The field operator corresponding to the above primary operator can be represented as $(\boldsymbol{q}_\alpha \boldsymbol{q}_\beta)(\boldsymbol{x})$ and not as $\boldsymbol{q}_\alpha(\boldsymbol{x})\boldsymbol{q}_\beta(\boldsymbol{x})$. All these problems will be discussed in detail on another occasion.

Appendix

Mathematical terms and symbols which are used in this paper are briefly explained in this Appendix. A detailed account of these matters will be found out in reference 6).

$\mathfrak{D}$: This is the space of complex-valued functions of N real variables, each of which is indefinitely differentiable and has a compact carrier. There is defined a topology in $\mathfrak{D}$, and $\mathfrak{D}$ is a topological vector space.

$\mathfrak{D}'$: (The space of distributions) This is the adjoint space of $\mathfrak{D}$, that is, a space of continuous linear functionals on $\mathfrak{D}$.

For any $T\in\mathfrak{D}'$ and $\varphi\in\mathfrak{D}$, we can write

$$\begin{aligned} T(\varphi) &= \int T(a_1, \cdots, a_N)\ \varphi(a_1, \cdots, a_N)\ da_1 \cdots da_N \\ &= \int T(a)\varphi(a)da \end{aligned}$$

in a symbolical sense. Under some appropriate conditions, three kinds of products are defined:

i) Product;

$$T_1 \cdot T_2(\varphi) = \int T_1(a) T_2(a) \varphi(a) da.$$

ii) Direct product;

$$T_a \times T_b(\varphi_a \times \varphi_b) = T_a(\varphi_a) \cdot T_b(\varphi_b).$$

iii) Convolution;

$$\begin{aligned} T_1 * T_2(\varphi) &= \int T_1(a-b) T_2(b) \varphi(a) da db \\ &= \int T_1(a) T_2(b) \varphi(a+b) da db. \end{aligned}$$

(A·1)

For physical purposes, $\mathfrak{D}$ and $\mathfrak{D}'$ are not appropriate, but the following spaces are very useful.

$\mathfrak{S}$: This is the topological vector space of functions, each of which is indefinitely differentiable and rapidly decreasing.

$\mathfrak{S}'$: (The space of slowly increasing distributions) This is the adjoint space of $\mathfrak{S}$.

$\mathfrak{D}_m$: This is the space of indefinitely differentiable and slowly increasing functions.

$\mathfrak{D}_c'$: (The space of rapidly decreasing distributions) This is the space of rapidly decreasing distributions, and a distribution T belongs to $\mathfrak{D}_c'$ if and only if $T*\varphi\in\mathfrak{S}$ for every $\varphi\in\mathfrak{D}$.

These spaces satisfy the relations

$$\begin{array}{ccccc}\mathfrak{D} & \subset & \mathfrak{S} & \subset & \mathfrak{D}_m \\ \cap & & \cap & & \\ \mathfrak{D}_c' & \subset & \mathfrak{S}' & \subset & \mathfrak{D}'\end{array},$$

and the following products can be defined:

$$\left.\begin{aligned} &T_1\cdot T_2\in\mathfrak{S}(\mathfrak{S}'),\quad T_1\in\mathfrak{D}_m,\quad T_2\in\mathfrak{S}(\mathfrak{S}'),\\ &T_1*T_2\in\mathfrak{S}(\mathfrak{S}'),\quad T_1\in\mathfrak{D}_c',\quad T_2\in\mathfrak{S}(\mathfrak{S}').\end{aligned}\right\}\quad(\text{A}\cdot 2)$$

In $\mathfrak{S}'$, Fourier transforms of all distributions can be defined, and this transformation determines one-to-one correspondences as follows:

$$\mathfrak{S}\leftrightarrow\mathfrak{S},\quad \mathfrak{D}_m\leftrightarrow\mathfrak{D}'_c,\quad \mathfrak{S}'\leftrightarrow\mathfrak{S}'.$$

It is well-known that

$$\left.\begin{aligned} &\Delta(x)=\Delta^+(x)+\Delta^-(x),\\ &\Delta^+(x)=-(i/(2\pi)^3)\int(1/2K_0)\exp(i\boldsymbol{k}\cdot\boldsymbol{x})\exp(-iK_0t)d\boldsymbol{k},\\ &\Delta^-(x)=(i/(2\pi)^3)\int(1/2K_0)\exp(i\boldsymbol{k}\cdot\boldsymbol{x})\exp(iK_0t)d\boldsymbol{k},\end{aligned}\right\}\quad(\text{A}\cdot 3)$$

where

$$K_0=\sqrt{\boldsymbol{k}^2+\kappa^2}.$$

This shows that Δ^+, Δ^-, and Δ belong to $\mathfrak{S}'$ for any time t, and if $\kappa\neq 0$, they belong to $\mathfrak{D}_c'$.

Let $\overline{\varphi}(a)$ be the conjugate complex of $\varphi(a)$, and

$$\check{\varphi}(a)=\varphi(-a),\quad \tilde{\varphi}(a)=\overline{\varphi}(-a).$$

Then we can define $\overline{T}$, $\check{T}$, and $\tilde{T}$ by the following equations,

$$\overline{T}(\overline{\varphi})=\overline{T(\varphi)},\quad \check{T}(\check{\varphi})=T(\varphi),\quad \tilde{T}(\tilde{\varphi})=\overline{T(\varphi)}.$$

If $T\in\mathfrak{S}'$ satisfies the relation

$$\check{T}(\tilde{\varphi}*\varphi)=\int T(a-b)\overline{\varphi}(a)\varphi(b)dadb\geq 0\qquad(\text{A}\cdot 4)$$

for any $\varphi\in\mathfrak{S}$, T is said to be positive definite. T is positive definite if and only if the Fourier transform of T is a slowly increasing positive measure. If $\varphi_1(a)$, $\varphi_2(a)\in\mathfrak{S}$, and if

$$\varphi_i(a)=\int\varphi_i(p)\ \exp\ (ipa)dp,\ i=1,2,$$

$$T(a)=\int T(p)\ \exp\ (ipa)dp,$$

then

$$\left.\begin{aligned}\check{T}(\tilde{\varphi}_1*\varphi_2)&=\int T(a-b)\bar{\varphi}_1(a)\varphi_2(b)dadb\\&=(2\pi)^{2N}\int T(p)\bar{\varphi}_1(p)\varphi_2(p)dp.\end{aligned}\right\}\quad(A\cdot5)$$

If $T\in\mathfrak{S}'$ is a given positve definite distribution, then (A·4) defines an inner product in $\mathfrak{S}$, and $\mathfrak{S}$ becomes a pre-Hilbert space. The completion $\mathfrak{S}_{\mathfrak{H}}$ of $\mathfrak{S}$ is a Hilbert space. (A·5) shows that, if the measure $T(p)$ is absolutely continuous, $\mathfrak{S}_{\mathfrak{H}}$ is separable. From (A·3), it is easy to show that the Fourier transforms of $\Delta^+(\boldsymbol{x},0)$, $-i\Delta^-(\boldsymbol{x},0)$, and $\delta(\boldsymbol{x})$ give absolutely continuous positive measures, that is, they are positive definite, and the corresponding $\mathfrak{S}_{\mathfrak{H}}$ are separable.

References

1) K. O. Friedrichs, *Mathematical Aspects of the Quantum Theory of Fields* (Interscience, New York, 1953).
2) H. Wakita, Prog. Theor. Phys. **20** (1958), 35; **21** (1959), 299.
3) W. Heisenberg and W. Pauli, Zeits. f. Physik **56** (1929), 1; **59** (1930), 168.
4) H. Wakita and T. Kimura, to be published.
5) J. von Neumann, Compositio Math. **6** (1939), 1.
6) L. Schwartz, *Théorie des Distributions* I, II (Hermann, Paris, 1950-51).

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Symmetry in Sakata's Model and Weak Interactions. I

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On the basis of the symmetry theory of Sakata's model we study some properties concerning various weak interactions. For baryon-meson systems, a scheme is proposed in which a decay can be described by the transitions between various configurations. In this scheme, the conventional selection rule concerning isospin can be understood quite naturally. It is also investigated how the symmetry property possessed by the basic particles is reflected in the β-decay of hyperons and in the K_{e3}-decay.

§ 1. Introduction

Through the recent studies both theoretical and experimental, we have accumulated a considerable amount of knowledge about the weak interactions. Now we consider that the main characteristics of the weak interactions can be summarized in the following three facts.

(1) Parity violation:[1] It is found in both the leptonic and the baryon-meson systems.

(2) Universality:[2),3)] For all the weak decay processes the effective coupling constants are almost of the same order of magnitude, i.e. $\sim 10^{-7}$ in the unit $\hbar = c = r_0 (= 10^{-13} \text{cm}) = 1$.

(3) Charge exchange[3] and V-A type:[4] The interactions are of charge exchange type in all the leptonic decay processes. From the universality of the weak interactions we expect this type also for the baryon-meson systems, although it has not necessarily been established. The V-A type interactions are confirmed or shown to be favourable in all the leptonic decay processes except K_{e3}, where some doubt is still left unsettled. There is no definite evidence against the V-A type in the decay processes in the baryon-meson systems.

These characteristic features are possessed in common by the phenomena of a fairly wide range. Yet one may have some objections to the arrangement mentioned above, particularly to the third. In fact, we know some phenomena which, at first sight, do not show the characteristics of the weak interactions. Of those phenomena we give three in the following.

(a) The so-called $|\Delta I| = 1/2$ rule,[5] which is fairly successful in the baryon-meson systems.

(b) The problem of the electron energy spectrum in the K_{e3}-decay,[6),7),8)] and of

the β-decay of hyperons,[9] although further experimental data are necessary.
(c) Asymmetry due to the parity violation in hyperon decays and the problem of branching ratios of hyperons and K-mesons into separate decay channels[10),11)] (excluding those due to the $|\Delta I|=1/2$ rule).

It seems of much interest whether these mean a deviation from the common features of the weak interactions or they are due to some other fundamental origin. On this question a new light may be thrown by the symmetry theory of Sakata's model,[12] which we are going to outline in the next paragraph.

In this theory, proton p, neutron n, and Λ-particle Λ are the basic particles which compose other baryons and mesons in the sense of Fermi and Yang.[13] These three particles have similar properties (spin, baryon number, strong interaction), and cannot be distinguished in the approximation that the mass difference is neglected and that the electromagnetic interactions are switched off. Therefore we may expect a certain symmetry[14),15)] to exist among the basic particles and to be realized in mutual interactions among baryons and mesons. This view has recently been formulated mathematically by Ikeda, Ogawa and Ohnuki[16] on the basis of the unitary group $U(3)$ of degree three. We shall hereafter call this the theory of full symmetry for the sake of simplicity. There the composite systems, classified into the irreducible representations of $U(3)$, have a good correspondence to the resonance states established in scattering experiments. We can see this from the mass formula which has been found by Sawada and Yonezawa[17] on the analogy of the case of atomic nuclei. Under this circumstance it seems that the theory of full symmetry involves very important suggestions as to the possible direction of the theory.

Now, in this paper, we shall make a concrete inquiry into the problems (a) and (b), considering them as reflections of the full symmetry possessed by the strong interactions that participate in the reactions. We expect that the problem (c) will be settled to a large extent when the dynamics of the strong interactions is made clear. In § **2** we shall discuss the weak interactions from the viewpoint of the full symmetry and propose a non-leptonic decay scheme, which will be analysed in § **3**. Because of our poor knowledge about higher configurations, we are obliged to limit the K-meson decays to those into two pions. In § **4** we shall examine the isospin selection rule on the basis of the above results. It is a remarkable result that from the only one assumption $|\Delta S|=1$ we can quite naturally deduce the results equivalent to the $|\Delta I|=1/2$ rule. In § **5** we shall investigate the problem of $K_{e3}\to\pi^0+e+\nu$ and the β-decay of hyperons. The concluding section will be devoted to the discussions on the direction of our future studies.

§ 2. Full symmetry and weak interactions

In the theory of full symmetry,[16] we assume a certain symmetry (full symmetry) to exist among p, n and Λ, which are the basic particles in Sakata's model.

This symmetry is mathematically expressed as the invariance of the theory for the unitary group $U(3)$ of degree three. On this assumption a state is associated with the six quantum numbers $(n_B, s_0, i_0; S, I, I_3)$. Here n_B, S, I and I_3 are familiar in the conventional theory, i.e. baryon number, strangeness, isospin and its third component respectively. s_0 and i_0, characteristic of the full symmetry theory, define together with n_B the irreducible representation of $U(3)$ to which the state under consideration belongs. s_0 is the maximum strangeness value for the states belonging to the representation, and i_0 is the maximum isospin value for those states with $S=s_0$. They cannot be ignored in order that the characteristic features of the full symmetry theory may be fully reflected.

There exist many composite states which have the same quantum numbers $(n_B, s_0, i_0; S, I, I_3)$. Having the same transformation character for $U(3)$, those states cannot be distinguished from the standpoint of the representation of $U(3)$. We shall interpret them as the various states of the same particle. Such an interpretation is closely connected with the concept of physical particles in the ordinary field theory. In fact, a physical particle is usually considered, from the viewpoint of perturbation, as a superposition of the bare particle states and the various clothed particle states. In accordance with such a picture in current physics, it is very natural in the present case to consider that one of the states with the same quantum numbers (e.g. the lowest configuration) corresponds to a bare particle and the others to the particle surrounded by various clouds. Thus we are led to the following important conclusion: In Sakata's model, a "physical particle" is represented by the totality of the composite states which have the same quantum numbers $(n_B, s_0, i_0; S, I, I_3)$.

Now it has been shown by the previous analyses[16),17)] that the concept of symmetry among the basic particles p, n and Λ is rather successful as regards the strong interactions. Here, we should like to note that, in the non-leptonic decays of hyperons and K-mesons, too, all participating particles are the strongly interacting particles only. Therefore, from the viewpoint of the full symmetry, the initial and final states of such decays should be classified according to the representations of $U(3)$ (i.e. in terms of n_B, s_0 and i_0), and the decays themselves may be regarded as the transformations between the representative vectors. We may expect that the symmetry existing among the basic particles is reflected fairly well in decay processes as well as in the phenomena induced by the strong interactions. Here we hit upon the idea that the weak decays conserve the full symmetry among the basic particles; to be more precise, they satisfy the selection rule

$$\Delta n_B=\Delta s_0=\Delta i_0=0. \qquad \text{(A)}$$

In this case a decay reaction can be attributed to a change of a vector in the same irreducible representation space of $U(3)$ (Fig. 1). It will be made clear that this assumption takes the place of the $|\Delta I|=1/2$ rule, which is known to be fairly

valid in the weak decay processes. One should notice that the new selection rule is a conservation law, which we believe is a salient feature of our theory. In the usual theory, the weak interactions are characterized by a non-conservation law such as $|\Delta I| \neq 0$ or $|\Delta S| \neq 0$, and a decay process gives rise to a transition between states which belong to the different (inequivalent) representation of isospin (Fig. 2).

Let us next study the form of the weak interactions that induce the non-leptonic decays of hyperons and K-mesons. As was previously made clear,[16] the mutual transitions between the basic particles are caused by the infinitesimal operators of the group $U(3)$. We presume that this fact will be also valid for higher configurations and that the changes of the representative vectors for decays will be expressed by the operators of the group $U(3)$. We remember, in this connection, the infinitesimal operators of the group, which are I_+, I_-, I_3, N_B, Q, S and N_{ij} $(ij=13, 23, 31, 32)$. The first three are the isospin operators, and the next three are baryon number, charge and strangeness respectively. I_3, N_B, Q and S are not linearly independent of each other but subject to the well-known relation $Q=I_3+\frac{1}{2}(N_B+S)$. The operators I_+, I_- and N_{ij} change the eigenvalues of S, Q and I_3 as in Table I, where we see that the strangeness undergoes a change only through N_{ij}. We know from experiments that the weak decay is subject to the rule $|\Delta S|=1$ and $|\Delta Q|=0$, so the decay interaction (which we hereafter denote by $\boldsymbol{O}$) must be expressed as an operator which satisfies this condition. N_{23} and N_{32} obviously fulfill such a requirement, while N_{13} and N_{31} do not. In order to make

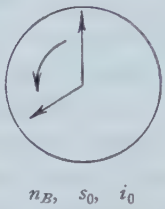

Fig. 1.

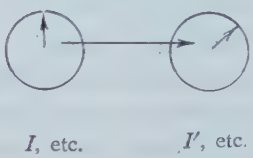

Fig. 2.

Table I. Changes of eigenvalues

	Δn_B	ΔS	ΔQ	ΔI_3	
I_+	0	0	1	1	$n\to p$, $\bar{p}\to-\bar{n}$
I_-	0	0	−1	−1	$p\to n$, $\bar{n}\to-\bar{p}$
N_{13}	0	1	1	$\frac{1}{2}$	$\Lambda\to p$, $\bar{p}\to-\bar{\Lambda}$
N_{23}	0	1	0	$-\frac{1}{2}$	$\Lambda\to n$, $\bar{n}\to-\bar{\Lambda}$
N_{31}	0	−1	−1	$-\frac{1}{2}$	$p\to\Lambda$, $\bar{\Lambda}\to-\bar{p}$
N_{32}	0	−1	0	$\frac{1}{2}$	$n\to\Lambda$, $\bar{\Lambda}\to-\bar{n}$

a proper operator of N_{13} or N_{31}, we must multiply them by I_- or I_+ respectively. Thus we have the following possibilities for $\boldsymbol{O}$:

For $\Delta S=+1$,

i) N_{23}, (2·1)

ii) $N_{13}I_-$ and I_-N_{13}. (2·2)

For $\Delta S=-1$,

$$\text{iii)} \quad N_{32}; \tag{2·3}$$

$$\text{iv)} \quad N_{31}I_+ \text{ and } I_+N_{31}. \tag{2·4}$$

This exhausts all the cases in which $\boldsymbol{O}$ is linear with respect to N_{ij}. Of course, it is equally possible to consider operators which are of higher degree in N_{ij}. They will, however, be reducible to the combinations of i) to iv) and operators with $\Delta S=\Delta Q=0$, hence we may consider that the operators i) to iv) play a fundamental role as the primary interaction in the weak decay processes in question.

In concluding this section we shall summarize our view on the weak interactions which has been explained above: In the theory of full symmetry, a physical particle consists of all the states with the same quantum numbers $(n_B, s_0, i_0; S, I, I_3)$. In the non-leptonic decays of the particle, these states play the role of the initial states and undergo the transitions through the primary interactions (2·1) to (2·4). In this case the full symmetry is conserved, that is, $\Delta n_B=\Delta s_0=\Delta i_0=0$.

In the next section, we shall analyse the decay processes from this point of view.

§ 3. Analysis of decay process

In order to carry our scheme forward, we must first assign the quantum numbers to each particle. No objection will be made to adopting the conventional values for the familiar quantum numbers n_B, S, I and I_3. As regards the new ones s_0 and i_0, a clue is provided by the aid of the empirical mass formula[17] which has been found recently. This formula predicts energy levels of various configurations for the baryon-meson systems and shows that there is a good correspondence between the theoretical levels and the experimental data on the mass of particles and on the resonance levels in scatterings. Taking this result into account, we shall assign the quantum numbers to each particle as in Tables II and III together with the corresponding configurations in the last column. There the symbols are the same as those in a previous paper:[17] $F_j^i(S, I)$ $(B_j^i(S, I))$ stands for a fermion (boson) in a state with strangeness S and isospin I; the index j is the number of constituent basic particles plus antiparticles, and i designates the class to which the (compound) particle belongs. The concrete expression for each configuration is given in the previous paper.[17]

As is seen in Table II, Λ has two configurations $F_3^2(-1, 0)$ and $F_3^4(-1, 0)$ in the three-body system. They are assigned the same quantum numbers $(n_B, s_0, i_0; S, I, I_3)$, but differ in the symmetry character of the constituent particles. These two states may or may not have the same spin, parity, etc., but we cannot say so much at present because we know nothing about the dynamics of

Table II. Quantum numbers and configurations for baryons

		n_B	s_0	i_0	S	I	I_3	$F_j^i(S, I)$
N	p	1	0	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$F_1(0, \frac{1}{2})$; $F_3^2(0, \frac{1}{2})$; $F_5^3(0, \frac{1}{2})$, $F_5^9(0, \frac{1}{2})$, $F_5^{13}(0, \frac{1}{2})$
	n	1	0	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$F_3^4(0, \frac{1}{2})$; $F_5^{16}(0, \frac{1}{2})$, $F_5^{19}(0, \frac{1}{2})$, $F_5^{21}(0, \frac{1}{2})$
Λ	Λ	1	0	$\frac{1}{2}$	-1	0	0	$F_1(-1, 0)$; $F_3^2(-1, 0)$; $F_5^3(-1, 0)$, $F_5^9(-1, 0)$, $F_5^{13}(-1, 0)$ $F_3^4(-1, 0)$; $F_5^{16}(-1, 0)$, $F_5^{19}(-1, 0)$, $F_5^{21}(-1, 0)$
Σ_1	Σ_1^+	1	1	1	-1	1	1	$F_3^1(-1, 1)$; $F_5^2(-1, 1)$, $F_5^5(-1, 1)$, $F_5^7(-1, 1)$ $F_5^{11}(-1, 1)$, $F_5^{14}(-1, 1)$, $F_5^{17}(-1, 1)$
	Σ_1^0	1	1	1	-1	1	0	
	Σ_1^-	1	1	1	-1	1	-1	
Σ_2	Σ_2^+	1	1	0	-1	1	1	$F_3^3(-1, 1)$; $F_5^8(-1, 1)$, $F_5^{12}(-1, 1)$, $F_5^{15}(-1, 1)$ $F_5^{18}(-1, 1)$, $F_5^{20}(-1, 1)$
	Σ_2^0	1	1	0	-1	1	0	
	Σ_2^-	1	1	0	-1	1	-1	
Ξ	Ξ^0	1	1	1	-2	$\frac{1}{2}$	$\frac{1}{2}$	$F_3^1(-2, \frac{1}{2})$; $F_5^2(-2, \frac{1}{2})$, $F_5^5(-2, \frac{1}{2})$, $F_5^7(-2, \frac{1}{2})$ $F_5^{11}(-2, \frac{1}{2})$, $F_5^{14}(-2, \frac{1}{2})$, $F_5^{17}(-2, \frac{1}{2})$
	Ξ^-	1	1	1	-2	$\frac{1}{2}$	$-\frac{1}{2}$	

Table III. Quantum numbers and configurations for bosons

		n_B	s_0	i_0	S	I	I_3	$B_j^i(S, I)$
K	K^+	0	1	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$B_2^1(1, \frac{1}{2})$; $B_4^2(1, \frac{1}{2})$, $B_4^5(1, \frac{1}{2})$, $B_4^7(1, \frac{1}{2})$, $B_4^8(1, \frac{1}{2})$
	K^0	0	1	$\frac{1}{2}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	
π	π^+	0	1	$\frac{1}{2}$	0	1	1	$B_2^1(0, 1)$; $B_4^2(0, 1)$, $B_4^5(0, 1)$, $B_4^7(0, 1)$, $B_4^8(0, 1)$
	π^0	0	1	$\frac{1}{2}$	0	1	0	
	π^-	0	1	$\frac{1}{2}$	0	1	-1	
$\pi^{0\prime}$	$\pi^{0\prime}$	0	1	$\frac{1}{2}$	0	0	0	$B_2^1(0, 0)$; $B_4^2(0, 0)$, $B_4^5(0, 0)$, $B_4^7(0, 0)$, $B_4^8(0, 0)$
$\overline{K}$	$\overline{K}^0$	0	1	$\frac{1}{2}$	-1	$\frac{1}{2}$	$\frac{1}{2}$	$B_2^1(-1, \frac{1}{2})$; $B_4^2(-1, \frac{1}{2})$, $B_4^5(-1, \frac{1}{2})$, $B_4^7(-1, \frac{1}{2})$, $B_4^8(-1, \frac{1}{2})$
	K^-	0	1	$\frac{1}{2}$	-1	$\frac{1}{2}$	$-\frac{1}{2}$	
$\pi^{0\prime\prime}$	$\pi^{0\prime\prime}$	0	0	0	0	0	0	$B_2^2(0, 0)$; $B_4^3(0, 0)$, $B_4^9(0, 0)$

the constituent particles. For this reason, we assume here that both configurations are equally possible for the initial state of the decays. A similar case occurs also in the four-body configurations of K-mesons or other higher configurations.

We have not given the unique quantum numbers s_0 and i_0 to Σ-particle on purpose. Correspondingly, there are two different (inequivalent) configurations in the three-body system, i.e. $F_3^3(-1, 1)$ and $F_3^1(-1, 1)$. We have once presumed[16] that the former is Σ and the latter is of spin 3/2, because $F_3^1(-1, 1)$ belongs to the same class as the well-known $(3/2, 3/2)$ resonance state in π-N scattering. But there is some room to doubt in this conclusion, as pointed out by Sawada and Yonezawa[17] on the basis of their mass formula. Further, the formula tells us that both configurations have the equal mass. Thus it seems impossible

at present to decide between the two possibilities, hence we shall examine them separately. It is to be noted that, within the framework of the full symmetry theory, Σ can take only one of the states $F_3^3(-1, 1)$ and $F_3^1(-1, 1)$ but not both, since they have different quantum numbers s_0 and i_0. This is contrary to the case of Λ, which may have the both states $F_3^2(-1, 0)$ and $F_3^4(-1, 0)$.

It follows from the above that there are many configurations capable of initial states for decays of a particle. But the higher configurations may correspond to the particle surrounded by complicated clouds and will possibly have little effect on decays. Accordingly, in this section, we shall discuss the non-leptonic decays of hyperons through three-body systems and the $K\to 2\pi$ decays through four-body systems. In order to treat the $K\to 3\pi$ decays we have to consider the six-body configurations. This problem will be discussed in the near future.

When the decay interactions (2·1) to (2·4) are switched on, the particle makes transitions from the above initial states to some final states. Tables IV and V show the final states through each interaction, where we have given only the results with $\Delta S=+1$ for hyperons and those with $\Delta S=-1$ for K-mesons. As is well known, in the case of a neutral K-meson decay, we should deal with the eigenstates of CP (i.e. K_1^0 and K_2^0), and not with those of S (i.e. K^0 and $\bar{K}^0$). On account of CP invariance the $K_2^0\to 2\pi$ decay is forbidden and only $K_1^0\to 2\pi$ is possible. Since

$$|K_1^0>=(|K^0>+|\bar{K}^0>)/\sqrt{2},$$

the $K_1^0\to 2\pi$ decay is a mixture of the $\Delta S=-1$ transition from K^0 and the $\Delta S=1$ transition from $\bar{K}^0$. As the decay interaction is to be Hermitian, the final state must be of the form,

$$(\boldsymbol{O}|K^0>+\boldsymbol{O}^*|\bar{K}^0>)/\sqrt{2},$$

where $\boldsymbol{O}$ is the weak interaction with $\Delta S=-1$ and $\boldsymbol{O}^*$ is its Hermite conjugate with $\Delta S=1$.* Therefore, if $\boldsymbol{O}$ is N_{32} or I_+N_{31} or $N_{31}I_+$, the corresponding $\boldsymbol{O}^*$ is N_{23} or $N_{13}I_-$ or I_-N_{13}, respectively. $\boldsymbol{O}|K^0>$ and $\boldsymbol{O}^*|\bar{K}^0>$ are given respectively in Tables V and VI.

We are now in a position to express the final states in terms of the decay products. For this purpose we must expand a state consisting of several particles in the linear combination of the eigenstates of the quantum numbers (n_B, s_0, i_0; S, I, I_3), but the coefficients** thereof are very complicated in general. Fortunately,

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* The relative phases of the states are chosen so that $CP|K^0>=|\bar{K}^0>$ and $CP|\bar{K}^0>=|K^0>$. It follows from the configurations embodied in the previous paper that under the operation CP

$$B_4^2(K^0)\leftrightarrow B_4^2(\bar{K}^0),\quad B_4^5(K^0)\leftrightarrow B_4^7(\bar{K}^0),$$
$$B_4^5(\bar{K}^0)\leftrightarrow B_4^7(K^0),\quad B_4^8(K^0)\leftrightarrow B_4^8(\bar{K}^0).$$

Thus the configurations of K_1^0 should be such combinations of K^0 and $\bar{K}^0$ that are compatible with these relations.

** This corresponds to the Clebsch-Gordan coefficients for the case of isospin or angular momentum.

in simple cases, we can calculate them directly by means of the concrete configurations in the previous paper.[17] In the Appendix we show the results which will be of use in what follows.

The decays of a particle take place through various channels, which may be specified by the indices i and j, and the values of S and I and, if necessary, the value of the angular momentum. Each decay reaction can be described in terms of the decay amplitudes $a_j^i(S, I)$ corresponding to the possible channels. We shall next analyse each reaction on the basis of Tables IV, V and VI and the Appendix.

1) The decay $\Lambda \to \pi + N$

We at once see the following:

i) The interaction $N_{13}I_-$ does not induce the decay of Λ.

ii) The interactions N_{23} and I_-N_{13} bring the same result. If A_- and A_0 denote the decay amplitude to $(p\pi^-)$ and $(n\pi^0)$ respectively, we have

$$A_- = (\sqrt{2}/4)a_3^2(0, \tfrac{1}{2}) - \tfrac{1}{2}\alpha a_3^4(0, \tfrac{1}{2}),$$

$$A_0 = -(1/4)a_3^2(0, \tfrac{1}{2}) + (\sqrt{2}/4)\alpha a_3^4(0, \tfrac{1}{2}),$$

Table IV. $\Delta S=1$ transitions for hyperons

initial state		final states		
particle	$F_j^i(S; I, I_3)$	N_{23}	I_-N_{13}	$N_{13}I_-$
Λ	$F_3^2(-1; 0, 0)$	$F_3^2\left(0; \frac{1}{2}, -\frac{1}{2}\right)$	$F_3^2\left(0; \frac{1}{2}, -\frac{1}{2}\right)$	0
	$F_3^4(-1; 0, 0)$	$-F_3^4\left(0; \frac{1}{2}, -\frac{1}{2}\right)$	$-F_3^4\left(0; \frac{1}{2}, -\frac{1}{2}\right)$	0
Σ_1^+	$F_3^1(-1; 1, 1)$	$\frac{2}{\sqrt{3}}F_3^1\left(0; \frac{1}{2}, \frac{1}{2}\right) + \frac{2}{\sqrt{6}}F_3^1\left(0; \frac{3}{2}, \frac{1}{2}\right)$	$\sqrt{6}F_3^1\left(0; \frac{3}{2}, \frac{1}{2}\right)$	$-\frac{2}{\sqrt{3}}F_3^1\left(0; \frac{1}{2}, \frac{1}{2}\right) + \frac{4}{\sqrt{6}}F_3^1\left(0; \frac{3}{2}, \frac{1}{2}\right)$
Σ_1^0	$F_3^1(-1; 1, 0)$	$\frac{\sqrt{6}}{3}F_3^1\left(0; \frac{1}{2}, -\frac{1}{2}\right) + \frac{2}{\sqrt{3}}F_3^1\left(0; \frac{3}{2}, -\frac{1}{2}\right)$	$-\frac{\sqrt{6}}{3}F_3^1\left(0; \frac{1}{2}, -\frac{1}{2}\right) + \frac{4}{\sqrt{3}}F_3^1\left(0; \frac{3}{2}, -\frac{1}{2}\right)$	$-\frac{2\sqrt{6}}{3}F_3^1\left(0; \frac{1}{2}, -\frac{1}{2}\right) + \frac{2}{\sqrt{3}}F_3^1\left(0; \frac{3}{2}, -\frac{1}{2}\right)$
Σ_1^-	$F_3^1(-1; 1, -1)$	$\sqrt{2}F_3^1\left(0; \frac{3}{2}, -\frac{3}{2}\right)$	$\sqrt{2}F_3^1\left(0; \frac{3}{2}, -\frac{3}{2}\right)$	0
Σ_2^+	$F_3^3(-1; 1, 1)$	$\sqrt{2}F_3^3\left(0; \frac{1}{2}, \frac{1}{2}\right)$	0	$-\sqrt{2}F_3^3\left(0; \frac{1}{2}, \frac{1}{2}\right)$
Σ_2^0	$F_3^3(-1; 1, 0)$	$F_3^3\left(0; \frac{1}{2}, -\frac{1}{2}\right)$	$-F_3^3\left(0; \frac{1}{2}, -\frac{1}{2}\right)$	$-2F_3^3\left(0; \frac{1}{2}, -\frac{1}{2}\right)$
Σ_2^-	$F_3^3(-1; 1, -1)$	0	0	0
Ξ^0	$F_3^1(-2; \frac{1}{2}, \frac{1}{2})$	$\sqrt{2}F_3^1(-1; 0, 0) + F_3^1(-1; 1, 0)$	$2F_3^1(-1; 1, 0)$	$-\sqrt{2}F_3^1(-1; 0, 0) + F_3^1(-1; 1, 0)$
Ξ^-	$F_3^1(-2; \frac{1}{2}, -\frac{1}{2})$	$\sqrt{2}F_3^1(-1; 1, -1)$	$\sqrt{2}F_3^1(-1; 1, -1)$	0

Table V. $\Delta S=-1$ transitions for K-mesons

initial state $B_j^i(S; I, I_3)$	final states		
K^+	N_{32}	$I_+ N_{31}$	$N_{31} I_+$
$B_4^2(1; \frac{1}{2}, \frac{1}{2})$	$B_4^2(0; 1, 1)$	$-B_4^2(0; 1, 1)$	0
$B_4^5(1; \frac{1}{2}, \frac{1}{2})$	$B_4^5(0; 1, 1)$	$-B_4^5(0; 1, 1)$	0
$B_4^7(1; \frac{1}{2}, \frac{1}{2})$	$B_4^7(0; 1, 1)$	$-B_4^7(0; 1, 1)$	0
$B_4^8(1; \frac{1}{2}, \frac{1}{2})$	$B_4^8(0; 1, 1)$	$-B_4^8(0; 1, 1)$	0
K^0			
$B_4^2(1; \frac{1}{2}, -\frac{1}{2})$	$\frac{\sqrt{6}}{2} B_4^2(0; 0, 0) + \frac{\sqrt{2}}{2} B_4^2(0; 1, 0)$	$-\sqrt{2}\, B_4^2(0; 1, 0)$	$\frac{\sqrt{6}}{2} B_4^2(0; 0, 0) - \frac{\sqrt{2}}{2} B_4^2(0; 1, 0)$
$B_4^5(1; \frac{1}{2}, -\frac{1}{2})$	$\frac{\sqrt{6}}{2} B_4^5(0; 0, 0) + \frac{\sqrt{2}}{2} B_4^5(0; 1, 0)$	$-\sqrt{2}\, B_4^5(0; 1, 0)$	$\frac{\sqrt{6}}{2} B_4^5(0; 0, 0) - \frac{\sqrt{2}}{2} B_4^5(0; 1, 0)$
$B_4^7(1; \frac{1}{2}, -\frac{1}{2})$	$\frac{\sqrt{6}}{2} B_4^7(0; 0, 0) + \frac{\sqrt{2}}{2} B_4^7(0; 1, 0)$	$-\sqrt{2}\, B_4^7(0; 1, 0)$	$\frac{\sqrt{6}}{2} B_4^7(0; 0, 0) - \frac{\sqrt{2}}{2} B_4^7(0; 1, 0)$
$B_4^8(1; \frac{1}{2}, -\frac{1}{2})$	$\frac{\sqrt{6}}{2} B_4^8(0; 0, 0) + \frac{\sqrt{2}}{2} B_4^8(0; 1, 0)$	$-\sqrt{2}\, B_4^8(0; 1, 0)$	$\frac{\sqrt{6}}{2} B_4^8(0; 0, 0) - \frac{\sqrt{2}}{2} B_4^8(0; 1, 0)$

Table VI. $\Delta S=1$ transitions for $\bar{K}^0$-mesons

initial state $B_j^i(S; I, I_3)$	final states		
	N_{23}	$I_- N_{13}$	$N_{13} I_-$
$B_4^2(-1; \frac{1}{2}, \frac{1}{2})$	$\frac{\sqrt{6}}{2} B_4^2(0; 0, 0) + \frac{\sqrt{2}}{2} B_4^2(0; 1, 0)$	$\sqrt{2}\, B_4^2(0; 1, 0)$	$-\frac{\sqrt{6}}{2} B_4^2(0; 0, 0) + \frac{\sqrt{2}}{2} B_4^2(0; 1, 0)$
$B_4^5(-1; \frac{1}{2}, \frac{1}{2})$	$\frac{\sqrt{6}}{2} B_4^5(0; 0, 0) + \frac{\sqrt{2}}{2} B_4^5(0; 1, 0)$	$\sqrt{2}\, B_4^5(0; 1, 0)$	$-\frac{\sqrt{6}}{2} B_4^5(0; 0, 0) + \frac{\sqrt{2}}{2} B_4^5(0; 1, 0)$
$B_4^7(-1; \frac{1}{2}, \frac{1}{2})$	$\frac{\sqrt{6}}{2} B_4^7(0; 0, 0) + \frac{\sqrt{2}}{2} B_4^7(0; 1, 0)$	$\sqrt{2}\, B_4^7(0; 1, 0)$	$-\frac{\sqrt{6}}{2} B_4^7(0; 0, 0) + \frac{\sqrt{2}}{2} B_4^7(0; 1, 0)$
$B_4^8(-1; \frac{1}{2}, \frac{1}{2})$	$\frac{\sqrt{6}}{2} B_4^8(0; 0, 0) + \frac{\sqrt{2}}{2} B_4^8(0; 1, 0)$	$\sqrt{2}\, B_4^8(0; 1, 0)$	$-\frac{\sqrt{6}}{2} B_4^8(0; 0, 0) + \frac{\sqrt{2}}{2} B_4^8(0; 1, 0)$

where α is the complex amplitude of F_3^4 in Λ relative to F_3^2, hence we have the following relation irrespective of the value of α:

$$A_-=-\sqrt{2}\,A_0.$$

This coincides with the consequence of the $|\Delta I|=1/2$ rule in the ordinary theory, and the fraction f_- of charged decays to all pionic decays is 2/3.*

2) The decay $\Sigma\to\pi+N$

In this case, the two configurations $F_3^1(-1, 1)$ and $F_3^3(-1, 1)$ do not correspond to the same particle, because they are associated with the different values of s_0 and i_0. We tentatively call them Σ_1 and Σ_2 respectively. They are of the equal mass, but Σ_2^- is forbidden to decay through the interactions (2·1) and (2·2). Thus, if we try to interpret the existing experiments on Σ in terms of either Σ_1 or Σ_2 we must identify Σ_1 with Σ. Then, from Table IV and the Appendix, we can write the decay amplitudes as follows:

	N_{23}	I_-N_{13}	$N_{13}I_-$
$A_0(\Sigma^+\to p+\pi^0)$	$-\frac{1}{6}a_3^1\left(0,\frac{1}{2}\right)+\frac{2}{3}a_3^1\left(0,\frac{3}{2}\right)$	$2a_3^1\left(0,\frac{3}{2}\right)$	$\frac{1}{6}a_3^1\left(0,\frac{1}{2}\right)+\frac{4}{3}a_3^1\left(0,\frac{3}{2}\right)$
$A_+(\Sigma^+\to n+\pi^+)$	$\frac{\sqrt{2}}{6}a_3^1\left(0,\frac{1}{2}\right)+\frac{\sqrt{2}}{3}a_3^1\left(0,\frac{3}{2}\right)$	$\sqrt{2}\,a_3^1\left(0,\frac{3}{2}\right)$	$-\frac{\sqrt{2}}{6}a_3^1\left(0,\frac{1}{2}\right)+\frac{2\sqrt{2}}{3}a_3^1\left(0,\frac{3}{2}\right)$
$A_-(\Sigma^-\to n+\pi^-)$	$\sqrt{2}\,a_3^1\left(0,\frac{3}{2}\right)$	$\sqrt{2}\,a_3^1\left(0,\frac{3}{2}\right)$	0

For the interaction N_{23} we have the identity

$$\sqrt{2}\,A_0+A_+=A_-,$$

as in the ordinary theory based on the $|\Delta I|=1/2$ rule. This is consistent with $|A_0|\approx|A_+|\approx|A_-|$ and with the behaviour of the asymmetry factor in the Σ decay.[10] The interaction I_-N_{13} gives $|A_0|^2:|A_+|^2:|A_-|^2=2:1:1$ and $N_{13}I_-$ forbids the Σ^- decay contrary to experiments. But their coexistence is not excluded because of the identity $N_{13}I_--I_-N_{13}=N_{23}$.

3) The decay $\Xi\to\Lambda+\pi$

As far as we consider the configurations of at most three bodies, there exists only one state $F_3^1(-2,\frac{1}{2})$ that can be identified with Ξ. Since the decay products have the $I=1$ state alone, we obtain

$$\frac{\Gamma(\Xi^0\to\Lambda+\pi^0)}{\Gamma(\Xi^-\to\Lambda+\pi^-)}=\begin{cases}1/2 & \text{for the interaction } N_{23},\\ 2 & \text{for the interaction } I_-N_{13},\end{cases}$$

and the Ξ^- decay is forbidden for the interaction $N_{13}I_-$.

* The decays other than $\Lambda\to\pi+N$ (e. g. $\pi^{0\prime}+n$, $\pi^{0\prime\prime}+n$, etc.) are energetically impossible, if we take $m_{\pi^{0\prime}}=615$ Mev, $m_{\pi^{0\prime\prime}}=377$ Mev.[17]

That is, the interaction N_{23} produces the same result as the ordinary $|\Delta I|=1/2$ rule, while I_-N_{13} does the reciprocal result. In this case our scheme is less restrictive than the $|\Delta I|=\frac{1}{2}$ rule.

4) The decay $K\to 2\pi$

The decays of K^+ are the $\Delta S=-1$ transitions induced by the interactions N_{32}, I_+N_{31} and $N_{31}I_+$. The K_1^0 decay is a mixture of the $\Delta S=-1$ transition from K^0 and the $\Delta S=1$ transition from $\bar{K}^0$. If the spin of the K-meson is zero, the system of two pions is in the S-state and its configuration should be symmetric, because the total wave function is symmetric for the system of two bosons. From the table in the Appendix it is easy to obtain the symmetrized configurations of two pions. These states are of $I=0$ or 2, which leads to the following conclusions.

i) The decay $K^+\to 2\pi$ is forbidden, as Table V shows that the final states are always of isospin one. (The $I=1$ final states may be $\pi^+ +\pi^{0\prime}$, $\pi^+ +\pi^{0\prime\prime}$, $K+\bar{K}$, etc. but they are all forbidden by the energy conservation.)

ii) Only the $I=0$ configurations contribute to the decay $K_1^0\to 2\pi$. We here give the decay amplitudes to $\pi^+ +\pi^-$ and to $2\pi_0$.

O	N_{32}	I_+N_{31}	$N_{31}I_+$
$K_1^0\to 2\pi^0$	$\sqrt{2}\left\{-\frac{\sqrt{5}}{10}a_4^2(0,0)+\frac{\beta}{2}a_4^8(0,0)\right\}$	$-\frac{\sqrt{2}}{2}\left\{-\frac{\sqrt{5}}{10}a_4^2(0,0)+\frac{\beta}{2}a_4^8(0,0)\right\}$	$\frac{\sqrt{2}}{2}\left\{-\frac{\sqrt{5}}{10}a_4^2(0,0)+\frac{\beta}{2}a_4^8(0,0)\right\}$
$K_1^0\to \pi^+ +\pi^-$	$-2\left\{-\frac{\sqrt{5}}{10}a_4^2(0,0)+\frac{\beta}{2}a_4^8(0,0)\right\}$	$-\frac{\sqrt{5}}{10}a_4^2(0,0)+\frac{\beta}{2}a_4^8(0,0)$	$\frac{\sqrt{5}}{10}a_4^2(0,0)-\frac{\beta}{2}a_4^8(0,0)$

Here β is the complex amplitude of B_4^8 in K_1^0 relative to B_4^2, hence the branching ratio of the K_1^0 decays is given by

$$\frac{\Gamma(K_1^0\to 2\pi^0)}{\Gamma(K_1^0\to \pi^+ +\pi^-)}=1/2,$$

which is independent of the decay interactions and of the relative probability amplitudes for the classes two and eight. This result is also the same as in the ordinary theory based on the $|\Delta I|=1/2$ rule.

It will be worthy of special mention that the configurations B_4^5 and B_4^7 make no contribution to the decay $K_1^0\to 2\pi$. In the lowest configuration B_2^1 for bosons, the charge conjugate of a state also belongs to B_2^1, whereas the case is different from B_4^5 and B_4^7. This behaviour in B_4^5 or B_4^7 is due to the difference between the symmetry character of the particles and that of the antiparticles.

Through the above analyses, we find that our selection rule (A) gives an

equivalent result to those obtained by the conventional $|\Delta I|=1/2$ rule. In the next section we shall give a general consideration concerning the isospin selection rule.

§ 4. Isospin selection rules

From the analysis in the preceding section, we see that our decay scheme is rather successful and that the $|\Delta I|=1/2$ rule in the conventional theory of weak interactions can be interpreted very naturally as the changes of states caused by the selection rule (A). Thus we may regard the $|\Delta I|=1/2$ rule as a reflection in the weak interactions of the full symmetry that exists among the basic particles. Some of the readers may raise an objection that our scheme is not so realistic and has little bearing upon those in the usual theory. In Sakata's model, for instance, we may understand the common features of weak interactions by Fig. 3. Our next task will be to study the relation between our interactions and Fig. 3. This problem has, in the opinion of the present writers, a close connection with the quantization of composite systems. We have at present no definite answer, but we here emphasize that our scheme is far more satisfactory than the conventional one (e.g. the spurion theory) in understanding weak decays.

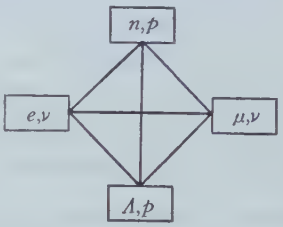

Fig. 3.

As is well known, the $|\Delta I|=1/2$ rule is consistent with experimental results on hyperons, but not necessarily with those on K-mesons. Some authors[10),18)] have tried to avoid this difficulty by introducing $\Delta I=3/2$ or $5/2$ transitions as a correction to $|\Delta I|=1/2$. If such transitions exist at all, how they should be treated in the framework of our theory? It may seem at a glance that the $|\Delta I|=3/2$ or $5/2$ transitions will be possible if we take account of other types of the interactions. But that is not the case, as far as K is assigned the quantum numbers as in Table III, which seems to be very reasonable from the investigations about the strong interactions. The reason is as follows: The final states belong to the same representation as the initial ones because of the conservation law $\Delta n_B=\Delta s_0=\Delta i_0=0$. In the representation to which K-meson belongs, the states with $S=0$ consists of those with $I=0$ and $I=1$. Therefore, under the $|\Delta S|=1$ transitions of K-meson, possible final states can be expressed by combinations of those with $I=0$ or 1. Note that this is true independently of the order of the configuration considered, particularly in the case of $K\to 3\pi$ decay. Thus, for the introduction of $|\Delta I|=3/2$ or $5/2$ transitions we must deal with the interaction which changes K-meson into the states of other (inequivalent) representations involving states with $I=2$ or 3. Such interactions cannot be expressed in terms of the infinitesimal operators of $U(3)$. Thus a new approach is required to introduce the $|\Delta I|=3/2$ or $5/2$ transitions in the scheme of our theory. In this sense there seems to be an intrinsic difference between the transitions with $|\Delta I|=1/2$ and those with $|\Delta I|=3/2$ or $5/2$.

From these circumstances, we are strongly inclined to consider that the $|\Delta I|=1/2$ rule is strictly valid, but then the question may arise by what mechanism the $K^+\to\pi^++\pi^0$ decay occurs. Good and Holladay[19] have recently suggested an interesting possibility for a natural resolution of this problem. Together with a strict $|\Delta I|=1/2$ weak interaction, the electromagnetic effects which permit $|\Delta I|=1$ will allow processes with $|\Delta I|=3/2$. Of course, they are still of insufficient strength to provide a required amount of $|\Delta I|=3/2$. The point is that the existence of a strong interaction between two pions in the $I=2$, $J=0$ state would enhance the decay rate of $K^+\to\pi^++\pi^0$. Thus it may not be necessary to introduce a $|\Delta I|=3/2$ weak interaction explicitly to account for the $K^+\to 2\pi$ decay. We emphasize here that the existence of such a strong interaction between two pions has already been predicted in the full symmetry theory of Sakata's model.[17] Although this explanation seems to be attractive, we should not disregard another possibility. That is, $|\Delta I|=3/2$ or $5/2$ is also possible in the case where the symmetry among the basic particles is violated by $m_N\neq m_\Lambda$ or by some origin characteristic of the weak interactions. On account of the kinematical character of our theory at the present stage, we do not go into detail, but point out the essential difference between the $\Delta I|=1/2$ and $3/2, 5/2, \cdots$ interactions mentioned in the last paragraph.

§ 5. The K_{e3}-decay and the β-decay of hyperons

So far we have confined ourselves to the non-leptonic decays of hyperons and K-mesons and studied how the symmetry character of the basic particles are reflected in these decay processes. We may expect that the case is similar in their leptonic decays, too. Since the treatment of leptons in our scheme is not so definite,[20] we are obliged to make a different approach, which will be given in what follows.

As is well known, the V-A type interactions are successful in the leptonic decay processes of wide range. But the electron spectrum in $K_{e3}\to\pi^0+e+\nu$ deviates to a certain degree from the result of such interactions, although it is not so definite on account of poor experimental data. In order to avoid this difficulty in the framework of V-A type interactions, there has been propounded[21] the possibility that in the K_{e3} decays $K_{e3}\to\pi^0+e+\nu$ coexists with a decay into a pseudoscalar neutral boson $\pi^{0\prime}$ which is somewhat heavier than π^0:

$$K'_{e3}\to\pi^{0\prime}+e+\nu.$$

In the following we show that this possibility can be deduced from the symmetry theory of Sakata's model.

First of all, we pay attention to the similarity between the vector type weak interaction proposed by Gell-Mann[22] and the current type interaction of the electromagnetic field and charged particles. We expect from this the following expression for the vector type interactions of leptons and baryon-meson system:

$$f\cdot(\bar{\psi}_L\gamma_\mu\psi_\nu)\{\bar{\psi}_p\gamma_\mu\psi_n-i\sqrt{2}\,(\tilde{\pi}^+\partial_\mu\pi^0-\tilde{\pi}^0\partial_\mu\pi^-)+(\tilde{K}^+\partial_\mu K^0-\partial_\mu\tilde{K}^+\cdot K^0)+\cdots\}+H.C. \tag{5·1}$$

We notice that we obtain from this the relation of the decays $n\to p+e+\nu$ and $\pi^-\to\pi^0+e^-+\nu$.

Now, Λ is given special treatment in (5·1). On the other hand, we cannot find any difference between n and Λ in the approximation of equal mass for Λ and nucleon. Therefore we may expect that the interaction obtained from (5·1) by the interchange of n and Λ is also a possible one. Under this interchange each particle is subject to the following transformation:

$$p\to p,\quad n\to\Lambda,\quad \Lambda\to n,$$

$$\pi^+\equiv-p\bar{n}\to-p\bar{\Lambda}\equiv-K^+,\quad K^+\equiv p\bar{\Lambda}\to p\bar{n}\equiv-\pi^+,$$

$$\pi^0\equiv\frac{1}{\sqrt{2}}(p\bar{p}-n\bar{n})\to\frac{1}{\sqrt{2}}(p\bar{p}-\Lambda\bar{\Lambda})=\frac{1}{2}\left\{\frac{1}{\sqrt{2}}(p\bar{p}-n\bar{n})+\frac{\sqrt{3}}{\sqrt{6}}(p\bar{p}+n\bar{n}-2\Lambda\bar{\Lambda})\right\}$$

$$\equiv\frac{1}{2}(\pi^0-\sqrt{3}\,\pi^{0\prime}),$$

$$K^0\equiv n\bar{\Lambda}\to\Lambda\bar{n}\equiv-\bar{K}^0.$$

Substituting this into (5·1), we have as the result of the interchange of n and Λ,

$$f'\cdot(\bar{\psi}_L\gamma_\mu\psi_\nu)\{\bar{\psi}_p\gamma_\mu\psi_\Lambda+i\sqrt{2}\,(\tilde{K}^+\partial_\mu[\pi^0/2-\sqrt{3}\,\pi^{0\prime}/2]-[\pi^0/2-\sqrt{3}\,\pi^{0\prime}/2]\partial_\mu\tilde{K}^-)$$
$$+(\tilde{\pi}^+\partial_\mu\tilde{K}^0-\partial_\mu\tilde{\pi}^+\cdot\tilde{K}^0)+\cdots\}+H.C. \tag{5·2}$$

This expression makes us expect that, in the limit $m_\Lambda=m_N$, if there exists

$$K^+_{e_0}\to\pi^0+e^++\nu,$$

then the reaction

$$K^{+\prime}_{e3}\to\pi^{0\prime}+e^++\nu$$

may be anticipated with the probability three times as large as that of K_{e3}. Practically, π^0 and $\pi^{0\prime}$ are not of equal mass because of $m_\Lambda\neq m_N$, and the ratio of coupling constants will deviate from $\sqrt{3}$. The problem of K'_{e3} should ultimately be settled by experiments, but here we have been able to give a support to the possible existence of K'_{e3}.

We further remark the following: In the limit $m_\Lambda=m_N$, if we express the interaction between baryons and mesons in the form of the Yukawa type (the notation being that in ref. 16))

$$GT_\mu T^\nu\cdot T_\nu{}^\mu,$$

the ratio between the couplings of π^0 and of $\pi^{0\prime}$ reduces to $\sqrt{3}:1$. In other words, by the perturbation calculation, the probability ratio will be $3:1$ between processes in which π^0 participates and the corresponding processes for $\pi^{0\prime}$, hence $\pi^{0\prime}$

becomes less contributive. Thus we should like to stress that the contribution of $\pi^{0\prime}$ to K_{e3} is not necessarily small, even if it is small in the phenomena (e.g. nuclear force) in the region where the perturbation treatment is valid to some extent.

From (5·2) we can also obtain the relation between $K_{e3}^{+}\to\pi^{0}+e^{+}+\nu$ and $\Lambda\to p+e^{-}+\nu$. If we assume that the observed K_{e3} processes are all due to $K_{e3}\to\pi^{0}+e+\nu$, we know the decay probability of $\Lambda\to p+e^{-}+\nu$[23] from that of K_{e3}. The result is given by

$$\frac{\Gamma(\Lambda\to p+e^{-}+\nu)}{\Gamma_{exp}(\Lambda\to p+\pi^{-})}=\frac{1}{1500},$$

which is about four percent of the magnitude when compared with that anticipated from the β-decay of n. Of course, we cannot ignore the violation of the symmetry caused by $m_{\Lambda}\neq m_{N}$.

Finally, we make the following remark. If we obtain the experimental ratio of relative decay probabilities for

$$\Lambda\to p+e^{-}+\nu,\ K_{e}^{+}\to\pi^{0}+e^{+}+\nu,\ K_{e}^{+\prime}\to\pi^{0\prime}+e^{+}+\nu,\ K_{2e}^{0}\to\pi^{+}+e^{-}+\nu,$$

we shall be given a criterion of the validity of our symmetry in the weak interactions. In particular, the phase volume being equal for K_{e}^{+} and for K_{2e}^{0}, their ratio is

$$\frac{\Gamma(K_{e}^{+}\to\pi^{0}+e^{+}+\nu)}{\Gamma(K_{2e}^{0}\to\pi^{+}+e^{-}+\nu)}=1.$$

The above results are applicable only to the vector type interactions. In reality we should take account of the contributions of the axial vector type interactions. In this sense, the above results are to be appreciated in their qualitative aspect rather than in their quantitative one.

§ 6. Concluding remarks

It is known that the full symmetry theory of Sakata's model is fairly successful in the strong interactions. The purpose of this paper is to examine how this symmetry is reflected in the decay processes induced by the weak interactions. We have proposed that, in Sakata's model, the decay processes should be regarded as those transitions between configurations that conserve the full symmetry in the sense $\Delta n_{R}=\Delta s_{0}=\Delta i_{0}=0$. Under this scheme we have obtained various favourable results, which are equivalent to those following from the so-called $|\Delta I|=1/2$ rule. There exists an essential difference between $|\Delta I|=1/2$ and $|\Delta I|=3/2,\ 5/2$, etc. This is a salient feature of our theory compared with the ordinary one, where $|\Delta I|=3/2$ is not generally forbidden in various decay processes when we proceed directly from the Fermi interaction with the scheme of Fig. 3.[24] We have given some reasoning, from the standpoint of full symmetry, concerning the possible

decay $K_{e3}' \to \pi^{0\prime} + e + \nu$ and the small probability of hyperon β-decay.

In concluding this paper we shall make some supplementary remarks.

1) In this paper the characteristics of weak interactions are not explicitly taken into account except $|\Delta S| = 1$. The reader will find that the results do not depend on particular (3- or 4-body) configurations.

2) There is the possibility that the full symmetry underlying our theory is violated to some extent because of $m_A \neq m_N$ or some other reason. In fact, although the configurations $F_5^6(0, 3/2)$, $F_5^{10}(0, 3/2)$, $F_5^4(0, 3/2)$ and $F_5^1(0, 3/2)$ are previously[17] identified with resonance states in π^+-p scatterings, they cannot be considered as π^+-p systems provided we adhere to the selection rule $\Delta s_0 = \Delta i_0 = 0$. Similarly, this rule may possibly be violated in the weak interactions, too. But the case may be different in decays, because i) the reactions are far slower than those induced by the strong interactions and ii) the weak decays are phenomena of low energies. The possible violation of $\Delta s_0 = \Delta i_0 = 0$ in decays is of interest in connection with the isospin selection rule discussed in § 4.

3) From the universality of the weak interactions stated in § 1, we hardly understand the smallness of the probability of the hyperon β-decays. If this smallness is really true, it follows that Λ and n differ not only in mass but in behaviours for the weak interactions. This is in marked contrast to the fact that μ-meson and electron, in spite of their large mass difference, bear striking similarities to each other with respect to the weak interactions.

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Appendix

Product of two configurations

a) $B_2^1(S\,;\, I, I_3) \times F_1(S'\,;\, I', I_3')$

i) π-N systems

$$B_2^1(0\,;\,1,1)\,F_1\left(0\,;\,-\frac{1}{2},\frac{1}{2}\right) \equiv \pi^+ p = F_3^1\left(0\,;\,\frac{3}{2},\frac{3}{2}\right)$$

$$B_2^1(0\,;\,1,0)\,F_1\left(0\,;\,\frac{1}{2},\frac{1}{2}\right) \equiv \pi^0 p = \frac{\sqrt{6}}{3} F_3^1\left(0\,;\,\frac{3}{2},\frac{1}{2}\right) - \frac{\sqrt{3}}{12} F_3^1\left(0\,;\,\frac{1}{2},\frac{1}{2}\right)$$

$$+ \frac{1}{4} F_3^2\left(0\,;\,\frac{1}{2},\frac{1}{2}\right) - \frac{\sqrt{2}}{4} F_3^3\left(0\,;\,\frac{1}{2},\frac{1}{2}\right) + \frac{\sqrt{2}}{4} F_3^4\left(0\,;\,\frac{1}{2},\frac{1}{2}\right)$$

$$B_2^1(0;1,-1)F_1\left(0;\ \frac{1}{2},\ \frac{1}{2}\right)\equiv\pi^- p=\frac{\sqrt{3}}{3}F_3^1\left(0;\ \frac{3}{2},\ -\frac{1}{2}\right)$$
$$-\frac{\sqrt{6}}{12}F_3^1\left(0;\ \frac{1}{2},\ -\frac{1}{2}\right)+\frac{\sqrt{2}}{4}F_3^2\left(0;\ \frac{1}{2},\ -\frac{1}{2}\right)-\frac{1}{2}F_3^3\left(0;\ \frac{1}{2},\ -\frac{1}{2}\right)$$
$$+\frac{1}{2}F_3^4\left(0;\ \frac{1}{2},\ -\frac{1}{2}\right)$$

$$B_2^1(0;1,1)F_1\left(0;\ \frac{1}{2},\ -\frac{1}{2}\right)\equiv\pi^+ n=\frac{\sqrt{3}}{3}F_3^1\left(0;\ \frac{3}{2},\ \frac{1}{2}\right)$$
$$+\frac{\sqrt{6}}{12}F_3^1\left(0;\ \frac{1}{2},\ \frac{1}{2}\right)-\frac{\sqrt{2}}{4}F_3^2\left(0;\ \frac{1}{2},\ \frac{1}{2}\right)+\frac{1}{2}F_3^3\left(0;\ \frac{1}{2},\ \frac{1}{2}\right)$$
$$-\frac{1}{2}F_3^4\left(0;\ \frac{1}{2},\ \frac{1}{2}\right)$$

$$B_2^1(0;1,0)F_1\left(0;\ \frac{1}{2},\ -\frac{1}{2}\right)\equiv\pi^0 n=\frac{\sqrt{6}}{3}F_3^1\left(0;\ \frac{3}{2},\ -\frac{1}{2}\right)$$
$$+\frac{\sqrt{3}}{12}F_3^1\left(0;\ \frac{1}{2},\ -\frac{1}{2}\right)-\frac{1}{4}F_3^2\left(0;\ \frac{1}{2},\ -\frac{1}{2}\right)+\frac{\sqrt{2}}{4}F_3^2\left(0;\ \frac{1}{2},\ -\frac{1}{2}\right)$$
$$-\frac{\sqrt{2}}{4}F_3^4\left(0;\ \frac{1}{2},\ -\frac{1}{2}\right)$$

$$B_2^1(0;1,-1)F_1\left(0;\ \frac{1}{2},\ -\frac{1}{2}\right)\equiv\pi^- n=F_3^1\left(0;\ \frac{3}{2},\ -\frac{3}{2}\right)$$

ii) π-Λ systems

$$B_2^1(0;1,1)F_1(-1;0,0)\equiv\pi^+\Lambda=\frac{\sqrt{2}}{2}F_3^1(-1;1,1)+\frac{\sqrt{2}}{2}F_3^3(-1;1,1)$$

$$B_2^1(0;1,0)F_1(-1;0,0)\equiv\pi^0\Lambda=\frac{\sqrt{2}}{2}F_3^1(-1;1,0)+\frac{\sqrt{2}}{2}F_3^3(-1;1,0)$$

$$B_2^1(0;1,-1)F_1(-1;0,0)\equiv\pi^-\Lambda=\frac{\sqrt{2}}{2}F_3^1(-1;1,-1)+\frac{\sqrt{2}}{2}F_3^3(-1,1,-1)$$

b) $B_2^1(S;I,I_3)\cdot B_2^1(S';I',I_3')$

iii) 2π systems

$$B_2^1(0;1,1)B_2^1(0;1,1)\equiv\pi^+\cdot\pi^+=B_4^1(0;2,2)$$

$$B_2^1(0;1,1)B_2^1(0;1,0)\equiv\pi^+\cdot\pi^0=\frac{\sqrt{2}}{2}B_4^1(0;2,1)-\frac{\sqrt{3}}{6}B_4^4(0;1,1)$$
$$+\frac{\sqrt{6}}{6}B_4^5(0;1,1)-\frac{\sqrt{3}}{6}B_4^6(0;1,1)-\frac{\sqrt{6}}{6}B_4^7(0;1,1)$$

$$B_2^1(0;1,0)B_2^1(0;1,1) \equiv \pi^0\cdot\pi^+ = \frac{\sqrt{2}}{2}B_4^1(0;2,1) + \frac{\sqrt{3}}{6}B_4^4(0;1,1)$$
$$-\frac{\sqrt{6}}{6}B_4^5(0;1,1) + \frac{\sqrt{3}}{6}B_4^6(0;1,1) + \frac{\sqrt{6}}{6}B_4^7(0;1,1)$$

$$B_2^1(0;1,0)B_2^1(0;1,0) \equiv \pi^0\cdot\pi^0 = \frac{\sqrt{30}}{60}B_4^1(0;0,0) + \frac{\sqrt{6}}{3}B_4^1(0;2,0)$$
$$-\frac{\sqrt{30}}{30}B_4^2(0;0,0) + \frac{\sqrt{6}}{12}B_4^3(0;0,0) + \frac{\sqrt{6}}{6}B_4^8(0;0,0) - \frac{\sqrt{3}}{6}B_4^9(0;0,0)$$

$$B_2^1(0;1,-1)B_2^1(0;1,1) \equiv \pi^-\cdot\pi^+ = -\frac{\sqrt{30}}{60}B_4^1(0;0,0) + \frac{\sqrt{6}}{6}B_4^1(0;2,0)$$
$$+\frac{\sqrt{30}}{30}B_4^2(0;0,0) - \frac{\sqrt{6}}{12}B_4^3(0;0,0) - \frac{\sqrt{3}}{6}B_4^4(0;1,0) + \frac{\sqrt{6}}{6}B_4^5(0;1,0)$$
$$-\frac{\sqrt{3}}{6}B_4^6(0;1,0) - \frac{\sqrt{6}}{6}B_4^7(0;1,0) - \frac{\sqrt{6}}{6}B_4^8(0;0,0) + \frac{\sqrt{3}}{6}B_4^9(0;0,0)$$

$$B_2^1(0;-1,1)B_2^1(0,1,-1) \equiv \pi^+\cdot\pi^- = -\frac{\sqrt{30}}{60}B_4^1(0;0,0) + \frac{\sqrt{6}}{6}B_4^1(0;2,0)$$
$$+\frac{\sqrt{30}}{30}B_4^2(0;0,0) - \frac{\sqrt{6}}{12}B_4^3(0;0,0) + \frac{\sqrt{3}}{6}B_4^4(0;1,0) - \frac{\sqrt{6}}{6}B_4^5(0;1,0)$$
$$+\frac{\sqrt{3}}{6}B_4^6(0;1,0) + \frac{\sqrt{6}}{6}B_4^7(0;1,0) - \frac{\sqrt{6}}{6}B_4^8(0;0,0) + \frac{\sqrt{3}}{6}B_4^9(0;0,0)$$

$$B_2^1(0;1,-1)B_2^1(0;1,0) \equiv \pi^-\cdot\pi^0 = \frac{\sqrt{2}}{2}B_4^1(0;2,-1) - \frac{\sqrt{3}}{6}B_4^4(0;1,-1)$$
$$+\frac{\sqrt{6}}{6}B_4^5(0;1,-1) - \frac{\sqrt{3}}{6}B_4^6(0;1,-1) - \frac{\sqrt{6}}{6}B_4^7(0;1,-1)$$

$$B_2^1(0;1,0)B_2^1(0;1,-1) \equiv \pi^0\cdot\pi^- = \frac{\sqrt{2}}{2}B_4^1(0;2,-1) + \frac{\sqrt{3}}{6}B_4^4(0;1,-1)$$
$$-\frac{\sqrt{6}}{6}B_4^5(0;1,-1) + \frac{\sqrt{3}}{6}B_4^6(0;1,-1) + \frac{\sqrt{6}}{6}B_4^7(0;1,-1)$$

$$B_2^1(0;1,-1)B_2^1(0;1,-1) \equiv \pi^-\cdot\pi^- = B_4^1(0;2,-2)$$

References

1) T. D. Lee and C. N. Yang, Phys. Rev. **105** (1956), 1671.
2) S. Ogawa, Prog. Theor. Phys. **13** (1955), 367.
3) S. Ogawa, Prog. Theor. Phys. **15** (1956), 487.
4) R. P. Feynman and M. Gell-Mann, Phys. Rev. **109** (1958), 193.
5) M. Kawaguchi and K. Nishijima, Prog. Theor. Phys. **15** (1956), 180.

6) S. Furuichi, T. Kodama, S. Ogawa, Y. Sugahara, A. Wakasa and M. Yonezawa, Prog. Theor. Phys. **17** (1957), 89.
S. Furuichi, Y. Sugahara, A. Wakasa and M. Yonezawa, Nuovo Cimento **4** (1956), 390.
7) S. Furuichi, S. Sawada and M. Yonezawa, Nuovo Cimento **10** (1959), 541.
8) Y. Miyachi, Prog. Theor. Phys. **19** (1958), 112.
M. Yonezawa, Nuclear Physics **12** (1959), 589.
9) R. H. Dalitz, Rev. Mod. Phys. **31** (1959), 823.
S. Oneda, Nuclear Physics **9** (1958/59), 476.
10) M. Gell-Mann and A. H. Rosenfeld, *Annual Review of Nuclear Science*, Vol. **7** (1957), 407.
11) G. Takeda and M. Kato, Prog. Theor. Phys. **21** (1959), 441.
12) S. Sakata, Prog. Theor. Phys. **16** (1956), 686.
13) E. Fermi and C. N. Yang. Phys. Rev. **76** (1948), 1739.
14) S. Ogawa, Prog. Theor. Phys. **21** (1959), 209.
15) Y. Yamaguchi, Prog. Theor. Phys. Supplement, No. **11** (1959), 1, 37.
16) M. Ikeda, S. Ogawa and Y. Ohnuki, Prog. Theor. Phys. **22** (1959), 715.
M. Ikeda, S. Ogawa and Y. Ohnuki, Prog. Theor. Phys. **23** (1960), 1073.
17) S. Sawada and M. Yonezawa, Prog. Theor. Phys. **23** (1960), 662.
18) G. Takeda, Prog. Theor. Phys. **19** (1958), 631.
19) M. L. Good and W. G. Holladay, Phys. Rev. Letters **4** (1960), 138.
20) Recently a model has been proposed by Nagoya Group, in which neutrino, electron, μ-meson and a new kind of matter B^+ are considered as basic particles.
Z. Maki, M. Nakagawa, Y. Ohnuki and S. Sakata, Prog. Theor. Phys. **23** (1960), 1174.
K. Matsumoto and M. Nakagawa, Prog. Theor. Phys. **23** (1960), 1181.
21) S. Sawada and M. Yonezawa, Prog. Theor. Phys. **22** (1959), 610.
22) M. Gell-Mann, Phys. Rev. **111** (1958), 362.
23) Z. Maki, Prog. Theor. Phys. **23** (1960), 853.
24) See, however, C. Iso, M. Kawaguchi and Y. Miyamoto, Prog. Theor. Phys. **24**(1960), 97.

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Theory of Classical Fluids and the Convolution Approximation (Note on Papers by Tohru Morita)

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A method employed by Morita for the approximate calculation of the free energy and pair distribution function of classical fluids is seen to be identical with the nodal expansion method. The limit of the nodal expansion sequence results in an approximate integral equation for the potential of average force. This *convolution approximation* is seen to be consistent with the Ornstein-Zernike theory of fluids, and thus provides a convenient means for investigating the behavior of fluids near the critical point. The convolution approximation results also from neglecting the non-convolutory terms in an *exact* integral equation for the pair distribution function.

Recently, several papers dealing with an approximate method for the calculation of the free energy and pair distribution function of fluids have been published by Morita.[1)-3)] In footnotes to the first two of these papers* Morita indicated a relationship between his methods and some of our own. It is the purpose of this note to clarify this relationship between the two developments, with particular reference to the nodal expansion method reported in several previous publications.[4)-7)]

In his first two papers[1),2)] Morita started from the Mayer cluster expansions of the free energy, distribution functions and potentials of average force in powers of particle number density, ρ. As is well known, the coefficients of ρ^n in these expansions are integrals of sums of products of Ursell-Mayer f-bonds, $f_{ij}=\exp[-U(\boldsymbol{r})/kT]-1$, where $U(\boldsymbol{r})$ is the direct potential of interaction of particles i and j separated by $\boldsymbol{r}$. These products are represented by graphs in which each vertex denotes a particle, and each line connecting two vertices i and j represents the corresponding factor f_{ij}. The coefficient of ρ^n involves integration over the coordinates of n particles. Morita noted that certain products in each coefficient contain only vertices connected either by direct f-bonds, by chains of f-bonds (that is, by rows of vertices each of which is connected only to the preceding one and to the following one by direct f-bonds), and by any number of f-bond chains between two vertices. Integrals of f-bond chains are readily evaluated by the n

* Reference 1), p. 932, and reference 2), p. 380.

fold convolution theorem for Fourier transforms.[8] Morita then defines any number of chains connecting two vertices as "identifiable lines" which can be replaced by a single line. The new lines again can form chains between any two vertices which themselves are not intermediate points in a chain, and any number of such chains is again replaced by a single line connecting the two vertices ("identified"). This process is repeated over and over, each step resulting in an n-th order approximation to the free energy, pair distribution function and potential of average force. Such an approximation is denoted by Morita as one obtained by "n times of identification". Morita then proceeds to give formulas for free energy, pair distribution function and potential of average force obtained in the limit $n\to\infty$. This procedure, however, involves complicated additions and substractions of graphs and, as has been noted by Morita,[3] results in omitting an entire class of graphs which should have been included in his approximation. Therefore comparison with the exact nodal expansion method[4)-7] is possible only in view of Morita's recently published correction.[3]

The nodal expansion method in its most general form[5)-7] likewise starts with the Mayer cluster expansion, using certain elementary concepts from graph theory. A vertex connected directly to n other vertices is known in topology as a *node* of n-th order. Thus an f-bond chain is simply a row of second-order nodes. Any number of f-bond chains between any two *branch points* (nodes of third or higher order) can be summed to give a single bond; this procedure can be carried out also with graphs which are not considered by Morita. We thus obtain *exact* expansions for the distribution function and potential of average force. These expansions involve two kinds of bonds: the summed multiple chains between any two branch points (F_1-bonds), and the summed single chains (k_1-bonds). This is the exact first-order nodal expansion for the distribution function, in which the "*zeroth order*" term is the corresponding first-order nodal *approximation*. In this expansion we again have chains, this time involving F_1-bonds and k_1-bonds. Using standard formulas of combinatory analysis[7] we are able to sum over such chains of two kinds of bonds, thus obtaining the exact second-order nodal expansion. A repetition of this procedure yields the exact general nodal expansion of n-th order; in each such expansion the "zeroth order" term is the n-th order nodal approximation to the potential of average force and the distribution function. This exact formal procedure *automatically precludes any possibility of omitting significant graphs*, thus assuring the correctness of all results. Since Morita's initial treatment,[1),2)] in which entire classes of convolutory ("hyper-netted chain") graphs were omitted, did not yield results identical with those of reference 7) beyond the first order, it was clear that the treatment was incorrect. When Morita's recent correction[3] is taken into account it becomes evident that, except for notation and terminology, not only his results but also his methods are *identical* with ours. Thus it is sufficient to note that $\Delta h^{(n-1)}(\boldsymbol{r}) = s_n(\boldsymbol{r})$ and $h^{(n-1)}(\boldsymbol{r}) = \sum_{m=1}^{n} s_m(\boldsymbol{r}) = -\omega_n(\boldsymbol{r})/kT$, where symbols on the left side of each equation

Table I. (Functions of t or k denote Fourier transforms of coordinate space functions in the same rectangle)

Nodal Expansions	Hyper-netted Chain Approximation
1. $s_n(r) = -w_n(r)/kT$; $S_n(t)$	$\Delta h^{(n-1)}(r)$; $\Delta H^{(n-1)}(k)$
2. $\sigma_n(r) = -\omega_n(r)/kT = \sum_{m=1}^{n} s_m(r)$; $p_n(t)$	$h^{(n-1)}(r)$; $H^{(n-1)}(k)$
3. $F_n(r)$; $\varphi_n(t)$	$f^{(n)}(r)+h^{(n-1)}(r)$; $F^{(n)}(k)+H^{(n-1)}(k)$
4. $k_n(r) = F_{n-1}(r) - w_n(r) = F_{n-1}(r) + s_n(r)$ $\kappa_n(t) = \varphi_{n-1}(t) + S_n(t)$	No equivalent
5. $S_{n+1}(t) =$ $\rho \dfrac{[\varphi_n(t)+\kappa_n(t)+\rho\kappa_n(t)\varphi_n(t)][\varphi_n(t)-\kappa_n(t)]}{1-\rho[\varphi_n(t)-\kappa_n(t)]-\rho^2\kappa_n(t)[\varphi_n(t)-\kappa_n(t)]}$ or[9)-12)]	No equivalent
6. $p_{n+1}(t) = \dfrac{\rho[\varphi_n(t)-p_n(t)]^2}{1-\rho[\varphi_n(t)-p_n(t)]}$ with $F_n(r) = \exp[-U(r)/kT+\sigma_n(r)]-1$	$H^{(n)}(k) = \dfrac{\rho F^{(n)}(k)^2}{1-\rho F^{(n)}(k)}$ with $f^{(n+1)}(r) = [f^{(0)}(r)+1]\exp h^{(n)}(r) - 1 - h^{(n)}(r)$
7. $\omega(r) = \lim_{n\to\infty} \omega_n(r)$, $F(r) = \lim_{n\to\infty} F_n(r)$ $g_e(r) = 1+F(r)$	$h(r) = \lim_{n\to\infty} h^{(n)}(r)$, no equivalent $[g(r)]_{\mathrm{HNC}} = [1+f^{(0)}(r)]\exp h(r)$
8. $\sigma_n(r) = -\omega_n(r)/kT$	Identifiable lines with n times of identification

denote Morita's corrected functions, and those on the right are the chain sums and sheaf potentials of reference 7), respectively. Detailed comparisons are given in Table I. It should be noted that Morita's work contains no equivalent to our recursion formula (Eq. (5) of Table I). However, Morita's recursion formula, equivalent to our Eq. (6), is readily obtained from (5) by induction, and has been reported elsewhere.[9)-14)] The recurrence formula (5) has been used to obtain an exact integral equation from which higher order approximations (in a topological sense) to the exact pair distribution function can be derived.[15)]

As has been noted already in the discussion section of reference 7), the limit of the nodal expansion sequence takes the form of an integral equation which yields all terms in the exact potential of average force (and the distribution function) which conform rigorously to the Kirkwood superposition principle. Alternatively, these terms can be defined as corresponding to graphs whose sub-duals are Cayley trees.[10),12)] However, the significant property, which enables us to evaluate the sum of these terms exactly in the form of an integral equation, consists of the fact that each of these terms can be wholly evaluated by repeated convolutions. Therefore the limiting integral equation is called the *convolution approximation*. In coordinate space it takes the form[9)-14)]

$$\sigma(\boldsymbol{R})=\rho\int F(\boldsymbol{R}-\boldsymbol{r})C(\boldsymbol{r})d\boldsymbol{r}=\rho\int F(\boldsymbol{R}-\boldsymbol{r})[F(\boldsymbol{r})-\sigma(\boldsymbol{r})]d\boldsymbol{r}. \qquad \text{(I)}$$

Here $F(\boldsymbol{x})=\exp[-U(\boldsymbol{x})/kT+\sigma(\boldsymbol{x})]-1$ is the convolution approximation to the total correlation function, $h(\boldsymbol{x})=g(\boldsymbol{x})-1$ (which correlates fluctuations in density at the two points separated by $\boldsymbol{x}$), and $C(\boldsymbol{x})=F(\boldsymbol{x})-\sigma(\boldsymbol{x})$ is the convolution approximation to the Ornstein-Zernike direct correlation function. Of course, $\sigma(\boldsymbol{x})$ is simply $-\omega(\boldsymbol{x})/kT$, where $\omega(\boldsymbol{x})$ is the limiting sheaf potential of reference 7). The convolution approximation has been derived also for multicomponent systems.[10),12)-14)] The corresponding exact expressions for distribution functions and potentials of average force, in terms of cluster integrals involving only branch points and F-bonds, are readily obtained from the general n-th order nodal expansion on letting $n\to\infty$, as well as by topological analysis.[9)-14)] The convolution approximation to the free energy and equation of state is likewise readily obtained from the first-order nodal expansion for pressure,[6)] on applying the repeated summation procedure of reference 7); furthermore, we also immediately obtain the corresponding exact expressions in which the convolution approximation is, formally, the "zeroth order" term.[10)-12),16)] As should be expected, Morita's corrected formula for the free energy[3)] is identical with that of the convolution approximation.*

The convolution approximation, Eq. (I), is formally identical with the exact Ornstein-Zernike relation defining the direct correlation function. In fact, the nodal expansion sequence is merely a method for solving Eq. (I) by repeated substitutions, and other methods are readily devised.[10)-13)] As is well known, the singularities of the Ornstein-Zernike integral equation yield important information concerning the critical points of fluids, while its properties in the non-singular regions of density provide some insight into the nature of the structure factors of liquids as determined experimentally by light, x-ray, and electron scattering.[17)] Thus the convolution approximation provides a convenient means for further development in this direction, and has been discussed from this standpoint.[12)-16)]

The interpretation of the function $F(\boldsymbol{r})-\sigma(\boldsymbol{r})$ as the Ornstein-Zernike direct correlation function yields an alternative formula for the free energy of interaction in the convolution approximation

$$-F/kTV=\int_0^{\rho}d\rho'\int_0^{\rho'}d\rho''\int[F(\boldsymbol{r})-\sigma(\boldsymbol{r})]d\boldsymbol{r}\quad(V=\text{volume}). \qquad \text{(II)}$$

The comparison of the value of F/kTV as calculated from Eq. (II) with that found as the limit of the nodal expansion sequence[3),10)-12),16)] would yield a measure

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* In this connection it is of interest to note that the "watermelon approximation" of R. Abe (J. Phys. Soc. Japan, Jan. 1959), on which Morita's initial development is based,[1)] is identical with the second term of the exact first-order nodal expansion of the free energy (Eq. (3), reference 6), errata). Abe's approximation for Coulombic potentials is likewise identical with our previously published results[4)].

of the internal consistency of the convolution approximation. However, the range of ρ for which the convolution approximation is valid has not yet been determined. Work in this direction has delayed the publication of the formal results for over one year. Papers dealing with this question are now in preparation.

Finally, we should like to note that the convolution approximation, Eq. (I), is also obtained from an exact integral equation for the pair distribution function[9),11),13),14)]

$$\tau(\boldsymbol{R}) = \rho \int h(\boldsymbol{R}-\boldsymbol{r})[h(\boldsymbol{r}) - \tau(\boldsymbol{r})]\, d\boldsymbol{r} \qquad \text{(III)}$$

with $h(\boldsymbol{x}) = g(\boldsymbol{x}) - 1$, $g(\boldsymbol{x}) = \exp[-U(\boldsymbol{x})/kT + \tau(\boldsymbol{x}) + \zeta(\boldsymbol{x})]$, and $\zeta(\boldsymbol{x})$ the sum of cluster integrals defined in the same way as the integrals of the density expansions of potentials of average force, except that each line corresponds to a density dependent function $h(\boldsymbol{x})$ (instead of the density independent Ursell-Mayer f-bond), and no graphs containing cutting points and bifocal points* are included.[9)-14)] On putting $\zeta(\boldsymbol{x}) = 0$ in Eq. (III) we obtain Eq. (I), the convolution approximation.

References

1) T. Morita, Prog. Theor. Phys. **20** (1958), 920.
2) T. Morita, Prog. Theor. Phys. **21** (1959), 361.
3) T. Morita, Prog. Theor. Phys. **23** (1960), 175.
4) E. Meeron, J. Chem. Phys. **28** (1958), 630; **26** (1957), 804.
5) E. Meeron, Phys. Fluids **1** (1958), 130.
6) E. Meeron, J. Chem. Phys. **28** (1958), 505; errata **29** (1958), 444.
7) E. Meeron, and E. R. Rodemich, Phys. Fluids **1** (1958), 246.
8) E. W. Montroll and J. E. Mayer, J. Chem. Phys. **9** (1941), 626; see also Appendix to reference 4).
9) American Physical Society Annual Meeting, Jan. 28-31, 1959, New York, N. Y.
10) E. Meeron, *Proceedings of the International Symposium on Electrolytes*, Trieste, Italy, June 6-12, 1959 (Pergamon Press, in press); BSRL report D1-82-0036, available on request.
11) E. Meeron, Physica **26** No. 6 (June 1960).
12) E. Meeron, submitted to Phys. Rev.
13) E. Meeron, J. Math. Phys., in press; BSRL report D1-82-0041 (Jan. 1960), available on request.
14) International Plasma Physics Institute, Seattle, Washington, Aug. 31-Sept. 5, 1959.
15) R. V. Hanks, submitted to Physica.
16) E. Meeron, Bull. Am. Phys. Soc. **5** (1960), 8.
17) See, e. g. L. Goldstein, Ann. Phys. **1** (1957), 33.

* A bifocal point is defined by the statement that a graph may be separated into two or more graphs, each containing at least one node, by cutting at *at least two* bifocal points. The two nodes which are not directly connected must remain in the same graph.

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On the Concept of Regraduation in General Relativity

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An attempt is made to develop a theory of regraduation of scales and clocks in the general theory of relativity as one of methodological approaches to examine the interrelationship between space-time and gravitation. The relation between the orthodox formalism of general relativity and its reinterpretations based on the flat space-time is made clear, and the problem of a particular ("conformal") regraduation is investigated in detail.

§ 1. Introduction

It is well known that the principle of regraduation of clocks and scales provides us with far-reaching consequences in kinematic relativity.[1] On the other hand, Robertson[2] and Walker[3] independently showed that the concept of Riemannian space-time is suitable for a kinematical basis of world models satisfying the so-called cosmological principle not only in the general relativistic cosmology but also in the kinematic cosmology. In addition to this, their researches offer, though implicitly, a clue to introducing the concept of regraduation mentioned above into any cosmological theory based on the concept of Riemannian space-time.[4]

But it seems to us that Walker[5] and McVittie[6] were the first to introduce consciously the concept of regraduation into the general theory of relativity, though their ultimate conclusions with respect to it were opposed to each other. Namely, in contrast with McVittie's opinion that the procedure of regraduation is ultimately reduced to a kind of coordinate transformation, Walker emphasized that it is not the case in the sense that the procedure of regraduation makes the structure of space-time alter entirely from the original one. Such a situation is of course due to the discrepancy in opinions as to how many postulates (of the general theory of relativity) should be preserved under the procedure of regraduation.

On the other hand, in contrast with the general theory of relativity in its orthodox form, not only there are various gravitational theories[7] based on the flat space-time, but also there exist several attempts[8] to reinterpret the general theory of relativity itself from the standpoint of the flat space-time. Moreover, in the aspect of observational tests, there is nothing at present to choose between these alternative theories including reinterpretations of general relativity.[9] Accordingly, it seems to be a hasty task to decide if we can observationally test the curvedness of a space-time in the presence of matter, i.e. there is a question

whether the concept of curved space-time is inevitable in stipulating the interrelationship between space-time and gravitation. In this respect, reference should be made to the following words of Carnap:[10] "*In physical geometry, there are two possible procedures for establishing a theory of physical space. First, the physicist may freely choose the rules for measuring length. After this choice is made, the question of the geometrical structure of physical space become empirical; it is to be answered on the basis of the results of experiments. Alternatively, the physicist may freely choose the structure of physical space, but then he must adjust the rules of measurement in view of the observational facts*···."

In order to examine the above question, the concept of regraduation in the style of Walker seems to serve as a powerful tool, since, according to him, regraduation is a kind of operation altering the structure of space-time, though his procedure is not necessarily the only permissible one. In this paper, we shall deal with the problem of regraduation in the general theory of relativity in a different manner from Walker's, showing the usefulness of such a methodological consideration in the examination of the interconnection between space-time and gravitation. In § 2 the essential points in Walker's[5] theory of regraduation is summarized, together with the enumeration of postulates in general relativity. In § 3 we discuss the problem of regraduation such that a given Riemannian space-time is changed into a flat one. Though the discussions in this section are not complete in that the gravitational field equations are not concretely taken into consideration, they may serve as a preliminary knowledge when we examine to what extent the concept of curved space-time is inevitable in the theory of gravitation. We consider in § 4 a special kind of regraduation such that a given Riemannian space-time is changed into another one whose metric tensor is conformal to the original one (hereafter we shall call this the conformal regraduation). Namely, we shall search for the regraduation properties of various field equations, the equations of motion for a test particle, and various physical constants including the natural constants such as h and c.

§ 2. Walker's theory of regraduation

Before proceeding to Walker's paper, we shall refer to McVittie's[6] study concerning the regraduation of clocks in a particular space-time. According to him, the basic postulates of general relativity can be enumerated as follows: "(a) The equations of mechanics, and of mathematical physics generally, are expressible in tensor form. ··· (b) Events are represented by points in a 4-dimensional space-time with metric

$$ds^2=g_{\mu\nu}dx^\mu dx^\nu,\ (\mu,\ \nu=0,\ 1,\ 2,\ 3), \tag{2·1}$$

$g_{\mu\nu}$ being functions of the (x^μ)···. (c) Material particles (and observers) trace

out the non-null geodesics of space-time. The integrated interval along the geodesic ……is called his proper-time. (d) The paths of light-rays are the null-geodesics of space-time. (e) The dynamical properties of the material content of space-time are reflected in the geometrical properties of space-time,…. The energy-tensor $(T_{\mu\nu})$ of the material…by Einstein's gravitational equations

$$-\kappa T_{\mu\nu}=R_{\mu\nu}-(1/2)(R-2\lambda)g_{\mu\nu}, \tag{2·2}$$

…… ". Then, demanding that all of the above five postulates should be preserved under the procedure of regraduation, he arrived at the conclusion that it is simply a kind of coordinate transformation in general.

Now, as is well known, at any point $E(x)$ in a Riemannian space-time with metric (2·1) we can introduce a set of local coordinates $X^{(\alpha)}$ such that

$$ds^2=\delta_{\alpha\beta}X^{(\alpha)}X^{(\beta)}, \tag{2·3}$$

where $\delta_{\alpha\beta}$ is the proper fundamental tensor specified by $\delta_{00}=1$, $\delta_{ii}=-c^{-2}$, $\delta_{\alpha\beta}=0$ $(\alpha\neq\beta)$, c being the velocity of light. Then the relation between $X^{(\alpha)}$ and the differential of basic coordinates x^{λ} can be written as

$$dx^{\lambda}=A^{\lambda}_{(\alpha)}(x)X^{(\alpha)}, \quad (\det|A^{\lambda}_{(\alpha)}|\neq 0), \tag{2·4}$$

where $A^{\lambda}_{(\alpha)}(x)$ stands for an orthogonal enupple such that

$$\left.\begin{aligned} g_{\mu\nu}&=\delta_{\alpha\beta}A^{(\alpha)}_{\mu}(x)A^{(\beta)}_{\nu}(x),\\ \delta_{\alpha\beta}&=g_{\mu\nu}A^{\mu}_{(\alpha)}(x)A^{\nu}_{(\beta)}(x),\end{aligned}\right\} \tag{2·5}$$

in which the matrix $(A^{(\alpha)}_{\mu})$ is the reciprocal of $(A^{\mu}_{(\alpha)})$.

After the above preparation, we shall summarize the essential points of Walker's paper. According to him, the procedure of regraduation should first of all be regarded as a local affair.* Thus the regraduation of scales and clocks together with that of mass at an event E is defined by

$$\bar{X}^{(0)}=uX^{(0)}, \ \bar{X}^{(i)}=vX^{(i)}, \ \bar{m}=wm, \ (i=1, 2, 3), \tag{2·6}$$

for some positive numbers u, v, w and all $X^{(\alpha)}$ and m. If we denote the regraduated value of any quantity due to (2·6) by the bar-symbol, we have $\bar{c}=cvu^{-1}$, $\bar{A}^{\lambda}_{(0)}=u^{-1}A^{\lambda}_{(0)}$, $\bar{A}^{\lambda}_{(\alpha)}=v^{-1}A^{\lambda}_{(\alpha)}$, etc. From these relations and (2·5) we obtain $\bar{g}_{\mu\nu}=u^2g_{\mu\nu}$. In a similar manner, we have $\bar{T}_{\mu\nu}=wv^{-3}u^2T_{\mu\nu}$. Now let us assume that a regraduation of the type (2·6) is carried out at each point of space-time. Then u, v and w in (2·6) are functions of x^{λ} and we obtain

$$\bar{g}_{\mu\nu}=e^{2\sigma}g_{\mu\nu}, \ \bar{T}_{\mu\nu}=e^{\psi}T_{\mu\nu}, \tag{2·7}$$

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* Such a standpoint is plausible, if we take it into consideration that the concept of scale and clock in general relativity is stipulated from the first in the local space-time with metric (2·3) at the event in question.

where σ and φ are functions (constructed from (u, v, w)) of x^λ. The space-time whose metric tensor is given by the first expression of (2·7), i.e.

$$d\bar{s}^2=\bar{g}_{\mu\nu}dx^\mu dx^\nu, \tag{2·8}$$

is conformal to the original one specified by the metric (2·1).

It is easily seen that the postulates (b) and (d) mentioned above are preserved under the conformal regraduation specified by the first expression of (2·7); there is also no obstacle with respect to the postulate (a). If we demand that the postulate (e) should also be preserved under the conformal regraduation (2·7), then we must have

$$-\bar{\kappa}\bar{T}_{\mu\nu}=\bar{R}_{\mu\nu}-(1/2)(\bar{R}-2\bar{\lambda})\bar{g}_{\mu\nu} \quad \text{for } \bar{g}_{\mu\nu}, \tag{2·9}$$

where $\bar{\kappa}$ and $\bar{\lambda}$ are constants* corresponding to κ and λ respectively. Then the problem of regraduation is reduced to whether or not there exist functions σ and φ satisfying (2·9) with (2·8) and (2·7), when (2·2) with (2·1) is given. Walker verified that (1), if the postulate (c) is preserved under the conformal regraduation, the regraduation itself is reduced to the uniform regraduation** specified by $\sigma=$const. and $\varphi=$const. and that (2) there exist some kinds of non-uniform regraduation if the postulate (c) is discarded.

Since the postulate (c) is generally violated under the conformal regraduation (2·7), it is not necessarily impossible to say that the regraduated version (based on the metric (2·8)) of general relativity is already different from general relativity itself. But it seems to us that such an opinion is rather narrow-minded, because a similar situation appears also in Rosen's[8] interpretation of general relativity with respect to the postulate (c), in addition to the fact that the second metric corresponding to the flat space-time is introduced together with the metric (2·1) in his treatment.

Now it must be remarked that the first expression of (2·7) is deduced without recourse to the regraduation of mass, i.e. the third expression of (2·6), though it is not the case with the second of (2·7); the regraduation of mass in (2·6) is entirely formal and artificial. Moreover, in his paper, Walker has exclusively treated Einstein's field equations, in spite of the situation that the Maxwell and Dirac equations can be treated in a covariant manner as well (cf. the postulate (a)) and that various physical constants such as Planck's constant h, and charge e and mass m of elementary particles are included in these field equations; c and κ have already appeared in (2·1) and (2·2). It is suggested in this connection

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* Remark that this is the opinion due to Walker.

** The uniform regraduation is substantially equivalent to the case arrived at by McVittie. Moreover, the problem of uniform regraduation has been investigated also by Hoffmann,[10] though his main concern is to establish a unified field theory from the standpoint of "similarity" invariance of physical laws.

that we can tackle the problem of conformal regraduation in a different manner from Walker's.

§ 3. Theories of gravitation based on the flat space-time and the concept of regraduation

It is clear that the conformal regraduation as specified by the first two expressions of (2·6) is incompetent to examine Rosen-Kohler's and Gupta's reinterpretations[8] of general relativity from the standpoint of regraduation, because, say, the Schwarzschild space-time is not conformal to the flat one. In this respect, let us for example consider such a regraduation that

$$\bar{X}^{(0)}=uX^{(0)},\ \bar{X}^{(a)}=v_aX^{(a)}\quad (a \text{ is non-dummy}), \tag{3·1}$$

in place of (2·6), where u and $v_a(a=1, 2, 3)$ are functions of x^λ. The first two expressions of (2·6) are, however, set up from the first with due regard to the postulate (d). If we use the local coordinates X's, this postulate can be represented as follows: (d′) A null curve specified by $\delta_{\alpha\beta}X^{(\alpha)}X^{(\beta)}=0$ is transformed into another one specified by $\bar{\delta}_{\alpha\beta}\bar{X}^{(\alpha)}\bar{X}^{(\beta)}=0$ under a regraduation.

Then, is the regraduation (3·1) consistent with the above postulate (d′)? As is easily seen, for (3·1) we obtain

$$d\bar{s}^2=\bar{g}_{\mu\nu}(x, dx)dx^\mu dx^\nu, \tag{3·2}$$

with

$$\bar{g}_{\mu\nu}=u^2[A^{(0)}_\mu A^{(0)}_\nu-c^{-2}J(x, dx)\sum_{a=1}^{3}\bar{C}_a^2A^{(a)}_\mu A^{(a)}_\nu], \tag{3·3}$$

where

$$J\equiv\left[\frac{\sum_{a=1}^{3}A^{(a)}_\mu A^{(a)}_\nu dx^\mu dx^\nu}{\sum_{a=1}^{3}\bar{C}_a^2A^{(a)}_\mu A^{(a)}_\nu dx^\mu dx^\nu}\right];\ \bar{C}_a\equiv cv_au^{-1}. \tag{3·4}$$

Since $\bar{g}_{\mu\nu}$ given by (3·3) is homogeneous of zeroth degree with respect to dx's, the metric (3·2) corresponds to a Finslerian 4-space, but not to a Riemannian space-time. Thus we can see that, so far as the postulate (b) is demanded, the postulate (d′) cannot be preserved under the regraduation (3·1). Such a situation should be taken into consideration, when we examine from the standpoint of regraduation any theory of gravitation based on the flat space-time.

(A) A regraduation of an arbitrary space-time into the flat one.

Now let us assume that a Riemannian space-time with metric (2·1) is given. Then, can we regraduate scales and clocks so that it may be converted into the flat space-time specified by

$$d\bar{s}^2=\bar{g}_{\mu\nu}dx^\mu dx^\nu, \tag{3·5}$$

with

$$\bar{R}_{\mu\nu\lambda\sigma}=0, \tag{3·6}$$

where $\bar{R}_{\mu\nu\lambda\sigma}$ is the Riemann-Christoffel tensor for $\bar{g}_{\mu\nu}$? To answer this question, we shall consider at each point of space-time a regraduation specified by

$$\bar{X}^{(\alpha)}=B^{(\alpha)}_{(\beta)}X^{(\beta)}, \tag{3·7}$$

where $B^{(\alpha)}_{(\beta)}$ are functions of x's. As is easily seen, a special case such that

$$\delta_{\alpha\beta}B^{(\alpha)}_{(\alpha')}B^{(\beta)}_{(\beta')}=\delta_{\alpha'\beta'} \tag{3·8}$$

corresponds simply to a Lorentz transformation at an event $E(x)$, i.e. such a degree of freedom is always included in the regraduation (3·7).

Now we shall introduce a new metric

$$d\bar{s}^2=\bar{\delta}_{\alpha\beta}\bar{X}^{(\alpha)}\bar{X}^{(\beta)}, \tag{3·9}$$

where $\bar{\delta}_{\alpha\beta}$ is given by $\bar{\delta}_{00}=1$, $\bar{\delta}_{ii}=-(\bar{c})^{-2}$, $\bar{\delta}_{\alpha\beta}=0(\alpha\neq\beta)$, $\bar{c}$ being a counterpart of c in (2·3). Then, by virtue of (3·7) and (2·4), (3·9) is reduced to

$$d\bar{s}^2=\bar{g}_{\mu\nu}(x)dx^\mu dx^\nu, \tag{3·10}$$

with

$$\begin{cases} \bar{g}_{\mu\nu}=b_{\alpha\beta}A^{(\alpha)}_{\mu}A^{(\beta)}_{\nu}, & (3·11)\\ b_{\alpha\beta}=\bar{\delta}_{\alpha'\beta'}B^{(\alpha')}_{(\alpha)}B^{(\beta')}_{(\beta)}. & (3·12) \end{cases}$$

Contracting (3·11) by the use of $A^{\mu}_{(\alpha')}A^{\nu}_{(\beta')}$, we obtain

$$b_{\alpha\beta}=\bar{g}_{\mu\nu}A^{\mu}_{(\alpha)}A^{\nu}_{(\beta)}, \tag{3·13}$$

which should be identified with the left-hand side of (3·12).

If we assume that $\bar{g}_{\mu\nu}$ given by (3·11) satisfies (3·6), then the left-hand side of (3·12) can be calculated by using (3·13), where A's are known functions of x's. To solve (3·12) with respect to B's is, however, similar asto solve the first equation of (2·5) with respect to A's. Accordingly, the general solution of B's can be written as

$$B^{(\alpha')}_{(\alpha)}=\bar{R}^{(\alpha')}_{(\beta)}\mathring{B}^{(\beta)}_{(\alpha)}, \tag{3·14}$$

where ($\mathring{B}$'s) is a solution of (3·12) and ($\bar{R}$'s) is the coefficient of a Lorentz transformation at $E(x)$ in the space-time with metric $d\bar{s}$, i.e.

$$\bar{\delta}_{\alpha\beta}\bar{R}^{(\alpha)}_{(\alpha')}\bar{R}^{(\beta)}_{(\beta')}=\bar{\delta}_{\alpha'\beta'}. \tag{3·15}$$

Substituting (3·14) in (3·7), we can absorb the above $\bar{R}$'s into $\bar{X}$'s by a suitable Lorentz transformation in such a way that

$$\bar{X}^{(\alpha)}=\mathring{B}^{(\alpha)}_{(\beta)}X^{(\beta)}. \tag{3·16}$$

Thus it is verified that there exists a regraduation in terms of which a given

space-time is converted into a flat one, up to any Lorentz transformation at each event.

There is however one important point to be remarked with respect to the above procedure. The required regraduation (3·16) is originally defined at an event $E(x)$ of the space-time with metric (2·1), x's being the basic coordinates fixed in some manner. But the system of coordinates employed cannot cover all the space-time region in general, so that the applicability of (3·16) is confined to a part of the region such that the coordinates in question are valid. Accordingly, we must stipulate that, for an event outside the region in which the basic coordinates is defined, a suitable transformation of coordinates ($x^\lambda \to {}'x^\lambda$) should be carried out prior to the procedure of regraduation. Such a situation is particularly conspicuous when the original space-time has a topologically different structure from that of the Minkowski space-time.

(B) The problem concerning the physical meaning of coordinates.

Though our discussion in (A) has been made with the basic coordinates x's fixed, it is clear that the formalism in (A) is covariant with respect to an arbitrary transformation of coordinates. We have, however, another point to be examined, i.e. what form of $\bar{g}_{\mu\nu}(x)$ should be adopted as a solution of (3·6). For example, we shall assume that the metric (2·1) corresponds to the Schwarzschild space-time. Then, the concrete form of (2·1) is, for example,

$$ds^2=(1-2m/r)dt^2-(1/c^2)\left[\frac{dr^2}{(1-2m/r)}+r^2(d\theta^2+\sin^2\theta\, d\varphi^2)\right];$$
$$(x^\lambda)=(t,\ r,\ \theta,\ \varphi), \tag{3·17}$$

or

$$ds^2=\left(\frac{1-m/2r^*}{1+m/2r^*}\right)^2 dt^2-(1+m/2r^*)^4(1/c^2)[dr^{*2}+r^{*2}(d\theta^2+\sin^2\theta d\varphi^2)];$$
$$(x^\lambda)=(t,\ r^*,\ \theta,\ \varphi), \tag{3·18}$$

where $r=r^*(1+m/2r^*)^2$. Now we shall assume that the procedure of regraduation elucidated in (A) is applied to the metric (3·17). Then we have

$$d\bar{s}^2=dt^2-(1/\bar{c}^2)[dr^2+r^2(d\theta^2+\sin^2\theta d\varphi^2)], \tag{3·19}$$

which is transformed into

$$d\bar{s}^2=dt^2-(1+m/2r^*)^4(1/\bar{c}^2)\left[\left(\frac{1-m/2r^*}{1+m/2r^*}\right)^2 dr^{*2}+r^{*2}(d\theta^2+\sin^2\theta d\varphi^2)\right], \tag{3·20}$$

if we use the relation between r and r^*. On the other hand, applying the procedure in (A) to the metric (3·18), we obtain

$$d\bar{s}^2=dt^2-(1/\bar{c}^2)[dr^{*2}+r^{*2}(d\theta^2+\sin^2\theta d\varphi^2)]. \tag{3·21}$$

Since the concept of regraduation is originally introduced without recourse to

the transformation of basic coordinates, each coordinate in (3·20) might be identified with the same symbol in (3·21). But, in spite of their different forms, these two metrics correspond equally to the flat space-time, so that one is obtained from the other by means of a suitable transformation of coordinates. Such an apparently contradictory situation is of course due to the fact that there exists a variety of arbitrariness for the functional form of $\bar{g}_{\mu\nu}(x)$ satisfying (3·6), even when the basic coordinates x's are fixed. At first sight, the metric (3·21) is superior to (3·20) as the regraduated form of the original metric (3·18), because the former is of the usual form for the Minkowski space-time (if we regard r^* as the radial coordinate). But it is not the case, if we take it into consideration that the original form (3·19), from which (3·20) is deduced, is also of such a type. A similar situation appears in Rosen-Kohler's reinterpretation of general relativity too, in spite of the device of the second metric introduced for stipulating the physical meaning of coordinates clearly.

Can we affirm, then, that in general relativity the system of coordinates is simply a set of mathematical labels specifying events? Such an opinion is not necessarily erroneous, if we take account of the following situations:

(1) At each point of a Riemannian space-time we can consider locally a set of flat ones, in each of which the local coordinates in the sense of special relativity are introduced.

(2) Strictly speaking, it is the so-called 'physical component,* which is a scalar constructed from the tensorial component of any physical quantity, that should be compared with the experimental data.

It must however be remarked that an important proviso should be added to the above consideration. For example, we shall take up the angular coordinates (θ, φ) in (3·17) and (3·18). These coordinates clearly have a direct physical meaning in that they characterize the spherical symmetry of the Schwarzschild space-time, in contrast to the other two coordinates.

In short, with respect to the problem of physical meaning of coordinates in general relativity, a much more detailed study is needed.**

(C) Theories of gravitation based on the flat space-time.

The last sub-sections have been devoted to show that there always exists such a regraduation that a given Riemannian space-time is converted into the flat one, together with an examination of the matter concerning the system of coordinates. But this is only the first step to establish the following scheme:

(*The general theory of relativity*) $\xrightarrow{(Translation)}$ (*Theory of gravitation based on the flat space-time*),

though it is not necessarily self-evident that the above scheme is possible. In

* In this respect, the problem of spatial distance provides us with a typical example.[12]

** In kinematic relativity the situation is rather different, irrespective of the introduction of the concept of Riemannian space-time.

order to examine the degree of validity of the above scheme, we shall adopt as the basic postulates of general relativity the ones (a)–(e) enumerated in § 2.

However, we have already known various theories of gravitation (together with several reinterpretations of general relativity) based on the flat space-time. Accordingly, we shall consider in the following the problem with respect to these theories and reinterpretations.

(i) Theory of Birkhoff and of Whitehead: The scheme in question is not satisfied by these theories, because the equations of motion for a test particle in Birkhoff's theory cannot be a geodesic in a Riemannian space-time and there are no field equations in Whitehead's theory.

(ii) Dicke's theory: His theory might be in principle one of the required type in that the correspondence of a curved space-time to the flat one is considered from the standpoint of altering "units of measurements". It seems to us, however, that there remains as yet a certain ambiguity in his concept of "local Lorentz covariance".*

(iii) Rosen-Kohler's reinterpretation: In their formalism, the second metric corresponding to the flat space-time is introduced in addition to the metric (2·1), so that the concept of regraduation is not included in it. But we may say that their formalism too is one of the required type, because we can regard the tensor $g_{\mu\nu}(x)$ as the gravitational potential defined covariantly in the flat space-time specified by the second metric.

(iv) Gupta's reinterpretation: In contrast with the above (iii), in which the postulate (a) in § 2 is preserved, Gupta showed that Einstein's field equations can be reformulated in a Lorentz covariant manner. Moreover, the postulates (c) and (d) in § 2 are no longer preserved, but we can still obtain the same (though practically) results as those of general relativity in the usual sense as for the equations of motion for particles and light rays. Accordingly, we see that Gupta's formalism is also one of the required type.

Summarizing (ii)–(iv), we may say that there are different manners for establishing the scheme** indicated at the top of this section, though the relation between (ii) and (iii) is not yet clear. There remains, however, one important point to be emphasized: Each theory of gravitation based on the flat space-time, even if it satisfies such a scheme, does not in principle predict the completely same result as that obtained from the orthodox formalism of general relativity for one and the same physical phenomenon. Because, even if the field equations and equations of motion for particles in the former are simply a different representation of those in the latter, the numerical value of a physical component (to be com-

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* Though his actual formalism is developed on the assumption that gravitation is tentatively described by a scalar field, it seems to us that he has implicitly introduced a tensor field of the second rank which is obscured by this concept.

** Whether such a scheme can be extended to the realm of cosmology is another problem to be iavestigated in future.

pared with the experimental result) depends upon the method of stipulation of the metric tensor.

§ 4. Theory of conformal regraduation

In § 2 we have summarized the essential points of Walker's paper and pointed out another possibility of approaching the problem of conformal regraduation. We shall in the following look into this problem in detail. Prior to the discussion, it should be emphasized once more that the conformal relation specified by the first expression of (2·7) is deduced without appealing to the regraduation of mass such as the third expression of (2·6). Moreover, so far as the conformal regraduation is concerned, we may regard $\bar{g}_{\mu\nu}$, instead of $g_{\mu\nu}$, as the original metric tensor, so that we shall here modify the symbol "bar" in § 2 by "°" for convenience's sake. Then (2·8) is reduced to

$$d\mathring{s}^2=\mathring{g}_{\mu\nu}dx^\mu dx^\nu. \tag{4·1}$$

Similarly we have (cf. (2·7))

$$\mathring{g}_{\mu\nu}=\chi^2 g_{\mu\nu}\,;\ \mathring{g}^{\mu\nu}=\chi^{-2}g^{\mu\nu}\,;\ (-\mathring{g})^{1/2}=\chi^4(-g)^{1/2}, \tag{4·2}$$

where $\mathring{g}^{\mu\lambda}\mathring{g}_{\mu\nu}=g^{\mu\lambda}g_{\mu\nu}=\delta_\nu^\lambda$, $\mathring{g}=\det|\mathring{g}_{\mu\nu}|$, and $\chi=e^\sigma$, from which we obtain

$$\mathring{\Gamma}^\gamma_{\alpha\beta}=\Gamma^\gamma_{\alpha\beta}+\frac{1}{\chi}(\delta_\alpha{}^\gamma\chi_{,\beta}+\delta_\beta{}^\gamma\chi_{,\alpha}-g_{\alpha\beta}\chi^{,\gamma}), \tag{4·3}$$

where $\mathring{\Gamma}$'s and Γ's are the Christoffel symbols for $\mathring{g}_{\mu\nu}$ and $g_{\mu\nu}$ respectively, and $\chi^{,\gamma}=g^{\gamma\alpha}\chi_{,\alpha}$. Then we can stipulate the conformal regraduation as follows: "a regraduation in terms of which a space-time with metric (4·1) is converted into another one with metric (2·1) in conformity to (4·2)".

(A) *Einstein's field equations*

Einstein's field equations in $d\mathring{s}$* can be written as

$$-\kappa_0\mathring{T}_{\mu\nu}=\mathring{R}_{\mu\nu}-(1/2)\mathring{g}_{\mu\nu}\mathring{R},\quad \text{for } \mathring{g}_{\mu\nu}, \tag{A·1}$$

where the cosmic term is neglected. Now let us assume that the postulate (e) in § 2 is preserved under the conformal regraduation (abbreviated as CR from now on). Then we must have

$$-\kappa T_{\mu\nu}=R_{\mu\nu}-(1/2)g_{\mu\nu}R,\quad \text{for } g_{\mu\nu}, \tag{A·2}$$

where we can regard κ as not only generally different from κ_0, but also a function of x's, because κ is a factor of $T_{\mu\nu}$ in (A·2). Contracting (A·1) and (A·2) with respect to $\mathring{g}^{\mu\nu}$ and $g^{\mu\nu}$ respectively, we obtain

$$\mathring{R}=\kappa_0\mathring{T}\,;\ R=\kappa T. \tag{A·3), (A·4}$$

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* We shall hereafter abbreviate the term "in the space-time with metric (4·1) (or (2·1))" as "in $d\mathring{s}$ (or ds)".

On the other hand, from (4·2) we have

$$\begin{cases} \mathring{R}_{\mu\nu}=R_{\mu\nu}+2\left[\chi^{-1}\chi_{;\mu\nu}-2\dfrac{\chi_{,\mu}\chi_{,\nu}}{\chi^2}+(1/2)g_{\mu\nu}\dfrac{\Delta_1\chi}{\chi^2}\right]+g_{\mu\nu}\left[\dfrac{\Delta_2\chi}{\chi}\right], & (A\cdot5) \\ \mathring{R}=\chi^{-2}[R+6\chi^{-1}\Delta_2\chi], & (A\cdot6) \end{cases}$$

with

$$\left.\begin{aligned} &\Delta_1\chi\equiv g^{\alpha\beta}\chi_{,\alpha}\chi_{,\beta}, \\ &\Delta_2\chi\equiv g^{\alpha\beta}\chi_{;\alpha\beta}=g^{\alpha\beta}(\chi_{,\alpha\beta}-\chi_{,\gamma}\Gamma^{\gamma}_{\alpha\beta})=\frac{1}{(-g)^{1/2}}\{(-g)^{1/2}\chi^{,\alpha}\}_{,\alpha}. \end{aligned}\right\} \quad (A\cdot7)$$

From (A·3) and (A·6), we obtain

$$\Delta_2\chi=(1/6)[\kappa_0\mathring{T}\chi^2-R]\chi. \quad (A\cdot8)$$

Substituting (A·5) and (A·6) into (A·1) and taking account of (A·2), we get

$$-\kappa_0\mathring{T}_{\mu\nu}=-\kappa T_{\mu\nu}+(2/\chi^2)[\chi(\chi_{;\mu\nu}-g_{\mu\nu}\Delta_2\chi)-2\chi_{,\mu}\chi_{,\nu}+(1/2)g_{\mu\nu}\Delta_1\chi], \quad (A\cdot9)$$

whose contraction by $\mathring{g}^{\mu\nu}=\chi^{-2}g^{\mu\nu}$ leads us to (A·8), by virtue of (A·4).

Thus, in order that Einstein's field equations (A·1) for $\mathring{g}_{\mu\nu}$ can be converted into the corresponding ones (A·2) for $g_{\mu\nu}$, the scalar χ appearing in (4·2) should obey the "field equation" (A·8), in addition to the occurrence of the relation (A·9) between $\mathring{T}_{\mu\nu}$ and $T_{\mu\nu}$. If we take into consideration these situations, our CR should be consistent with the following procedure: Now let both the metric tensor $g_{\mu\nu}(x)$ in ds and the spur of the energy tensor, $\mathring{T}$, in $d\mathring{s}$ be given. Then (A·8) is the field equation for χ in ds, so that it can be solved under suitable initial and boundary conditions. On the other hand, $\kappa T_{\mu\nu}$ can be calculated by making use of (A·2). Substituting χ and $\kappa T_{\mu\nu}$ thus calculated in the right-hand side of (A·9), we obtain $\mathring{T}_{\mu\nu}$.

(B) *Lagrangian formalism in ds*

It is suggested from the preceding paragraph that various relations deduced in (A) could be revived from the standpoint of a Lagrangian formalism for χ in ds. Since $g_{\mu\nu}(x)$ is already known, the Lagrangian density for χ should generally be of the form

$$\mathfrak{L}=\mathfrak{L}(\chi, \chi_{,\alpha}, \chi_{,\alpha\beta}; x), \quad (B\cdot1)$$

where the arguments $\chi_{,\alpha\beta}=\partial^2\chi/\partial x^\alpha\partial x^\beta$ are inserted for our subsequent discussion of (A·9) in which the second derivatives of χ are included. Then the action integral to be variated is

$$I=\int\mathfrak{L}(\chi, \cdots; x)d^4x. \quad (B\cdot2)$$

Now let an infinitesimal transformation of coordinates be

$$x^\alpha\to{}'x^\alpha=x^\alpha+\xi^\alpha(x). \quad (B\cdot3)$$

Then the transformation properties of χ and $\chi_{,\alpha}$ under (B·3) can be represented by

$$\delta\chi=0\,;\ \delta(\chi_{,\alpha})=-\chi_{,\beta}\xi^{\beta}{}_{,\alpha}, \tag{B·4}$$

so that, for the usual variation specified by the symbol δ^*, we have

$$\delta\chi=0=\delta^*\chi+\chi_{,\alpha}\xi^{\alpha}, \text{ etc.,} \tag{B·5}$$

where $\delta^*(\chi_{,\alpha})=(\delta^*\chi)_{,\alpha}$. Taking account of (B·3), (B·4) and (B·5), we obtain from (B·2)

$$0=\delta I=\int d^4x[[\mathfrak{L}]\delta^*\chi+\mathfrak{A}^{\alpha}{}_{,\alpha}], \tag{B·6}$$

with

$$\begin{cases} [\mathfrak{L}]\equiv\partial\mathfrak{L}/\partial\chi-\mathfrak{p}^{\alpha}{}_{,\alpha}, & \text{(B·7)}\\ \mathfrak{A}^{\alpha}\equiv-\mathfrak{t}_{\beta}{}^{\alpha}\xi^{\beta}-\mathfrak{p}^{\alpha\beta}\chi_{,\gamma}\xi^{\gamma}{}_{,\beta}\equiv-\mathfrak{J}_{\beta}{}^{\alpha}\xi^{\beta}-\mathfrak{p}^{\alpha\beta}\chi_{,\gamma}\xi^{\gamma}{}_{;\beta}, & \text{(B·8)} \end{cases}$$

where

$$\begin{cases} \mathfrak{t}_{\beta}{}^{\alpha}\equiv\chi_{,\beta}\mathfrak{p}^{\alpha}+\chi_{,\beta\gamma}\mathfrak{p}^{\alpha\gamma}-\delta_{\beta}{}^{\alpha}\mathfrak{L}, & \text{(B·9)}\\ \mathfrak{J}_{\beta}{}^{\alpha}\equiv\chi_{,\beta}\mathfrak{p}^{\alpha}+\chi_{;\beta\gamma}\mathfrak{p}^{\alpha\gamma}-\delta_{\beta}{}^{\alpha}\mathfrak{L}, & \text{(B·10)}\\ \mathfrak{p}^{\alpha}\equiv\dfrac{\partial\mathfrak{L}}{\partial\chi_{,\alpha}}-\mathfrak{p}^{\alpha\beta}{}_{,\beta}\,;\ \mathfrak{p}^{\alpha\beta}\equiv\dfrac{\partial\mathfrak{L}}{\partial\chi_{,\alpha\beta}}, & \text{(B·11)} \end{cases}$$

in which $\mathfrak{p}^{\alpha}$ and $\mathfrak{p}^{\alpha\beta}$ are contravariant vector and tensor (of the second rank) densities respectively.

The expression (B·6) leads us to the following field equation for χ:

$$[\mathfrak{L}]\equiv\partial\mathfrak{L}/\partial\chi-\mathfrak{p}^{\alpha}_{,\alpha}=0. \tag{B·12}$$

As is easily seen, in contrast with $\mathfrak{t}_{\beta}^{\alpha}$, $\mathfrak{J}_{\beta}^{\alpha}$ is a tensor density such that $\chi_{,\beta\gamma}$ in $\mathfrak{t}_{\beta}^{\alpha}$ is replaced by $\chi_{;\beta\gamma}$. We have no a priori reason to assert that $\mathfrak{J}_{\beta}^{\alpha}$ is superior to the pseudo-tensor density $\mathfrak{t}_{\beta}^{\alpha}$ as the quantity specifying the content of energy and momentum for the χ-field; this is analogous to the situation in the problem concerning the energy-momentum of the gravitational field. We shall however adopt as the required one the expression $\mathfrak{J}_{\beta}^{\alpha}$, which is in conformity with the postulate (a) in § 2. Moreover, it should be remarked that the covariant divergence of $\mathfrak{J}_{\beta}^{\alpha}$ does not vanish on account of the arguments x's in $\mathfrak{L}$.

After the above general consideration, let us put

$$\mathfrak{L}=(1/6)(-g)^{1/2}\left[(\Delta_1\chi-2\chi\Delta_2\chi)+\frac{1}{4}(\kappa_0\mathring{T}\chi^2-2R)\chi^2\right], \tag{B·13}$$

which is clearly of the type (B·1). Inserting this into (B·11), we optain

$$\mathfrak{p}^{\alpha}=(2/3)(-g)^{1/2}\chi,^{\alpha}\,;\ \mathfrak{p}^{\alpha\beta}=(-1/3)(-g)^{1/2}g^{\alpha\beta}\chi, \tag{B·14}$$

so that

$$\mathfrak{p}^{\alpha}+2\mathfrak{p}^{\alpha\beta}{}_{;\beta}=0. \tag{B·15}$$

Substituting (B·13) and (B·14) in (B·7), we get

$$[\mathfrak{L}]=-(-g)^{1/2}[\Delta_2\chi-(1/6)(\kappa_0\mathring{T}\chi^2-R)\chi], \tag{B·16}$$

so that the field equation (A·8) is deduced as Euler's equation (B·12) for $\mathfrak{L}$ given by (B·13).

Now let $\boldsymbol{T}_\beta^\alpha$ be a tensor corresponding to the tensor density defined by (B·10), i.e.

$$\mathcal{T}_\beta^\alpha=(-g)^{1/2}\boldsymbol{T}_\beta^\alpha,\quad \boldsymbol{T}_{\alpha\beta}=g_{\gamma\beta}\boldsymbol{T}_\alpha{}^\gamma. \tag{B·17}$$

Then, from (B·14) and (B·10), we have

$$\boldsymbol{T}_{\mu\nu}=-(1/3)(\chi\chi_{;\mu\nu}-2\chi_{,\mu}\chi_{,\nu})-g_{\mu\nu}L,\quad (L=\mathfrak{L}/(-g)^{1/2}). \tag{B·18}$$

Substituting (B·13) in the above expression, we obtain

$$\begin{aligned}\boldsymbol{T}_{\mu\nu}=&-(1/3)[\chi(\chi_{;\mu\nu}-g_{\mu\nu}\Delta_2\chi)-2\chi_{,\mu}\chi_{,\nu}+(1/2)g_{\mu\nu}\Delta_1\chi]\\&-(1/24)(\kappa_0\mathring{T}\chi^2-2R)\chi^2g_{\mu\nu}.\end{aligned} \tag{B·19}$$

If we take (A·2) into consideration, a comparison of the above expression with (A·9) enables us to deduce

$$\boldsymbol{T}_{\mu\nu}=(1/6)\chi^2[\kappa_0(\mathring{T}_{\mu\nu}-(1/4)\mathring{T}\mathring{g}_{\mu\nu})+R_{\mu\nu}], \tag{B·20}$$

which is the relation between the energy-tensor $\boldsymbol{T}_{\mu\nu}$ for χ, defined by (B·17) with (B·10) from the Lagrangian density $\mathfrak{L}$ given by (B·13), and the energy-tensor $\mathring{T}_{\mu\nu}$ (and its spur $\mathring{T}$) in Einstein's field equations (A·1) for $\mathring{g}_{\mu\nu}$. Multiplying (B·20) by $g^{\mu\nu}=\chi^2\mathring{g}^{\mu\nu}$ and contracting, we have

$$\boldsymbol{T}\equiv g^{\mu\nu}\boldsymbol{T}_{\mu\nu}=(1/6)R\chi^2, \tag{B·21}$$

so that $\boldsymbol{T}$ does not vanish, except when $R=0$.

Next, we shall look into the covariant divergence of $\mathcal{T}_\mu^\nu$. From (B·19) and (B·17), we obtain

$$\begin{aligned}\mathcal{T}_{\mu;\nu}^{\nu}&=(-g)^{1/2}\boldsymbol{T}_{\mu;\nu}^{\nu}\\&\doteq(1/3)(-g)^{1/2}[\chi\chi_{,\nu}R_\mu{}^\nu-(1/8)\{\chi^2(\kappa_0\mathring{T})_{,\mu}-2R_{,\mu}\}\chi^2],\end{aligned} \tag{B·22}$$

by virtue of (B·12), so that $\mathcal{T}_{\mu;\nu}^{\nu}\not\equiv 0$ as mentioned previously. To what expression in $d\mathring{s}$ does the above (B·22) correspond, then? If we make use of (B·20), the above expression is written as

$$[\kappa_0(-\mathring{g})^{1/2}(\mathring{T}_\mu{}^\nu-(1/4)\mathring{T}\delta_\mu{}^\nu)]_{;\nu}=-(1/4)(-\mathring{g})^{1/2}(\kappa_0\mathring{T})_{,\mu}. \tag{B·23}$$

On the other hand, we have from (4·3)

$$0\equiv(-\mathring{g})^{1/2*}_{|\nu}=(-\mathring{g})^{1/2}_{;\nu}-4(\ln\chi)_{,\nu}(-\mathring{g})^{1/2}, \tag{B·24}$$

so that (B·23) is reduced to

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* Hereafter we shall use the symbol "|" for the covariant differentiation with respect to $\mathring{g}_{\mu\nu}$.

$$\mathring{T}^{\nu}_{\mu|\nu} \equiv \mathring{T}^{\nu}_{\mu;\nu} + 4(\ln\chi)_{,\nu}(\mathring{T}^{\nu}_{\mu} - (1/4)\mathring{T}\delta^{\nu}_{\mu}) = 0, \tag{B·25}$$

where we have omitted the factor κ_0 (=const., by our prescription). Since the above expression is nothing but the conservation law of general relativity due to the contracted Bianchi identities, it may be said that (B·22) is a regraduated version in ds of the conservation law (B·25) in $d\mathring{s}$.

Summarizing the above, we may say that various relations in (A) are revived from the standpoint of a Lagrangian formalism in ds based on $\mathfrak{L}$ given by (B·13). It is remarkable that the constancy of κ is not assumed, though it is not the case with respect to κ_0.

(C) *Equations of motion for a test particle*

It is well known that a null geodesic in $d\mathring{s}$ corresponds to a null geodesic in ds, so far as the conformal relation (4·2) is satisfied. Accordingly, the postulate (d) in § 2 is preserved under our CR. What is the matter with the postulate (c), then ?

In order to attack this problem, we shall start with the geodesic equations of motion for a test particle in $d\mathring{s}$, i.e.

$$D\mathring{\lambda}^{\alpha}/D\mathring{s} \equiv d\mathring{\lambda}^{\alpha}/d\mathring{s} + \mathring{\Gamma}^{\alpha}_{\beta\gamma}\mathring{\lambda}^{\beta}\mathring{\lambda}^{\gamma} = 0, \tag{C·1}$$

whose first integral is (cf. (4·1))

$$1 = \mathring{g}_{\alpha\beta}\mathring{\lambda}^{\alpha}\mathring{\lambda}^{\beta}, \quad (\mathring{\lambda}^{\alpha} = dx^{\alpha}/d\mathring{s}). \tag{C·2}$$

On the other hand, from (4·1), (2·1) and (4·2), we have

$$d\mathring{s} = \chi ds. \tag{C·3}$$

If we make use of (C·3) and (4·3), (C·1) is reduced to

$$\delta\lambda^{\alpha}/\delta s \equiv d\lambda^{\alpha}/ds + \Gamma^{\alpha}_{\beta\gamma}\lambda^{\beta}\lambda^{\gamma} = F^{\alpha}, \tag{C·4}$$

with

$$F^{\alpha} \equiv (g^{\alpha\beta} - \lambda^{\alpha}\lambda^{\beta})(\ln\chi)_{,\beta}, \tag{C·5}$$

where

$$1 = g_{\alpha\beta}\lambda^{\alpha}\lambda^{\beta}, \quad (\lambda^{\alpha} = dx^{\alpha}/ds). \tag{C·6}$$

Contraction of (C·5) by $\lambda_{\alpha}(=g_{\alpha\beta}\lambda^{\beta})$ leads us to

$$\lambda_{\alpha}F^{\alpha} = 0, \tag{C·7}$$

so that the 4-vector F^{α} in ds is orthogonal to the unit tangential vector λ^{α} to the curve in ds corresponding to the geodesic curve in $d\mathring{s}$. Since the left-hand side of (C·4) denotes a covariant definition of acceleration, (C·4) can be regarded as a relativistic version of Newtonian equations of motion, if we interpret F^{α} as the 4-force (due to the scalar field χ) divided by mass m_0, say, of the test particle. At any rate, except in the case χ=const., the postulate (c) in § 2 is not preserved under the CR, similar to the situation in Rosen's reinterpretation of general

relativity.

(D) *Maxwell's equations and the equations of motion for a charged particle*

In the remaining sections, we shall show first how the behaviors of various physical constants under our CR are intimately connected with those of the field equations in which these constants are included, and then that our CR is ultimately stipulated by the behaviors of these physical constants.

Now we shall begin with Maxwell's equations in $d\mathring{s}$, i.e.

$$\begin{cases} \mathring{F}_{\mu\nu,\lambda}+\mathring{F}_{\nu\lambda,\mu}+\mathring{F}_{\lambda\mu,\nu}=0, & (D\cdot1) \\ \mathring{F}^{\alpha\nu}_{\ \ |\nu}=-\mathring{J}^{\alpha}, & (D\cdot2) \end{cases}$$

using the usual notation. As is well known, (D·1) is equivalent to the existence of a 4-vector $\mathring{A}_{\mu}$ (determined up to the gauge-transformation) such that

$$\mathring{F}_{\mu\nu}=\mathring{A}_{\mu|\nu}-\mathring{A}_{\nu|\mu}=\mathring{A}_{\mu,\nu}-\mathring{A}_{\nu,\mu}. \qquad (D\cdot3)$$

If we use (4·3), (D·2) is reduced to

$$\mathring{F}^{\alpha\nu}_{\ \ ;\nu}+4(\ln\chi)_{,\nu}\mathring{F}^{\alpha\nu}=-\mathring{J}^{\alpha}. \qquad (D\cdot4)$$

Moreover, the energy-tensor of the electromagnetic field is of the form

$$\mathring{E}_{\mu\nu}=-\mathring{F}_{\mu\alpha}\mathring{F}_{\nu\cdot}^{\ \alpha}+(1/4)\mathring{g}_{\mu\nu}\mathring{F}_{\alpha\beta}\mathring{F}^{\alpha\beta}, \qquad (D\cdot5)$$

so that $\mathring{E}\equiv\mathring{g}^{\mu\nu}\mathring{E}_{\mu\nu}=0$.

In the next, we shall consider a charged particle with charge e_0 and mass m_0. The equations of motion for it in $d\mathring{s}$ are

$$D\mathring{\lambda}^{\alpha}/D\mathring{s}-(e/m)_0\mathring{F}_{\cdot\beta}^{\alpha}\mathring{\lambda}^{\beta}=0, \qquad (D\cdot6)$$

where $(e/m)_0$ is an abbreviation of e_0/m_0, and other symbols are of the same meaning as those in (C). By means of the procedures in (C), the above equations are transformed into

$$\delta\lambda^{\alpha}/\delta s-(e/m)_0(\chi\mathring{F}_{\cdot\beta}^{\alpha})\lambda^{\beta}=F^{\alpha}, \qquad (D\cdot7)$$

where F^{α} is given by (C·5). Since the origin of F^{α} is independent of whether the particle in question is charged or not, in contrast to the Lorentz force, it seems natural that we can regard (D·7) as the regraduated version in ds of (D·6) by rewriting it as follows:

$$\delta\lambda^{\alpha}/\delta s-(e/m)F_{\cdot\beta}^{\alpha}\lambda^{\beta}=F^{\alpha}, \qquad (D\cdot8)$$

where e and m stand for the regraduated values of e_0 and m_0 respectively, and $F_{\cdot\beta}^{\alpha}$ is the mixed tensor in ds specifying the strength of the electromagnetic field. However, such a rewriting should hold for any λ^{β}. Then we must have

$$(e/m)F_{\cdot\beta}^{\alpha}=(e/m)_0(\chi\mathring{F}_{\cdot\beta}^{\alpha}). \qquad (D\cdot9)$$

On account of the physical meaning of $F^{\alpha}_{\cdot\beta}$ given above, $F_{\mu\nu}=g_{\mu\lambda}F^{\lambda}_{\cdot\nu}$ should satisfy the same equation as (D·1) for $\mathring{F}_{\mu\nu}$, i.e. at least we must have*

$$F_{\mu\nu}=\mathring{F}_{\mu\nu},\ (A_{\mu}=\mathring{A}_{\mu}+\text{grad. term}). \tag{D·10}$$

Then, by virtue of (4·2), we have

$$F^{\alpha}_{\cdot\beta}=\chi^{2}\mathring{F}^{\alpha}_{\cdot\beta};\ F^{\alpha\beta}=\chi^{4}\mathring{F}^{\alpha\beta}, \tag{D·11}$$

so that (D·9) is reduced to

$$(e/m)=(e/m)_{0}\chi^{-1}, \tag{D·12}$$

which is the relation between $(e/m)_0$ and its regraduated value (e/m). The above expression shows that there is no CR under which both charge and mass are preserved. Moreover, (D·10) is the regraduation law of the field strength under our CR.

Substituting the second expression of (D·11) in (D·4), we have

$$F^{\alpha\nu}{}_{;\nu}=-J^{\alpha}, \tag{D·13}$$

with

$$J^{\alpha}\equiv\chi^{4}\mathring{J}^{\alpha}, \tag{D·14}$$

which is interpreted as the regraduation of current density. Both $\mathring{J}^{\alpha}$ and J^{α}, however, are put phenomenologically as follows:

$$\mathring{J}^{\alpha}=\mathring{\rho}\mathring{\lambda}^{\alpha};\ J^{\alpha}=\rho\lambda^{\alpha}, \tag{D·15}$$

where $\mathring{\rho}$ and ρ are proper densities of charge in $d\mathring{s}$ and in ds respectively. Then (D·14) is reduced to

$$\rho=\mathring{\rho}\chi^{3}, \tag{D·16}$$

since $\lambda^{\alpha}=\chi\mathring{\lambda}^{\alpha}$ (cf. (C)).

From the definition of $\mathring{\rho}$ and ρ, we must have

$$de_{0}=\mathring{\rho}(-\mathring{g})^{1/2}(dt/d\mathring{s})d^{3}x;\ de=\rho(-g)^{1/2}(dt/ds)d^{3}x, \tag{D·17}$$

from which, together with (4·2), (C·3) and (D·16), we obtain $de_0=de$. Accordingly, we have

$$e=e_{0}, \tag{D·18}$$

which is essentially due to (D·10) and our interpretation of (D·14) mentioned before. Inserting (D·18) into (D·12), we obtain

$$m=m_{0}\chi, \tag{D·19}$$

which is the regraduation of mass for a charged particle.

As is immediately seen, (D·13) with (D·10) are not only the regraduated

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* More generally, we can consider the case $F_{\mu\nu}=\text{const.}\times\mathring{F}_{\mu\nu}$, but (D·10) is adopted for simplicity's sake.

version in ds of the original Maxwell equations in $d\mathring{s}$, but also Maxwell equations themselves in ds; this is nothing but the conformal invariancy of Maxwell's equations. Moreover, from (D·5), (D·10) and (4·2), we have

$$E_{\mu\nu}=\chi^2\mathring{E}_{\mu\nu}, \tag{D·20}$$

where $E_{\mu\nu}$ is the energy-tensor constructed from $F_{\mu\nu}$ and $g_{\mu\nu}$. It is to be remarked that the discussion in (A) is valid even when $\mathring{E}_{\mu\nu}$ and $E_{\mu\nu}$ satisfying (D·20) are inserted into the left-hand side of (A·1) and (A·2) respectively, though $\mathring{T}=T=0$ in such a case.

(E) *Dirac's equations in a curved space-time*

For a system consisting of the electromagnetic and electron, say, fields in $d\mathring{s}$, the right-hand side of (D·2), $\mathring{J}^\alpha$, is described by a spinor field $\mathring{\psi}$. Accordingly, it is suggested that the regraduation law (D·14) of current density enables us to deduce the regraduation of $\mathring{\psi}$.

Dirac's equations for the spinor field $\mathring{\psi}$ in $d\mathring{s}$ can be written as

$$\{-i\mathring{\gamma}^\mu\mathring{\nabla}_\mu+(mc/\hbar)_0\}\mathring{\psi}=0, \tag{E·1}*$$

with

$$\mathring{\nabla}_\mu\mathring{\psi}\equiv\partial\mathring{\psi}/\partial x^\mu-\mathring{\Gamma}_\mu\mathring{\psi}, \tag{E·2}$$

where $\mathring{\Gamma}_\mu$ is, within a freedom of an arbitrary multiple of the unit matrix I, given by a solution of $\mathring{\nabla}_\mu\mathring{\gamma}_\nu=0$, i.e.

$$\partial\mathring{\gamma}_\nu/\partial x^\mu=\mathring{\Gamma}^\lambda_{\mu\nu}\gamma_\lambda+\mathring{\Gamma}_\mu\mathring{\gamma}_\nu-\mathring{\gamma}_\nu\mathring{\Gamma}_\mu, \tag{E·3}$$

and $\mathring{\gamma}_\mu$ is a 4–4-matrix such that

$$\mathring{\gamma}_\mu\mathring{\gamma}_\nu+\mathring{\gamma}_\nu\mathring{\gamma}_\mu=2\mathring{g}_{\mu\nu}I. \tag{E·4}$$

Similarly, we have for the adjoint of $\mathring{\psi}$, $\bar{\mathring{\psi}}$,

$$\bar{\mathring{\psi}}[i\bar{\mathring{\nabla}}_\mu\mathring{\gamma}^\mu+(mc/\hbar)_0]=0. \tag{E·5}$$

If we introduce a 4–4-matrix such that

$$\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=2g_{\mu\nu}I, \tag{E·6}$$

then (4·2) leads us to

$$\mathring{\gamma}_\mu=\chi\gamma_\mu\,;\ \mathring{\gamma}^\mu=\chi^{-1}\gamma^\mu, \tag{E·7}$$

within a freedom of spin-transformation; we shall neglect such a freedom here, since it is not essential to our discussion. Similarly, we introduce a matrix Γ_μ such that

$$\partial\gamma_\nu/\partial x^\mu=\Gamma^\lambda_{\mu\nu}\gamma_\lambda+\Gamma_\mu\gamma_\nu-\gamma_\nu\Gamma_\mu. \tag{E·8}$$

Then, by virtue of (4·3), (E·7) and (E·8), (E·3) can be rewritten as

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* The suffix "0" is attached to $(m, c, \hbar)$, in order to emphasize that (E.1) is described in $d\mathring{s}$,

$$\Lambda_\mu\gamma_\nu-\gamma_\nu\Lambda_\mu+[\delta_\mu{}^\alpha(\ln\chi)_{,\nu}-g_{\mu\nu}(\ln\chi)_{,}{}^\alpha]\gamma_\alpha=0, \tag{E·9}$$

where

$$\Lambda_\mu\equiv\mathring{\Gamma}_\mu-\Gamma_\mu. \tag{E·10}$$

Solving (E·9) with respect to Λ_μ, we obtain

$$\Lambda_\mu=(1/2)(\ln\chi)_{,}{}^\alpha\delta_\mu{}^\beta\gamma_{[\alpha}\gamma_{\beta]}, \tag{E·11}$$

where $\gamma_{[\alpha}\gamma_{\beta]}=(\gamma_\alpha\gamma_\beta-\gamma_\beta\gamma_\alpha)/2$, and a freedom of the unit matrix I is absorbed by Γ_μ. Substituting (E·7), (E·10) and (E·11) in (E·1), we have

$$\{-i\gamma^\mu(\nabla_\mu-\Lambda_\mu)+(mc/\hbar)_0\chi\}\mathring{\psi}=0, \tag{E·12}$$

where $\nabla_\mu=\partial/\partial x^\mu-\Gamma_\mu$.

On the other hand, (E·1), (E·5) and (E·3) lead us to

$$\{(-\mathring{g})^{1/2}\bar{\mathring{\psi}}\mathring{\gamma}^\mu\mathring{\psi}\}_{,\mu}=0, \tag{E·13}$$

i.e. $(\bar{\mathring{\psi}}\mathring{\gamma}^\mu\mathring{\psi})$ is a contravariant vector in $d\mathring{s}$ whose covariant divergence vanishes automatically, in a way similar to the current density $\mathring{J}^\mu$ in (D·2). Accordingly, we can put

$$\mathring{J}^\alpha=e_0(\bar{\mathring{\psi}}\mathring{\gamma}^\alpha\mathring{\psi}). \tag{E·14}$$

If we rewrite (D·14) by using (D·18), (E·7) and (E·14), it becomes

$$J^\alpha=e(\bar{\psi}\gamma^\alpha\psi), \tag{E·15}$$

with

$$\psi\equiv\chi^{3/2}\mathring{\psi}, \tag{E·16}$$

which, if we regard ψ as the spinor field in ds, is the regraduation of spinor in question; it is easy to verify that the covariant divergence of J^α vanishes automatically. Inserting (E·16) into (E·12), we get after some reductions

$$[-i\gamma^\mu\nabla_\mu+(mc/\hbar)_0\chi]\psi=0. \tag{E·17}$$

Similarly, (E·5) is reduced to

$$\bar{\psi}[i\bar{\nabla}_\mu\gamma^\mu+(mc/\hbar)_0\chi]=0, \tag{E·18}$$

where $\bar{\nabla}_\mu=\partial/\partial x^\mu+\Gamma_\mu$.

(E·17) is of course mathematically equivalent to the original one (E·1), but it is described by the quantities in ds, except in the factor $(mc/\hbar)_0$. We may regard this as Dirac's equations (in ds) for ψ interacting with χ. In the following we shall, however, interpret the factor χ in the mass term from the standpoint of regraduation. Rewriting (E·17) as

$$[-i\gamma^\mu\nabla_\mu+(mc/\hbar)]\psi=0, \tag{E·19}$$

with

$$(mc/\hbar)=(mc/\hbar)_0\chi, \tag{E·20}$$

then we can regard (E·20) as the regraduation law of the Compton wavelength, where $(\hbar, c, m)$ stand for the regraduated values of $(\hbar_0, c_0, m_0)$. However, we have (D·19) as the regraduation law of mass for a charged particle, so that (E·20) leads us to

$$(\hbar/c) = (\hbar/c)_0. \tag{E·20}$$

Moreover, for a system consisting of the spinor field interacting with the electromagnetic field specified by the 4-vector $\mathring{A}_\mu$, the operator $\mathring{\Gamma}_\mu$ in (E·1) should be replaced by $[\mathring{\Gamma}_\mu - i(e/\hbar c)_0 \mathring{A}_\mu]$. In order that the regraduated version of these equations can be of the form (E·19) with $[\Gamma_\mu - i(e/\hbar c)A_\mu]$ in place of Γ_μ, we must have (cf. (D·10) and (D·18))

$$(\hbar c) = (\hbar c)_0. \tag{E·22}$$

From (E·21) and (E·22), we obtain

$$\hbar = \hbar_0 \text{ and } c = c_0, \tag{E·23}$$

which means that both $\hbar_0$ and c_0 should preserve their numerical values under our CR.

Remarks. To sum up the above, it may be said that Dirac's equations are formally "invariant" (cf. (E·1) and (E·19)) under our CR, in which (E·23), (D·18) and (D·19) are demanded. As is easily seen, however, even such a formal invariancy is not valid for other field eqautions such as the one for a neutral scalar field. This is because (D·19) should be applied to the mass of a neutral particle too, on account of its intimate connection with the demand (E·23) which is very desirable in view of the universal character of $\hbar_0$ and c_0.

(F) *Natural constants, charge and mass*

According to the discussions hitherto developed, the behaviors of various physical constants under our CR are as follows:

(i) The natural constants c_0 and h_0, characteristic of the relativistic and quantum theories respectively, preserve their numerical values ((E·23)).

(ii) This is also the case with respect to the coupling constant, e_0, for the electromagnetic field ((D·18)).

(iii) The regraduation of mass for a charged particle is stipulated by (D·19) and it should also be applied to a neutral particle in view of the situation mentioned in the remark of (E).

(iv) We have not yet any decisive statement as for the gravitation constant κ_0, though the assumption $\kappa = \text{const.}$ is not used in (A) and (B). Though the first three items are set up independently of the last one, there would be an intimate connection between them, because we have only three physical dimensions independent of each other.

In order to examine this problem, we shall come back to the original definition of regraduation, from which the relation (4·2) is deduced. In our notation, the regraduation in question can be represented by (cf. (2·6))

$$X^{(0)}=\chi\overset{\circ}{X}{}^{(0)},\quad X^{(i)}=\chi\overset{\circ}{X}{}^{(i)}, \tag{F·1}$$

since the universal validity of (i) is accepted. On the other hand, as quantities (constructed from physical constants) whose physical dimensions are [length], we have the Compton wavelength of an elementary particle, $(\hbar/mc)_0$, and the classical electron radius, $(e^2/mc^2)_0$, etc. So far as both (i) and (ii) are accepted, these quantities can suffer from the procedure of regraduation only through the factor $(m_0)^{-1}$. In this respect, we shall adopt as the values of $[\sum_{i=1}^{3}(\overset{\circ}{X}{}^{(i)})^2]^{1/2}$ and $[\sum_{i=1}^{3}(X^{(i)})^2]^{1/2}$ respectively in (F·1) the following ones:

$$l_0=(\hbar/mc)_0\,;\ l=(\hbar/mc). \tag{F·2}$$

From (F·2) and the second expression of (F·1), we easily obtain (D·19), irrespective of whether a particle is charged or not.

Now the physical dimensions of κ_0 are $[\kappa_0]=[l_0/m_0]$, where l_0 is specified by the first expression of (F·2). Accordingly, if we further take the standpoint of dimensional analysis, we obtain

$$\kappa=\kappa_0\chi^{-2}, \tag{F·3}$$

by virtue of (F·2) and (D·19). The above relation is to be regarded as the regraduation law of κ_0 in question.

Remark. If we insert (F·3) into (A·2), it becomes

$$-\kappa_0 T^{*}_{\mu\nu}\equiv -\kappa_0\chi^{-2}T_{\mu\nu}=R_{\mu\nu}-(1/2)g_{\mu\nu}R. \tag{F·4}$$

The discussion in (A) and (B) remain valid, even when we regard $T^{*}_{\mu\nu}=\chi^{-2}T_{\mu\nu}$ as the energy-tensor in *ds*. Moreover, if we reinterpret (F·3) as the definition of χ, (A·8) can be rewritten as

$$\Delta_2\kappa-(3/2)\frac{\Delta_1\kappa}{\kappa}=-(1/3)(\kappa_0{}^2\overset{\circ}{T}-\kappa R), \tag{F·5}$$

which is similar to one of the field equations in Jordan's[13] gravitational theory with variable gravitation "constant". This suggests that his theory can be reinterpreted from the standpoint of our CR. This problem will be treated in another paper.

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References

1) E. A. Milne and G. J. Whitrow, Zs. f. Ap. **15** (1938), 263.
E. A. Milne, *Kinematic Relativity* (Oxford, 1948).
2) H. P. Robertson, Ap. J. **82** (1935), 284; **83** (1936), 187.
3) A. G. Walker, Proc. Lond. Math. Soc. **42** (1937), 90.
4) See, for example, W. Rindler, M. N. **116** (1956), 355.
5) A. G. Walker, Proc. Roy. Soc. Edinburgh **62** (1946), 164.

6) G. C. McVittie, Proc. Roy. Soc. Edinburgh **62** (1945), 147.
7) (i) G. D. Birkhoff, Proc. Nat. Acad. Sci. **29** (1943), 231; **30** (1944), 324.
A. Barajas, Proc. Nat. Acad. Sci. **30** (1944), 54.
(ii) J. L. Synge, Proc. Roy. Soc. **A211** (1952), 303.
A. Schild, Proc. Roy. Soc. **A235** (1956), 202.
(iii) R. H. Dicke, Rev. Mod. Phys. **29** (1957), 355, 363.
8) N. Rosen, Phys. Rev. **57** (1940), 147, 150.
M. Kohler, Z. Physik **131** (1952), 571.
S. N. Gupta, Phys. Rev. **96** (1954), 1683; Rev. Mod. Phys. **29** (1957), 334.
9) H. Nariai and Y. Ueno, Prog. Theor. Phys. **20** (1958), 703.
10) R. Carnap, Introductory remarks of "*The Philosophy of Space and Time*" by H. Reichenbach (New York, 1956).
11) B. Hoffmann, Phys. Rev. **89** (1953), 49, 52.
12) Various papers by E. T. Whittaker, W. O. Kermack, W. H. McCrea, I. M. H. Etherington and A. G. Walker. (See, G. Temple, Proc. Roy. Soc. **A168** (1938), 122).
E. Newmann and J. N. Goldberg, Phys. Rev. **114** (1959), 1391.
13) P. Jordan, *Schwerkraft und Weltall* (Braunschweig, 1955).

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Non-Locality in the Second Order Potential†

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The non-local properties of the second order two nucleon interaction are investigated using *ps-ps* meson theory and the Tamm-Dancoff approach. None of the customary transformations are applied to the potential or wave function, so the second order potential derived is both momentum dependent and energy dependent. The potential is hermitian, however, so that the two nucleon component of the wave function can be normalized in the usual manner, conserving probability. The size of the non-local corrections on scattering states is estimated as a function of energy and as a function of the internucleon separation distance. The Born approximation is used to show that the lowest order velocity correction accounts for almost all of the non-local corrections at 300 Mev laboratory energy where it decreases the static scattering amplitude by about 25 per cent. The energy dependence of the potential is instrumental for the Born approximation scattering amplitude being the same as given by lowest order S-matrix theory. The non-local corrections are estimated as a function of internucleon distance by defining an "effective" scattering potential which disguises most of the velocity terms as a static fourth order potential. Velocity corrections are found to be very large in triplet states, dominating over the static second order potential at about 1.5~3.0 f. Comparison with potentials of others where various transformations have been employed is discussed. There is no spin orbit component in the second order *ps-ps* potential, but the corresponding potential derived from scalar meson theory does have such a component. The latter calculation suggests that a spin orbit potential should really be regarded as a quadratic momentum dependence and therefore does not have an apriori preference over other admissible quadratic dependences.

§ 1. Introduction

The recent trend in phenomenological studies of the two-nucleon problem has been to emphasize the velocity dependence of the two nucleon potential.[1),2)] The present situation may be summarized as follows. Using static potentials with widely varying radial shapes and a general spin angle dependence, the experimental data available for two-nucleon systems can not be adequately fit at all energies. This is true even though hard cores (or some other cutoff parameter) are introduced separately for even and odd states.[3),4)] Thus, if one wishes to retain the usual formulation of nuclear physics where only non-relativistic nucleons occur explicitly,

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one is forced to admit rather general interactions, i.e. certain velocity dependent (or non-local) potentials.

How one should proceed in introducing a velocity dependence into phenomenological investigations of the two-nucleon potential is really open to question. Present phenomenological investigations have taken the attitude that the familiar spin-orbit potential is the natural velocity dependence to consider since it is the only scalar invariant which depends linearly on the relative momentum.[5] Although this idea is bolstered by the fact that phenomenological potentials with a spin-orbit dependence are considerably more successful than those without a spin-orbit potential, the special role of the spin-orbit potential is not so readily established from a theoretical point of view. In fact, other velocity dependences seem to be preferred. Furthermore, the spin-orbit potential may well arise as a correction not linear but quadratic in the relative momentum.

Most current theories of the two nucleon potential are based on the relativistic formulation of interacting fields, and extraction of accurate predictions from a relativistic theory of interacting fields corresponding to particles of finite mass has proven to be quite formidable. The basic difficulty is that there is no rapidly convergent method of calculation available. Consequently, the problem of finding the non-relativistic interaction of two nucleons at all separation distances in co-ordinate space (the operator which will yield the correct scattering phase shifts) is still not satisfactorily solved. If the separation distance of the two nucleons is kept large enough, however, the correctness of the potential can be insured.

In the present work, the Tamm-Dancoff method is used to investigate the velocity dependence of the two-nucleon potential. No innovation is introduced into the *method* of calculating the potential and, primarily for simplicity, the calculation is carried through only for the second order potential (the interaction arising from the exchange of a single meson).[6] For a realistic example, the forms of non-locality that occur and their effective strength are investigated. Further, the velocity and energy dependent contributions are compared with potentials where these contributions have been transformed into local interactions.

§ 2. Second order potential in momentum space

The field equation for the strongly interacting nucleon and meson fields is

$$H\psi=E\psi, \tag{1}$$

where

$$H=H_N+H_M+H_I$$

is the system Hamiltonian and ψ is its state vector for the eigenvalue E. H_N and H_M are the free nucleon and free meson field Hamiltonians, respectively, and H_I is the interaction energy between these fields. Explicitly,

$$H_I = G\int d\boldsymbol{x}\bar{\psi}(\boldsymbol{x})I\psi(\boldsymbol{x}), \tag{2}$$

where G is the coupling constant between the nucleon and pion fields and I is the operator coupling these fields:

$$I=\phi(\boldsymbol{x}) \quad \text{scalar meson field,} \tag{2a}$$

$$I=\gamma_5\sum_\alpha \tau_\alpha\phi_\alpha(\boldsymbol{x}) \quad \text{pseudoscalar meson field.} \tag{2b}$$

The amplitude ϕ describes the meson field and $\boldsymbol{\tau}$ is the nucleon isospin vector.

Really the derivation of the nuclear potential is only the process of finding the non-relativistic form of Eq. (1) for the case of two interacting nucleons. This non-relativistic equation will take the form of the familiar Schrödinger equation for two particles in which the two nucleon potential can be readily identified. We shall follow this procedure using the Tamm-Dancoff technique to account for the two-nucleon interaction only in the lowest approximation of single meson exchanges.[7),8),9)]

Using the center of mass system and the momentum representation, the second order relativistic equation of motion for the two-nucleon system is given by the following:

$$(E-2E(\boldsymbol{p}))F(\boldsymbol{p})=\frac{G^2M^2}{4(2\pi)^3}\frac{E(\boldsymbol{p})+M}{E^2(\boldsymbol{p})}\int\frac{d\boldsymbol{k}}{\omega(\boldsymbol{k})}(E-E(\boldsymbol{p})-E(\boldsymbol{p}-\boldsymbol{k})-\omega(\boldsymbol{k}))^{-1}$$

$$\times\frac{E(\boldsymbol{p}-\boldsymbol{k})+M}{E^2(\boldsymbol{p}-\boldsymbol{k})}T(\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\boldsymbol{p},\boldsymbol{k})F(\boldsymbol{p}-\boldsymbol{k}). \tag{3}$$

$F(\boldsymbol{p})$ is the two nucleon wave function (i.e. the projection of ψ in (1) onto the state of two nucleons with no pairs or mesons), $E(\boldsymbol{p})=\sqrt{p^2+M^2}$, $\omega(\boldsymbol{k})=\sqrt{k^2+\mu^2}$, and $\boldsymbol{\sigma}_{1,2}$ are the Pauli spin operators for nucleons 1 and 2, respectively. In the scalar meson theory (Eq. (2a))

$$T=\left[1-\frac{p^2-(\boldsymbol{p}\cdot\boldsymbol{\sigma}_1)(\boldsymbol{k}\cdot\boldsymbol{\sigma}_1)}{(E(\boldsymbol{p})+M)(E(\boldsymbol{p}-\boldsymbol{k})+M)}\right]\left[1-\frac{p^2-(\boldsymbol{p}\cdot\boldsymbol{\sigma}_2)(\boldsymbol{k}\cdot\boldsymbol{\sigma}_2)}{(E(\boldsymbol{p})+M)(E(\boldsymbol{p}-\boldsymbol{k})+M)}\right], \tag{4a}$$

whereas in the pseudoscalar meson theory (Eq. (2b))

$$T=\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\left[\frac{(\boldsymbol{p}\cdot\boldsymbol{\sigma}_1)(\boldsymbol{p}\cdot\boldsymbol{\sigma}_2)}{(E(\boldsymbol{p})+M)^2}+\frac{((\boldsymbol{p}-\boldsymbol{k})\cdot\boldsymbol{\sigma}_1)((\boldsymbol{p}-\boldsymbol{k})\cdot\boldsymbol{\sigma}_2)}{(E(\boldsymbol{p}-\boldsymbol{k})+M)^2}\right.$$

$$\left.-\frac{(\boldsymbol{p}\cdot\boldsymbol{\sigma}_1)((\boldsymbol{p}-\boldsymbol{k})\cdot\boldsymbol{\sigma}_2)}{(E(\boldsymbol{p})+M)(E(\boldsymbol{p}-\boldsymbol{k})+M)}-\frac{((\boldsymbol{p}-\boldsymbol{k})\cdot\boldsymbol{\sigma}_1)(\boldsymbol{p}\cdot\boldsymbol{\sigma}_2)}{(E(\boldsymbol{p})+M)(E(\boldsymbol{p}-\boldsymbol{k})+M)}\right]. \tag{4b}$$

In each case, the function T contains all the interesting spin dependence in the potential. Equation (3) is well known and has been derived by several authors.[10)]

While only one meson is exchanged between the nucleons, the separate nucleons, their interaction with the meson field and the meson are treated exactly.

Thus G, M, and μ are the renormalized quantities. Unfortunately we do not know how to conveniently describe the exact dependence of the pion-nuclear interaction on, say, the momentum transferred to the pion. This momentum dependence of the pion-nucleon interaction as compared with the unrenormalized interaction arises from two sources: renormalization of the interaction involving one nucleon, and the presence of the pion field associated with the second nucleon.[11] The unrenormalized form of the pion-nucleon interaction is used here, considering this as an example of what the actual momentum dependence might be.

§ 3. Non-relativistic transformation to coordinate space

In order to obtain the two nucleon potential, Eq. (3) is reduced to its non-relativistic value. The Fourier transform of the resulting equation will have the form of the usual non-relativistic Schrödinger equation, and the second order potential can be readily identified.

In taking the non-relativistic limit, $\boldsymbol{p}$ and $\boldsymbol{k}$ (and therefore $\boldsymbol{p}-\boldsymbol{k}$) are treated on the same footing and $(p/M)^2$ is used as the expansion parameter. The non-relativistic energy (kinetic plus potential) is

$$\varepsilon = E - 2M. \tag{5}$$

The essential energy denominator is

$$(E-E(\boldsymbol{p})-E(\boldsymbol{p}-\boldsymbol{k})-\omega(\boldsymbol{k}))^{-1} \approx \frac{-1}{\omega(\boldsymbol{k})}\left(1-\frac{1}{2\omega(\boldsymbol{k})}\right.$$
$$\left.\times\left(\frac{p^2}{M}+\frac{(\boldsymbol{p}-\boldsymbol{k})^2}{M}\right)+\frac{\varepsilon}{\omega(\boldsymbol{k})}\right). \tag{6}$$

In the usual "static" theories, the right side of Eq. (6) is replaced by $-1/\omega(\boldsymbol{k})$. In calculating the Born approximation scattering amplitude (see § 5 below), Eq. (6) reduces exactly to the "static" result of $-1/\omega(\boldsymbol{k})$; this is perhaps an indication that the expansion in (6) is quite accurate.

The non-relativistic energy, ε, is treated as being of order p^2/M. (For large separation distances this is clearly correct, and for small separation distances the approximation depends on the sign of potential. In the region where the potential is accurately predicted ($r \gtrsim 2\mathrm{f}$), the approximation applies.)

In the non-relativistic (but non-static) limit, Eq. (3) has the form

$$(\varepsilon - p^2/M)F(\boldsymbol{p}) = \int d\boldsymbol{k}\, K_\varepsilon(\boldsymbol{p}, \boldsymbol{k})F(\boldsymbol{p}-\boldsymbol{k}).$$

Transforming to coordinate space, this equation becomes

$$(\varepsilon + \nabla^2/M)\psi(\boldsymbol{r}) = \int d\boldsymbol{r}'\, U_\varepsilon(\boldsymbol{r}, \boldsymbol{r}')\psi(\boldsymbol{r}'), \tag{7}$$

where

$$U_\varepsilon(\boldsymbol{r}, \boldsymbol{r}') = \int d\boldsymbol{k} (\boldsymbol{r}|K_\varepsilon(\boldsymbol{p}, \boldsymbol{k})|\boldsymbol{r}')$$

$$= \int d\boldsymbol{k} \int d\boldsymbol{p} \frac{e^{-i\boldsymbol{p}\cdot\boldsymbol{r}}}{(2\pi)^{3/2}} K_\varepsilon(\boldsymbol{p}, \boldsymbol{k}) \frac{e^{i(\boldsymbol{p}-\boldsymbol{k})\cdot\boldsymbol{r}'}}{(2\pi)^{3/2}}. \tag{8}$$

The integration over $\boldsymbol{p}$ may be formally carried out as follows. First move K_ε to the left in the integrand in (8) and replace its dependence on $\boldsymbol{p}$ with the operator $i\boldsymbol{\nabla}$. The remaining integration over $\boldsymbol{p}$ is then simply a delta function and Eq. (7) becomes

$$(\varepsilon + \boldsymbol{\nabla}^2/M)\psi(\boldsymbol{r}) = V_\varepsilon(\boldsymbol{r}, -i\boldsymbol{\nabla})\psi(\boldsymbol{r}), \tag{9}$$

where

$$V_\varepsilon(\boldsymbol{r}, -i\boldsymbol{\nabla}) = \int d\boldsymbol{k} K_\varepsilon(i\boldsymbol{\nabla}, \boldsymbol{k}) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} \tag{10}$$

and

$$V_\varepsilon(\boldsymbol{r}, -i\boldsymbol{\nabla})\psi(\boldsymbol{r}) = \int d\boldsymbol{r}' U_\varepsilon(\boldsymbol{r}, \boldsymbol{r}')\psi(\boldsymbol{r}'). \tag{11}$$

From Eq. (7) it is clear that the scalar and pseudoscalar meson theories predict a non-local, energy dependent potential which, from Eq. (11), is equivalent to an ordinary multiplicative type potential with explicit energy and velocity dependence. The restriction only to lowest order velocity corrections (to order $(p/M)^2$), however, is a short range limitation on the amount of non-locality. As long as one uses the usual non-relativistic kinetic energy operator, i.e. p^2/M, this limitation must be imposed for overall consistency. Complete non-locality corresponds to a relativistic interaction which accounts for all orders of $(p/M)^2$; in this case, the kinetic energy operator is $E(\boldsymbol{p}) = \sqrt{p^2 + M^2}$.

§ 4. Second order scalar potential

Following the procedure outlined above, the non-relativistic second order scalar potential is given by

$$V_\varepsilon = -\frac{G^2}{4\pi}\left[\mu \frac{e^{-x}}{x}\left(1 + \frac{3}{4}\left(\frac{\mu}{M}\right)^2\right) + \frac{2}{\pi}\left(\varepsilon + \frac{M}{2}\left(\frac{\mu}{M}\right)^2\right) K_0(x)\right.$$

$$- \frac{M}{\pi}\left(\frac{\mu}{M}\right)^2 \frac{K_1(x)}{x} - \frac{2}{M^2}\boldsymbol{p}\cdot\left(\mu\frac{e^{-x}}{x} + \frac{M}{\pi}K_0(x)\right)\boldsymbol{p}$$

$$\left. - \frac{1}{4}\left(\frac{\mu}{M}\right)^2 \frac{\mu}{x}\left(\frac{e^{-x}}{x}\right)' \boldsymbol{L}\cdot\boldsymbol{\sigma}\right], \tag{12}$$

where $x = \mu r$, $\boldsymbol{L}$ is the total orbital angular momentum relative to the center of mass, and $\boldsymbol{\sigma} = \boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2$ is twice the total spin operator of the two-nucleon system. K_0 and K_1 are the Bessel functions of imaginary argument of zero and first order, respectively.

In the limit $\varepsilon/M \to 0$ and $p/M \to 0$, the scalar potential in (12) reduces to the familiar Yukawa potential.

It is of particular interest to note the presence of the spin-orbit term in the scalar potential. Explicitly,[12]

$$V_{L\sigma}=\frac{G^2}{4\pi}\left(\frac{\mu}{M}\right)^2\frac{\mu}{4x}\left(\frac{e^{-x}}{x}\right)'\boldsymbol{L}\cdot\boldsymbol{\sigma}.$$

This potential arises from the spin, angle dependence of the nucleon spinors, a fact revealed by the $1/M^2$ dependence on the nucleon mass. $V_{L\sigma}$ is written with an explicit differentiation to be performed *in order to emphasize that this potential is really of order* p^2 *in the nucleon momentum.* The spin-orbit potential is characterized by one derivative acting on the wave function and the other only on the radial shape of the potential. The easiest way to keep account of the order of a given term with respect to the nucleon momentum is by the power of the dependence on $1/M$.

§ 5. Second order pseudoscalar potential

The non-relativistic second order pseudoscalar potential is given by

$$V_\varepsilon=V_0+\frac{\varepsilon}{M}K-[p^2/4M^2,\ 4V_0+2K]_+ +[p^2/4M^2,\ V_0-\Lambda]_-, \tag{13}$$

where $[A, B]_\pm = AB \pm BA$,

$$V_0=\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\frac{G^2}{4\pi}\left(\frac{\mu}{2M}\right)^2\frac{\mu}{3}\frac{e^{-x}}{x}\left[\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2+S\left(\frac{3}{x^2}+\frac{3}{x}+1\right)\right], \tag{14}$$

$$K=\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\frac{G^2}{4\pi}\left(\frac{\mu}{2M}\right)^2\frac{2M}{3\pi}\left[\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\left(K_0(x)-\frac{K_1(x)}{x}\right)+S\left(K_0(x)+2\frac{K_1(x)}{x}\right)\right],$$

$$\Lambda=\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\frac{G^2}{4\pi}\left(\frac{\mu}{2M}\right)^2 i\left[(\boldsymbol{p}\cdot\boldsymbol{\sigma}_1)\,(\boldsymbol{r}\cdot\boldsymbol{\sigma}_2)+(\boldsymbol{p}\cdot\boldsymbol{\sigma}_2)\,(\boldsymbol{r}\cdot\boldsymbol{\sigma}_1)\right]\frac{\mu}{x}\left(\frac{e^{-x}}{x}\right)',$$

$$S=3(\boldsymbol{\sigma}_1\cdot\boldsymbol{r})\,(\boldsymbol{\sigma}_2\cdot\boldsymbol{r})/r^2-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2.$$

Equation (13) is a convenient way to write the second order potential because the commutator, which contains the most complicated spin, angle dependence of the potential, vanishes in the Born approximation.[13] From this viewpoint, an even more useful form to write the second order potential is given below in Eq. (17).

The familiar static form of the potential is obtained by taking the simultaneous limits $\varepsilon/M \to 0$, $p/M \to 0$; hence the static potential is given by (14).

The operator Λ introduces a new spin, angle dependence into the nuclear potential, but it does not involve any spin-orbit coupling. (All other terms of the potential are clearly of the central force or tensor force variety.) To show this, only diagonal matrix elements of V_ε in the quantum numbers J, L, and S need to be considered, i.e.

$$(j_L(pr)\,Y_{JLS}|\,V_\varepsilon|v_L(pr)\,Y_{JLS}). \qquad (15)$$

Here j_L is the spherical Bessel function of order L, Y_{JLS} is the two particle spin, angle function, and v_L is the radial part of the exact wave function projected onto Y_{JLS}. Only diagonal states need to be considered in (15) because the spin-orbit operator is diagonal in L (the whole nuclear potential is diagonal in J and S).

The spin, angle dependence of V_ε in diagonal states as (15) can be uniquely separated into three parts for $L\neq 0$: central, tensor, and spin-orbit. This statement is obvious in singlet spin states where the potential is spin independent and, therefore, all central. In triplet states the statement is readily proven by noting there are three triplet states for each non-zero L, and the operators 1, $\boldsymbol{L}\cdot\boldsymbol{S}$, S are linearly independent in these three states. Upon calculation, however, the coefficient of $\boldsymbol{L}\cdot\boldsymbol{S}$ is found to be zero.

§ 6. Born approximation scattering amplitude

A simple criterion for the second order potential to satisfy is that it gives the same scattering amplitude in the Born approximation as does the lowest order diagram using S-matrix theory. It is easy to show that the exact second order relativistic interaction in (3) satisfies this criterion. The non-relativistic form of the Born approximation result is then obtained by expanding the factor $E^{-4}(\boldsymbol{p})$ in powers of $(p/M)^2$:

$$E^{-4}(\boldsymbol{p})=(1/M^4)\,(1-2(p/M)^2+3(p/M)^4\cdots). \qquad (16)$$

The Born approximation scattering amplitude resulting from the second order potential of Eq. (13) includes the first two terms of this expansion; thus (16) gives a simple way of seeing exactly how much the non-locality contributes to the scattering at various energies. For a lab. energy of 300 Mev, $p^2/M=150$ Mev, the series in (16) becomes

$$1-.320+.063=1-.320+.077-\cdots$$

The left side of this equation is the numerical value of $(M/E(\boldsymbol{p}))^4$ at 300 Mev written to easily compare it with the term by term evaluation of the series in (16). Thus the non-locality of the interaction decreases the static Born approximation result by about 25 per cent at 300 Mev, and the non-locality is quite accurately represented by the lowest order velocity terms.*

The energy dependence of V_ε plays an interesting role in calculating the Born approximation scattering amplitude. For in the Born approximation, $\varepsilon=p^2/M$ and the operator $\boldsymbol{p}^2$ becomes a numerical factor; thus the terms involving K in Eq. (13) cancel identically! This cancellation is instrumental in obtaining agreement with

* Signell also pointed out that the corrections due to the long range component of the lowest-order velocity-dependent potential effectively weaken the second order potential. See P. Signell, Prog. Theor. Phys. **22** (1959), 492.

the lowest order S-matrix scattering amplitude and emphasizes the necessity of retaining an explicit energy dependence in the potential.

Writing Eq. (13) as

$$V_\varepsilon = V_0\left(1-2\frac{\varepsilon}{M}\right)+\left[\frac{\varepsilon}{M}-\frac{p^2}{M^2},\ U_1\right]_+ +\left[\frac{p^2}{M^2},\ U_2\right]_-, \tag{17}$$

where V_0 is given in (14) and

$$U_1 = V_0+\frac{1}{2}K,$$

$$U_2 = \frac{1}{4}(V_0-\Lambda),$$

only the first term of (17) contributes to the Born approximation scattering amplitude.

§ 7. Hermitian property

It is easy to see that the condition for the potential to be hermitian is satisfied in both the scalar and pseudoscalar theories. The hermitian conjugate potential is

$$V_\varepsilon^\dagger(\boldsymbol{r}, \boldsymbol{p}) = \int d\boldsymbol{k} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} K_\varepsilon{}^T(i\boldsymbol{\nabla}, \boldsymbol{k}) = \int d\boldsymbol{k} K_\varepsilon{}^T(i\boldsymbol{\nabla}+\boldsymbol{k}, \boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{r}}.$$

Simultaneously reversing the sign of $\boldsymbol{k}$ and the limits of integration gives

$$V_\varepsilon^\dagger(\boldsymbol{r}, \boldsymbol{p}) = \int d\boldsymbol{k} K_\varepsilon{}^T(i\boldsymbol{\nabla}-\boldsymbol{k}, -\boldsymbol{k}) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}. \tag{18}$$

The dependence of K_ε on spin and charge operators has been suppressed since these operators are hermitian and essentially play no role other than as specified by the superscript "T". The superscript "T" stands for transposing the order of these spin and charge operators whenever they appear in non-commuting combinations, and accounting for anything else that would change under transposition (as cross products). Comparing Eqs. (10) and (18) it follows that the potential is hermitian in case

$$K_\varepsilon{}^T(\boldsymbol{p}-\boldsymbol{k}, -\boldsymbol{k}) = K_\varepsilon(\boldsymbol{p}, \boldsymbol{k})$$

or that K_ε is invariant under the simultaneous operations (along with any necessary transposition):

$$\boldsymbol{k}\to-\boldsymbol{k},$$

$$\boldsymbol{p}\to\boldsymbol{p}-\boldsymbol{k}.$$

The kernel in Eq. (3) satisfies this condition using either (4a) or (4b) at all energies.

One can also see directly that the potential is hermitian. The scalar potential

in (12) is obviously hermitian, and the pseudoscalar potential in (17) is readily shown to be hermitian using the relation $U_2^\dagger = -U_2$.

§ 8. Estimate of velocity corrections

In this section a crude estimation of the size of the velocity corrections in the two-nucleon interaction as a function of the separation distance of the nucleons is presented.

The non-relativistic scattering amplitude is proportional to

$$(e^{iq'\cdot r}|V_\varepsilon|\psi(\boldsymbol{r})), \tag{19}$$

where

$$(-\nabla^2/M+V_\varepsilon)\psi(\boldsymbol{r})=\varepsilon\psi(\boldsymbol{r}).$$

Approximating the kinetic energy of the particles by

$$(-\nabla^2/M)\psi(\boldsymbol{r})\approx(\varepsilon-V_0)\psi(\boldsymbol{r}), \tag{20}$$

where V_0 is the static potential, the "effective" second order potential is

$$V_{eff}=(1-2(p/M)^2)\,V_0+(U_1+U_2)\,V_0/M. \tag{21}$$

It is emphasized that this effective potential is defined only within the matrix element (19). The approximation in (20) amounts to removing the velocity dependence in the second order potential by a single iteration of the Schrödinger equation, retaining only the leading term in powers of $1/M$. Thus the second term in (21), which is the correction to the Born approximation result, naturally manifests itself as a potential of fourth order range. Since it is assumed that the non-local effects have a perturbation effect on the scattering, it is clear that the effective potential becomes completely meaningless at small distances where the fourth order term of (21) begins to dominate over the second order term. One can then make a meaningful quantitative statement only from a detailed treatment of the original Schrödinger equation. A detailed treatment suitable for comparing various potentials could take the form of integrating from large distances into some small distance.

In diagonal states, the effective potential may now be given by

$$V_{eff}=V_C+SV_T. \tag{22}$$

The appearance of a spin-orbit term in (22) is regarded as spurious, and has been dropped, since the exact second order potential does not have such a term. To facilitate matters, the approximation

$$\boldsymbol{r}\cdot\nabla\approx\frac{1}{2}\left(r\frac{\partial}{\partial r}-3\right)$$

is used. This approximation is understood to be made within the matrix element (19) between diagonal states; it is exact in the Born approximation.

In terms of spin and parity eigenstates, the six potentials defined by (22)

have been numerically evaluated at 0 and 300 Mev lab. energies and compared with the respective static potentials. Table I is based on these calculations. Triplet states are affected more by the velocity dependence than are singlet states, and even parity triplet states are affected earlier (i.e. at larger ranges) than odd parity triplet states. Singlet even and singlet odd states are roughly affected to the same degree. At larger distances the change in any of the potentials from the static case is greater with increasing energy. At smaller distances, the velocity corrections (which are cast into a static mold in the second term of (21)) are predominant.

Table I. Ratio of effective to static potentials in each spin and parity state. Values are tabulated at various separation distances for lab. energies of 0 and 300 Mev. The separation distance is expressed in units of $x=\mu r$; thus $x=1$ corresponds to a separation distance of 1.414 f.

x	Triplet Even States				Triplet Odd States				Singlet States			
	tensor		central		tensor		central		even		odd	
	0 Mev	300 Mev	0 Mev	300 Mev	0 Mev	300 Mev	0 Mev	300 Mev	0 Mev	300 Mev	0 Mev	300 Mev
0.7	4.75	4.43			−.251	−.570			1.25	.934	.239	−.081
1.0	2.34	2.02			.554	.234			1.04	.726	.862	.542
1.2	1.77	1.45			.744	.424			1.01	.687	.981	.661
1.4	1.47	1.15	−9.23	−9.55	.844	.524	4.41	4.09	.991	.671	1.03	.708
1.6	1.30	.980	−4.93	−5.25	.900	.580	2.98	2.66	.985	.666	1.04	.725
1.8	1.20	.878	−2.64	−2.96	.934	.614	2.21	1.89	.984	.665	1.05	.727
2.0	1.13	.814	−1.34	−1.66	.955	.635	1.78	1.46	.985	.666	1.04	.724
2.5	1.06	.735	.117	−.202	.982	.662	1.29	.974	.989	.670	1.03	.711
3.5	1.01	.692	.825	.505	.996	.676	1.06	.738	.996	.676	1.01	.692
∞	1.00	.680	1.00	.680	1.00	.680	1.00	.680	1.00	.680	1.00	.680

In the triplet even state, the effective tensor force is more attractive than the static result (it is less attractive at 300 Mev at large distances); in the triplet odd state, the effective tensor force is less repulsive than the static result. For $r \lesssim 1.4$f, the velocity corrections begin to predominate and, therefore, the approximations begin to fail. In the triplet odd state, the effective tensor force actually "changes" the sign of the static result, becoming strongly attractive near the origin. As mentioned before, this change in sign is an indication of a complete breakdown of the approximation (20) and at best indicates the static potential is severely weakened.

In triplet states, the effective central potentials are drastically different from the static potentials. With the approximations made, the effective triplet even central potential "changes sign" to become repulsive at about 3f! The effective

triplet odd central potential is always repulsive but more than doubles the static result at even 2.5 f.

In singlet states, the approximations seem to apply quite well since velocity corrections are not very large even at $r=1.0$ f. In the singlet odd state, the effective potential is slightly less repulsive than the static potential. In the singlet even state, the effective potential is slightly less attractive. So in both singlet states, the velocity corrections tend to slightly weaken the static potential.

(Comparisons of the effective potential with the work of others are discussed below.)

§ 9. Conclusions and discussion

The second order scalar and pseudoscalar potentials have been calculated taking account of first order recoil effects that the nucleons undergo in the process of interaction. Because of recoil, the two-nucleon interaction is of course non-local. Defining the two-nucleon probability amplitude by the Tamm-Dancoff method, the non-locality of the interaction is expressed by a "potential" which is both velocity dependent and energy dependent. The energy dependence in the potential is perhaps undesirable from the standpoint of simplicity, but if the Tamm-Dancoff technique is regarded as exact for exchanges of a given number of mesons, then the energy occurs in a natural application of the theory.

With local potentials, the stationary state continuity equation is expressed by

$$\nabla \cdot \boldsymbol{S}=0, \tag{23}$$

where $\boldsymbol{S}$ is the familiar probability flux density. Non-local potentials, however, provide a source of probability at a point $\boldsymbol{r}$ from all other points $\boldsymbol{r}'$. Thus, for non-local potentials, the stationary state continuity equation is

$$\nabla \cdot \boldsymbol{S}=-i(\psi^* V\psi-\psi(V\psi)^*). \tag{24}$$

The same flux density, $\boldsymbol{S}$, is used in (23) and (24). (The right side of (24) of course vanishes for local potentials.) As long as the non-local potential in (24) is hermitian, the total probability will be conserved. This follows by integrating (24) over all space and using the hermitian property of the potential. Thus, for hermitian potentials, the stationary state wave function can be normalized in the usual way, conserving the total probability.

The energy dependence of the second order potential implies that non-degenerate eigenfunctions of the non-relativistic Hamiltonian are not orthogonal. The energy dependence in the two-nucleon interaction arises from the fact that the interaction Hamiltonian, H_I, is non-diagonal in the particle number representation. If H_I were diagonal in the particle number representation, so that it could not cause transitions between states with different numbers of particles, then each of the Tamm-Dancoff amplitudes would satisfy an independent integral equation instead of coupled equations with the same energy. That the non-relativistic energy, ε, is real is consis-

tent with the hermitian property of the Hamiltonian but actually follows from the relativistic formulation of the theory. (In the proof that the second order interaction is hermitian (§ 7), it is assumed that ε is real.) The important completeness property of the two nucleon wave functions also follows from the original formulation of the theory. For an arbitrary state of two nucleons is merely the projection of ψ in (1), which must be complete, onto the subspace of two nucleons and no anti-nucleons or pions.

Since the ideal purpose of calculations as in the present work is to arrive at the non-relativistic equation of motion for two nucleons starting from a postulated relativistic formulation involving any number of particles, and since this purpose is only partially accomplished, a consistent attitude would be to regard the non-relativistic energy as a *given* number and the non-relativistic Schrödinger equation as an equation for the wave function for each *given* eigenvalue (i.e. not as an eigenvalue equation). Since it is known empirically that none of the two nucleon stationary states is degenerate, the complication of finding independent degenerate states from an approximate Schrödinger equation is not present.

It is interesting to note that the essential difference among the static potentials calculated by other investigators arises from the way the energy dependence of the two-nucleon interaction is handled. Brueckner and Watson (BW), for example, simply set the energy equal to zero to calculate the static potential.[14] Thus, their potentials give the local approximation to the two-nucleon potential. Others have chosen to remove the energy dependence according to some prescription. Feldman does this in a general way using canonical transformations (each written as an infinite series of operators), but his theory is not exact in the sense that the Tamm-Dancoff procedure is exact.[15] Fukuda, Sawada, and Taketani (FST) regard the energy dependence as an inconsistency in the method of deriving the two-nucleon potential.[16] They interpret the energy dependence of the interaction as expressing a possible absorption between, say, the state of two nucleons and the state of two nucleons and one meson.[17] They also remove the energy dependence by employing canonical transformations, again in approximate form, and redefining the two-nucleon wave function. As a result, the fourth order static potentials of both Feldman and FST differ from those of BW; this is because Feldman and FST have partially accounted for the non-locality of the two-nucleon interaction and have introduced a fourth order potential very much as was done in the "effective" potential considerations of the present work (§ 8).

In the present work, no transformation is applied to the momentum and energy dependence of the potential in order to remove or partially remove these dependences as by FST, Feldman, and more recently by Sugawara and Okubo (SO).[18] Although energy dependent, our potential is hermitian and it is of interest to contrast the present approach with others where various transformations have been applied. We are not sure these transformations are necessary and they raise additional doubts as to convergence. As a consequence of not making any transfor-

mations, the Schrödinger equation describes the two physical nucleon component of the full state vector. This component fully describes the asymptotic wave function and, therefore, all the scattering. At small distances it will differ from the wave functions of other theories. There is considerable arbitrariness as to the interpretation of a phenomenological wave function relative to these theories.

The role of the non-locality of the second order two-nucleon interaction in scattering states was estimated in two ways. As a function of energy, it was shown that at 300 Mev the static Born approximation scattering amplitude is reduced about 25 per cent by the velocity corrections. As a function of separation distance, an "effective" potential was defined. Here much of the velocity dependence of the interaction was "transformed" into a static fourth order potential. Because of the large magnitude of the corrections to the static potential, the approximations made to define the effective potential break down at rather large distances in triplet states (1.5～3.0 f) and somewhat smaller distances in singlet states (1 f). The large corrections in triplet states are expected when noting that the highly singular tensor part of the second order static potential contributes to the effective potential as a multiplicative factor in both the central and tensor parts of the fourth order term of (21). Although they are of fourth order range, the effective velocity corrections to the second order static potential are so large in triplet states that they exhibit themselves, from a practical point of view, as potentials of second order range. Accordingly, in triplet states, the fourth order static potentials (purely local), which are the BW fourth order potentials, are quite negligible at distances where the second order velocity corrections change the second order static potential, say, 50 per cent. In singlet states, the fourth order BW potentials are as large or larger than the effective second order velocity corrections. It is clear, however, that the large changes in triplet states cannot be regarded as spurious and that corrections to the static potential are likely to be important even at large separation distances.

It is interesting to compare the effective potential in the present work with the potentials of FST[16] and SO.[18] The difference of the fourth order potentials of FST and BW corresponds to non-local corrections to the static second order potential in the present work (especially at large distances). In the region $1 \lesssim x \lesssim 1.5$, the non-local corrections of the effective potential at zero energy are found to be generally larger than the difference of the FST and BW potentials, sometimes by as much as an order of magnitude. The signs of the non-local corrections agree except for the triplet odd tensor force where the effective potential gives a negative correction.

The SO potentials at zero energy differ only slightly from the second order static potentials (14). Thus in triplet states the effective potential gives much larger corrections than SO, and only in the singlet even state is the SO correction larger. In all states, the signs of the non-local corrections agree.

All the BW and FST potentials converge to the second order static potential,

V_0, at large distances. The effective potential (21), however, converges to $V_0(1-2(p/M)^2)$, showing a long range velocity correction. The correction factor is the same as found in the Born approximation scattering amplitude, Eq. (16), a physically plausible result. The SO potentials also have a long range velocity correction and converge to $V_0(1-(p/M)^2)$. This discrepancy is related to the manner in which the energy dependence has been removed from the SO potentials.

Phenomenological work with the two-nucleon problem projects the spin-orbit coupling into a favored position over the long range velocity components of the second order potential. This fact results from the phase shift fit to the 310 Mev proton-proton scattering data that is available.[19),20),21)] In order to achieve the indicated ^3P phase shift splitting, a potential must have a spin-orbit component satisfying a particular strength relation with the tensor force. Since the usual phenomenological tensor force is quite strong and long ranged, the phenomenological spin-orbit force is required to be quite strong—much stronger than expected from meson theory.

The velocity and energy dependence of the meson theoretic potential offers a possible explanation of this situation. The reason is that in the rough considerations made with the effective potential above, it was observed that the velocity corrections in the triplet odd states weakened the tensor force, particularly as the energy is increased If this weakening were taken into account in phenomenological work, the strength of the appropriate spin-orbit force may fall into line with theoretical expectations.

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References

1) J. L. Gammel and R. M. Thaler, Phys. Rev. **107** (1957), 291, 1337.
2) P. S. Signell and R. E. Marshak, Phys. Rev. **109** (1958), 1229.
3) Gammel, Christian, and Thaler, Phys. Rev. **105** (1957), 311.
4) S. Gartenhaus, Phys. Rev. **100** (1955), 900.
5) A summary of all the possible velocity dependences permitted by invariance principles is given by S. Okubo and R. E. Marshak, Annals of Physics **4** (1958), 166.
6) The second order potential seems to be the only component of the two-nucleon interaction which is universally of good repute. A review of the discrepancies among the various techniques of calculating two-nucleon potentials is given in Supplement Prog. Theor. Phys. **No. 3** (1956).
7) S. M. Dancoff, Phys. Rev. **78** (1950), 382.
8) I. Tamm, J. Phys. (U.S.S.R.) **9** (1945), 449.
9) M. M. Levy, Phys. Rev. **88** (1952), 72.
10) For example: M. M. Levy, Phys. Rev. 88 (1952), 725; K. A. Brueckner and K. M. Watson, Phys. Rev. **92** (1953), 1023; A. Klein, Phys. Rev. **90** (1953), 1101.

11) R. E. Cutkosky, Phys. Rev. **112** (1958), 1027.
12) This result for $V_{L\sigma}$ has also been obtained recently by S. N. Gupta, Phys. Rev. Letters **2** (1959), 124.
13) This was first pointed out by M. Sugawara.
14) K. A. Brueckner and K. M. Watson, Phys. Rev. **92** (1953), 1023.
15) D. Feldman, Phys. Rev. **98** (1955), 1456.
16) Fukuda, Sawada, and Taketani, Prog. Theor. Phys. **12** (1954), 156.
17) This interpretation is, of course, inconsistent with the present work where it is shown that the energy dependent second order two-nucleon interaction is hermitian and, therefore, probability is conserved.
18) M. Sugawara and S. Okubo, Phys. Rev. **117** (1960), 605.
19) Stapp, Ypsilantis, and Metropolis, Phys. Rev. **105** (1957), 302.
20) Cziffra, MacGregor, Moravcsik, and Stapp, Phys. Rev. **114** (1959), 880.
21) MacGregor, Moravcsik, and Stapp, Phys. Rev. **116** (1959), 1248.

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The Hyperon and Nucleon Interactions. I

——*Potentials*——

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Two- and three-body Y-N potentials including the rescattering corrections are calculated by using Miyazawa's method. The Y-N systems are treated by the two-channel formalism in the case of $T=1/2$: The one is the channel of the Σ-N system and the other is of the Λ-N system. Then the potential is defined as the 2×2 matrix form. The three-body force vanishes, in which no direct two-nucleon interaction is considered, if the π-Λ rescattering corrections are not taken into account. The other type of the three-body force in which one pion is exchanged between each pair of the particles is also investigated.

§ 1. Introduction

Since about 1950, successful results[1] on the nuclear force problems have been obtained by many efforts based upon Taketani's methodology. At present the main interests are in overcoming the frame of the static theory, that is, the attack on the region II. As for these subjects several attempts have already been proposed.[2]

Now, it may be worthwhile to point out the possibility of another approach to the above subjects from the different point of view, though it is rather conservative. It is an approach through the investigation of the hyperon (Y) and nucleon (N) interactions, that is, the difference of the interaction being considered, the nuclear force problems can be reinvestigated. For this object it is most convenient to consider the Λ-N system. The investigation of the Y-N systems, however, is theoretically and experimentally more difficult than that of the two-nucleon systems; for example, the important freedoms such as the relative parity of a Σ- and Λ-hyperon, or the variety of the coupling constants are left undetermined. As for the coupling constants of the π-Y interactions, no direct π-Y scattering experiments can be made, contrary to the π-N scattering experiments, so we are forced to investigate the indirect processes such as $\bar{K}$-N absorption, Y-N scattering and hyperfragments for the determination of the coupling constants. Therefore this is also the main aim of our research.

We summarize the difference of the Y-N interactions from the two-nucleon interactions as follows.

1⁰) In the case of $T=1/2$ (T means the total isospin quantum number), one must treat Λ and Σ as the two-channel system.

2°) As Λ is an iso-singlet, there exists no one-pion exchange potential. (In this case the charge independence is assumed.)

3°) ${}_{\Lambda}H^{2}$ (or a (Λ-n) bound state) has not yet been found, so ${}_{\Lambda}H^{3}$ must play the same role as the deuteron does in the two-nucleon system.

4°) Pauli's exclusion principle does not act.

5°) The hyperons are the unstable particles which have very short lifetimes. Especially as for the point 2°), one must notice that the region II plays an essential role from the first, contrary to the two-nucleon force, so we can expect that the investigation of the Λ-N systems may throw light upon the dynamical properties of the potential in the region II. But unfortunately there have been but little Λ-N scattering data, and we know only the binding energies of the Λ-hyperfragments as the phenomena in which Λ-N interactions participate. We therefore expect that the Λ-N scattering data will be accumulated in the near future.

The simplest hyperfragment which have up to now been found for sure is ${}_{\Lambda}H^{3}$, in which Λ-binding energy is less than about 1 Mev. This means that ${}_{\Lambda}H^{3}$ is more loosely bound than H^{3} or He^{3}, and the mean distance of any pair of constituent particles is about 3×10^{-13}cm and then it is much larger than the force range. The force range of the two-body Λ-N force is $1/(2\mu)$ (μ is a pion mass, $\hbar=c=1$), but three-body forces have the force range $1/\mu$ as will be discussed in § 4. Three-body force, therefore, may be more important in ${}_{\Lambda}H^{3}$ than in H^{3} or He^{3}. In the latter case, the two-nucleon force of the classical region prevails over the three-body force.

Finally, we shall briefly argue on the kaon exchange potential. In the first step, we shall not consider even the one-kaon exchange potential (OKEP) for the following reasons.

a) As $m_K>3\mu$, OKEP may not have the definite meaning as OPEP (one pion exchange potential) has in the two-nucleon case, for OKEP acts in the region II or in the region III and the static calculation is not reliable.

b) OKEP is the exchange force which has the longest force range in the Y-N systems, but such a property as the exchange force, as well as its net effects, may give small effects for the hyperfragments in which the binding energy is very low and then the mean distance of any pair is larger than the force range.

c) As for the low energy scattering, it may also give little effects except for the very large angle scattering, for the exchange effects are largest for the backward scattering.

In the second step, we shall investigate this exchange effects by using the phenomenological potentials.

In § 2 the definition of the potentials in the two-channel formalism will be discussed. In § 3 we shall argue the two-body potentials up to the two-pion exchange ones and § 4 will be devoted to the calculation of the three-body potentials of the two different types.

§ 2. Two-channel formalism

In order to treat the Y-N system as the two-channel one, we use the formalism[3] familiar in the theory of nuclear reactions. We first start with the following Schrödinger equation as the basic equation, i.e.

$$(H^0+V)\Psi=E\Psi, \tag{2·1}$$

$$H^0=\begin{pmatrix} H_\Lambda^0 & 0 \\ 0 & H_\Sigma^0 \end{pmatrix}, \tag{2·2}$$

$$V=\begin{pmatrix} V_\Lambda & V_{\Lambda\Sigma} \\ V_{\Sigma\Lambda} & V_\Sigma \end{pmatrix}, \tag{2·3}$$

$$\Psi=\begin{pmatrix} \psi_\Lambda & \psi_{\Lambda\Sigma} \\ \psi_{\Sigma\Lambda} & \psi_\Sigma \end{pmatrix}, \tag{2·4}$$

where H^0 is the free Hamiltonian and includes the rest masses corresponding to the relevant systems, and E is the initial energy including the rest masses, too. $\psi_{\Lambda\Sigma}$ and $\psi_{\Sigma\Lambda}$ are the wave functions that correspond to the hyperon-exchange scattering such as $\Lambda+N\rightleftarrows\Sigma+N$.

Before the calculation of the potentials, let us generally investigate the so-called repetition terms of the potential. From Eq. (2·1), one can construct the S-matrix by Lippmann-Schwinger's prescription,[4] i.e.

$$S=-2\pi i\delta(E_a-E_b)\,(\Phi_b|V|\Psi_a^{(+)}), \tag{2·5}$$

$$\Psi_a^{(+)}=\Phi_a+\frac{1}{E-H^0+i\epsilon}V\Psi_a^{(+)}, \tag{2·6}$$

where Φ_a is the eigenstate of H^0 with the energy E and has a form of the diagonal matrix, and $(\Phi_b, V\Psi_a^{(+)})$ is the very T-matrix. The integral equation for the T-matrix is

$$(\Phi_b|T|\Phi_a)=(\Phi_b|V|\Phi_a)+\sum_n\frac{(\Phi_b|V|\Phi_n)\,(\Phi_n|T|\Phi_a)}{E-E_n+i\epsilon} \tag{2·7}$$

and if we identify this T-matrix with the one calulated from the field theoretical prescriptions, we can regard Eq. (2·7) as the integral equation for the potential V. Hence the second term of the right-hand side of Eq. (2·7) gives the repetition terms. In the case of the static theory, E and E_n in Eq. (2·7) are merely the rest masses of the initial and intermediate states respectively, and we have

$$(b|V|a)=(b|T|a)-\sum_n\frac{(b|V|n)\,(n|T|a)}{M_0-M_n+i\epsilon}, \tag{2·8}$$

where $|a)$, $|b)$ and $|n)$ are the state vectors specifying the kind of the particles which contribute to the matrix elements. As for the two-pion exchange potentials, the repetition term is

$$-\sum_{n}\frac{(b|V^{(1)}|n)(n|V^{(1)}|a)}{M_0-M_n+i\epsilon} \tag{2·9}$$

where $V^{(1)}$ is the one-pion exchange potential. In the one channel formalism the repetition term vanishes for the Λ-N potential, for the intermediate states $|n)$ do not contain the Σ-N state in this case. But the mass difference ΔM of M_Σ and M_Λ turns out to play an essential role in removing the divergency of the potential. To see this circumstance more accurately, we reduce the Schödinger equation (2·1) to the equation which contains only ψ_Λ, under the boundary condition in which $\psi_{\Sigma\Lambda}$ has only an out-going wave, then we have

$$(E-H_\Lambda^0)\psi_\Lambda(r)=V_\Lambda(r)\psi_\Lambda(r)+\int V_{\Lambda\Sigma}(r)\,(r|\frac{d^3r'}{E+i\epsilon-H_\Sigma-V_\Sigma}|r')\,V_{\Sigma\Lambda}(r')\psi_\Lambda(r'). \tag{2·10}$$

In the two-channel formalism, the effective "potential" becomes non-local and energy dependent. On the other hand, in the case of the one channel formalism there are of course no off-diagonal potentials, so the one channel potential is local and energy independent, and we can express the one channel potential by the two-channel ones if one restricts the potential up to the two-pion exchange as a whole, and we have

$$(E-H_\Lambda^0)\psi_\Lambda(r)=\left\{V_\Lambda(r)-\frac{1}{\Delta M}V_{\Lambda\Sigma}(r)\,V_{\Sigma\Lambda}(r)\right\}\psi_\Lambda(r), \tag{2·11}$$

where $V_{\Lambda\Sigma}$ and $V_{\Sigma\Lambda}$ are restricted to the one-pion exchange. The second term has ΔM in the denominator, so $\Delta M\neq 0$ is essential.

A few years ago Ferrari and Fonda[5] calculated the Λ-N potential, which corresponds to our V_Λ only, by Brueckner-Watson's method.[6] Then, from the standpoint of the two-channel formalism their Schrödinger equation corresponds to the one in which the non-local potential part of Eq. (2·10) is neglected, on the other hand from the standpoint of the one-channel formalism they omitted the second term of Eq. (2·11) which is larger than the first term. On the contrary, Lichtenberg and Ross[7] treated the Y-N system by the two-channel formalism for the first time. But their application of the potential to the actual Y-N system is less cautious; for example, their result on the binding of the Σ^+p system may not be reliable, because the form of the inner potential is the most important in order to bind two particles; or as the S-wave scattering, too, reflects the form of the inner potential, the situation is the same as in the case of the binding of two particles.

Moreover, they took account of the off-diagonal potential up to the two-pion exchange ones, but in such a case as the elastic components, ψ_Λ or ψ_Σ, of the wave function are considered, it is clear from Eq. (2·10) that this effectively introduces the three- or four-pion exchange potential into the Schrödinger equation, so if one restricts the potential up to the two-pion exchange as a whole, it is

sufficient to take only the one-pion exchange potential. On the contrary, for the inelastic components, $\psi_{\Lambda\Sigma}$ or $\psi_{\Sigma\Lambda}$, of the wave function we have the following equations,

$$(E-H_\Sigma^0)\psi_{\Sigma\Lambda}(r)=V_\Sigma(r)\psi_{\Sigma\Lambda}(r)+\int V_{\Sigma\Lambda}(r)\ (r|\frac{d^3r'}{E+i\epsilon-H_\Lambda^0-V_\Lambda}|r')\ V_{\Lambda\Sigma}(r')\psi_{\Sigma\Lambda}(r)$$
$$+V_{\Sigma\Lambda}(r)\varphi_\Lambda(r), \qquad (2\cdot12)$$

where φ_Λ is a solution of the equation

$$(H_\Lambda^0+V_\Lambda(r))\varphi_\Lambda(r)=E\varphi_\Lambda(r) \qquad (2\cdot13)$$

and for $\psi_{\Lambda\Sigma}$ we have similar equations in which the suffices Σ and Λ are interchanged. Therefore due to the existence of the last term in Eq. (2·12) one must consider the two-pion exchange potential for the off-diagonal ones, too, in order to take account of the effects up to the two-pion exchange as a whole. Thus, whether one needs to take the one-pion exchange potential or the one- plus two-pion exchange one depends on what procss one will consider.

The authors mentioned above used Breuckner and Watson's method for the calculation of the potentials, but according to the analysis of the low energy two-nucleon phenomena the most favourable potential is the one obtained by Konuma, Miyazawa and Otsuki[8] (KMO potential). Accordingly we shall use Miyazawa's method[9] to calculate the Y-N potential in the following sections.

§ 3. Two-body potential

Throughout this paper we assume that the charge independence of the π-Y interactions holds and the members of a charge multiplet have the same intrinsic parity and mass, the spin of the hyperon being also assumed 1/2. Therefore in the static theory our vertex operator of the π-Y interactions are as follows,

$$\Lambda\Sigma\pi \quad V_k^\Lambda=\begin{cases}(f_\Lambda^0/\mu)i\boldsymbol{\sigma}\cdot\boldsymbol{k}(t_k+t_k^\dagger)\,e^{i\boldsymbol{k}\cdot\boldsymbol{x}} & \text{for } P_{\Sigma\Lambda}=\text{even}\\ g_\Lambda^0(t_k+t_k^\dagger)\,e^{i\boldsymbol{k}\cdot\boldsymbol{x}} & \text{for } P_{\Sigma\Lambda}=\text{odd}\end{cases} \qquad (3\cdot1)$$

$$\Sigma\Sigma\pi \quad V_k^\Sigma=\frac{f_\Sigma^0}{\mu}i\boldsymbol{\sigma}\cdot\boldsymbol{k}\rho_k e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \qquad (3\cdot2)$$

and

$$NN\pi \quad V_k^N=\frac{f_N^0}{\mu}i\boldsymbol{\sigma}\cdot\boldsymbol{k}\tau_k e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \qquad (3\cdot3)$$

where t_k is the 3×1 matrix which picks up $\bar{\Sigma}_k$ from $\bar{\Sigma}=(\bar{\Sigma}_1\bar{\Sigma}_2\bar{\Sigma}_3)$ and $\dagger$ means to take hermite conjugation, and ρ_k's are the 3×3 matrix with $\sum_k{}'\rho_k\rho_k=2$ and hermite, f^0's being the unrenormalized and unrationalized coupling constants.

One can immediately construct the potentials up to the two-pion exchange from the above assumptions.

3–1. *One-pion exchange potentials*

We write the results as follows, for the diagonal potential we have

$$V_A(r)=0$$

$$V_\Sigma(r)=\frac{1}{4\pi}\left(\frac{f_\Sigma f_N}{\mu^2}\right)(\boldsymbol{\rho}\cdot\boldsymbol{\tau})(\boldsymbol{\sigma}_\Sigma\cdot\boldsymbol{\nabla})(\boldsymbol{\sigma}_N\cdot\boldsymbol{\nabla})\frac{e^{-\mu r}}{r} \tag{3·4}$$

and for the off-diagonal potentials

$$V_{\Sigma A}(r)=\frac{1}{4\pi}\left(\frac{f_A f_N}{\mu^2}\right)(\boldsymbol{t}\cdot\boldsymbol{\tau})(\boldsymbol{\sigma}_A\cdot\boldsymbol{\nabla})(\boldsymbol{\sigma}_N\cdot\boldsymbol{\nabla})\frac{e^{-\mu r}}{r} \quad \text{for } P_{\Sigma A}=\text{even} \tag{3·5}$$

$$V_{A\Sigma}(r)=V_{\Sigma A}^{\dagger}(r)$$

$$V_{\Sigma A}(r)=-\frac{1}{4\pi}g_A\frac{f_N}{\mu}(\boldsymbol{t}\cdot\boldsymbol{\tau}),(\boldsymbol{\sigma}_N\cdot\boldsymbol{\nabla})\frac{e^{-\mu r}}{r} \quad \text{for } P_{\Sigma A}=\text{odd} \tag{3·6}$$

$$V_{A\Sigma}(r)=V_{\Sigma A}^{\dagger}(r),$$

where f's are renormalized but unrationalized coupling constants. The eigenvalue of $(\boldsymbol{t}\cdot\boldsymbol{\tau})$ is $-\sqrt{3}$ for $T=1/2$ and 0 for $I=3/2$. As for the off-diagonal part, we neglected the mass difference $\Delta M=M_\Sigma-M_A$ between the initial and final states.

3–2. *Two-pion exchange potentials*

We calculate at first the S-matrix of such a type as in Fig. 1, where the shaded area means π-Y and π-N rescattering corrections. In the case of the static theory one can use the π-N and π-Y S-matrix elements for the shaded area. Then one has

$$S=-\frac{1}{2}\frac{1}{(2\pi)^8}\int d^4p\,d^4q\frac{\langle p,i|S_Y|q,j\rangle\langle -p,i|S_N|-q,j\rangle}{(p^2+\bar{\mu}^2)(q^2+\bar{\mu}^2)}, \tag{3·7}$$

where the bar means that μ^2 has a small negative imaginary part $-i\epsilon$ $(\epsilon>0)$, and p and q are four momenta of the exchanged pions. The factor 1/2 comes out from the fact that as π-Y or π-N scattering amplitude includes both crossed and uncrossed graphs, then one counts the topologically equivalent graphs twice. As for the scattering amplitudes $\langle p,i|S_Y|q,j\rangle$ and $\langle p,i|S_N|q,j\rangle$, we use the formula derived from the Low equation.

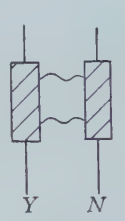


Fig. 1. Feynman graph for Eq. (3·7).

Now, we substitute the scattering amplitude thus obtained into Eq. (3·7) and integrate by p_0 at first. The p_0-integration generally has the following form,

$$\int_{-\infty}^{\infty}dp_0\frac{f^{(\pm)}(p_0)}{(p_0^2-\bar{\omega}_k^2)(p_0^2-\bar{\omega}_q^2)},$$

where $f^{(\pm)}$ is essentially the scattering amplitude and

$$f^{(+)}(p_0)=[\lambda\pm p_0-i\epsilon]^{-1}\cdot[\mu\pm p_0-i\epsilon]^{-1}$$

and

$$f^{(-)}(p_0)=[\lambda \pm p_0-i\epsilon]^{-1}\cdot[\mu \mp p_0-i\epsilon]^{-1}.$$

In the case of $f^{(+)}(p_0)$, one can take the closed path in which no poles of $f^{(+)}(p_0)$ exist. But in the case of $f^{(-)}(p_0)$, one cannot do so, and either of the two poles of $f^{(-)}(p_0)$ must be included in the closed path. When both factors of $f^{(-)}$ come out from the Born term which make either λ or μ equal to zero, it is convenient to take the closed path in which the pole at $p_0=0$ exists, for the residue of the pole at $p_0=0$ just gives the repetition term discussed in § 2. Then one must subtract this term from Eq. (3·7) to obtain the two-pion exchange potential.

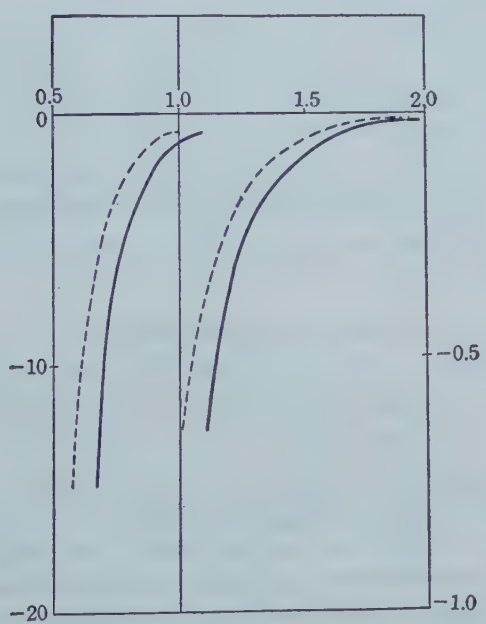


Fig. 2a. V_A: Singlet state.

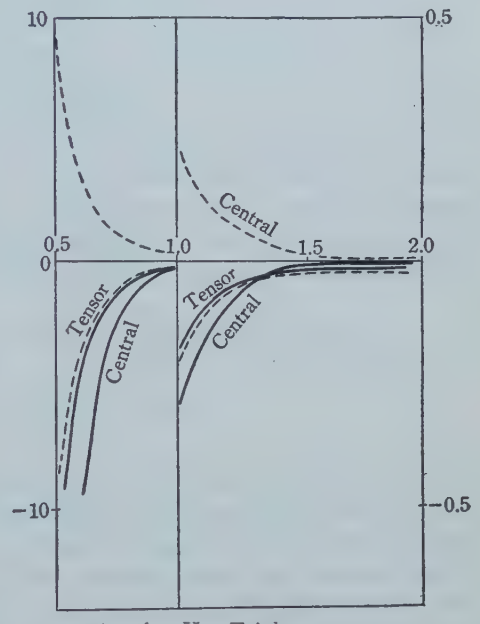


Fig. 2b. V_A: Triplet state.

V_A's divided by f_A^2 are plotted. The dashed lines are the Born terms $\Delta M=0$ and the real lines are the potentials including the π-N rescattering and the mass difference correction. When π-Λ rescattering corrections are added, the potentials are more strengthened. The unit of the ordinate is μ (pion mass)/f_A^2, and that of the abscissa is pion compton wave length.

3–3. *The Born terms of the two-pion exchange potentials*

At first we shall discuss the Born terms of the potentials in the case of $\Delta M=0$. In the case of $P_{\Sigma\Lambda}$=even, as for V_A we have (hereafter we use $\mu=1$ unit)

$$V_c^{\Lambda}(r)=0$$

$$V_\sigma^{\Lambda}(r)=\left(\frac{f_A^2}{4\pi}\right)\left(\frac{f_N^2}{4\pi}\right)\frac{8}{\pi}\left[\frac{3}{r^3}K_0(2r)+\left(\frac{2}{r^2}+\frac{3}{r^4}\right)K_1(2r)\right] \qquad (3\cdot8)$$

$$V_T^{\Lambda}(r) = -\left(\frac{f_{\Lambda}^2}{4\pi}\right)\left(\frac{f_N^2}{4\pi}\right)\frac{8}{\pi}\left[\frac{3}{r^3}K_0(2r) + \left(\frac{1}{r^2}+\frac{15}{4r^4}\right)K_1(2r)\right],$$

where K_i's are the modified Bessel function. This potential is identical with the two-nucleon potential derived by Taketani, Machida and Ohnuma[10] (TMO potential), provided one replaces f_N^4 into $f_{\Lambda}^2 f_N^2$. Fukuda, Sawada and Taketani[11] discussed the relation between the TMO potential and the BW (Brueckner and Watson[6]) potential and obtained the FST potential, that is given as follows,

$$V_{FST} = \frac{\left\langle H'\frac{1-\eta}{-H^0}H'\right\rangle + \left\langle H'\frac{1-\eta}{-H_0}H'\frac{1-\eta}{-H_0}H'\frac{1-\eta}{-H_0}H'\right\rangle}{1+\left\langle H'\frac{1-\eta}{(-H_0)^2}H'\right\rangle}$$

$$= \frac{V_{BW}}{1+\left\langle H'\frac{1-\eta}{(-H_0)^2}H'\right\rangle}, \qquad (3\cdot 9)$$

where $\langle\ \rangle$ denotes the meson vacuum expectation value and η is the projection operator on the meson vacuum state and H_0 and H' are the free and interaction Hamiltonian, respectively. If we expand the denominator by H' up to the second order, then we obtain the TMO potential, say,

$$V_{TMO} = \left\langle H'\frac{1-\eta}{-H_0}H'\right\rangle + \left\langle H'\frac{1-\eta}{-H_0}H'\frac{1-\eta}{-H_0}H'\frac{1-\eta}{-H_0}H'\right\rangle$$
$$-\left\langle H'\frac{1-\eta}{-H_0}H'\right\rangle\left\langle H'\frac{1-\eta}{(-H_0)^2}H'\right\rangle. \qquad (3\cdot 10)$$

If this is the case, as for the Λ-N potential the TMO potential has to coincide with the BW potential, for $\langle H'(1-\eta)/(-H_0)^2H'\rangle$ in the denominator of (3·9) vanishes owing to Λ being an isosinglet. But if we consider TMO expression (3·10) as a base and permit the existence of such a matrix element as $\langle H'(1-\eta)/(-H_0)H'\rangle_{\Sigma N}\cdot{}_{\Sigma N}\langle H'(1-\eta)/(-H_0)^2H'\rangle$, that is to say, if we use the two channel formalism, then not only the ΛN but the other two-pion exchange potentials thus obtained are identical with the two-nucleon potential, provided the appropriate replacement of the coupling constants or the iso-spin operators is made. This prescription, however, cannot be applied to the FST method. On the other hand, the method of canonical transformation given by Nishijima[12] is valid in these multi-channel cases.

As for the Σ-N potential we have the same expression as Eq. (3·8) for the spin dependent and tensor forces, provided f_{Λ}^2 be replaced by $(2f_{\Sigma}^2+f_{\Lambda}^2)/3$, and for V_c we have

$$V_c^{\Sigma}(r) = -(\boldsymbol{\rho}\cdot\boldsymbol{\tau})\left(\frac{f_N^2}{4\pi}\right)\frac{1}{4\pi}\left(\frac{f_{\Sigma}^2+f_{\Lambda}^2}{2}\right)\frac{8}{\pi}$$
$$\times\left[\left(\frac{1}{r}+\frac{23}{4r^3}\right)K_0(2r) + \left(\frac{3}{r^2}+\frac{23}{4r^4}\right)K_1(2r)\right]. \qquad (3\cdot 11)$$

Similarly we have

$$V_c^{\Sigma\Lambda}(r) = -(\boldsymbol{t}\cdot\boldsymbol{\tau})\left(\frac{f_N^2}{4\pi}\right)\left(\frac{f_\Lambda f_\Sigma}{4\pi}\right)\frac{8}{\pi}$$
$$\times\left[\left(\frac{1}{r}+\frac{23}{4r^3}\right)K_0(2r)+\left(\frac{3}{r^2}+\frac{23}{4r^4}\right)K_1(2r)\right], \qquad (3\cdot12)$$

for the off-diagonal potential for which we have no spin dependent and tensor forces.

When $P_{\Sigma\Lambda}$=odd, as for the Λ-N diagonal potential we have no Born terms. This is due to the fact that the S-wave π-Λ scattering amplitude has neither spin nor isospin dependency, so the contributions from the Born terms of the scattering amplitude cancel each other when we substitute the amplitude into Eq. (3·7). In the case of Σ-N potential, as for the part to which the P-wave π-Σ scattering amplitude contributes we have the same expressions as $P_{\Sigma\Lambda}$=even provided that f_Λ^2 be of course set equal to zero, and as for the part to which the S-wave π-Σ scattering amplitude contributes, we have

$$V_c(r) = (\boldsymbol{\rho}\cdot\boldsymbol{\tau})\left(\frac{g_A^2}{4\pi}\right)\left(\frac{f_N^2}{4\pi}\right)\frac{2}{\pi}\left[\frac{2}{r}K_0(2r)+\frac{5}{r^2}K_1(2r)\right]. \qquad (3\cdot13)$$

This has the opposite sign to the central potential which comes out from P-wave π-Σ scattering. The potential in the case of $P_{\Sigma\Lambda}$=odd is made rather weak, though it depends on the value of g_A^2.

3–4. *Qualitative properties of the potentials*

First, we shall consider the effect of the mass difference of Σ and Λ. It has been concluded in the works[13] so far presented that this effect is almost 10% and then the mass difference can be neglected. But generally the effect of the mass difference depends on the inter-particle distance; indeed in the case of V_Λ expressing the ratio of [Born$(\Delta M=0)$ − Born $(\Delta M\neq 0)$] to the Born term where $\Delta M=0$ in percentage, it is less than 20% when $r<1$, but it becomes about 40% when $r\gtrsim 2$, thus the mass difference is more important in the outer region than in the inner region. As for V_Σ, however, the situation is rather different from the case of V_Λ, that is, if f_Σ is not zero, V_Σ has the one-pion exchange potential and moreover in the two-pion exchange potential there is the part which does not include Λ in the intermediate states, so by the very reasons this effect of the mass difference is not so important for V_Σ. As for the off-diagonal potentials the mass difference between the initial and final state must be set equal to zero, because the definition of the potentials in the static theory includes $\delta(0)$.

Next, we shall discuss the rescattering corrections. In Figs. 2 to 4 only the π-N rescattering corrections are taken into account, for we cannot know of the π-Y scattering cross sections at all. But there may be no room to have a doubt about the importance of the rescattering corrections, especially for the central potential of V_Λ. The rescattering corrections to the tensor force is not so large, i.e. it

is less than 20% when $r>1$ and becomes less and less as r increases, but when $r\lesssim 0.7$ it is about 30~40%. Thus these effects are, contrary to the effect of the mass difference, more important in the inner region than in the outer region. The π-Y rescattering corrections generally participate additively with the π-N rescattering corrections in the results. Moreover, it has been pointed out[14] that the π-Y resonances corresponding to the 3-3 resonances in the π-N scattering may occur near a few hundred Mev in the various states. Then the uncertainty will not be avoidable.

In Figs. 3 we have drawn the curves of $\alpha^2=0, 1/5$ and 1 when $\alpha=f_\Sigma/f_\Lambda$ and α is taken to be positive, and moreover f_N/f_Λ is taken to be one. The reason why we take the ratio of f_Σ to f_Λ is unfounded, but it is considered in a somewhat interesting philosophy given by Miyazawa, Nakano and Umezawa.[15] They have insisted that all the pion or the kaon coupling constants are respectively one or zero in the certain units, and conjectured that $|f_\Lambda|$ is one and $|f_\Sigma|$ is zero. If f_Λ has the sign opposite to f_N, the sign of the potentials in the region I is reversed, but in the region where the two-pion exchange potentials exceed the one-pion exchange potentials the sign is naturally invariant.

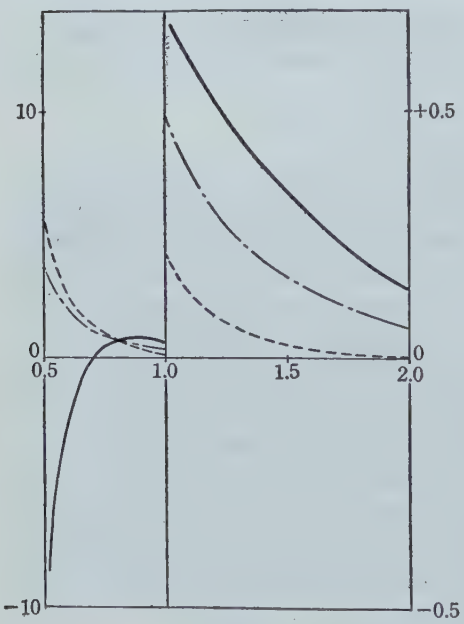


Fig. 3a. V_Σ: Singlet state with T=1/2.

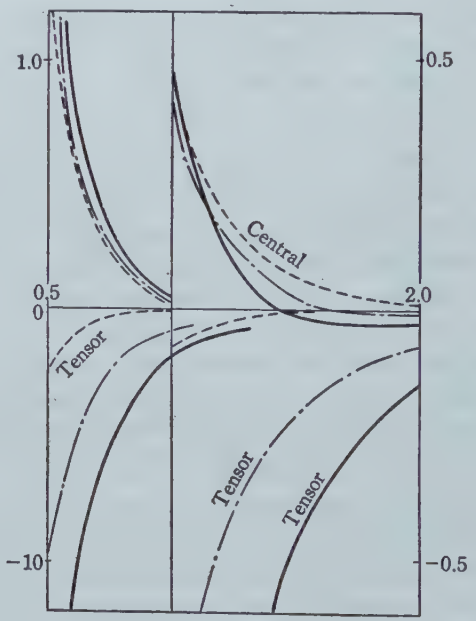


Fig. 3b. V_Σ: Triplet state with T=1/2.

V_Σ's divided by f_Λ^2 are plotted. The dashed lines are for $\alpha^2=0$, the broken lines for $\alpha^2=1/5$, and the real lines for $\alpha^2=1$, where $\alpha=f_\Sigma/f_\Lambda$ and is taken to be positive. Moreover f_Λ is taken to be equal to f_N in these figures. The units are the same as in Fig. 2.

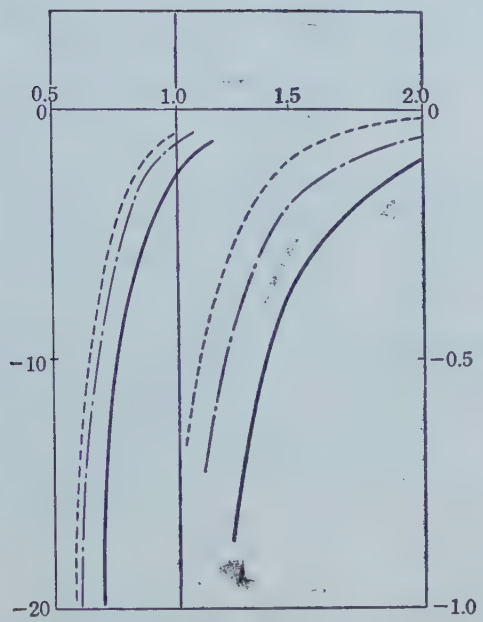


Fig. 3c. V_{Σ}: Singlet state with T=3/2.

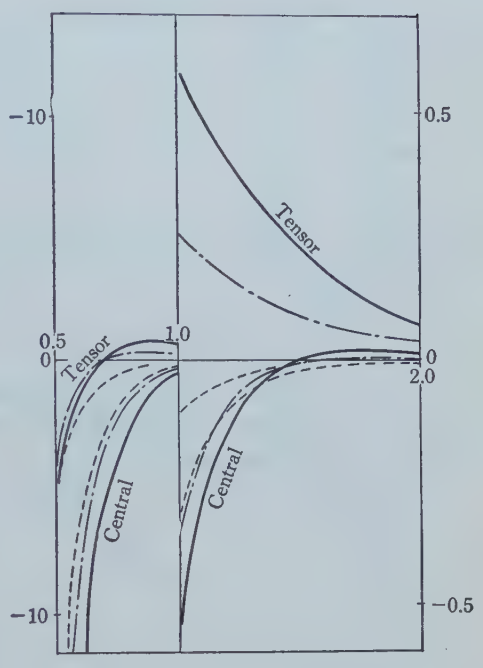


Fig. 3d. V_{Σ}: Triplet state with T=3/2.

The meaning of the lines is the same as in Fig. 3a and 3b. The V_{Σ}'s with T=3/2 resemble to the two-nucleon potentials of the triplet odd and the singlet even state. The units are the same as in Fig. 2.

On the contrary, the off-diagonal potentials are not sensitive to the value of α, because there appears f_{Λ} instead of f_{Σ} in the one-pion exchange potential; they depends on the value of f_N/f_{Λ}.

The Σ-N potential in the outer region is not so sensitive to the relative parity, provided f_{Σ} is not zero, but as for V_{Λ} the relative parity is more important than in V_{Σ}, for V_{Λ} is constructed through the interaction $\Sigma\Lambda\pi$. When $P_{\Sigma\Lambda}$=odd, not only the spin dependence but also the depth of V_{Λ} are much weakened, though they depend on the value of g_A^2. The off-diagonal potentials, of course, depend sensitively on the relative parity and include the factor $(\boldsymbol{\sigma}\cdot\boldsymbol{r})$ which makes the orbital angular momentum of the final state different by one from that of the initial state and at the same time makes the spin of either of two baryons flipped. The potential generally has the following form,

$$V(r)=(\boldsymbol{\sigma}_N\cdot\boldsymbol{r})\,V_N(r)+(\boldsymbol{\sigma}_Y\cdot\boldsymbol{r})\,V_Y(r), \qquad (3\cdot14)$$

where $V_N(r)$ has the one-pion exchange tail but V_Y has not. The lowest order graph of V_Y is the two-pion exchange one and then the Born term of V_Y has the following form,

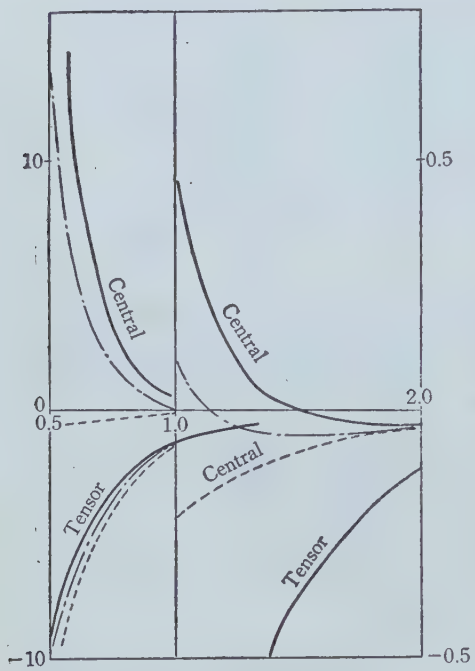


Fig. 4b. $V_{\Sigma\Lambda}$: Triplet state.

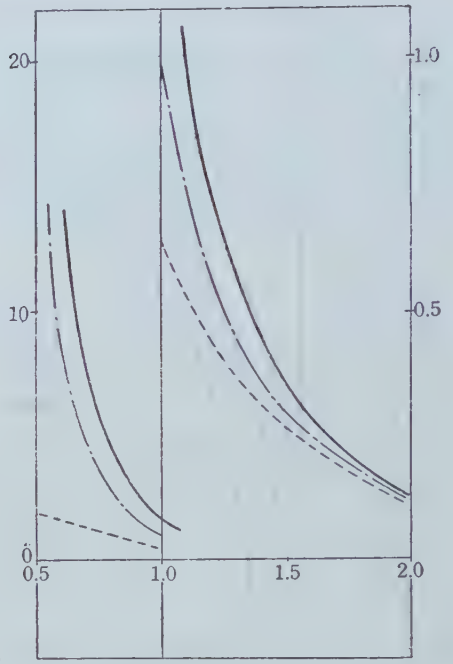


Fig. 4a. $V_{\Sigma\Lambda}$: Singlet state.

The meaning of the lines is the same as in Fig. 3. The units are the same as in Fig. 2. These potentials convert Λ into Σ and vice versa. The α-dependency is small in the outer region but become large in the inner region.

$$V_T(r) = -\left(\frac{g_A f_\Sigma}{4\pi}\right)\left(\frac{f_N^2}{4\pi}\right)\frac{2}{\pi}(\boldsymbol{t}\cdot\boldsymbol{\tau})\left[\frac{8}{r^3}K_0(2r)+\left(\frac{4}{r^2}+\frac{9}{r^4}\right)K_1(2r)\right]. \tag{3·15}$$

On the other hand, V_N has not the two-pion exchange part. Thus the spin-flip of a nucleon is more liable to occur than that of a hyperon in the outer region, but in the inner region the situation is reversed.

§ 4. Three-body forces

The necessity for the three-body force is not so strong in the ordinary nuclei, because the two-nucleon force has the one-pion exchange part whose force range is $1/\mu$. On the contrary, it may be expected that the three-body force is more important in the Λ-hypernuclei than in the ordinary nuclei.

We shall calculate the three-body forces of two different types in this section. In the case of the three-body problem which includes a Λ-hyperon, it is also necessary to use the two-channel formalism, i.e. we must solve the Schrödinger equation like Eq. (2·1) or Eq. (2·10). Then the three-body potential is defined

as the 2×2 matrix form similar to Eq. (2·3) but we calculate only the diagonal part in the case of $P_{\Sigma\Lambda}=$even.

We write the three-body potential as follows,

$$V(x, y, z)=V^{(1)}(x, y, z)+V^{(2)}(x, y, z)+V^{(3)}(x, y, z), \tag{4·1}$$

where x, y and z are the distances of three pairs and the suffices 1, 2 and 3 denote the number of exchanged pions. Then $V^{(1)}$ can be decomposed as in Fig. 6,

$$V^{(1)}(x, y, z)=V_1^{(1)}(x)+V_2^{(1)}(y)+V_3^{(1)}(z). \tag{4·2}$$

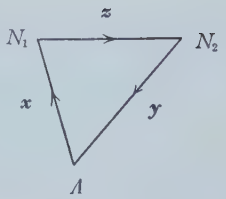


Fig. 5. Triangle Λ and two nucleon construct. The arrows denote the directions of vectors x, y and z.

Though $V^{(2)}$ generally includes such potentials as two pions are exchanged between the same pair, it is possible to make them included in $V^{(1)}$ and then we do not consider them, but we deal with the potentials where only one pion is exchanged between the same pair. Thus we can decompose $V^{(2)}$ as in Fig. 7,

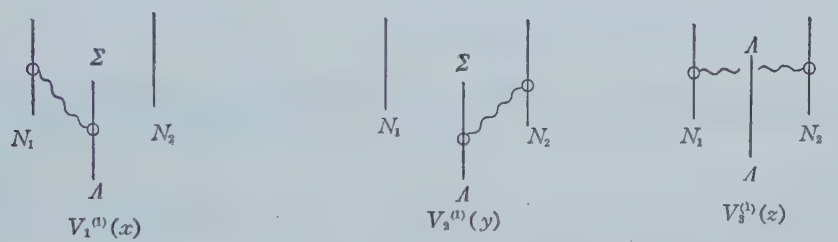


Fig. 6. Feynman graphs for $V^{(1)}$.

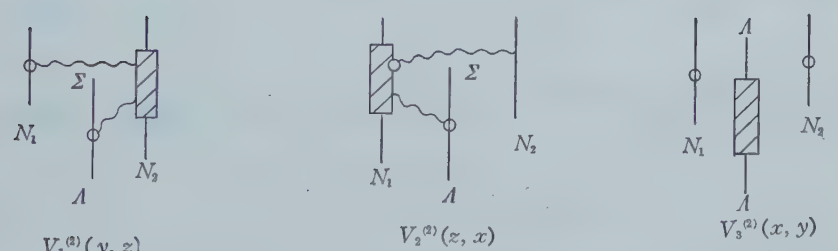


Fig. 7. Feynman graphs for $V^{(2)}$.

$$V^{(2)}(x, y, z)=V_1^{(2)}(y, z)+V_2^{(2)}(z, x)+V_3^{(2)}(x, y). \tag{4·3}$$

$V^{(3)}$ corresponds to Fig. 8 provided our consideration is restricted to the diagonal potential and we do not consider the potentials where more than one pion are exchanged between the same pair.

Now, we can separate the repetition terms from the S-matrix by the same prescription as discussed in § 2:

$$(b|V_i^{(2)}|a)=(b|T^{(2)}|a)-\Big[\frac{(b|V_j^{(1)}|n)\,(n|V_k^{(1)}|a)}{M_0-M_n+i\epsilon}$$

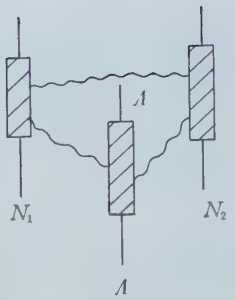
$$+\frac{(b|V_k^{(1)}|n)\,(n|V_j^{(1)}|a)}{M_0-M_n+i\epsilon}\Big] \quad (i,j,k \text{ cyclic.}) \qquad (4\cdot4)$$

and for $V^{(3)}$ we have

$$(b|V^{(3)}|a)=(b|T^{(3)}|a)-\sum_i\Big[\frac{(b|V_i^{(2)}|n)\,(n|V_i^{(1)}|a)}{M_0-M_n+i\epsilon}$$

$$+\frac{(b|V_i^{(1)}|n)\,(n|V_i^{(2)}|a)}{M_0-M_n+i\epsilon}\Big]$$

$$-\sum_P\frac{(b|V_i^{(1)}|n)\,(n|V_j^{(1)}|m)\,(m|V_k^{(1)}|a)}{(M_0-M_n+i\epsilon)\,(M_0-M_m+i\epsilon)}\Big], \qquad (4\cdot5)$$



$V^{(3)}(x,y,z)$

Fig. 8. Feynman graph for $V^{(3)}$.

where $\sum_P$ means to take all permutation of i, j and k. We have not taken the summation on $|n)$ and $|m)$ in the above, for $|n)$ or $|m)$ is uniquely determined by the suffices i, etc.

4–1. *Calculation of* $V^{(2)}$

The S-matrix for Fig. 6 is given as

$$S=\frac{1}{(2\pi)^6}\int d^4p\,d^4q\,\delta(p_0)\,\delta(q_0)\cdot f_N^{\,2}\tau_i^{(1)}\tau_j^{(2)}\,\frac{(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{p})\,(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{q})}{(p^2+\bar{\mu}^2)\,(q^2+\bar{\mu}^2)}\langle p,\,i|S_\Lambda|q,\,j\rangle\,e^{i(px_1-qx_2)}, \qquad (4\cdot6)$$

where suffices 1 and 2 refer to the nucleons 1 and 2. Subtracting the repetition terms from Eq. (4·6), we have

$$V^{(2)}(x,\,y)=-\Big(\frac{f_N^{\,2}}{4\pi}\Big)\Big(\frac{\widetilde{A}_\Lambda(0)}{4\pi}\Big)(\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)})[(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\nabla}_x\boldsymbol{\sigma}^{(0)}\cdot\boldsymbol{\nabla}_x),(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{\nabla}_y\boldsymbol{\sigma}^{(0)}\cdot\boldsymbol{\nabla}_y)]_+\frac{e^{-(x+y)}}{xy}, \qquad (4\cdot7)$$

where suffix 0 refers to Λ and $\widetilde{A}_\Lambda$ is the π-Λ scattering amplitude subtracted by the Born term and then

$$\widetilde{A}_\Lambda(0)=\frac{1}{\pi}\int dk\,\frac{\sigma_{\mathrm{tot}}^{\Lambda}(k)}{\omega^2}. \qquad (4\cdot8)$$

After the differentiation, Eq. (4·7) turns out to be as follows,

$$V^{(2)}(x,\,y)=-\Big(\frac{f_N^{\,2}}{4\pi}\Big)\Big(\frac{\widetilde{A}_\Lambda(0)}{4\pi}\Big)(\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)})[V_1^{(1)}(x),\,V_2^{(1)}(y)]_+, \qquad (4\cdot9)$$

where

$$V_1^{(1)}(x)=(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(0)})\,\frac{e^{-x}}{3x}+S_{10}\Big(1+\frac{3}{x}+\frac{3}{x^2}\Big)\,\frac{e^{-x}}{3x}, \qquad (4\cdot10)$$

and $[\ \ ]_+$ denotes anticommutator. Thus $V^{(2)}$ has a very simple structure and it

can be clearly understood that the force range of $V^{(2)}$ is just $1/\mu$. In this two-channel formalism the Born terms do not exist, contrary to the one channel formalism.[16] $\widetilde{A}_A(0)$ is, however, not zero but may be rather larger than the coupling constant f_A^2, for the resonance may occur near a few hundred Mev in the state with the total angular momentum 3/2. If this resonance has the same structure as the 3-3 resonance, that is, in the case of the universal pion coupling,[14] $\widetilde{A}_A(0)$ is nearly equal to 0.4 and 5 times as large as the coupling constant $f_A^2(=f_N^2)$. If this is not the case, $\widetilde{A}_A(0)$ may at least be comparable to f_A^2.

The " central " potential is simple and

$$V_c^{(2)}(x, y) = -2\left(\frac{f_N^2}{4\pi}\right)\left(\frac{\widetilde{A}_A(0)}{4\pi}\right)(\tau^{(1)}\cdot\tau^{(2)})\,(\sigma^{(1)}\cdot\sigma^{(2)})\,\frac{e^{-(x+y)}}{9xy}. \quad (4\cdot11)$$

Then $V_c^{(2)}$ is repulsive when two nucleons are both in the singlet even and triplet even state, and is attractive both in the singlet odd and triplet odd state.

4–2. *Calculation of* $V^{(3)}$

The S-matrix for Fig. 8 is given as

$$S=\frac{i}{(2\pi)^{12}}\int d^4p\,d^4q\,d^4k\,\frac{\langle p,\ i|S_1|k,\ l\rangle\langle k,\ l|S_2|q,\ j\rangle\langle q,\ j|S_A|pi\rangle}{(p^2+\mu^2)\,(q^2+\mu^2)\,(k^2+\mu^2)}. \quad (4\cdot12)$$

Although one can take into account the rescattering corrections in this case, too, if one spares no trouble, we do not consider them here and only investigate the qualitative feature of the Born terms.

Subtracting the contributions of the pole at $p_0=0$ from Eq. (4·12), we have

$$\begin{aligned}V^{(3)}(x, y, z) = &\left(\frac{f_N^2}{4\pi}\right)^2\left(\frac{f_A^2}{4\pi}\right)\tau_i^{(1)}\tau_i^{(1)}\tau_i^{(2)}\tau_l^{(2)}\big[(\sigma^{(0)}\cdot\nabla_x\sigma^{(0)}\cdot\nabla_y)\,(\sigma^{(1)}\cdot\nabla_z\sigma^{(1)}\cdot\nabla_x)\\
&\times(\sigma^{(2)}\cdot\nabla_y\sigma^{(2)}\cdot\nabla_z)+(\sigma^{(0)}\cdot\nabla_y\sigma^{(0)}\cdot\nabla_x)\,(\sigma^{(1)}\cdot\nabla_x\sigma^{(1)}\cdot\nabla_z)\,(\sigma^{(2)}\cdot\nabla_z\sigma^{(2)}\cdot\nabla_y)\\
&+(\sigma^{(0)}\cdot\nabla_y\sigma^{(0)}\cdot\nabla_x)\,(\sigma^{(1)}\cdot\nabla_z\sigma^{(1)}\cdot\nabla_x)\,(\sigma^{(2)}\cdot\nabla_y\sigma^{(2)}\cdot\nabla_z)\\
&+(\sigma^{(0)}\cdot\nabla_x\sigma^{(0)}\cdot\nabla_y)\,(\sigma^{(1)}\cdot\nabla_x\sigma^{(1)}\cdot\nabla_z)\,(\sigma^{(2)}\cdot\nabla_z\sigma^{(2)}\cdot\nabla_y)\big]F_{\Delta M,0,0}(x, y, z)\\
&-\left(\frac{f_N^2}{4\pi}\right)^2\left(\frac{f_A^2}{4\pi}\right)(\tau^{(1)}\cdot\tau^{(2)})^2\big[(\sigma^{(0)}\cdot\nabla_x\sigma^{(0)}\cdot\nabla_y)\,(\sigma^{(1)}\cdot\nabla_z\sigma^{(1)}\cdot\nabla_x)\\
&\times(\sigma^{(2)}\cdot\nabla_z\sigma^{(2)}\cdot\nabla_y)+(\sigma^{(0)}\cdot\nabla_y\sigma^{(0)}\cdot\nabla_x)\,(\sigma^{(1)}\cdot\nabla_z\sigma^{(1)}\cdot\nabla_x)\,(\sigma^{(2)}\cdot\nabla_z\sigma^{(2)}\cdot\nabla_y)\\
&+(\sigma^{(0)}\cdot\nabla_y\sigma^{(0)}\cdot\nabla_x)\,(\sigma^{(1)}\cdot\nabla_x\sigma^{(1)}\cdot\nabla_z)\,(\sigma^{(2)}\cdot\nabla_y\sigma^{(2)}\cdot\nabla_z)\\
&+(\sigma^{(0)}\cdot\nabla_y\sigma^{(0)}\cdot\nabla_x)\,(\sigma^{(1)}\cdot\nabla_x\cdot\sigma^{(1)}\cdot\nabla_z)\,(\sigma^{(2)}\cdot\nabla_y\sigma^{(2)}\cdot\nabla_z)\big]F_{\Delta M,0,0}(x, y, z),\end{aligned} \quad (4\cdot13)$$

where

$$F_{\lambda,\mu,\nu}(x, y, z)=\frac{1}{\pi}\,\frac{1}{(xyz)}\int pdp\,\frac{\sin p(x+y+z)+\sin(px)\sin(py)\sin(pz)}{\omega(\lambda+\omega)\,(\mu+\omega)\,(\nu+\omega)}. \quad (4\cdot14)$$

We neglect the mass difference in Eq. (4·13) and $F_{0,0,0}(x, y, z)$ is

$$F_{0,0,0}(x, y, z) = 3G(x, y, z) - [G(-x, y, z) + G(x, -y, z) + G(x, y, -z)] \tag{4·15}$$

$$G(x, y, z) = \frac{1}{16} \frac{x+y+z}{xyz} \exp[-(x+y+z)],$$

where the triangle inequality appears explicitly in brackets, i.e. $x+y-z>0$, etc. From this the potential with such structure as $f(x, y, z) \exp[-(x+y-z)]$ appears, where $f(x, y, z)$ is a rational function. Such structure does not exist in the ordinary potentials we have so far considered, and this gives the longest force range.

The "central" potential of $V^{(3)}$ has the following structure,

$$V_c^{(3)}(x, y, z) = (3+2\boldsymbol{\tau}^{(1)} \cdot \boldsymbol{\tau}^{(2)}) V_\tau^{(3)}(x, y, z) + (3-2\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)}) V_\sigma^{(3)}(x, y, z) \tag{4·16}$$

and

$$V_\tau^{(3)} = 8\left(\frac{f_N^2}{4\pi}\right)^2 \left(\frac{f_A^2}{4\pi}\right) \Big[(\boldsymbol{i} \cdot \boldsymbol{j})(\boldsymbol{j} \cdot \boldsymbol{k})(\boldsymbol{k} \cdot \boldsymbol{i}) xyz \frac{\partial^3 f}{\partial x \partial y \partial z} + (\boldsymbol{i} \cdot \boldsymbol{j})^2 xy \frac{\partial^2 f}{\partial x \partial y}$$

$$+ (\boldsymbol{j} \cdot \boldsymbol{k})^2 yz \frac{\partial^2 f}{\partial y \partial z} + (\boldsymbol{k} \cdot \boldsymbol{i})^2 zx \frac{\partial^2 f}{\partial z \partial x} + \left(x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z}\right) + 3f \Big], \tag{4·17}$$

$$V_\sigma^{(3)} = -8\left(\frac{f_N^2}{4\pi}\right)^2 \left(\frac{f_A^2}{4\pi}\right) \Big[\frac{1}{9} xyz \frac{\partial^3 f}{\partial x \partial y \partial z} + \frac{1}{3}\left(xy \frac{\partial^2 f}{\partial x \partial y} + yz \frac{\partial^2 f}{\partial y \partial z}\right.$$

$$\left. + zx \frac{\partial^2 f}{\partial z \partial x}\right) + \left(x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z}\right) + 3f \Big], \tag{4·18}$$

where $\boldsymbol{i} = \boldsymbol{x}/x$, $\boldsymbol{j} = \boldsymbol{y}/y$ and $f = 1/(xyz) \cdot \partial^3 F_{0,0,0}(x, y, z)/\partial x \partial y \partial z$. Thus this "central" potential depends on the angles of the triangle which A and $2N$ construct, and has no simple structure as $V^{(2)}$. Then we shall consider the asymptotic forms where x, y and z are large, i.e.

$$V_\tau^{(3)} \to \frac{1}{2}\left(\frac{f_N^2}{4\pi}\right)^2 \left(\frac{f_A^2}{4\pi}\right) (\boldsymbol{i} \cdot \boldsymbol{j})(\boldsymbol{j} \cdot \boldsymbol{k})(\boldsymbol{k} \cdot \boldsymbol{i}) A(x, y, z)$$

$$V_\sigma^{(3)} \to -\frac{1}{18}\left(\frac{f_N^2}{4\pi}\right)^2 \left(\frac{f_A^2}{4\pi}\right) A(x, y, z), \tag{4·19}$$

where

$$A(x, y, z) = \frac{1}{xyz}[3(x+y+z) e^{-(x+y+z)} + (-x+y+z) e^{-(-x+y+z)}$$

$$+(x-y+z)\,e^{-(x-y+z)}+(x+y-z)\,e^{-(x+y-z)}]. \qquad (4\cdot 20)$$

Morcover if we add the condition that Λ and $2N$ construct the regular triangle the asymptotic forms become

$$V_\tau^{(3)}\to-\frac{3}{16}\left(\frac{f_N^2}{4\pi}\right)^2\left(\frac{f_A^2}{4\pi}\right)\left[\frac{e^{-3r}}{r^2}+\frac{e^{-r}}{r^2}\right] \qquad (4\cdot 21)$$

and

$$V_\sigma\to-\frac{1}{6}\left(\frac{f_N^2}{4\pi}\right)^2\left(\frac{f_A^2}{4\pi}\right)\left[\frac{e^{-3r}}{r^2}+\frac{e^{-r}}{r^2}\right], \qquad (4\cdot 22)$$

where r is a side of the regular triangle. Thus the "force range" is rather longer than $V^{(2)}$ when x is taken to be equal to y in $V^{(2)}$.

The tensor part of $V^{(3)}$ is more and more complicated and we no longer discuss it here.

§ 5. Discussions

It could be understood that by the use of the two-channel formalism one can treat both the Y-N potentials of the two-body system and those of the three-body system in the parallel fashion as the two-nucleon potential. However, we have so far calculated the potentials in the frame of the static theory, in spite of the existence of the attempts[2] to overcome the static theory. In addition, it has recently been pointed out[17] that the π-π interactions are very important in the pion physics, but we have not taken into account the π-π interactions at all. In the case of the two-nucleon system the potential which is constructed by the S-wave scattering amplitudes was very small because of the smallness of the S-wave π-N scattering, but this may not be the case in the Y-N system. Thus in the various senses our approach to the problems is conservative, but considering the success in the two-nucleon potentials it will be significant at the first step to investigate the Y-N systems by the use of the static potentials as in the two-nucleon systems. The system which corresponds just to the two-nucleon system is the one with $T=3/2$ such as Σ^+p or Σ^-n; in this case there is not the Λ-N channel, so it is able to treat the system by using the same method as in the two-nucleon system with $T=1$. In this sense it will be most suitable to determine the coupling constant f_Σ and the sign, though these experiments are difficult because of the short lifetime of Σ^+ which is almost half of the lifetime of Σ^- and of the impossibility of the neutron target in the case of Σ^-n system. Therefore the existence of the bound state of Σ^-n or Σ^+p is very interesting, but the theoretical prediction on the existence of these bound states is not significant from the meson theoretical point of view. This is indeed evident from the fact that even the binding of the deuteron has never been accomplished by the meson theory. The experimental data[18] which exist up to now are on the low energy Σ^-p reactions. It may make us confused to consider the various reaction modes as in the same level; for example,

in the hyperon exchange scattering such as $\Sigma^- + p \to \Lambda + n$ the final energy is larger than the mass difference of Σ and Λ, that is, almost 80 Mev, and then it is already a high energy phenomenon in the sense of the nuclear force.[1] The hyperon exchange scattering is, however, advantageous to test the off-diagonal potential which has the one-pion exchange tail and then to determine the type of the coupling and the coupling constant f_A or g_A. There are the experimental results on the Λp system,[19] but they as well as the experimental results on the $\Sigma^- p$ system only indicate the order of the cross section. In the case of the Λp scattering, the elastic scattering below the threshold energy of a Σ-production will be interesting. But even in the energy region below the threshold energy of a Σ-production it will be rather difficult to separate the outer region and the inner region due to the existence of the Σ-N channel. This situation will also be suggested by considering Eq. (2·10), i.e. the non-local potential part includes the behaviour of the Σ-N system in the inner region.

Now, we proceed to the problems of the hyperfragments which have been analysed phenomenologically by Dalitz and Downs.[20] The analysis made so far, however, is in the one-channel formalism. As pointed out in § 4, it is impossible to ignore the Σ-component in the analysis of the hyperfragments, even if the analysis is phenomenological. Moreover, the three-body potentials have long force ranges, so the three-body correlation becomes important.

We are now in a position to proceed with the investigations of the above subjects and the succeeding paper will be devoted to them.

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References

1) J. Iwadare, S. Otsuki, R. Tamagaki, S. Machida, T. Toyoda and W. Watari, Supplement of Prog. Theor. Phys. No. **3** (1956).
2) Y. Hara and H. Miyazawa, Prog. Theor. Phys. (to be published). H. Yamamoto, Soryushiron Kenkyu, **26** (1959), 303 (mimeographed circular in Japanese). His method is the one by using the Bethe-Salpeter equation. This possibility is also pointed out by M. Ida (private communication). G. Chew, *Proceedings of the Ninth Conference on High Energy Nuclear Physics at Kiev*, U. S. S. R., 1959.
3) H. Feshbach, Annals of Physics **5** (1958), 357.
L. Fonda, Nuovo Cimento **13** (1959), 956.
Y. Yamaguchi, Supplement of Prog. Theor. Phys. No. **7** (1959).
4) B. A. Lippman and J. Schwinger, Phys. Rev. **79** (1950), 469.
5) F. Ferrari and L. Fonda, Nuovo Cimento **9** (1959), 842.
6) K. A. Brueckner and K. Watson, Phys. Rev. **92** (1953), 1023; **96** (1954), 508.

7) D. B. Lichtenberg and M. Ross, Phys. Rev. **107** (1957), 1714.
8) M. Konuma, H. Miyazawa and S. Otsuki, Prog. Theor. Phys. **19** (1958), 17.
9) H. Miyazawa, Phys. Rev. **104** (1956), 1741.
10) M. Taketani, S. Machida and S. Ohnuma, Prog. Theor. Phys. **7** (1952), 45. See also reference 1).
11) N. Fukuda, K. Sawada and M. Taketani, Prog. Theor. Phys. **12** (1954), 156. See also reference 1).
12) K. Nishijima, Supplement of Prog. Theor. Phys. No. **3** (1956).
13) R. H. Dalitz, *1958 Annual International Conference on High Energy Physics at CERN.* D. B. Lichtenberg and M. Ross, Phys. Rev. **107** (1957), 1714.
14) M. Gell-Mann, Phys. Rev. **106** (1957), 1296.
A. Komatsuzawa, R. Sugano and Y. Nogami, Prog. Theor. Phys. **21** (1959), 151.
Y. Nogami, Prog. Theor. Phys. **22** (1959), 25.
15) H. Miyazawa, T. Nakano and H. Umezawa, Prog. Theor. Phys. **22** (1959), 643.
16) R. H. Dalitz, *Proceedings of the Ninth Conference on High Energy Physics at Kiev* U.S.S.R., 1959.
17) L. I. Schiff, *Proceeedings of the Ninth Conference on High Energy Nuclear Physics at Kiev*, U.S.S.R., 1959. G. Chew, reference 2).
K. Kawarabayashi and H. Miyazawa, Prog. Theor. Phys. (1960), S. Ishida, to be published. In the last two papers the effects on the S-wave π-N scattering are discussed.
18) L. W. Alvarez, *Proceedings of the Ninth Conference on High Energy Nuclear Physics at Kiev*, U.S.S.R., 1959.
Leitner, Nardin, Rosenfeld, Solmitz and Tripp, Phys. Rev. Letters **3** (1959), 238.
19) Crawford, Gresti, Good, Solmitz, Stevenson and Ticho, Phys. Rev. Letters **2** (1959), 174.
20) R. H. Dalitz and B. W. Downs, Phys. Rev. **110** (1958), 842; **111** (1958), 967.
R. H. Dalitz, reference 16).
S. Iwao and E. C. G. Sudarshan, Phys. Rev. Letters **4** (1960), 140.

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Vibrational Thermodynamic Properties of Lattices with Defects. II

——*Two- and Three-Dimensional Simple Cubic Lattices**——

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This work is a continuation and application to two- and three-dimensional crystals of the methods of reference 1) for the evaluation of the changes in thermodynamic properties of a lattice due to the presence of defects of various kinds. In particular we consider the effects of an isotope defect and a pair of isotope defects on the zero point energy and free energy of two- and three-dimensional cubic lattices. In addition we evaluate the free energy and self entropy of a vacancy in a three-dimensional cubic lattice.

§ 1. Introduction

In an earlier paper[1] to be referred to as (I), some general methods were developed for analyzing the vibrational properties of lattices with defects, and the methods were applied to the linear lattice. In this paper these methods have been extended to two- and three-dimensional lattices.

The techniques to be used in this paper are modifications of those described in § 2 of (I). In order to make this paper self-contained we recapitulate some of the formulae obtained in (I). We showed in that paper that the change in free energy due to defects could be written as

$$\Delta A=\Delta E_0-kT\sum_{n=1}^{\infty}\frac{\Delta I_n}{n}. \tag{1·1}$$

The quantities in this equation are defined in terms of the matrix $\boldsymbol{\Delta}(f)=\boldsymbol{I}+\boldsymbol{M}_0^{-1}(f)$.

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$\delta \boldsymbol{M}(f)$, where $\boldsymbol{M}_0(f)$ is the matrix of the secular determinant for the unper turbed lattice, $\delta \boldsymbol{M}(f)$ represents the change in $\boldsymbol{M}_0(f)$ due to defects, and we have written $\omega/\omega_L=f$, where ω_L is the maximum normal mode frequency. If $|\Delta(if)|$ represents the determinant of $\boldsymbol{\Delta}(if)$ and we define $\Omega(f)$ by the equation

$$\Omega(f)=\frac{d}{df}\ln|\Delta(if)|, \tag{1·2}$$

we showed in (I) that ΔE_0, the change in zero-point energy, and the ΔI_n are given by the following expressions,

$$\begin{aligned}\Delta E_0&=-\frac{\hbar\omega_L}{2\pi}\int_0^\infty f\Omega(f)\,df\\ \Delta I_n&=\frac{1}{\pi}\int_0^\infty \sin(\alpha_n f)\,\Omega(f)\,df\end{aligned} \tag{1·3}$$

where $\alpha_n=n\hbar\omega_L/(kT)$. We further showed that when $\Omega(f)$ can be expanded as a Taylor series in even powers of f in a neighborhood of $f=0$ one can derive an asymptotic expression for ΔI_n at low temperatures by repeated integrations by parts. Although this expansion is valid in one and three dimensions it is not valid in two dimensions because $\Omega(f)$ has a nonanalytic behavior near $f=0$. However, we shall show that it is still possible to derive an asymptotic expansion for ΔI_n in two dimensions which is valid at low temperatures and which yields the proper low temperature behavior of the various thermodynamic functions.

Since the only singularity of $\Omega(f)$ for one-, two-, or three-dimensional lattices occurs at $f=0$, as is demonstrated in Appendix I, the asymptotic behavior of the integral for ΔI_n in the limit as $T\to 0$ will be determined by the nature of this one singularity.[7] We further show, in Appendix II, that the leading term in the expansion of $\Omega(f)$ for a two-dimensional lattice as $f\to 0$ is proportional to $f\ln|f|$. Consequently we must evaluate the asymptotic behavior of ΔI_n given this type of small f behavior. The asymptotic properties of such integrals are most conveniently summarized in the book by Lighthill.[2] Hence it is possible to discuss the low temperature properties of A for two-dimensional lattices as well as those of an odd number of dimensions.

We have also shown in (I) that given the expression for $\Omega(f)$ one can discuss the high temperature properties of the free energy. The expression for ΔA at high temperatures is

$$\Delta A=\frac{kT}{2}\ln|\Delta D(0)|-\frac{kT}{2}\sum_{n=1}^{\infty}\frac{B_n}{n(2n)!}\left(\frac{\hbar\omega_L}{kT}\right)^{2n}\alpha_{2n+1}, \tag{1·4}$$

where the B_n are Bernoulli numbers. In this expression α_{2n+1} is the coefficient of $2(-1)^n f^{-(2n+1)}$ in the expansion of $\Omega(f)$ for f large. The matrix $\boldsymbol{D}(0)$ is obtained from $\boldsymbol{M}(\omega)=\boldsymbol{M}_0(\omega)+\delta\boldsymbol{M}(\omega)$ by dividing every row and column of $\boldsymbol{M}(\omega)$

by the square root of the mass appearing in the diagonal element corresponding to that row and column and changing the sign of each term. The resulting matrix can be written as $\boldsymbol{D}(0)-\omega^2\boldsymbol{I}$ which defines $\boldsymbol{D}(0)$.

§ 2. Effect of defects in a monatomic square lattice

In this section we will consider the effect of (i) an isotopic impurity and (ii) a pair of isotopic impurities on the thermodynamic properties of a monatomic square lattice with nearest neighbor central and noncentral force interactions.

(i) *An isotopic defect*: The defect determinant $|\Delta(f)|$ is given by

$$|\Delta(f)|=1-\frac{2\varepsilon(1+\sigma)f^2}{\pi^2}\iint_0^{\pi}\frac{d\phi_1\,d\phi_2}{(1+\sigma)(2f^2-1)+\cos\phi_1+\sigma\cos\phi_2} \tag{2·1}$$

with $\sigma=\gamma_2/\gamma_1$, $\varepsilon=1-M'/M$, where γ_1, and γ_2 are the central and non-central force constants, respectively, and M' is the impurity mass.

One can obtain the discrete frequency which comes out of the band by solving the equation

$$|\Delta(f)|=0.$$

We have to explore the region $f>1$. In this region the integral on the right-hand side of Eq. (2·1) becomes

$$\frac{1}{\pi^2}\iint_0^{\pi}\frac{d\phi_1\,d\phi_2}{(1+\sigma)(2f^2-1)+\cos\phi_1+\sigma\cos\phi_2}$$

$$=\frac{1}{\pi}\frac{1}{\sqrt{[(1+\sigma)f^2-1][(1+\sigma)f^2-\sigma]}}K\left(\sqrt{\frac{\sigma}{[(1+\sigma)f^2-1][(1+\sigma)f^2-\sigma]}}\right) \tag{2·2}$$

where $K(k)$ is the complete elliptic integral of the first kind.

Substituting this result in the equation for the discrete frequency, we get the following transcendental equation for f:

$$K\left(\sqrt{\frac{\sigma}{[(1+\sigma)f^2-1][(1+\sigma)f^2-\sigma]}}\right)$$

$$=\frac{\pi}{2\varepsilon(1+\sigma)f^2}\sqrt{[(1+\sigma)f^2-1][(1+\sigma)f^2-\sigma]}. \tag{2·3}$$

It is easy to show that this equation has a solution only for $0<\varepsilon<1$, i.e. for a light isotope defect. For $|\varepsilon|\ll 1$, the discrete frequency is given by

$$\omega_{\mathrm{imp}}\cong\omega_L\left[1+\frac{16\sigma}{(1+\sigma)^2}\exp\left(-\frac{\pi}{(1+\sigma)\varepsilon}\right)\right]. \tag{2·4}$$

To evaluate the self-energy of the defect we substitute the expression

$$|\Delta(if)|=1-\frac{2\varepsilon(1+\sigma)}{\pi}\frac{f^2}{\sqrt{[(1+\sigma)f^2+1][(1+\sigma)f^2+\sigma]}}\times K\left(\sqrt{\frac{\sigma}{[(1+\sigma)f^2+1][(1+\sigma)f^2+\sigma]}}\right) \tag{2·5}$$

into Eq. (1·3) to obtain

$$\Delta E_0=-\frac{\hbar\omega_L}{2\pi}\int_0^\infty f\,d\ln\left[1-\frac{2\varepsilon(1+\sigma)}{\pi}\frac{f^2}{\sqrt{[(1+\sigma)^2f^2+1][(1+\sigma)f^2+\sigma]}}\times K\left(\sqrt{\frac{\sigma}{[(1+\sigma)f^2+1][(1+\sigma)f^2+\sigma]}}\right)\right]. \tag{2·6}$$

This integral cannot be evaluated in closed form in terms of elementary functions, but one can obtain its numerical value by a combination of numerical and analytical methods. The result in the case $\sigma=1$ is

$$\Delta E_0=\frac{\hbar\omega_L}{2\pi}[0.83\varepsilon+0.73\varepsilon^2+O(\varepsilon^3)]. \tag{2·7}$$

To obtain the low temperature expansion for the change in free energy, we note that (see Appendix II) for f small and ε small

$$\Omega(f)=\frac{4\varepsilon(1+\sigma)}{\pi\sqrt{\sigma}}f\ln|f|+O(\varepsilon^2). \tag{2·8}$$

It can be shown that the functional form of the behavior of $\Omega(f)$ for small $|f|$ is sufficient to determine the low temperature behavior of ΔI_n. The result of the asymptotic expansion is best summarized in the book by Lighthill[2] and yields, for the leading term in the low temperature behavior of ΔI_n:

$$\Delta I_n\sim\frac{\varepsilon(1+\sigma)}{2\pi^3\sqrt{\sigma}}\left(\frac{kT}{\hbar\omega_L}\right)^2\frac{1}{n^2}. \tag{2·9}$$

Finally, substituting this result in Eq. (1·1), we obtain the following leading terms in the low temperature expansion of the change in the free energy:

$$\Delta A(T)_{L.T.}\cong\Delta E_0-\frac{\varepsilon(1+\sigma)}{2\pi^3\sqrt{\sigma}}\zeta(3)\left(\frac{kT}{\hbar\omega_L}\right)^2kT \tag{2·10}$$

where $\zeta(n)$ is the Riemann zeta-function.

The high temperature expansion for the change in the free energy is readily obtained through the use of Eq. (1·4) and is given in the weak defect limit by

$$\Delta A(T)_{H.T.}=\frac{kT}{2}\ln\left(\frac{M}{M'}\right)-\varepsilon kT\left\{\frac{B_1}{2}\left(\frac{\hbar\omega_L}{kT}\right)^2+\frac{B_2}{16}\left(\frac{\hbar\omega_L}{kT}\right)^4|3(1+\sigma)^2+4\sigma|+\cdots\right\}. \tag{2·11}$$

(ii) *A pair of isotopic defects*: We consider now the case of two defects located at lattice points $\boldsymbol{m}$ and $\boldsymbol{m}'$. In this case the defect determinant is ($g(\boldsymbol{m}, \boldsymbol{m}')$ is an element of the inverese matrix $M_0^{-1}(\omega)$, see I, Eq. (2·12)):

$$|\Delta(\omega)|=1-\frac{\varepsilon^2 M^2\omega^4 g^2(\boldsymbol{m},\boldsymbol{m}')}{[1-\varepsilon M\omega^2 g(0,0)]^2}\approx 1-\varepsilon^2 M^2\omega^4 g^2(\boldsymbol{m},\boldsymbol{m}') \quad |\varepsilon|\ll 1 \qquad (2\cdot 12)$$

The function $g(\boldsymbol{m}, \boldsymbol{m}')$ can be expressed as a generalized hypergeometric function of two variables but simpler results are obtained by making $|\boldsymbol{m}-\boldsymbol{m}'|$ large, when for $\omega\to i\omega$

$$\begin{aligned} g(\boldsymbol{m},\boldsymbol{m}') &= -\frac{1}{2\pi^2\gamma_1}\int_0^{\pi}\!\!\int \frac{\cos l_1\phi_1\cos l_2\phi_2\, d\phi_1 d\phi_2}{(1+\sigma)(2f^2+1)-\cos\phi_1-\sigma\cos\phi_2} \\ &\approx -\frac{1}{\pi^2\gamma_1}\int_0^{\infty}\!\!\int \frac{\cos l_1\phi_1\cos l_2\phi_2}{4(1+\sigma)f^2+\phi_1^2+\sigma\phi_2^2}\, d\phi_1 d\phi_2 \\ &= -\frac{1}{2\pi\gamma_1\sqrt{\sigma}} K_0\left(2f\sqrt{(1+\sigma)(l_1^2+l_2^2/\sigma)}\right), \end{aligned} \qquad (2\cdot 13)$$

where $l_i=|m_i-m_i'|$ and $K_0(x)$ is a modified Bessel function of the second kind. The first non-vanishing contribution to the interaction zero-point energy is

$$\begin{aligned} \Delta E_0 &= -\frac{\varepsilon^2\hbar\omega_L}{2\pi}\,\frac{4(1+\sigma)^2}{\sigma\pi^2}\int_0^{\infty} f^4 K_0^2\left(2f\sqrt{(1+\sigma)(l_1^2+l_2^2/\sigma)}\right) df \\ &= -\frac{27}{2^{13}}\,\frac{\varepsilon^2\hbar\omega_L}{\pi\sigma\sqrt{1+\sigma}}\,\frac{1}{(l_1^2+l_2^2/\sigma)^{5/2}}. \end{aligned} \qquad (2\cdot 14)$$

The low temperature interaction free energy is given by

$$\begin{aligned} \Delta A(T)_{L.T.} &= \Delta E_0 + \frac{24\varepsilon^2(1+\sigma)}{\pi^2\sigma}\,\frac{(kT)^5}{(\hbar\omega_L)^4}\zeta(5)\left[\psi(3)-\ln\frac{\hbar\omega_L}{2kT\sqrt{(1+\sigma)(l_1^2+l_2^2/\sigma)}}\right] \\ &\quad -\frac{24\varepsilon^2(1+\sigma)}{\pi^2\sigma}(0.0285)\,\frac{(kT)^5}{(\hbar\omega_L)^4}+\cdots, \end{aligned} \qquad (2\cdot 15)$$

where

$$\psi(z)=\frac{d}{dz}\ln\Gamma(z).$$

For evaluating the interaction free energy at high temperature it is inappropriate to use the asymptotic form of the inverse matrix elements in $|\Delta(\omega)|$ for large distance of separation of the defects since at high temperatures the details of the upper end of the frequency spectrum are important. Since the frequency distribution function is directly related to $|\Delta(\omega)|$,* it is necessary to use the exact

* See the Appendix to (I). This point will be discussed in detail in a subsequent paper. For preliminary reports, the reader is referred to Maradudin and Weiss, J. Phys. Chem. Solids **7** (1958), 327 and J. Mahanty, Ph. D. Thesis, University of Maryland, (1959).

form of the inverse matrix elements in forming $|\Delta(\omega)|$ and $\Omega(f)$. This offers no difficulty because we need to evaluate only the large f expansion of $\Omega(f)$.

For $x=i\omega$ and large x, we have

$$g(\boldsymbol{m},\boldsymbol{m}')=-\frac{1}{M\pi^2x^2}\int_0^\infty\!\!\int\cos l_1\phi_1\cos l_2\phi_2\left\{1+\frac{4\gamma_1}{Mx^2}\left[\sin^2\frac{\phi_1}{2}+\sigma\sin^2\frac{\phi_2}{2}\right]\right\}^{-1}d\phi_1\,d\phi_2$$

$$=-\frac{\sigma^{l_2}}{M\pi}\left(\frac{\omega_L^2}{4(1+\sigma)}\right)^{l_1+l_2}\frac{(l_1+l_2)!}{l_1!\,l_2!}\frac{1}{x^{2(l_1+l_2+1)}}$$

$$-\frac{2\sigma^{l_2}}{M\pi}\left(\frac{\omega_L^2}{4(1+\sigma)}\right)^{l_1+l_2+1}\frac{(1+\sigma)(l_1+l_2+2)!}{l_1!\,l_2!}\left(\frac{1}{(1+\sigma)(l_1+1)}\right.$$

$$\left.+\frac{\sigma}{(1+\sigma)(l_2+1)}\right)\frac{1}{x^{2(l_1+l_2+2)}}+\cdots. \qquad (2\cdot16)$$

Using this result, we obtain the following expansion for $\Omega(x)$:

$$\lim_{x\to\infty}\Omega(x)=\varepsilon^2\left[\frac{4\sigma^{2l_2}}{\pi^2}\left(\frac{\omega_L^2}{4(1+\sigma)}\right)^{2(l_1+l_2)}\frac{(l_1+l_2)[(l_1+l_2)!]^2}{(l_1!)^2(l_2!)^2}\frac{1}{x^{4(l_1+l_2)+1}}\right.$$

$$+\omega_L^2\frac{\sigma^{2l_2}}{\pi^2}\left(\frac{\omega_L^2}{4(1+\sigma)}\right)^{2(l_1+l_2)}\frac{(l_1+l_2)!\,(l_1+l_2+2)![4(l_1+l_2)+2]}{(l_1!)^2(l_2!)^2}$$

$$\left.\times\left[\frac{1}{(\sigma+1)(l_1+1)}+\frac{\sigma}{(\sigma+1)(l_2+1)}\right]\frac{1}{x^{4(l_1+l_2)+3}}+\cdots.\right. \qquad (2\cdot17)$$

The high temperature expansion of the interaction free energy can now be obtained in the following form, using Eq. (1·4):

$$\Delta A(T)_{H.T.}=-\varepsilon^2kT\left[\frac{B_{2(l_1+l_2)}\sigma^{2l_2}}{2\pi^2}\left(\frac{1}{4(1+\sigma)}\right)^{2(l_1+l_2)}\frac{[(l_1+l_2)!]^2}{[4(l_1+l_2)]!\,(l_1!)^2(l_2!)^2}\right.$$

$$\left.\times\left(\frac{\hbar\omega_L}{kT}\right)^{4(l_1+l_2)}+\cdots\right]. \qquad (2\cdot18)$$

We note that the exponent of T in the first non-vanishing term in the expansion of the interaction free energy in inverse powers of temperature depends on the separation between the defects. This feature has already been noted for the linear chain and is a characteristic of lattices of the type under consideration independent of the dimensionality of the lattice.

The results for the low temperature expansions of the free energy serve to illustrate the peculiar difficulties that arise in the analysis of a 2-D lattice, difficulties that are related to the occurrence of logarithmic terms in the inverse matrix elements when $if\to0$ for f real.

§ 3. Effect of isotope defects in a simple cubic lattice

In this section we will consider the effect of isotope defects in a monatomic

simple cubic lattice with nearest neighbor central and non-central force interactions. The equations of motion and the inverse matrix elements have been described in § 2 of (I). We will assume that the noncentral force constants are equal.

(i) *An isotopic impurity*: The defect determinant is given by

$$|\Delta(\omega)|=1-\frac{\varepsilon M\omega^2}{\pi^3}\int_0^{\pi}\int\int\frac{d\varphi_1 d\varphi_2 d\varphi_3}{M\omega^2-\sum_{j=1}^{3}2\gamma_j(1-\cos\varphi_j)}. \tag{3·1}$$

The self energy is given by

$$\Delta E_0=-\frac{\hbar\omega_L}{2\pi}\int_0^{\infty}x\,d\ln\left[1-\frac{\varepsilon x^2}{\pi^3}\int_0^{\pi}\int\int\frac{d\phi_1 d\phi_2 d\phi_3}{x^2+\sum_{j=1}^{3}\omega_j^2\sin^2(\phi_j/2)}\right] \tag{3·2}$$

with $\omega_i^2=4\gamma_i/M$, and $\omega_L^2=\sum_i\omega_i^2$.

This integral can be evaluated by a combination of numerical and analytical methods. We give here the results for two cases specified by two values of α, where α is defined by the equation $\alpha=\gamma_1/\gamma_2=\gamma_1/\gamma_3$:

for $\alpha=8$: $$\Delta E_0=\frac{\hbar\omega_L}{\pi}[0.53\varepsilon+0.40\varepsilon^2+\cdots];$$

for $\alpha=16$: $$\Delta E_0=\frac{\hbar\omega_L}{\pi}[1.43\varepsilon+0.26\varepsilon^2+\cdots].$$

The low temperature value of the change in the free energy is obtained from the small x expansion of $\Omega(x)$. Since the integral in the inverse matrix element has an expansion of the form

$$g(0,0,0;ix)=a_0+a_1x+a_2x^2+\cdots$$

it can be shown by simple algebraic manipulations that

$$\Delta A(T)_{L.T.}=\Delta E_0+\varepsilon\frac{kT}{\pi}\left[6\zeta(4)a_1\left(\frac{kT}{\hbar}\right)^3+(20a_3+120a_0a_1\varepsilon)\zeta(6)\left(\frac{kT}{\hbar}\right)^5+\cdots\right]. \tag{3·3}$$

The first few a_i are calculated in reference 6).

The high temperature expansion of the change in free energy is given by

$$\Delta A(T)_{H.T.}=\frac{kT}{2}\ln\frac{M}{M'}-kT\left\{\frac{B_1}{4}\left(\frac{\hbar\omega_L}{kT}\right)^2\frac{\varepsilon+\varepsilon^2}{2}\right.$$

$$\left.-\frac{B_2}{96}\left[\frac{\varepsilon}{2}\left(\omega_L^4+\frac{\sum_i\omega_i^4}{2}\right)+\frac{\varepsilon^2}{4}(3\omega_L^4+\sum_i\omega_i^4)\right]\left(\frac{\hbar}{kT}\right)^4+\cdots\right\}. \tag{3·4}$$

(ii) *A pair of isotopic defects*: In order to determine the interaction zero-point energy and the low temperature value of ΔA, we will use the asymptotic

form of the Green's function for large separation between the defects given by Montroll and Potts.[3] We give below the results for the interaction zero-point energy and low temperature value of ΔA for $|\varepsilon| \ll 1$:

$$\Delta E_0 \simeq -\frac{3}{256}\frac{\varepsilon^2}{\pi^3}\frac{\alpha^2\hbar\omega_1}{a^7},$$

$$\Delta A(T)_{L.T.} = \Delta E_0 - \frac{\varepsilon^2 kT}{\pi^3}\left[\frac{480}{a\omega_1\omega_2^2\omega_3^2}\zeta(6)\left(\frac{kT}{\hbar}\right)^5 + \frac{53760}{\omega_1^3\omega_2^2\omega_3^2}a\zeta(8)\left(\frac{kT}{\hbar}\right)^7 + \cdots\right], \tag{3·5}$$

where $a=(l_1^2+\alpha l_2^2+\alpha l_3^2)^{1/2}$.

The procedure for obtaining the interaction free energy at high temperatures is the same as was followed in obtaining Eq. (2·16). Only the result for the leading term will be given here.

$$\Delta A_{H.T.} \cong -\varepsilon^2 kT\left[\frac{B_{2n}}{(4n)!\,4n}\left(\frac{\hbar}{kT}\right)^{4n} 2\lambda^2 n\right] \tag{3·6}$$

where

$$n=l_1+l_2+l_3, \quad \lambda=\frac{n!\,\omega_1^{2l_1}\omega_2^{2l_2}\omega_3^{2l_3}}{4^n l_1!\,l_2!\,l_3!}.$$

§ 4. Vacancies in the three-dimensional lattice

Vacancies in a lattice are of particular interest in view of the role they play in current theories of melting,[5] and also in processes involving diffusion in solids.[4] The concentration of vacancies at any temperature depends on the self-entropy of a vacancy.[4] With the present approach one can obtain an explicit expression for the vibrational self-entropy of a vacancy since at high temperatures

$$\Delta A(T) \sim \frac{kT}{2}\ln|\Delta \boldsymbol{D}(0)|,$$

whence

$$\Delta S = -\left(\frac{\partial \Delta A(T)}{\partial T}\right)_v = -\frac{k}{2}\ln|\Delta \boldsymbol{D}(0)|. \tag{4·1}$$

The determinant $|\Delta \boldsymbol{D}(0)|$ can be evaluated in a straightforward way.* We will illustrate this by computing the self-entropy of a substitutional impurity and a vacancy in the simple cubic lattice with nearest neighbor central and noncentral force interactions.

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* It may be mentioned here that a similar approach for evaluating the change in free energy at high temperatures due to a substitutional impurity has been adopted by Yamahuzi and Tanaka, Prog. Theor. Phys. **20** (1958), 327.

The defect determinant $|\Delta \boldsymbol{D}_1(0)|$ of a general substitutional impurity whose mass is $M'=(1-\varepsilon)M$ and whose nearest neighbor spring constants are denoted by Γ'_j is given in Appendix III. When all of the spring constants are equal $|\Delta D_1(0)|$ can be factored as follows:

$$|\Delta \boldsymbol{D}_1(0)|=\frac{M}{M'}\left\{1-\frac{\gamma}{M}(1-\delta)[g(0,0,0)-g(2,0,0)]\right\}^3$$
$$\times\left\{1+\frac{\gamma}{M}(1-\delta)[2g(1,1,0)-g(2,0,0)-g(0,0,0)]\right\}^2$$
$$\times\left\{1+\frac{\gamma}{M}(1-\delta)[12g(1,0,0)-4g(1,1,0)-7g(0,0,0)-g(2,0,0)]\right\} \tag{4·2}$$

where $\delta=\Gamma'/\gamma$ and $g(i,j,k)$ stands for the ijk'th Green's function. These Green's functions have been tabulated for a wide range of parameters.[6] Using the values contained in the tables of reference 6), we obtain the following expression for the vibrational self-entropy of the defect,

$$\Delta S_{H.T.}=-\frac{k}{2}\ln\frac{M}{M'}[0.7848+0.2152\delta]^3[0.8091+0.1909\delta]^2$$
$$\times[0.0356+0.9644\delta]. \tag{4·3}$$

The self-entropy of a vacancy is obtained from Eq. (4·3) by subtracting from it the change of entropy due to the change of mass associated with it, viz., $-k/2\cdot\ln M/M'$, and then setting $\delta=0$. The result is

$$\Delta S_{H.T.}=2.246k. \tag{4·4}$$

The calculation of the self-entropy of a vacancy in the lattice in which the spring constants are not all equal is more involved. We give below the results for the two cases $\alpha=8$, and $\alpha=16$:

$$\alpha=8: \quad \Delta S_{H.T.}=1.913k,$$
$$\alpha=16: \quad \Delta S_{H.T.}=2.110k.$$

We note that the high temperature value of the vibrational self-entropy of a vacancy is positive. This is in agreement with the results of Stripp and Kirkwood[5] who used a perturbation theoretic approach.

In the case of the defect pair, $|\Delta \boldsymbol{D}_2(0)|$ is a 14×14 determinant. The computation of the high temperature value of the interaction entropy is rather involved.* For a large distance of separation between the defects, one can use the asymptotic forms of the Green's functions. It turns out that the leading term in the expansion

* For the details of the calculation the reader is referred to pp. 46-51 of Technical Report No. 145 (1959), Department of Physics, University of Maryland.

of $\ln|\Delta D_2(0)|$ in inverse powers of the distance of separation R is of the form constant$\times R^{-6}$. For the case when all spring constants are equal the interaction entropy is

$$\Delta S_{H.T.} = \frac{0.113}{L^6}k, \quad L^2=(l_1^2+l_2^2+l_3^2). \tag{4·5}$$

This result is based on the use of the asymptotic forms of the Green's functions which are dependent on the dimensionality of the lattice and are rather insensitive to the model of the lattice. It is reasonable to presume that for all three-dimensional lattices the interaction entropy and the interaction free energy at high temperatures of two substitutional defects or vacancies would depend on the distance of separation between them in the form R^{-6}.

Appendix I

Proof that the Green's functions have no singularities except on the real axis

In this appendix we examine the analytic behavior of integrals of the form

$$I_n(\omega^2)=\int_0^{\pi}\cdots\int\frac{\prod_j^n \cos m_j\theta_j\, d\theta_j}{M\omega^2-2\sum_j^n \gamma_j(1-\cos\theta_j)}$$

where $n=1, 2, 3$. The variable ω^2 will be considered as a complex variable $\omega^2=\sigma+i\tau$. Of particular importance is the case $\sigma=0$ which is useful in the low temperature expansion of thermodynamic functions and in other applications. We show, in particular, that the only possible singularities of $I_n(\omega^2)$ lie on a strip $-\omega_L\leq\omega\leq\omega_L$ on the real axis. To do this we need only to note that for $\tau\neq 0$ the functions $\{d^k/d(\omega^2)^k\}I_n(\omega^2)$ are bounded in absolute value for all k. This follows from the elementary estimates

$$\left|\frac{d^k}{d(\omega^2)^k}I_n(\omega^2)\right|=M^k\left|\int_0^{\pi}\cdots\int\frac{\prod_j^n \cos m_j\theta_j\, d\theta_j}{[M(\sigma+i\tau)-2\sum_j^n \gamma_j(1-\cos\theta_j)]^{k+1}}\right|\leq\frac{\pi^k}{M|\tau|^{k+1}}. \tag{A·1}$$

Therefore there are no singularities lying off the real axis. Furthermore for ω^2 on the real axis and obeying the relation $M\omega^2\geq\sum_j 4\gamma_j=M\omega_L^2$, the derivatives of $I_n(\omega^2)$ satisfy the inequality

$$\left|\frac{d^k}{d(\omega^2)^k}I_n(\omega^2)\right|=M^k\left|\int_0^{\pi}\cdots\int\frac{\prod_j^n \cos m_j\theta_j\, d\theta_j}{[M\omega^2-2\sum_j^n \gamma_j(1-\cos\theta_j)]^{k+1}}\right|\leq\frac{1}{M|\omega^2-\omega_L^2|^{k+1}}; \tag{A·2}$$

hence there can be no singularity in this region.

Appendix II

The leading term in the small f expansion of the Green's functions

The leading term in the behavior of $\Omega(f)=d/df\cdot\ln|\Delta(if)|$ for small f is determined by the behavior of the Green's functions for small f. In this Appendix we obtain the leading terms in the small f expansions of the Green's functions for two- and three-dimensional simple cubic lattices. In general we find for the Green's function in n dimensions[3]

$$g(m_1, m_2, \cdots, m_n; if)=-\frac{2}{M\omega_L{}^2\pi^n}\int_0^{\pi}\cdots\int\frac{\prod_j^n \cos m_j\theta_j\, d\theta_j}{2f^2+\sum_j^n f_j^2(1-\cos\theta_j)} \qquad \text{(B·1)}$$

where $f_j^2=4\gamma_j/M\omega_L{}^2$. This expression can be rewritten as

$$g(m_1, m_2, \cdots, m_n; if)=-\frac{2}{M\omega_L{}^2\pi^n}\int_0^{\pi}\cdots\int_0^{\infty}\cdots\int e^{-(2f^2+\sum_j f_j^2)t}\prod_j^n e^{tf_j^2\cos\theta_j}\cos m_j\theta_j\, d\theta_j$$

$$=-\frac{2}{M\omega_L{}^2}\int_0^{\infty} e^{-(2f^2+\sum_j f_j^2)t}\prod_j I_{m_j}(f_j^2 t)\, dt, \qquad \text{(B·2)}$$

where $I_m(x)$ is the Bessel function of pure imaginary argument. Since we deal with the Laplace transform of the function $\exp(-\sum_j f_j^2 t)\prod_j I_{m_j}(f_j^2 t)$, we may use the Abelian theorem[7] which states:

Let $F(t)$ be representable in the form

$$F_1(t)=At^m+F_1(t) \quad \text{for } t\geq 1$$

where the Laplace transform of $F_1(t)$ has an abscissa of convergence $\beta<0$. Then the Laplace transform of $F(t)$ exists in the half plane Re $s>0$ and can be represented there as

$$f(s)=\frac{A\Gamma(m+1)}{s^{m+1}}+g(s) \qquad m\neq -1, -2, \cdots,$$

$$f(s)=\frac{A(-1)^n s^{n-1}}{(n-1)!}\ln s+g(s) \qquad m=-n,\ n=1,2, \cdots$$

and $g(s)$ is regular for Re $s>0$. These results give only the singular terms in the small s behavior of $f(s)$.

In order to apply this theorem we note that $I_m(x)$ has the representation

$$I_m(x)=\frac{e^x}{\sqrt{2\pi x}}+F_1(x) \qquad \text{(B·3)}$$

where $F_1(x)$ satisfies the conditions of the theorem. Thus, the leading singular

term for $|f|$ small, in the expression for $g(\boldsymbol{m}; if)$ is:

$$\begin{aligned} &2\text{–}D \qquad g(m_1, m_2; if) \sim \frac{2}{M\omega_L^2} \frac{1}{\pi f_1 f_2} \ln|f| \\ &3\text{–}D \qquad g(m_1, m_2, m_3; if) \sim \frac{2}{M\omega_L^2} \frac{1}{\pi f_1 f_2 f_3} |f|, \end{aligned} \tag{B·4}$$

which was to be proved.

Appendix III

The defect determinant $|\Delta \boldsymbol{D}_1(\omega)|$

$$|\Delta \boldsymbol{D}_1(\omega)| = \begin{vmatrix} s_1 a(100;000)+1, & A_1, & s_1 a(100;200), & s_2 a(100;110), & s_2 a(100;110), \\ & & & & s_3 a(100;101), \; s_3 a(100;101) \\ s_1 a(000;100), & A_2+1, & s_1 a(000;100), & s_2 a(000;010), & s_2 a(000;010), \\ & & & & s_3 a(000;001), \; s_3 a(000;001) \\ s_1 a(100;200), & A_1, & s_1 a(100;000)+1, & s_2 a(100;110), & s_2 a(100;110), \\ & & & & s_3 a(100;101), \; s_3 a(100;101) \\ s_1 a(010;110), & A_3, & s_1 a(010;110), & s_2 a(010;000)+1, & s_2 a(010;020), \\ & & & & s_3 a(010;011), \; s_3 a(010;011) \\ s_1 a(010;110), & A_3, & s_1 a(010;110), & s_2 a(010;020), & s_2 a(010;000)+1, \\ & & & & s_3 a(010;011), \; s_3 a(010;011) \\ s_1 a(001;101), & A_4, & s_1 a(001;101), & s_2 a(001;011), & s_2 a(001;011), \\ & & & & s_3 a(001;000)+1, \; s_3 a(001;002) \\ s_1 a(001;101), & A_4, & s_1 a(001;101), & s_2 a(001;011), & s_2 a(001;011), \\ & & & & s_3 a(001;002), \; s_3 a(001;000)+1 \end{vmatrix}$$

$$a(ijk; i'j'k') = g(ijk) - g(i'j'k')$$

$$s_i = \frac{\gamma_i - \Gamma_i}{m}$$

$$\begin{aligned} A_1 &= s_1[g(000)+g(200)] + 2s_2 g(110) + 2s_3 g(101) \\ &\quad - g(100)[2s_1 + 2s_2 + 2s_3 - \epsilon\omega^2], \\ A_2 &= 2s_1 g(100) + 2s_2 g(010) + 2s_3 g(001) \\ &\quad - g(000)[2s_1 + 2s_2 + 2s_3 - \epsilon\omega^2], \end{aligned}$$

$$A_3 = 2s_1 g(110) + s_2[g(000) + g(020)] + 2s_3 g(011) - g(010)[2s_1 + 2s_2 + 2s_3 - \epsilon\omega^2],$$

$$A_4 = 2s_1 g(101) + 2s_2 g(011) + s_3[g(000) + g(002)] - g(001)[2s_1 + 2s_2 + 2s_3 - \epsilon\omega^2].$$

References

1) J. Mahanty, A. Maradudin, G. Weiss, Prog. Theor. Phys. **20** (1958), 369.
2) M. Lighthill, *Fourier Analysis and Generalized Functions*, (Cambridge University Press, 1958).
3) E. Montroll, R. B. Potts, Phys. Rev. **100** (1955), 525; **102** (1956), 72.
4) H. Huntington, G. Shirn, E. Wajda, Phys. Rev. **99** (1955), 1085.
5) K. Stripp, J. Kirkwood, J. Chem. Phys. **22** (1954), 1579.
6) A. Maradudin, E. Montroll, G. Weiss, R. Herman, H. Milnes, to appear in the Memoirs of the Roy. Belg. Acad. Sci.
7) G. Doetsch, *Theorie und Anwendung der Laplace Transformation*, (Dover press, New York, 1945).

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On a Unified Description of Elementary Particles

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A scheme of approach to the unified description of elementary particles is proposed by using the "neutrino" and the concept of "charge-on". The electron and muon are considered as two kinds of electrically (ε-) charged-on neutrino, and the proton, neutron and Λ-particle are baryonically (b-) charged-on neutrino, electron and muon, respectively.

The interactions are constructed from the "currents of charge": The fundamental strong interaction is the b-charged and conserved current-current type where the full (Ikeda-Ogawa-Ohnuki's) symmetry among three baryons automatically holds and the fundamental electromagnetic interaction is the ε-charged and conserved current-current type (action at a distance), while the weak interactions consist of the ε-charge fading current-current type (usual weak), the b-charge fading current-current type, and the b- and ε-charge fading current-current type (both are extremely weak). Their relation is considered genetically by using the b-charge-on process (Gamba-Marshak-Okubo's symmetry) and the ε-charge-on process.

§ 1. Introduction

Recent advances in the theory of elementary particles enforce us to consider the unified description of various particles. We summarize several points of this trend.

The first is the successes in the theory of composite model which have been attained by Sakata's school. In 1956, Sakata[1] proposed the method which treats the elementary particles strongly interacting as the fundamental particles, proton, neutron and Λ-particle, and their entities (Sakata's model). This method can be interpreted as an approach which clarifies the substantial features in the theory of elementary particles and constructs the corresponding logics between present and future theories. Following his idea, the mass-formula for the composite entities which may give phenomenological law for interactions of the fundamental particles has been derived by Matumoto.[2] And a remarkable progress followed when Ogawa[3] introduced the full symmetry among the three fundamental particles. The refined theory of Ikeda, Ogawa and Ohnuki[4] could predict the existence of resonance states in the poin- and kaon-nucleon scatterings and also the mass formula was arranged by Sawada and Yonezawa[5] so as to cover all existing elementary particles excluding the leptons and photon. (We shall call it the "I-O-O symmetry").

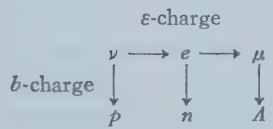
The second is the advance following the fact that a symmetry between baryons (proton, neutron and Λ-particle) and leptons (neutrino, electron and muon) holds in the weak interaction as pointed out by Gamba, Marshak and Okubo.[6] (We shall call it the "G-M-O symmetry.") This fact is easily interpreted by a model proposed by the Nagoya group[7] in which the proton, neutron and Λ-particle are entities of the neutrino, electron and muon coupled with unknown "B-matters" respectively; the strong interactions can be understood as those of B-matters themselves, while the weak interactions are those between leptons. (We shall call it the "Nagoya model.") Following this idea, the fundamental elementary particles are reduced to the neutrino, electron, muon and B-matter and consequently our considerations must be concentrated on the problems of leptons.

For the leptons which may be the third point, we have got a nearly definite conclusion for the scheme of their interactions; the interactions have almost the same magnitudes of coupling (weak) and completely parity violating terms (left-handed) and are realized by the electric charge exchanging V-A current-current Fermi coupling.[8] The difference of nature among three leptons, however, has not been discovered with the exception of electric charge and mass, though the history went a long way round the problem.

The authors[9] have been expressing the opinion that the difference among leptons does not come from any particular observable interaction with other particles, but from some inner structure which is revealed in the common (weak and electromagnetic) interactions. If it were likely, the difference between electron and muon would be expected in the weak interactions as a shift of local V-A theory. Detailed examinations, however, clarified that the effects would be confirmed when the experiments are made with one more decimal order than the present ones. At present, we must conclude that the difference cannot be detected in the weak interaction except the electric charge, the diffenence in the mass also being not expected by the reaction of weak interaction with V-A coupling.

The difference in the structures of electron and muon would probably appear in the electromagnetic interactions.[10] If the discrimination of electron and muon from neutrino was caused by the electric charge, the difference in electromagnetic structure would classify three leptons together with their mass level. An analysis of getting the electron mass through the electromagnetic interaction starting with the zero bare mass was made by the present authors and L. Ferreira,[11] and then the similar idea was applied to the muon mass.[12] We have a model in which the electron and muon are originated from the neutrino by attaching the electric charge. (We shall call it the "São Paulo model".) Though the muon might be strange particle, its characteristic feature would be covered owing to the absence of strong interaction, but should be revealed in the electromagnetic structure so as to produce a large mass compared with electron. We can expect that future experimental information about the electromagnetic structure of leptons will clarify its characteristics and the meaning of the strangeness concept.

The B-matter introduced in the Nagoya model has a phenomenologically similar nature to the electric charge. One of the authors (M.T.)[13] has proposed that it is suitable in the present stage to interpret the B-matter as a kind of charge (we shall call it "b-charge") similar to the electric charge (we shall call it "ε-charge") and to consider both on the same footing: The consideration shows the following model:



where six particles are related with neutrino by both charges, that is, ε-charge relates the leptons, while b-charge does the baryons with the leptons.

Our considerations based on this model are not to construct the so-called unitary theory of elementary particles but to get some foundations to the unified theory in future. As we must solve many fundamental problems in order to construct the unified theory, the urgent task we must take up in the present state is to find the corresponding logic to the future theory by searching for it within the frame of the present theory. Our attempt, therefore, must be taken in this sense, even if we use the results of present quantum field theory.* The reader will not be content in the point where the mechanism of charge is described only phenomenologically, but it will be solved in the further progress.

The final picture of our attempt has a very similar nature to the unified theory of Heisenberg's school,[15] but the way of attack may be just opposite. We have neither the fundamental equation with many symmetries nor the closed system, which are considered as our final point through arrangements of various materials entangled in the present theory of elementary particles.

The main part of our considerations is devoted to the arrangement of phenomenologically known interactions through the concept of ε- and b-charges so as to point to our direction (§ 2). Then we propose the genetical viewpoint about the results (§ 3) and give some justifications from the present field theory (§ 4).

§ 2. Concept of charge-on process and the interactions

Before going in to the unified scheme, we shall rearrange the usual phenomenological theory so as to point to our direction. The essential idea proposed here is to interpret the fundamental particles as the resultants of "charge-on (loading) processes" onto neutrino, that is, the electron and muon are the ε(electric)-charged neutrinos, while the proton, neutron and Λ-particle are the b(baryonic)-charged neutrino, electron and muon, respectively. The word "charge-on process" is used phe-

* A preliminary consideration following the above proposed model was published afterwards independently by Matumoto and Nakagawa.[14] Their analysis based on the viewpoint that the problem might be completely beyond the present quantum theory.

nomenologically until when its mechanism is clarified: We cannot yet present any definite idea of whether it may be interpreted as the dynamical process between particles and a kind of matter (ε-matter and b-matter), or as the consequences of the space-time structure. The symbolical use of this word, however, gives some important conclusions in the following.

The fundamental features of charge-on processes assumed phenomenologically are:

I) The ε-charge-on process makes the particle-(antiparticle-)neutrino negatively (positively) charged with a unit of charge and its modes are at least two-folds; one corresponds to electron and the other to muon.

II) The b-charge-on process makes the particle-(antiparticle-)leptons positively (negatively) charged with a unit of charge and is necessarily accompanied by a positive (negative) unit ε-charge. Its modes may be one or two-folds depending on the models sketched in the following section, but in any case modes of the both charge-on processes will be harmonized so as to discriminate two particles, neutron and Λ-particle.

Expressing the particles with $\varepsilon_i(0, -1)$-charge and $b_j(1, 0)$-charge (automatically followed $\varepsilon_j(1, 0)$-charge) $N_{\varepsilon_i+\varepsilon_j, b_j}$, we can define the charge-on processes as

$$\begin{aligned} &\varepsilon\text{-charge-on process}: \quad N_{0,0}\to N_{-1,0},\ N^*_{-1,0},\\ &b\text{-charge-on process}: \quad N_{0,0}\to N_{1,1},\ (N_{-1,0}\to N_{0,1},\ N^*_{-1,0}\to N^*_{0,1}), \end{aligned} \tag{2·1}$$

where we identify

$$\begin{aligned} &N_{0,0}\equiv\nu,\quad N_{-1,0}\equiv e,\quad N^*_{-1,0}\equiv\mu,\\ &N_{1,1}\equiv p,\quad N_{0,1}\equiv n,\quad N^*_{0,1}\equiv\Lambda. \end{aligned}$$

When the particles are ε- and/or b-charged, the transitions between particles show the rate of maintenance of some charge, which is classically well known as the word "current". We can construct such quantities in the notation

$$J_{1/2\cdot(\varepsilon_i+\varepsilon_I+\varepsilon_J+\varepsilon_F),\,1/2\cdot(b_I+b_F)}\equiv\bar{N}_{\varepsilon_J+\varepsilon_F,b_F}ON_{\varepsilon_i+\varepsilon_I,b_I}, \tag{2·2}$$

where O means some linear combination of Dirac matrices;

b-with ε current:

$$J_{1,1}=\bar{N}_{1,1}ON_{1,1} \tag{2·3}$$

b-without ε current:

$$J^{(1)}_{0,1}=\bar{N}_{0,1}ON_{0,1},\quad J^{(2)}_{0,1}=\bar{N}^*_{0,1}ON^*_{0,1},\quad J^{(3)}_{0,1}=\bar{N}_{0,1}ON^*_{0,1} \tag{2·4}$$

ε-without b current:

$$J^{(1)}_{-1,0}=\bar{N}_{-1,0}ON_{-1,0},\quad J^{(2)}_{-1,0}=\bar{N}^*_{-1,0}ON^*_{-1,0},\quad J^{(3)}_{-1,0}=\bar{N}_{-1,0}ON^*_{-1,0} \tag{2·5}$$

neutral current:

$$J_{0,0}=\bar{N}_{0,0}ON_{0,0} \tag{2·6}$$

ε-charge fading current:

$$J^{(1)}_{-1/2,0}=\bar{N}_{0,0}ON_{-1,0},\quad J^{(2)}_{-1/2,0}=\bar{N}_{0,0}ON^{*}_{-1,0},$$
$$J^{(1)}_{1/2,1}=\bar{N}_{1,1}ON_{0,1},\quad J^{(2)}_{1/2,1}=\bar{N}_{1,1}ON^{*}_{0,1} \tag{2·7}$$

b-charge fading current:

$$J_{1/2,1/2}=\bar{N}_{0,0}ON_{1,1},$$
$$J^{(1)}_{-1/2,1/2}=\bar{N}_{-1,0}ON_{0,1},\quad J^{(2)}_{-1/2,1/2}=\bar{N}^{*}_{-1,0}ON^{*}_{0,1}, \tag{2·8}$$
$$J^{(3)}_{-1/2,1/2}=\bar{N}^{*}_{-1,0}ON_{0,1},\quad J^{(3)}_{-1/2,1/2}=\bar{N}_{-1,0}ON^{*}_{0,1}$$

ε- and b-charge fading current:

$$J^{(1)}_{0,1/2}=\bar{N}_{0,0}ON_{0,1},\quad J^{(2)}_{0,1/2}=\bar{N}_{0,0}ON^{*}_{0,1},$$
$$J^{(3)}_{0,1/2}=\bar{N}_{-1,0}ON_{1,1},\quad J^{(4)}_{0,1/2}=\bar{N}^{*}_{-1,0}ON_{1,1}. \tag{2·9}$$

The currents defined by (2·3), (2·4) and (2·5) can be interpreted as the streams in classical sense, while the latter four (2·6), (2·7), (2·8) and (2·9) do not correspond to any classical quantities and are understood only quantum theoretically. The ways of our consideration, then, become two-step-wise: The first is to use the full analogy with the classical current picture for the former and the second is to reinforce the latter with a kind of limiting procedure which will be proposed later so as to utilize a classical picture.

The classical picture of current connects with the conservation law which guarantees the carrying quantity (charge) as the constant of motion in the system considered. If we use the word "current" analogously, we must select the quantities in conformity with this requirement.

The system we are now considering is composed of six particles, neutrino ν, electron e, muon μ, proton p, neutron n and Λ-particle Λ with masses 0, M_e, M_μ, M_p, M_n and M_Λ respectively, and there exist the following interactions among them:

i) the strong interactions among n, p and Λ under the conditions

$$\Delta n_p=\Delta n_n=\Delta n_\Lambda=0, \tag{2·10}$$

where Δn_i means the difference between i-particle numbers in initial and final states.

ii) the electromagnetic interactions with e, μ and p under the conditions

$$\Delta n_e=\Delta n_\mu=\Delta n_p=0, \tag{2·11}$$

and

iii) the weak interaction

$$\frac{f^2}{2M^2}j_\lambda{}^{H}j_\lambda:\quad j_\lambda=\bar{\nu}\gamma_\lambda\frac{1+\gamma_5}{2}(e+\mu)+\bar{p}\gamma_\lambda\frac{1+\gamma_5}{2}(n+\Lambda), \tag{2·12}$$

where f^2/M^2 means the coupling constant and H is the hermitian conjugation.

The details of the strong and electromagnetic interactions are unnecessary in our considerations.

The system can uniquely determine the conserved b-current and ε-current

$$\partial B_\lambda/\partial x_\lambda=0: \quad B_\lambda=J^V_{\lambda 1,1}+J^{V(1)}_{\lambda 0,1}+J^{V(2)}_{\lambda 0,1}, \tag{2·13}$$

$$\partial Q_\lambda/\partial x_\lambda=0: \quad Q_\lambda=J^V_{\lambda 1,1}-J^{V(1)}_{\lambda -1,0}-J^{V(2)}_{\lambda -1,0}, \tag{2·14}$$

where we take $J^V_\lambda=\bar{N}\gamma_\lambda N$, regardless of the divergence,

$$\bar{N}\gamma_\lambda N\to\bar{N}\gamma_\lambda N-i\cdot\text{const}\cdot\partial(\bar{N}\sigma_{\lambda\varkappa}N)/\partial x_\varkappa, \tag{2·15}$$

$$\sigma_{\lambda\varkappa}=(\gamma_\lambda\gamma_\varkappa-\gamma_\varkappa\gamma_\lambda)/2i.$$

With these conserved currents, we can construct the interactions from the standpoint of universal "current-current theory" as follows:

[A] The strong interactions come from the b-charged and conserved current-current inreraction

$$\frac{b^2}{2M^2}B_\lambda B_\lambda: \quad B_\lambda=J^V_{\lambda 1,1}+J^{V(1)}_{\lambda 0,1}+J^{V(2)}_{\lambda 0,1}=\bar{N}_{1,1}\gamma_\lambda N_{1,1}+\bar{N}_{0,1}\gamma_\lambda N_{0,1}+\bar{N}^*_{0,1}\gamma_\lambda N^*_{0,1}, \tag{2·16}$$

where b^2/M^2 means the coupling constant. The full (I-O-O) symmetry, then, holds automatically. The b-charged current has the vector character.

[B] The electromagnetic interactions come from the ε-charged and conserved current-current interaction

$$\frac{\varepsilon^2}{2M^2}Q_\lambda Q_\lambda: \quad Q_\lambda=J^V_{\lambda 1,1}-J^{V(1)}_{\lambda -1,0}-J^{V(2)}_{\lambda -1,0}=\bar{N}_{1,1}\gamma_\lambda N_{1,1}-\bar{N}_{-1,0}\gamma_\lambda N_{-1,0}-\bar{N}^*_{-1,0}\gamma_\lambda N^*_{-1,0}, \tag{2·17}$$

where ε^2/M^2 means the coupling constant. Then the photon field appears as a secondary entity. The ε-charged current has the vector character.

Though the coupling constants b and ε are written with the same notation as that used for charges, the charge-on processes are not restricted to the introduction of the above interactions.

Correspondingly, we can also express that

[C] The weak interaction is the coupling between the ε-charge fading currents so as to conserve ε-charge as a whole

$$\frac{f^2}{2M^2}j_\lambda{}^\dagger j_\lambda: \quad j_\lambda=\frac{1}{2}(J^V+J^A)^{(1)}_{\lambda -1/2,0}+\frac{1}{2}(J^V+J^A)^{(2)}_{\lambda 1/2,0}+\frac{1}{2}(J^V+J^A)^{(1)}_{\lambda 1/2,1}$$

$$+\frac{1}{2}(J^V+J^A)^{(2)}_{\lambda 1/2,1}=\bar{N}_{0,0}\gamma_\lambda\frac{1+\gamma_5}{2}(N_{-1,0}+N^*_{-1,0})+\bar{N}_{1,1}\gamma_\lambda\frac{1+\gamma_5}{2}(N_{0,1}+N^*_{0,1}), \tag{2·18}$$

where we take $J_\lambda{}^A=\bar{N}\gamma_\lambda\gamma_5 N$ and f^2/M^2 is the coupling constant. The charge-fading current has the vector and also the axial vector characters with one to one ratio.

We have also some possibilities of introducing the new (perhaps extremely) weak interactions:

[D] The extremely weak interaction with the coupling between the b-charge fading currents.

[E] The similar one with the coupling between the ε- and b-charge fading currents.

[F] The self-interaction of neutral currents.

Although their forms cannot be determined, they give the following reactions, which are considered to be so weak that we have yet unobserved:

$$\Lambda+e^-\to n+\mu^{-*}\quad (K^0\to\mu^++e^-)\qquad \text{from } D,$$
$$p+\mu^-\to p+e^-,\quad \Lambda+\nu\to n+\nu\qquad \text{from } E,$$
$$\nu+\nu\to\nu+\nu\qquad \text{from } F.$$

Assumed interactions might be

$$\frac{f'^2}{2M^2}j_\lambda^H j_\lambda' :\quad j_\lambda'=\frac{1}{2}(J^V+J^A)_{\lambda\,1/2,1/2}+\frac{1}{2}(J^V+J^A)^{(1)}_{\lambda-1/2,1/2}+\frac{1}{2}(J^V+J^A)^{(2)}_{\lambda-1/2,1/2}$$
$$=\bar N_{0,0}\gamma_\lambda\frac{1+\gamma_5}{2}N_{1,1}+\bar N_{-1,0}\gamma_\lambda\frac{1+\gamma_5}{2}N_{0,1}+\bar N^*_{-1,0}\gamma_\lambda\frac{1+\gamma_5}{2}N^*_{0,1}\qquad (2\cdot19)$$

for [D] and

$$\frac{f''^2}{2M^2}(j_\lambda^{''H}j_\lambda''+j_\lambda^{'''H}j_\lambda''') :\quad j_\lambda''=\frac{1}{2}(J^V+J^A)^{(1)}_{\lambda0,1/2}+\frac{1}{2}(J^V+J^A)^{(2)}_{\lambda0,1/2}$$
$$=\bar N_{0,0}\gamma_\lambda\frac{1+\gamma_5}{2}(N_{0,1}+N^*_{0,1}),\qquad (2\cdot20)$$
$$j_\lambda'''=\frac{1}{2}(J^V+J^A)^{(3)}_{\lambda0,1/2}+\frac{1}{2}(J^V+J^A)^{(4)}_{\lambda0,1/2}$$
$$=(\bar N_{-1,0}+\bar N^*_{-1,0})\gamma_\lambda\frac{1+\gamma_5}{2}N_{1,1}\qquad (2\cdot21)$$

for [E] where f'^2/M^2 and f''^2/M^2 are their coupling constants. The self-interaction of the neutral currents would be

$$\frac{f_0^2}{2M^2}[(J^V_{\lambda0,0})^2+(J^A_{\lambda0,0})^2]\qquad (2\cdot22)$$

so as to assure the zero mass and the form invariance

$$\sum_A C_A :\bar\psi O_A\psi\cdot\bar\phi O_A\phi :=\sum_A C_A :\bar\psi O_A\phi\cdot\bar\phi O_A\psi :,\qquad (2\cdot23)$$

where: : means Wick's product and f_0^2/M^2 is the coupling constant. The intro-

* The possibility of this process has been pointed also by Maki et al.[7]

duction of these three interactions cannot destroy the conservation laws (2·13) and (2·14). Symbolic pictures of the interactions are raised in Fig. 1.

A characteristic feature of the interaction is the magnitude of coupling constant. Among the various constants b^2/M^2, ε^2/M^2, f^2/M^2, and so on, we have the definite value only for the weak interaction

$$\frac{f^2}{M^2}\sim 10^{-5}\frac{1}{M_N{}^2} \qquad (2\cdot 24)$$

in unit of nucleon mass M_N and other constants, especially b^2/M^2 and ε^2/M^2, are unsettled until when we realize the meson and photon. However, if we identify b^2 and ε^2 with the strong and electromagnetic couplings $G_{\pi N}^2/4\pi$ and α

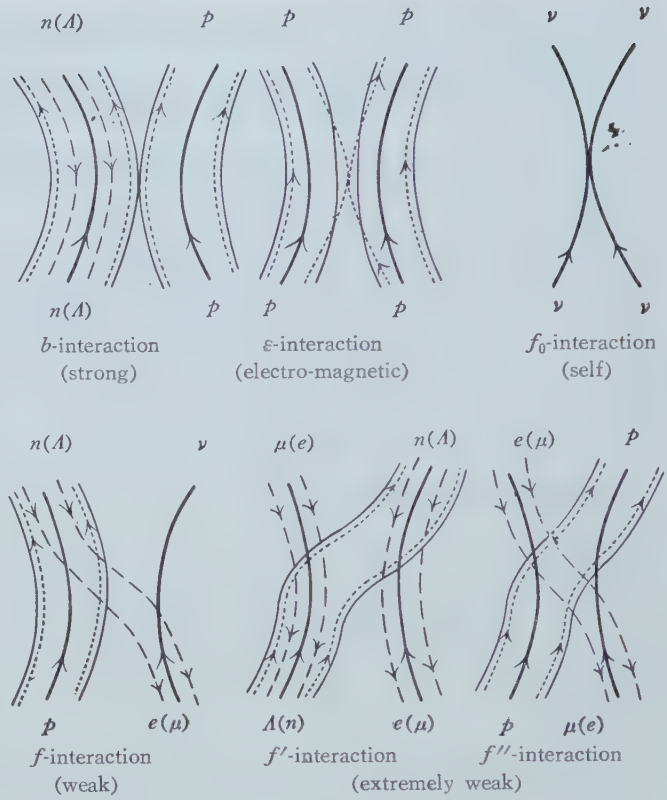


Fig. 1. Symbolic pictures of interactions. The width does not mean space spreading.

═══ b-charge (without ε) ⋮⋮⋮ ε-charge accompanied with b

⋮⋮⋮ ε-charge → neutrino core

$$b^2\sim\frac{G_{\pi N}^2}{4\pi}\sim 10,\quad \varepsilon^2\sim\alpha\sim 10^{-2} \qquad (2\cdot 25)$$

and take M, the nucleon mass, we have the relation

$$\frac{f^2}{\varepsilon^2}\simeq\frac{\varepsilon^2}{b^2}\simeq 10^{-3}, \qquad (2\cdot 26)$$

which seems to be very suggestive. Further, the fact that $K^0\to\mu^+ + e^-$ has yet unobserved tells us

$$\frac{f'^2}{f^2}\lesssim 10^{-3} \qquad (2\cdot 27)$$

which is in harmony with the relation (2·26). The relations (2·26) and (2·27)

show that the stronger the coupling between conserved charged currents becomes, the weaker the coupling between its charge fading currents.

The charge fading currents appearing in the interactions (2·18), (2·19), (2·20) and (2·21) cannot satisfy the conservation law which guarantees the universality of coupling strength. Searches for the conserving current in the weak interaction have been reported by many authors[16] but with negative answers. This fact can be understood that the current in question does not correspond to the classical picture, its essential feature being the charge fading nature.

In order to realize the classical picture, we must utilize the inverse processes of charge-on, i.e. the " charge-off (leaking) processes " which eliminate the charge fading natures. Using the ε-charge-off process

$$N_{-1,0},\ N^*_{-1,0}\to N_{0,0}, \qquad N_{0,1},\ N^*_{0,1}\to N_{1,1} \tag{2·28}$$

with

$$M_e,\ M_\mu\to M_\nu=0, \qquad M_n,\ M_\Lambda\to M_p, \tag{2·29}$$

we can show that the vector part of current in the weak interaction satisfies

$$\partial j_\lambda^{\,V}/\partial x_\lambda\to 0. \tag{2·30}$$

This shows that the universality of vector part holds in the limit of the ε-charge-off process. Also if we use the b-charge-off process

$$N_{1,1}\to N_{0,0}, \qquad N_{0,1}\to N_{-1,0}, \qquad N^*_{0,1}\to N^*_{-1,0} \tag{2·31}$$

with

$$M_p\to M_\nu=0, \qquad M_n\to M_e, \qquad M_\Lambda\to M_\mu, \tag{2·32}$$

we have

$$\partial j_\lambda''^{V}/\partial x_\lambda\to 0 \tag{2·33}$$

for the vector part of current in the extremely weak interaction (2·19). And successive use of (2·28) and (2·31) gives

$$\partial j_\lambda'^{\prime}/\partial x_\lambda\to 0, \qquad \partial j_\lambda'''/\partial x_\lambda\to 0$$

for the currents in the other interaction (2·20). As the b-charge-off process is not so good approximation as the ε-charge-off process, the universality of their coupling constants f'^2 and f''^2 is less reliable.

§ 3. Genetical approach to the elementary particles

The consideration made in the preceding section makes us have a genetical scheme from the single fundamental particle, neutrino, with the cooperation of the charge-on processes. The six particles, neutrino, electron, muon, proton, neutron and Λ-particle are constructed along the following scheme:

$$\begin{array}{ccc} & \varepsilon\text{-charge-on} & \\ & \nu(N_{0,0}) \longrightarrow e(N_{-1,0}),\ \mu(N^*_{-1,0}) & \\ b\text{-charge-on} & \downarrow & \\ & p(N_{1,1}) \longrightarrow n(N_{0,1}),\ \Lambda(N^*_{0,1}). & \end{array}$$

Since the mechanism of charge-on processes is unknown, we tentatively call it the "genetical scheme" distinguishing it from the so called "unitary" theory. A profit of the genetical consideration is to get the logic of analogy and difference for both concepts, electric charge and baryonic nature, that is, we consider both as the charges in the usual sense or some kinds of matter.

According to this genetical scheme, we may expect that the masses of leptons and of baryons must appear as the resultant of the charge-on processes. A suggestion[17] comes from

$$\frac{\varepsilon^2}{b^2} \sim \frac{M_e}{M_p}, \tag{3·1}$$

which can be interpreted as that the masses of electron and proton (nucleon) have the origin of field reaction. Though the masses of muon and Λ-particle may be understood on the more profound base in the charge-on processes, we can take the circumstance as the other modes of the charge-on processes.

Corresponding to this circumstance, we have the three possible schemes raised in Fig. 2: (a) the case in which only the ε-charge-on process has two modes, (b)

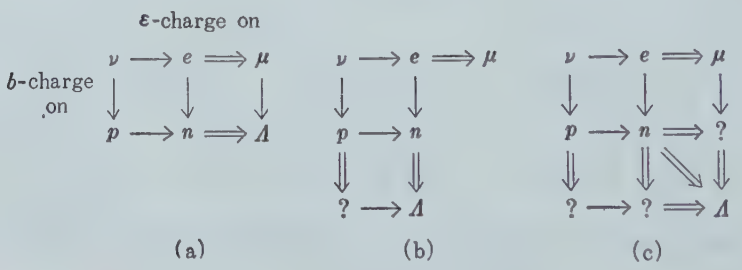


Fig. 2. Possible genetical schemes.
⟶ usual charge-on process
⟹ another mode of charge-on process

the case in which the ε-charge-on process gives two modes only for lepton, while the b-charge-on process gives two modes for neutral baryons and (c) the case in which both charge-on processes simaltaneously give different modes. The decision of scheme cannot be made at present, but it is useful to consider the experimental fact

$$M_\Lambda - M_n > M_\mu - M_e, \tag{3·2}$$

because we must have any amplification of the difference of ε-charge-on process when we introduce the b-charge-on process. For instance, the case (a) must automatically give amplification, the case (b) may separate the problem and the case

(c) is intermediate between the two former extreme cases.

With these schemes, we can imagine the hypothetical worlds and their alternations: We have the world of neutrino, the world of leptons, the world of baryons and the real world. Their alternations are caused by the new elements, the charge-on processes. The freedoms of the elementary particles are not due to the neutrino, but to the mechanism of the processes. The circumstnace is very similar to the case of hydrogen atom. When an alternation occurred, the modifications and introductions of interactions should follow. Then the interactions can be interpreted genetically. We shall outline the circumstances as follows:

i) *Weak interactions*

The usual weak interaction is constructed in two steps: The original form will be caused when the ε-charge-on process acts:

$$\frac{f^2}{2M^2}\int dx j_\lambda^{H} j_\lambda(x): \quad j_\lambda=\bar{N}_{0,0}\gamma_\lambda\frac{1+\gamma_5}{2}(N_{-1,0}+N^*_{-1,0}) \tag{3·3}$$

and its modification follows by the b-charge-on process:

$$j_\lambda=\bar{N}_{0,0}\gamma_\lambda\frac{1+\gamma_5}{2}(N_{-1,0}+N^*_{-1,0})+\bar{N}_{1,1}\gamma_\lambda\frac{1+\gamma_5}{2}(N_{0,1}+N^*_{0,1}). \tag{3·4}$$

The one of the extremely weak interactions is caused by the b-charge-on process:

$$\frac{f^2}{2M^2}\int dx j_\lambda'^{H} j_\lambda'(x): \quad j_\lambda'=\bar{N}_{0,0}\gamma_\lambda\frac{1+\gamma_5}{2}N_{1,1} \tag{3·5}$$

which is modified by the succeeding ε-charge-on process

$$j_\lambda'=\bar{N}_{0,0}\gamma_\lambda\frac{1+\gamma_5}{2}N_{1,1}+\bar{N}_{-1,0}\gamma_\lambda\frac{1+\gamma_5}{2}N_{0,1}+\bar{N}^*_{-1,0}\gamma_\lambda\frac{1+\gamma_5}{2}N^*_{0,1}, \tag{3·6}$$

while other extremely weak interaction is created only when the both charge-on processes act simultaneously:

$$\frac{f'^2}{2M^2}\int dx\Big[\Big\{\bar{N}_{1,1}\gamma_\lambda\frac{1+\gamma_5}{2}(N_{-1,0}+N^*_{-1,0})\Big\}\Big\{(N_{1,0}+N^*_{-1,0})\gamma_\lambda\frac{1+\gamma_5}{2}N_{1,1}\Big\}$$
$$+\Big\{(\bar{N}_{0,1}+\bar{N}^*_{0,1})\gamma_\lambda\frac{1+\gamma_5}{2}N_{0,0}\Big\}\Big\{\bar{N}_{0,0}\gamma_\lambda\frac{1+\gamma_5}{2}(N_{0,1}+N^*_{0,1})\Big\}\Big]. \tag{3·7}$$

ii) *Electromagnetic interaction*

The electromagnetic interaction is originated when the ε-charge-on process goes on. Whatever the charge-on process may be, it will be probable to expect the different charge distributions for each mode of charge-on process. We can express the fact at the non-local effect phenomenologically. The ε-charge-on process, then, gives the interaction

$$\frac{\varepsilon^2}{2M^2}\int dx_1 dx_2 dx_3 Q_\lambda(x_1, x_3)Q_\lambda(x_2, x_3): \tag{3·8}$$
$$Q_\lambda(x_1, x_3)=-\bar{N}_{-1,0}(x_1)\Gamma_\lambda(x_1-x_3)N_{-1,0}(x_1)-\bar{N}^*_{-1,0}(x_1)\Gamma_\lambda^*(x_1-x_3)N^*_{-1,0}(x_1),$$

where we have

$$\Gamma_\lambda^{(*)}(x)=\gamma_\lambda g_1^{(*)}(x)+\frac{i\lambda^{(*)}}{2M}\sigma_{\lambda\kappa}\frac{\partial g_2^{(*)}}{\partial x_\kappa}(x) \tag{3·9}$$

with a constant $\lambda^{(*)}$ and unknown functions $g_i^{(*)}(x)$ $(i=1, 2)$.

The succeeding b-charge-on process which gives other distributions may modify it with the current

$$\begin{aligned}Q_\lambda(x_4, x_3)=&-\bar{N}_{-1,0}(x_1)\Gamma_\lambda(x_1-x_3)N_{-1,0}(x_1)-\bar{N}^*_{-1,0}(x_1)\Gamma_\lambda^*(x_1-x_3)N^*_{-1,0}(x_1)\\&+\bar{N}_{0,1}(x_1)(\Gamma_\lambda-\Gamma_\lambda)(x_1-x_3)N_{0,1}(x_1)+\bar{N}^*_{0,1}(x_1)(\Gamma_\lambda^*-\Gamma_\lambda^*)(x_1-x_3)N^*_{0,1}(x_1)\\&+\bar{N}_{1,1}(x_1)\Gamma_\lambda(x_1-x_3)N_{1,1}(x_1),\end{aligned} \tag{3·10}$$

where we express the b-charge distributions as

$$\Gamma_\lambda^{(*)}(x)=\gamma_\lambda G_1^{(*)}(x)+\frac{iL^{(*)}}{2M}\sigma_{\lambda\kappa}\frac{\partial G_2^{(*)}}{\partial x_\kappa}(x) \tag{3·11}$$

with a constant $L^{(*)}$ and unknown functions $G_i^{(*)}(x)$ $(i=1,2)$. The functions introduced in baryons depend on the scheme taken, i.e. for the case (a), we have

$$\Gamma_\lambda^*(x)=\Gamma_\lambda(x), \tag{3·12}$$

and for the case (b) the term of $N^*_{0,1}$ must be understood

$$\bar{N}^*_{0,1}(x_1)(\Gamma_\lambda^*-\Gamma_\lambda)(x_1-x_3)N^*_{0,1}(x_1). \tag{3·13}$$

The local limit of the interaction tends to the form of (2·17).

iii) *Strong interaction*

The strong interaction is caused by the b-charge-on process. Its original form is

$$\frac{b^2}{2M^2}\int dx_1dx_2dx_3B_\lambda(x_1, x_3)B_\lambda(x_2, x_3):$$

$$B_\lambda(x_1, x_3)=\bar{N}_{1,1}(x_1)\Gamma_\lambda(x_1-x_3)N_{1,1}(x_1) \tag{3·14}$$

and a modification by the succeeding ε-charge-on process gives

$$\begin{aligned}B_\lambda(x_1, x_3)=&\bar{N}_{1,1}(x_1)\Gamma_\lambda(x_1-x_3)N_{1,1}(x_1)\\&+\bar{N}_{0,1}(x_1)\Gamma_\lambda(x_1-x_3)N_{0,1}(x_1)+\bar{N}^*_{0,1}(x_1)\Gamma_\lambda^*(x_1-x_3)N^*_{0,1}(x_1).\end{aligned} \tag{3·15}$$

The other baryons and their interactions are the resultant of this interaction.

§ 4. Considerations from the quantum field theory

The scheme proposed here includes many difficulties which may be beyond the validity of the present quantum field theory. Notwithstanding this circumstance, it may be instructive to see some problems from the present standpoint so as to seek the corresponding logic. The consideration in the following is to see whether we can understand the statement that the whole masses of elementary

particles have an origin in the field reactions. Can the electron and also the muon get the sufficient masses through the electromagnetic interaction (São Paulo model) and the baryon through the strong interaction?

For the qualitative discussion, we define the mass term of a fermion

$$\Delta M\int dx\,\bar{N}N(x) \tag{4·1}$$

which is known to destroy the invariance under Touschek's transformation[18]

$$N\to e^{i\beta\gamma_5}N. \tag{4·2}$$

Then it is necessary to have the conditions in order to get the mass qualitatively:

1^0 The interaction which destroys the invariance under Touschek's transformation must exist.

2^0 From the dimensional analysis, it is necessary to have a constant of the linear dimension of length [L] in order to get a non-zero reaction mass.

The weak interactions cannot satisfy the former condition, while the electromagnetic and strong interactions can satisfy the both conditions only if

$$\lambda^{(*)}\neq 0,\quad L^{(*)}\neq 0. \tag{4·3}$$

The alternative couplings, if we propose for the strong interaction, satisfy the former condition, but not the latter unless we modify them as similar to ours. (4·3) means that the charge distributions are unspherical,

$$\Gamma_\lambda^{(*)}(x)=\frac{1}{(2\pi)^4}\gamma_\lambda\int dq\,e^{iqx}\left[g_1^{(*)}(q^2)+\frac{i\lambda^{(*)}}{2M}\gamma_\kappa q_\kappa g_2{}^{(*)}(q^2)\right]. \tag{4·4}$$

Before entering into the semi-qualitative discussions, we propose the quantizations as follows: For the neutrino field $N_{0,0}$, though we have not definitely defined its equation, we have for the sake of simplicity

$$\langle N_{0,0}(x),\,\bar{N}_{0,0}(x')\rangle_0=\frac{iZ}{(2\pi)^4}\int dq\,e^{iq(x-x')}\frac{-i\gamma q}{q^2}, \tag{4·5}$$

$$Z=\int dm^2\rho_0(m^2)$$

where $\rho_0(m^2)$ means the spectral function in the first approximation, and we can normalize Z to unit. For the electron and muon fields, $N_{-1,0}$ and $N^*_{-1,0}$, we propose the new quantization

$$\langle N_{-1,0}(x),\,\bar{N}_{-1,0}(x')\rangle_0=-\frac{i\langle m^2\rangle}{(2\pi)^4}\int dq\,e^{iq(x-x')}\frac{-i\gamma q}{q^2}\cdot\frac{1}{q^2}, \tag{4·6}$$

$$\langle N^*_{-1,0}(x),\,\bar{N}^*_{-1,0}(x')\rangle_0=-\frac{i\langle m^{*2}\rangle}{(2\pi)^4}\int dq\,e^{iq(x-x')}\frac{-i\gamma q}{q^2}\cdot\frac{1}{q^2}, \tag{4·7}$$

$$\langle m^2\rangle=\int dm^2m^2\rho_1(m^2),\quad \langle m^{*2}\rangle=\int dm^2m^2\rho_1{}^*(m^2),$$

where $\rho_1{}^{(*)}(m^2)$ means the spectral function under the influence of the ε-charge-on process and $\langle m^{(*)2}\rangle$ is the mean square mass. The new quantization which is

connected with the so-called dipole-ghost nature,[19]

$$-\frac{1}{q^2}\cdot\frac{1}{q^2}=\lim_{\sigma\to 0}\frac{1}{2\sigma}\left[\frac{1}{q^2+\sigma}-\frac{1}{q^2-\sigma}\right], \tag{4·8}$$

can be interpreted as the propagator of particle accompanied with the photon. The difference between $\rho_1(m^2)$ and $\rho_1{}^*(m^2)$ comes from the difference of mode in the ε-charge-on process. The quantization of the baryons, $N_{1,1}$, $N_{0,1}$ and $N_0{}^*_1$ is assumed roughly in parallel

$$\langle N_{1,1}(x),\ \bar{N}_{1,1}(x')\rangle_0=\langle N_{0,1}(x),\ \bar{N}_{0,1}(x')\rangle_0=-\frac{i\langle M^2\rangle}{(2\pi)^4}\int dq\, e^{iq(x-x')}\,\frac{-i\gamma q}{q^2}\cdot\frac{1}{q^2}, \tag{4·9}$$

$$\langle N^*_{0,1}(x),\ \bar{N}^*_{0,1}(x')\rangle_0=-\frac{i\langle M^{*2}\rangle}{(2\pi)^4}\int dq\, e^{iq(x-x')}\,\frac{-i\gamma q}{q^2}\cdot\frac{1}{q^2}, \tag{4·10}$$

$$\langle M^2\rangle=\int dm^2 m^2 P_1(m^2),\quad \langle M^{*2}\rangle=\int dm^2 m^2 P_1{}^*(m^2),$$

where $P_1{}^{(*)}(m^2)$ means the spectral function under the influence of the b-charge on process.

The reactive masses in the first approximation of the electromagnetic interaction are given by

$$(\Delta M^{(*)})_\varepsilon=-\frac{\varepsilon^2}{4\pi}\,\frac{3\lambda^{(*)}}{2\pi M}\,\frac{\langle m^{(*)2}\rangle}{M^2}\int dm_1{}^2 dm_2{}^2 g_1{}^{(*)}(m^2)\,g_2{}^{(*)}(m^2)\,\frac{\log(m_1{}^2/m_2{}^2)}{m_1{}^2-m_2{}^2}, \tag{4·11}$$

where we use

$$\Gamma_\lambda{}^{(*)}(x)=\frac{1}{(2\pi)^4}\int dm^2\int dq\, e^{iqx}\,\frac{1}{q^2+m^2}\left[\gamma_\lambda g_1{}^{(*)}(m^2)-\frac{\lambda^{(*)}}{2M}\sigma_{\lambda\kappa}q_\kappa g_2{}^{(*)}(m^2)\right]. \tag{4·12}$$

Since the result for the electron ΔM coincides with the one starting from the interaction

$$-i\sqrt{4\pi\alpha}\int dx_1 dx_2\,\bar{N}_{-1,0}(x_1)\,\Gamma_\lambda(x_1-x_3)\,N_{-1,0}(x_1)\,A_\lambda(x_3) \tag{4·13}$$

and following the usual quantization like (4·5), except for the numerical constant, where A_λ is the photon field and α the fine structure constant, we can tentatively adjust the parameter until when the photon field is deduced

$$\alpha\simeq\frac{\varepsilon^2}{4\pi}\,\frac{\langle m^2\rangle}{M^2} \tag{4·14}$$

and

$$\lambda \simeq 10^{-3} \tag{4·15}*$$

from the discrepancy in the electron moment

$$\mu_{exp} - \mu_{theor} \simeq 10^{-6} \frac{e}{2M_e} \simeq 10^{-3} \frac{e}{2M}, \quad (M=M_n). \tag{4·16}$$

Therefore if we could expect

$$\Delta_\varepsilon \equiv -\int dm_1^2 dm_2^2 g_1(m_1^2) g_2(m_2^2) \frac{\log(m_1^2/m_2^2)}{m_1^2 - m_2^2} \simeq 10 M^2, \tag{4·17}$$

we would have the desired order of electron mass

$$(\Delta M)_\varepsilon \sim 10^{-3} M. \tag{4·18}$$

This expectation is not so difficult if we use the form factors with the mean square radius $\langle r^2 \rangle \simeq 1/M^2$. For the muon mass, however, as the universality of the fine structure constant makes us have

$$\langle m^{*2} \rangle = \langle m^2 \rangle \tag{4·19}$$

and the maximum allowance for the discrepancy in the muon moment

$$\mu_{exp} - \mu_{theor} \simeq 10^{-3} \frac{e}{2M_\mu} \simeq 10^{-2} \frac{e}{2M} \tag{4·20}$$

gives

$$\lambda^* \simeq 10^{-2}, \tag{4·21}$$

we are obliged to expect

$$\Delta_\varepsilon^* / \Delta_\varepsilon \sim 10 \tag{4·22}$$

in order to get the desired order

$$(\Delta M)_\varepsilon \sim 10^{-1} M. \tag{4·23}$$

The corresponding form factors become more singular than the electron. The other possibility is to interpret the muon as the excited state having the original mass m_0 in the ε-charge-on process, its value being $2/3 \cdot \alpha^{-1} \cdot M_e$ as pointed out by Nambu[20] and its deviation from the muon mass being just the electron mass,**

$$M_\mu = \frac{3}{2} \alpha^{-1} M_e + M_e. \tag{4·24}$$

Similar discussion can be devoted to the masses of the baryons. For the sake of simplicity, taking the scheme (a), i.e.

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* The value $\lambda \simeq 10^{-2}$ used in the previous paper[11),12)] may be somewhat large, even though it is also probable.

** The idea and the relation (4·24) have been proposed by Matumoto. We express our thanks for his private communication.

$$P_1(m^2)=P_1^*(m^2), \quad T_\lambda(x)=T_\lambda^*(x) \tag{4·25}$$

and the similar structure for the b-charge distributions as the electron ε-charge distribution

$$L=L^*\simeq\lambda, \quad \langle M^2\rangle=\langle M^{*2}\rangle\simeq\langle m^2\rangle, \quad \Delta_b=\Delta_b^*\simeq\Delta_\varepsilon, \tag{4·26}$$

where

$$T_\lambda(x)=\frac{1}{(2\pi)^4}\int dm^2\int dq\, e^{iqx}\frac{1}{q^2+m^2}\left[\gamma_\lambda G_1(m^2)-\frac{L}{2M}\sigma_{\lambda\kappa}q_\kappa G_2(m^2)\right], \tag{4·27}$$

$$\Delta_b=-\int dm_1^2\, dm_2^2\, G_1(m_1^2)G_2(m_2^2)\frac{\log(m_1^2/m_2^2)}{m_1^2-m_2^2}, \tag{4·28}$$

we have

$$(\Delta M)_b/(\Delta M)_\varepsilon\simeq b^2/\varepsilon^2. \tag{4·29}$$

Then we can get the main part of the baryon in virtue of the relation (3·1).

The difference between the masses of Λ-particle and nucleon comes from the resultant of the three steps: i) It is originated by the difference between masses of muon and electron, ii) a part of this original difference is cancelled out by the opposite ε-charge followed in the b-charge-on process and iii) finally a remaining difference produces the effective difference by the strong interaction. The difference between the b-charge and ε-charge distributions in Λ-particle is essential in this discussion, that is, with the assumptions (4·25) and (4·26), we must take

$$G_1=G_1^*=g_1\neq g_1^*, \quad LG_2=L^*G_2^*=\lambda g_2\neq\lambda^* g_2^*. \tag{4·30}$$

Assumig

$$g_2\doteqdot g_2^*, \quad \lambda\neq\lambda^*, \tag{4·31}$$

we get the remaining difference at the second step from (3·10)

$$(\Delta M_\Lambda)_\varepsilon=\frac{\lambda^*-\lambda}{\lambda^*\lambda}[\lambda(\Delta M^*)_\varepsilon-\lambda^*(\Delta M)_\varepsilon]. \tag{4·32}$$

Therefore it is probable that we may expect the remaining mass

$$O\lesssim(\Delta M_\Lambda)_\varepsilon\lesssim M_\mu \tag{4·33}$$

only when we can get the muon mass. The estimation at the final step is impossible within the approximation used here, but it is not improbable to expect the desired mass difference.

Consequently we can conclude that the statement about the origin of mass is not inconsistent with the present standpoint.

§ 5. Discussions and conclusions

An attempt proposed here stands on the unsolved points, which we consider to belong to the future theory. We cannot reach the final stage of the unified

description of elementary particles along this line of thought until when they are clarified. In spite of this defect, however, we can get some conclusions as follows:

1) The parallel considerations with ε-charge and b-charge are suitable according to the present knowledge of elementary particles.

2) The interactions can be classified from the classical picture of current and their couplings are uniquely determined through the already established weak interaction. All the interactions are current-current types.

3) The strong interaction becomes to satisfy the full (I.O.O.) symmetry. A similar procedure (ε-charge-on) as G.M.O. symmetry in the weak interaction (b-charge-on) gives the new (extremely) weak interactions.

4) The coupling constants of interaction have a simple regularity which is connected with the ratio of electron mass to nucleon mass.

5) The masses of all elementary particles are understood as the result of charge-on processes and six particles can be connected genetically.

6) The ε-charge process causes the weak interaction together with the electromagnetic interaction, while the b-charge-on process causes the extremely weak interaction together with the strong interaction.

Among the unsolved problems, the photon field and the fine structure constant will be considered in connection with the meson fields and their interaction constants by pushing this attempt. The large mass defects in constructing the photon and mesons as the composite entities can be more easily understood from the viewpoint of the reactive mass than of the bound state.

Though the mechanism of charge-on processes would belong to an unknown theory, the weak interaction might give some definite information about ε-charge-on process and the extremely weak interaction which will be observed in high energy phenomena might do about b-charge-on process, because they are characterized as the charge exchange interactions. An interpretation[21] of the weak interaction with the intermediate boson may be understood as a substantial approach to the ε-charge. The mechanism may be not only due to the dynamical process of some substances, but also related to the space-time structure.[22] The nature of neutrino having only spin can be interpreted as a sort of vortex, which is a consequence of the curved space time. The ε-charge-on process may reveal the conservation of such structure, while the b-charge-on process may give the origin of singular region which is taken as the massive particle. This imagination is suggested by the gauge invariance and by the common spreading $1/M$ by which the particle can be explained.

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References

1) S. Sakata, Prog. Theor. Phys. **16** (1956), 686.
M. A. Markov, *6th Rochester Conf. High Energy Physics* (1956).
2) K. Matumoto, Prog. Theor. Phys. **16** (1956), 583.
3) S. Ogawa, Prog. Theor. Phys. **21** (1959), 209.
4) M. Ikeda, S. Ogawa and Y. Ohnuki, Prog. Theor. Phys. **22** (1959), 715.
Y. Yamaguchi, Suppl. Prog. Theor. Phys. **11** (1959), 1.
5) S. Sawada and M. Yonezawa, Prog. Theor. Phys. **23** (1960), 662.
6) A. Gamba, R. E. Marshak and S. Okubo, Proc. Nat. Acad. Sci. **45** (1959), 881.
7) Z. Maki, M. Nakagawa, Y. Ohnuki and S. Sakata, Prog. Theor. Phys. **23** (1960), 1174.
8) R. P. Feynman and M. Gell-Mann, Phys. Rev. **109** (1958), 192.
E. C. G. Sudarshan and R. E. Marshak, Phys. Rev. **109** (1958), 1860.
J. J. Sakurai, Nuovo Cimento **7** (1958), 649.
9) M. Taketani, Y. Katayama, P. L. Ferreira, G. W. Gund and P. R. de Paula e Silva, Prog. Theor. Phys. **21** (1959), 799.
10) K. Hiida, Prog. Theor. Phys. **21** (1959), 98.
S. C. Drell, Ann. Phys. **4** (1958), 75.
J. D. Bjorken and S. D. Drell, Phys. Rev. **114** (1959), 1368.
11) Y. Katayama, M. Taketani and J. L. Ferreira, Prog. Theor. Phys. **21** (1959), 818.
12) J. L. Ferreira and Y. Katayama, Prog. Theor. Phys. **23** (1960), 776.
13) M. Taketani, Report at the meeting of weak interaction held at Kyoto in December 1959; Kagaku **30** (1960), 118; Suppl. Prog. Theor. Phys. to be published.
14) K. Matumoto and M. Nakagawa, Prog. Theor. Phys. **23** (1960), 1181.
15) W. Heisenberg, Nachr. d. Gött. Akad. Wiss. **8** (1953); Z. Naturforschg. **9a** (1954), 292.
W. Heisenberg, F. Kortel and H. Mitter, Z. Naturforschg. **10a** (1955), 425.
R. Ascoli and W. Heisenberg, Z. Naturforschg. **12a** (1957), 177.
H. P. Dürr, W. Heisenberg, H. Mitter, S. Schlieder and K. Yamazaki, Z. Naturforschg. **14a** (1959), 441.
16) R. P. Feynman and M. Gell-Mann, loc. cit.
J. C. Polkinghorne, Nuovo Cimento **8** (1958), 179.
M. L. Goldberger and S. B. Treiman, Phys. Rev. **110** (1958), 1458.
S. Weinberg, R. E. Marshak, S. Okubo, E. C. G. Sudarshan and W. B. Teutsh, Phys. Rev. Letters **1** (1958), 25.
17) G. Stephenson, Nuovo Cimento **5** (1957), 1009.
A. Salam and J. Tiomno, Nucl. Phys. **9** (1958/59), 585.
18) B. Toushek, Nuovo Cimento **5** (1957), 1281.
19) W. Heisenberg, Nucl. Phys. **4** (1957), 532.
H. Kita, Prog. Theor. Phys. **15** (1956), 83.
20) Y. Nambu, Prog. Theor. Phys. **7** (1952), 595.
H. Fröhlich, Nucl. Phys. **7** (1958), 148.
21) S. Ogawa, Prog. Theor. Phys. **15** (1956), 487.
J. Schwinger, Ann. Phys. **2** (1957), 407.
J. L. Lopes, Nucl. Phys. **8** (1958), 234.
G. Feinberg, Phys. Rev. **110** (1958), 1482.
N. Byers and R. E. Peierls, Nuovo Cimento **10** (1958), 520.
A. Salam and J. C. Ward, Nuovo Cimento **11** (1959), 568.
22) C. W. Misner and J. A. Wheeler, Ann. Phys. **2** (1957), 525.
J. A. Wheeler, Ann. Phys. **2** (1957), 604.

Letters to the Editor

The opinions expressed in these columns do not necessarily reflect those of the Board of Editors. Communications should be submitted in duplicate and should be held to within 100 lines (pica tipe) on standard size letter paper (approx. 21×30 cm.), so that each letter may be arranged into two pages when printed. Do not forget to count in enough space for formulas, figures or tables.

A New Approach to the Theory of Classical Fluids. IIIa

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June 11, 1960

In the preceding papers of this series[1),2)] as well as in several other works,[3)] analysis has been made by starting from the density expansion formulas. Therefore we have had no sufficient reason for the applicability of the results in the densities of liquid even when the resultant expressions are convergent. It is desirable to develop a theory of classical fluids without using the density expansion formulas. Here we consider a system enclosed in a large but finite volume V which is kept constant in the whole course of analysis, and show the condition under which the results in I and II may be applicable to liquids. Moreover, we obtain the expressions which may be applicable to solids as well as gases and liquids.

We consider a one-component gas of fugacity z in volume V and at temperature T. In the following $\int d\boldsymbol{r}$ means an integration over $\boldsymbol{r}$ within V. In the grand canonical ensemble the probability that the volume V contains exactly N particles and these take a configuration $\boldsymbol{r}^N \equiv \boldsymbol{r}_1, \boldsymbol{r}_2 \cdots, \boldsymbol{r}_N$, is:

$$\rho(N, \boldsymbol{r}^N) = \Xi^{-1} \frac{z^N}{N!} \exp\{-\Phi(\boldsymbol{r}^N)/kT\}, \tag{1}$$

where $\Xi = \sum_{N=0}^{\infty} (z^N/N!) \int d\boldsymbol{r}^N \exp\{-\Phi(\boldsymbol{r}^N)/kT\}$. We assume that $\Phi(\boldsymbol{r}^N)$, the interaction potential, is a sum of the pair interaction potentials, namely $\Phi(\boldsymbol{r}^N) = \sum_{N \geq i > j \geq 1} \phi(\boldsymbol{r}_i, \boldsymbol{r}_j)$.

The one particle distribution function is defined by

$$\rho(\boldsymbol{r}_1) = \Xi^{-1} \sum_{N=0}^{\infty} \frac{z^N}{N!} \cdot N \int \cdots \int d\boldsymbol{r}_2 \cdots d\boldsymbol{r}_N \exp\{-\Phi(\boldsymbol{r}^N)/kT\}. \tag{2}$$

Using Salpeter's method,[4)] we can reduce the above expression, without introducing any approximation, to the form

$$\rho(\boldsymbol{r}_1) = z \sum_{l=0}^{\infty} \frac{z^l}{l!} \int \cdots \int d\boldsymbol{r}_2 \cdots d\boldsymbol{r}_{l+1} \sum_{l+1 \geq i > j \geq 1} \prod b_{ij}, \tag{3}$$

Sum over all products which are connected with each other.

where $b_{ij} \equiv b(\boldsymbol{r}_i, \boldsymbol{r}_j) \equiv \exp\{-\phi(\boldsymbol{r}_i, \boldsymbol{r}_j)/kT\} - 1$. In the analogous way, the

pair distribution function is expressed as

$$\rho^{(2)}(\boldsymbol{r}_1, \boldsymbol{r}_2) = z^2 \sum_{l=0}^{\infty} \frac{z^l}{l!} \int \cdots \int d\boldsymbol{r}_3 \cdots d\boldsymbol{r}_{l+2} \sum_{l+2 \geq i > j \geq 1}^{\prime} \prod b_{ij}. \tag{4}$$

Sum over all products for which each of $\boldsymbol{r}_3, \cdots, \boldsymbol{r}_{l+2}$ is connected to $\boldsymbol{r}_1$ or $\boldsymbol{r}_2$ or both.

If the diagrams having articulation points are eliminated from the right-hand side of (4), it can be shown, with the aid of (3), that $\rho^{(2)}(\boldsymbol{r}_1, \boldsymbol{r}_2)$ is written in the form

$$\rho^{(2)}(\boldsymbol{r}_1, \boldsymbol{r}_2) = \rho(\boldsymbol{r}_1)\rho(\boldsymbol{r}_2)g(\boldsymbol{r}_1, \boldsymbol{r}_2),$$

$$g(\boldsymbol{r}_1, \boldsymbol{r}_2) = (1+b_{12}) \exp\{w(\boldsymbol{r}_1, \boldsymbol{r}_2)\} \tag{5}$$

and $w(\boldsymbol{r}_1, \boldsymbol{r}_2)$ is given by

$$w(\boldsymbol{r}_1, \boldsymbol{r}_2) = \sum_{l=0}^{\infty} \frac{1}{l!} \int \cdots \int d\boldsymbol{r}_3 \cdots d\boldsymbol{r}_{l+2} \rho(\boldsymbol{r}_3) \cdots \rho(\boldsymbol{r}_{l+2}) \sum_{l+2 \geq i > j \geq 1}^{(W)} \prod b_{ij}. \tag{6}$$

In the right-hand side of (6), (W) means the restriction about the summation that the sum is taken over all products for which each particle of $\boldsymbol{r}_3, \cdots, \boldsymbol{r}_{l+2}$ is connected to $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ by an independent path and $\boldsymbol{r}_3, \cdots, \boldsymbol{r}_{l+2}$ are connected independently of $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$.

By making a black circle correspond to an integral $\int d\boldsymbol{r}\rho(\boldsymbol{r}) \cdots$ and a line $\overset{b}{\text{———}}$ connecting circles i and j to a factor b_{ij}, we can express an integral in (6), including a numerical factor, by a diagram as in reference 2) and the Appendix of I. In such a graphical repesentation expression (6) is identical with (6″) of reference 2) or (A·3) of I. Therefore the analysis of the pair distribution function can be made in quite the same way as in the Appendix of I. We obtain, corresponding to (A·4), (A·5), (A·7) and (A·8) of I,

$$v(\boldsymbol{r}_1, \boldsymbol{r}_2)[\equiv z(\boldsymbol{r}_1, \boldsymbol{r}_2) + z_s(\boldsymbol{r}_1, \boldsymbol{r}_2)] = z(\boldsymbol{r}_1, \boldsymbol{r}_2) + \int d\boldsymbol{r}\, \rho(\boldsymbol{r}) z(\boldsymbol{r}_1, \boldsymbol{r}) v(\boldsymbol{r}, \boldsymbol{r}_2), \tag{7}$$

$$v(\boldsymbol{r}_1, \boldsymbol{r}_2) = g(\boldsymbol{r}_1, \boldsymbol{r}_2) - 1 = \{1 + b(\boldsymbol{r}_1, \boldsymbol{r}_2)\} \exp\{w(\boldsymbol{r}_1, \boldsymbol{r}_2)\} - 1, \tag{8}$$

$$w(\boldsymbol{r}_1, \boldsymbol{r}_2) = v(\boldsymbol{r}_1, \boldsymbol{r}_2) - z(\boldsymbol{r}_1, \boldsymbol{r}_2) + x(\boldsymbol{r}_1, \boldsymbol{r}_2), \tag{9}$$

(A·8) *in* I *expressing* $x(\boldsymbol{r}_1, \boldsymbol{r}_2)$ *in terms of* $v(\boldsymbol{r}_1, \boldsymbol{r}_2)$. (10)

The set of equations (7)–(10) determines the pair distribution function when we know $\rho(\boldsymbol{r})$ and $b(\boldsymbol{r}_1, \boldsymbol{r}_2)$. In the analogous way, (3) is further reduced to

$$\rho(\boldsymbol{r}_1) = z \exp\{-\mu_1(\boldsymbol{r}_1)/kT\}, \tag{11}$$

where, in the graphical representation, $-\mu_1(\boldsymbol{r}_1)/kT$ is expressed in a form identical with $-\mu_1/kT$ in I and reference 2). The graphical analysis in the Appendix of I leads us to (A·14) which is here to be interpreted as

$$\ln\rho(\boldsymbol{r}_1) - \ln z = \int d\boldsymbol{r}\, \rho(\boldsymbol{r}) \{z(\boldsymbol{r}_1, \boldsymbol{r}) - x(\boldsymbol{r}_1, \boldsymbol{r}) - \frac{1}{2} v^2(\boldsymbol{r}_1, \boldsymbol{r}) - v(\boldsymbol{r}_1, \boldsymbol{r}) x(\boldsymbol{r}_1, \boldsymbol{r}) + \frac{1}{2} \cdot v(\boldsymbol{r}_1, \boldsymbol{r}) z(\boldsymbol{r}_1, \boldsymbol{r})\} + \textit{Series in terms of } v(\boldsymbol{r}, \boldsymbol{r}'). \tag{12}$$

By means of the set of equations (7)–(10) and (12), we can, in principle, determine $\rho(\boldsymbol{r})$ and $w(\boldsymbol{r}_1, \boldsymbol{r}_2)$ from the knowledges of fugacity z and $b(\boldsymbol{r}_1, \boldsymbol{r}_2)$.

Now, when the series of (12) and the series appearing in (12) converge uniformly with respect to the volume V, we can take the limit $V \to \infty$ in these equations. If we have such a solution that $\rho(\boldsymbol{r})$ tends to a constant independent of $\boldsymbol{r}$ in the limit of $V \to \infty$, it will correspond to gases or liquids. Then, the set of equations (7)–(10) is reduced to the form which was given in the previous report.[1] If we have such a solution that $\rho(\boldsymbol{r})$ tends to a periodic function of $\boldsymbol{r}$, it will correspond to the solid state.

In this way, the possibility is indicated that the formalism developed in the preceding reports[1),2),3)] is applicable to liquids as well as gases, and the formalism is generalized to the case of solids.

A detailed account will be given shortly in this journal, with inclusion of the straightforward extension to a multi-component system in external field. The discussions about the expression for the free energy and its variational character will be given in a separate report.

1) T. Morita and K. Hiroike, Prog. Theor. Phys. **23** (1960), 385, 1006 (to be referred to as I).
K. Hiroike, Prog. Theor. Phys. **24** (1960), 317 (to be referred to as II).
2) T. Morita, Prog. Theor. Phys. **23** (1960), 829.
3) T. Morita, Prog. Theor. Phys. **20** (1958), 920; **21** (1959), 361.
[Errata, **23** (1960), 843]; 23 (1960), 175.
J. M. J. Van Leeuwen, J. Groeneveld and J. de Boer, Physica **25** (1959), 792.
E. Meeron, preprints.
4) E. E. Salpeter, Ann. Physics **5** (1958), 183.

Charge Distribution in Be^9

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June 13, 1960

Scattering of electrons by nuclei has been a very useful tool for determining the charge distributions in various nuclei. The general procedure has been to choose a model distribution with some empirical parameters, evaluate the scattering cross-sections, and compare them with the experimental results. It is possible thus to eliminate certain types of charge distributions, such as uniform charge distribution, etc., but in general, one is still left with a family of models which all give similar results. For the purpose of studying the systematics of variation of parameters of nuclear charge distributions it would be desirable to use a single model for a large range of nuclei. For heavy nuclei the Fermi distribution[1] (or a very similar "folded distribution"[2]) has been successfully used to analyse the electron scattering data. Elton, Hiley and Price[3] have analysed the C^{12} data also in terms of the Fermi distribution. Partly in order to extend this analysis to other light nuclei, we have calculated the electron scattering of Be^9 on the basis of the Fermi model. This nucleus is also of particular interest, since it is in the middle of the p-shell, hence one may expect a certain amount of quadrupole deformation of the nucleus to manifest itself. Recently Meyer-Berkhout, Ford and Green[4] have carried out a fairly com-

plete analysis of the p-shell nuclei; however, their results for Be^9 parameters are somewhat different from ours, and since we have consistently employed the Fermi distribution even for evaluation of the quadrupole scattering effects, we present here our results for Be^9.

The Fermi distribution for a spherical nucleus is given by

$$\rho_0(r)=\rho_0\left[1+\exp\left(\frac{r-c}{z}\right)\right]^{-1}.$$

If we take into account the quadrupole deformation of the nucleus by replacing r by $r(1+\epsilon Y_2^0(\cos\theta))$, we may write

$$\rho(r)=\rho_0(r)+\frac{d\rho_0(r)}{dr}\epsilon r Y_2^0(\cos\theta)$$

$$=\rho_0(r)+\rho_2(r)\,Y_2^0(\cos\theta).$$

The scattering cross-section in the Born approximation is given by the well-known results:

$$\sigma=\left(\frac{ze^2}{2E}\right)^2\frac{\cos^2(\theta/2)}{\sin^4(\theta/2)}[|F_0|^2+|F_2|^2]$$

where

$$F_0=4\pi\int\rho_0(r)\,j_0(qr)\,r^2dr$$

$$F_2=-\sqrt{20\pi}\int\rho_2(r)\,j_2(qr)\,r^2dr.$$

The quadrupole moment Q is given by

$$Q=\sqrt{\frac{16\pi}{5}}\int\rho_2(r)\,r^4dr.$$

The Born approximation may be expected to give a valid result for such light nuclei except in the immediate neighbourhood of the diffraction minimum.

First we take $\epsilon=0$, and determine the

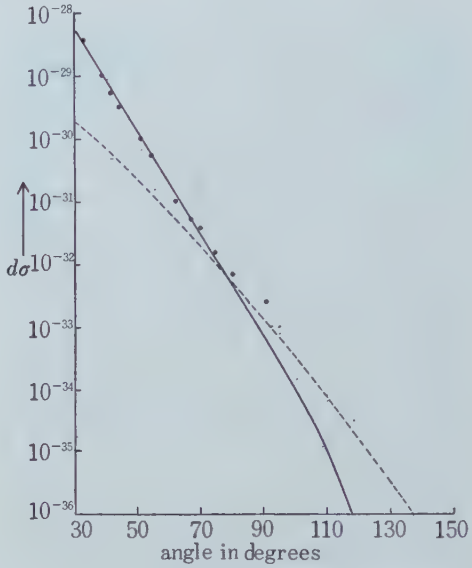

Fig. 1. Elastic scattering of 300 Mev electrons by Be^9

The continuous curve shows the monopole scattering, the dotted curve the quadrupole contribution and the dots are the experimental points.

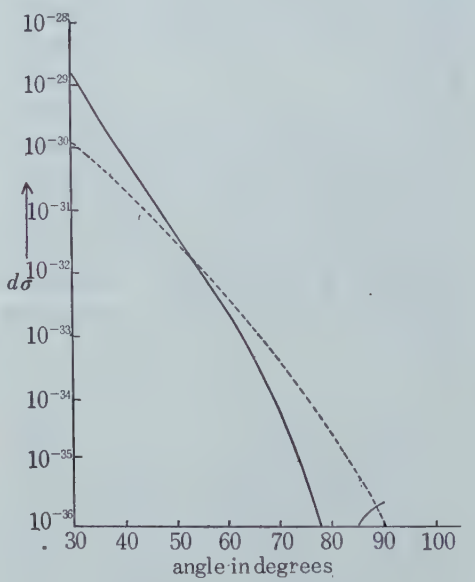

Fig. 2. Calculated curve for 420 Mev electron scattering by Be^9

The continuous curve represents the monopol scattering and the dotted curve shows th quadrupole scattering.

best values of the parameters c, z by fitting the experimental data at small angles for scattering of 300 Mev electrons as given in reference 4). Since we later find that the quadrupole scattering at these angles is very small, the analysis at small angles in terms of monopole scattering alone is valid. We find the best choice of the parameters to be $c=1.4f$, $z=0.6f$. The experimental results and the contribution of the monopole scattering to the cross-section given by $|F_0|^2$ alone is shown in Fig. 1. It is clear that for larger angles ($\theta>75°$), the monopole scattering alone is inadequate and no variation of parameters can improve the situation.

The value of ϵ is now fixed by evaluating the quadrupole moment of Be^9 with the above values of c, z and comparing with the experimental value, which we take to be $Q=2\times10^{-26}\,cm^2$. This gives us $\epsilon=0.256$. Now all the parameters are determined and the quadrupole scattering cross-section as given by $|F_2|^2$ is evaluated and the plot is shown in Fig. 1.

Thus the Fermi charge distribution with the inclusion of quadrupole distortion appears to fit the experimental results at 300 Mev fairly satisfactorily. Perhaps an exact evaluation of the monopole cross-section would improve the agreement near the diffraction minimum. At this energy the diffraction minimum is not clearly seen. It would however be expected to be seen clearly at the electron energy of 420 Mev. In view of the possibility that these experiments may be performed, we also give in Fig. 2 the results of our model for this energy. Here again the calculations are made in the Born approximation.

The values c, z obtained here are different from those of reference 4), who obtain $c=0.9f$, $z=0.79f$, values that appear to be somewhat extreme. In view of the values $c=2.60f$, $2.35f$ and $z=0.43f$, $0.42f$ for O^{16} and C^{12} respectively, our value of z appears rather high, indicating that Be^9 nucleus is almost all surface. The r.m.s. radius obtained with our values of the charge distribution parameters is $2.48f$, compared to $2.99f$ reported by Meyer-Berkhout et al. for their values of the parameters, and $2.2f$ for the Harmonic oscillator model.

The deformation parameter ϵ as determined here for Be^9 has the value $\epsilon=0.256$, which indicates quite a sizable quadrupole deformation.

In conclusion, we may say that the Fermi charge distribution model including the effects of possible quadrupole deformation satisfactorily accounts for the observed scattering cross-section of Be^9 for 300 Mev electrons.

The author is greatly indebted to Dr. S. P. Pandya for suggesting the problem and helpful discussions during the course of this work. He also extends his thanks to the Ministry of Education, Government of India, for a grant during the period of this work.

1) R. Hofstadter, Ann. Revs. Nuc. Sc. **7** (1957), 231.
2) R. Helm, Phys. Rev. **104** (1956), 1466.
3) L. R. B. Elton, B. J. Hiley and R. Price, Proc. Phys. Soc. **73** (1959), 112.
4) U. Meyer-Berkhout, K. W. Ford and A. E. S. Green, Ann. Phys. 8 (1959), 119.

Causality and Local Commutativity

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June 23, 1960

The scattering amplitude can be expressed in terms of retarded commutators in the following form:

$$\boldsymbol{t}=i\int dx \exp\left[-i\frac{k+k'}{2}x\right] \times\langle p'|[j(x/2),\ j(-x/2)]|p\rangle\theta(x). \tag{1}$$

When we want to prove the dispersion relation, it is usual to require the local commutativity only for the special commutators, that of (1), and so on. We believe that this requirement corresponds to causality, since this is necessary for the microscopic causality to hold. However, it is not clear whether this requirement is sufficient for the causality in the original meaning to hold. The aim of this note is to show that, from the only requirement mentioned above, except for the case where the mass of the bombarding particle is zero, we cannot prove that the scattering amplitude $\boldsymbol{t}$ satisfies the macroscopic causality discussed in the previous paper.[1] Therefore, there is no fact denying that the special case exists where the macroscopic causality is the stronger restriction than the local commutativity mentioned above.

For simplicity, we shall confine ourselves to the case where $\boldsymbol{t}$ is independent of the momentum transfer. Considering the forward scattering in the laboratory system, we see that $\boldsymbol{t}$ is written as

$$\boldsymbol{t}=\int dx \exp[-iqx] \times f(x^2, x_0)\theta(-x^2)\theta(x) \tag{2}$$

where $q=(\boldsymbol{e}\sqrt{\omega^2-M_B^2}, \omega)$, $\boldsymbol{e}$ is unit vector, ω is energy of the bombarding particle in the laboratory system.

The restrictions on $\boldsymbol{p}$, $\boldsymbol{p}'$, $\boldsymbol{R}$, $\boldsymbol{r}$, $\boldsymbol{R}'$, $\boldsymbol{r}'$ for getting a non-vanishing transition amplitude (3·1) in the previous paper[1] are

$$\boldsymbol{R}'-\boldsymbol{R}-\boldsymbol{V}_A(\boldsymbol{p}')(t-T-\tau) -\boldsymbol{V}_A(\boldsymbol{p})T\approx 0, \tag{3}$$

$$\boldsymbol{R}'-\boldsymbol{r}'-\boldsymbol{R}-\boldsymbol{V}_B(-\boldsymbol{p}')(t-T-\tau) -\boldsymbol{V}_A(\boldsymbol{p})T\approx 0. \tag{4}$$

$$\boldsymbol{r}-\boldsymbol{V}_A(\boldsymbol{p})T+\boldsymbol{V}_B(-\boldsymbol{p})T\approx 0, \tag{5}$$

where

$$\tau=[E_A(\boldsymbol{p})+E_B(-\boldsymbol{p})] \times[x_0-\boldsymbol{e}\cdot\boldsymbol{x}w/\sqrt{w^2-M_B^2}]/M_A \tag{6}$$

and

$$[E_A(\boldsymbol{p})+E_B(-\boldsymbol{p})]^2 =M_A^2+M_B^2+2M_Aw. \tag{7}$$

In order that, for arbitrary $f(x^2, x_0)$, $\boldsymbol{t}$ in (2) satisfies the macroscopic causality, τ must be positive or τ must be of microscopic order if τ is negative, for all values of x and ω restricted by $-x^2>0$, $x_0>0$, $\omega>M_B$. However, this is not the case except for $M_B=0$.

One might think that the above requirement is too strong for the macroscopic causality, because the causality condition can be satisfied if the contribution from the parts of the transition matrix which violates the causality vanishes after the integration with re-

spect to x. However, this is not the case, since $f(x^2, x_0)$ is arbitrary. Therefore we reach the conclusion that the requirement of the local commutativity only for the commutator of the right-hand side of (1) is not sufficient for the causality in the original meaning to hold.

1) K. Yamamoto, Prog. Theor. Phys. **23** (1960), 859.

Synthesis of Iron Group Elements by the Rapid Nuclear Process

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June 30, 1960

The cosmical abundance curve of chemical elements as a function of atomic weight based on the data of Suess and Urey[1] shows that there exists a strongly-marked peak centered in Fe^{56}, which lies near the minimum of the packing-fraction curve. To explain this feature in the abundance curve, Hoyle[2] discussed the so-called equilibrium process in which the elements comprising the iron peak in the abundance curve (V, Cr, Mn, Fe, Co, and Ni) are synthesized under conditions of high temperature and density in the stellar interior. Recently, Burbidge, Burbidge, Fowler and Hoyle[3] have re-examined this process using the more accurate nuclear and abundance data and have concluded that the best fit of the theoretical abundance curve of the iron peak to the observational one is obtained for $T=3.78\times10^9$ °K and the ratio of protons to neutrons near 300.

On the other hand, Hayashi, Nishida, Ohyama and Tsuda[4] have investigated the stellar conditions under which the cosmical abundances of α-particle nuclei, Mg^{24}, Si^{28}, S^{32}, A^{36} and Ca^{40}, can be accounted for. Their conclusions are as follows. The elements in question can be synthesized by the so-called α-process at the high temperature ($2\sim3\times10^9$ °K) but relatively low density ($1\sim10^2$g/cm^3), the latter being different from that suggested by Burbidge et al., and it is necessary for this temperature to continue for a rather short time, $1\sim10^2$ sec, with rapid heating-up and subsequent cooling. Such stellar conditions are expected in the envelope of the supernova.

In this note, we investigate whether or not the cosmical abundances of the elements in the iron peak can be explained in terms of the *rapid* nuclear process which is supposed to take place during the initial phases of the supernova explosion.

First, the rates for various nuclear reactions induced by protons and α-particles and their reverse ones, which are considered to take place in the circumstance of the supernova envelope at its earliest exploding stage (i.e. at the temperature near $1\sim2\times10^9$ °K and the density of $1\sim10^2$ g/cm^3), are calculated through the well-known procedures.[3] In calculating the reaction rates involving the proton-rich nuclei whose masses are not yet determined experimentally, we estimated them using

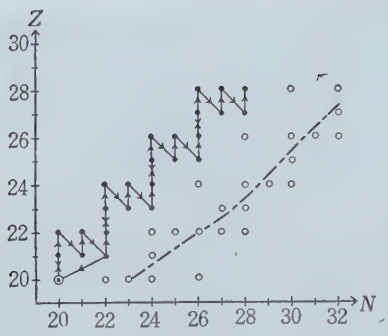

Fig. 1.

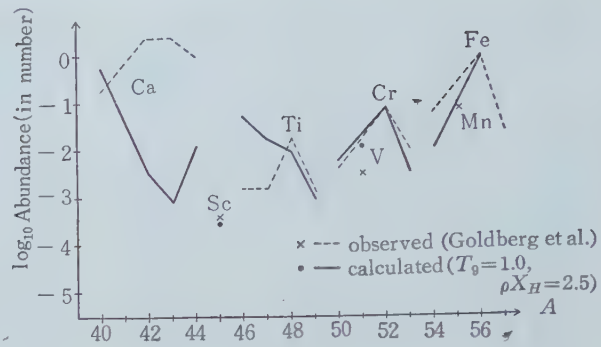

Fig. 2.

the semi-empirical mass-formula derived by Cameron.[5] However, its terms representing the shell- and pairing-effects were modified in such a way that the differences between estimated and observed values of nuclear masses for the stable isotopes near the iron peak become smaller than those obtained by Cameron's formula. The calculated results show that the reactions induced by α-particles, such as (α, γ), (α, p), (α, n), etc., occur much slower than those by protons at the temperature below 1.5×10^9 °K and for the number ratio of the α-particles to protons less than 10^7. Therefore, we can neglect the α-particle reactions in our rapid process. Since, as seen later, our rapid process is supposed to take place in the outermost part of the envelope of a supernova in which the so-called r-process[3] occurs in the inner envelope, the presence of free neutrons might be neglected. Moreover, the photo-proton reactions and β^+-decays slower than $10^2\sim10^3$ sec., which is the time scale of duration of nuclear synthesis in the exploding envelope of a supernova, were neglected. Thus the route of formation of elements in the iron peak, which starts from Ca^{40}, is determined by (p, γ) and (γ, p) reactions and β^+-decays faster than this characteristic time scale. This is shown in Fig. 1.

The abundances of nuclei in the route of this rapid process have been calculated on the assumption that a steady state is reached in which the abundance of each nuclide is proportional to its proton-capturing or β^+-decay time. This circumstance under which the steady state is a good approximation can be elucidated by considering the following two facts. (i) The estimated total time for our rapid process is comparable to the $\sim$100 sec expansion and cooling time for the supernova envelope. (ii) The establishment of steady proton-flow requires about 16 protons per capturing Ca^{40} since the average atomic weight of elements in the region of interest is $\simeq$56, and this amount of protons seems to remain in the outermost part of supernova envelope. However, this assumption is not valid for the synthesis beyond Ni^{56}, because the formation of Ni^{57} through $Ni^{56}(p, \gamma)$ $Cu^{57}(\beta^+)$ Ni^{57} takes a time of $10^5\sim10^6$ sec due to the fast $Cu^{57}(\gamma, p)$ reaction, whose reaction rate is five or six orders

of magnitude larger than that for $Ni^{56}(p, \gamma)$ under the conditions considered above. As to the abundance of Ni^{56}, which is not determined in the steady-state approximation, it has been assumed to have such a value that the abundance ratio Ni^{56}/Fe^{52} is equal to the observed ratio Fe^{56}/Cr^{52}. Calculations of the abundances have been carried out for values of temperature ranging from 0.8×10^9 to 1.5×10^9 °K and values of proton density, ρX_H, from 1.0 to 10^3 g/cm^3. In determining the final abundances of stable isotopes, only β^+-decays have been considered as the freezing reaction in the cooling stage of supernova envelope. The best fit is obtained for temperature about 1.0×10^9 °K and $\rho X_H \sim 2.5$ in comparing the calculated abundances with observed ones.[6] This result is shown in Fig. 2. Taking account of uncertainty of nuclear masses of proton-rich nuclide, we may conclude that the abundances of elements in the iron peak can be explained in terms of the rapid process.

Finally, the authors wish to thank Professors C. Hayashi and S. Hayakawa for their valuable discussions.

1) H. E. Suess and H. C. Urey, Rev. Mod. Phys. **28** (1956), 53.
2) F. Hoyle, M. N. Roy. Astron. Soc. **106** (1946), 343.
F. Hoyle, Ap. J. Suppl. **2** (1954), 1.
3) E. M. Burbidge, G. R. Burbidge, W. A. Fowler and F. Hoyle, Rev. Mod. Phys. **29** (1957), 547.
4) C. Hayashi, M. Nishida, N. Ohyama and H. Tsuda, Prog. Theor. Phys. **20** (1958), 110; **22** (1959), 101.
5) A. G. W. Cameron, Can. Jour. Phys. **35** (1957), 1021.
6) L. Goldberg, L. H. Aller and E. Müller, to be published.

Errata

Electromagnetic Structure of the Nucleon. V

——Numerical Results of Three-Pion-State Contributions——

Kichiro HIIDA and Noboru NAKANISHI

Prog. Theor. Phys. **24** (1960), 414

1. Fig. 1 on p. 415 should read Fig. 1 (a).
2. The following figure should be added.

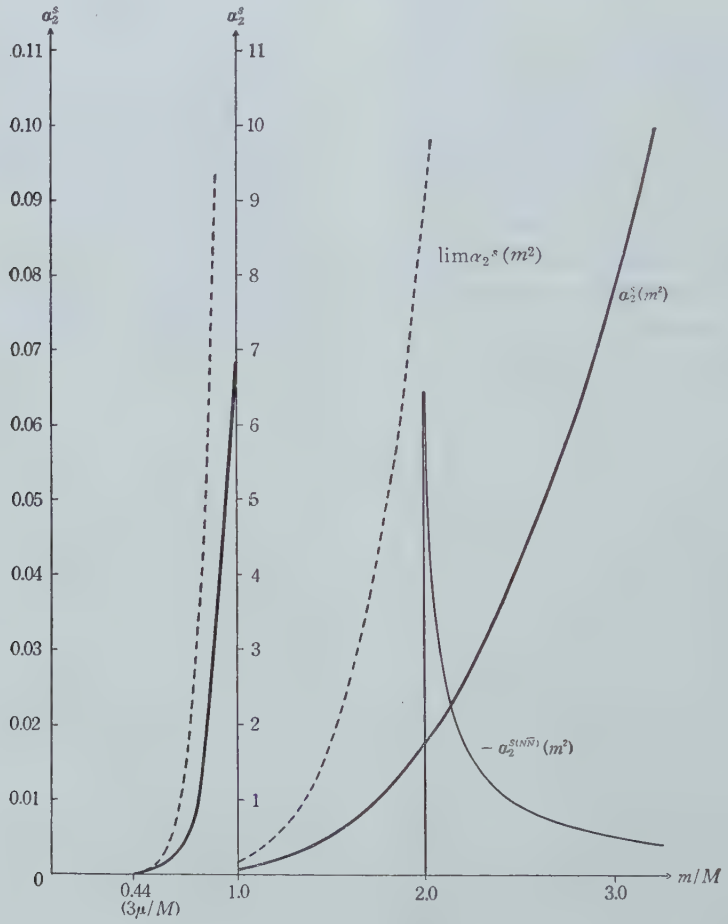

Fig. 1 (b).

We regret that this figure has been dropped out without our knowledge during the course of publication.

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Energy Spectrum and Scattering Problem in Quantized Field Theory

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(Received December 23, 1959)

It is shown that the scattering problem can be formulated in terms of a generating operator with respect to the variation of the coupling parameter which is contained in the Hamiltonian. This formalism makes the rigorous treatment of such problems possible concerning the self energies of closed systems as well as the scattering between bound states. One can further obtain a certain general relation concerning the phase shifts and levels, which includes the field theoretical generalization of Levinson's theorem.

Introduction

The field concept, in classical physics, was incompatible with the particle concept in their essence. In quantum mechanics the wave character which had been original with a field theory was unified consistently with the corpuscular character. However, the particle concept is again not clear in quantum theory of field owing to its self field ambiguities. When there is no interaction or when the interaction is infinitesimally small, quantized fields certainly describe an assembly of quanta each of which has definite energy and momentum. But, when the interaction is finite, quantized fields in themselves do not correspond to such a situation. A clear-cut definition of a particle picture for the interacting fields was first introduced by the interaction representation of Tomonaga and Schwinger. This particle is not a real particle (with a definite energy and momentum) but a virtual particle (propagating in accordance with the free Hamiltonian). To formulate a scattering problem one must have, in addition to the basic formalism of the field theory, a prescription which tells us how to describe the real particles at $t\sim\mp\infty$. Feynman-Dyson's prescription is an assumption that the virtual particles of the interaction representation asymptotically coincide with real particles at $t\sim-\infty\,(+\infty)$. In Green's function formalism developed after Yang-Feldman's[1] and Gell-Mann and Low's[2] works the propagators are considered to represent the propagation of particles.[3]

Let us now confine our interests to the description of real particles at $t\sim\mp\infty$ only, which is necessary for the formulation of the scattering problem, and consider it more axiomatically. At $t\sim\mp\infty$ the particles are spatially separated far distant from each other, each of them being localized and flying without mutual interaction. In Heisenberg representation a constant state vector represents a certain physical process. Since an incoming plane plus outgoing scattered wave stationary

state (which shall hereafter be abbreviated as "in-state") is characterized by its incoming particles, it is described by the real particle (which may be elementary as well as composite) annihilation and creation operators (that is, by operating the creation operators on the vacuum state), which satisfy

$$[P_\mu, a^{in}(k)] = -k_\mu a^{in}(k), \tag{1}$$

$$[a^{in}(k), a^{in\dagger}(k')] = \delta(\boldsymbol{k}-\boldsymbol{k}'), \tag{2}$$

where P_μ is the total energy-momentum operator. (2) guarantees the orthonormality of the in-states, and (1) means the particle (at $t\sim-\infty$) described by $a^{in}(k)$ is a real quantum with the definite energy-momentum of k_μ. The conditions (1) and (2) for $a^{in}(k)$'s are necessary but not sufficient.

In Yang-Feldman's[1] and Green's function formalism[3),4)] one put

$$a^{in(out)}(k) = \lim_{x\to-\infty(+\infty)} \varphi(x)\overleftrightarrow{O}e^{-ikx} \equiv \lim_{x\to-\infty(+\infty)} (i/2)[2k_0(2\pi)^3]^{-1/2}\int d\boldsymbol{x}\,\varphi(x)\,(\overleftarrow{\partial}_t-\overrightarrow{\partial}_t)\,e^{-ikx}, \tag{3}$$

which is restricted to elementary particles. The extension to a compound particle in this form was done by Haag[5)] by the introduction of "almost local operator field $\varphi'(x)$":

$$a^{in(out)}(k) = \lim_{x\to-\infty(+\infty)} \varphi'(x)\overleftrightarrow{O}e^{-ikx}. \tag{4}$$

Almost local operator field $\varphi'(x)$ is defined as such a function of basic field variables which satisfies

a) $\varphi'(x+y) = e^{-iPy}\varphi'(x)e^{iPy}$,

b) $\varphi'(x)$ is expressed as a (finite or infinite) polynomial of field variables which are contained in essentially a finite space-time region including x.

One may as well adopt the form (4) also for elementary particles. A quantity which satisfies (a) and (b) is not unique and really there are infinitely many different possible forms. Rigorously speaking, different $\varphi'(x)$'s give different $a^{in(out)}(k)$'s and, therefore, different S-matrices. One must then inquire which one of $\varphi'(x)$'s has a legitimate asymptotic behavior. For a compound particle the form of $\varphi'(x)$ will relate to its structure. The same will be true for an elementary particle with a structure.

The present situation of the particle description on the field theory being so, it will be seen that there exists a new method of treating scattering problems* which is apparently quite different from these conventional formalisms. Studying how a scattered system changes when the interaction Hamiltonian is changed infinitesimally, one obtains a relation which prescribes the way of this change of the system. Conversely, if one regards this relation to be a basic equation, it can

* There is a similar treatment[6)] which is less general than that presented here.

be shown that it provides a mathematical foundation prescribing a particle picture at the infinite past and future.

In § 1 we will study how the energy spectrum and the state vectors describing each energy levels change when the parameter g is varied. The variations of state vectors belonging to a complete set of stationary states are described by a generating operator D_g which is determined one to one by a given complete set. We will obtain a relation for D_g which restricts the way of the change of the stationary states. This relation will, under certain additive requirements, determine a pair of operators D_g^{in} and D_g^{out} (introduced in § 2) which are the generating operators describing the changes of the in- and the out-stationary states respectively:

$$\left.\begin{aligned} da^{in}(k)/dg &= i[D_g^{in}, a^{in}(k)], \text{ etc.,} \\ da^{out}(k)/dg &= i[D_g^{out}, a^{out}(k)], \text{ etc.} \end{aligned}\right\} \tag{5}$$

The determination of $D_g^{in,out}$ is identical with the determination of the boundary condition of the states in a differential form with respect to the variation of g. The determined forms of $D_g^{in,out}$ are simply related to the interaction Hamiltonian. The scattering problem is now reduced formally to integrate the differential equation (5) for particle operators with respect to g. We will discuss Lee's model in § 3 as an illustration of our general scheme. It is further discussed in § 4 under a reasonable assumption that the Hilbert space which is spanned by the complete set of the incoming or the outgoing stationary states does not change abruptly at a threshold of the coupling parameter where a new bound state appears.

We can further exhibit a fundamental character about phase shifts which could not hitherto been taken into account so clearly by the formalism in the field theory. In § 5 a general relation connecting the g-derivatives of phase shifts with $D_g^{in,out}$ will be given, for which the spur on an eigen-space of constants of motion has especially simple results, and which is a field theoretical generalization of a well recognized fact that the phase shift of a scattering system is equal to the infinitesimal energy shift of the system multiplied by the volume of the box which encloses the system. Now this energy shift produces a finite change on the level density, since the level density itself is infinite of the order of the volume V of the box and each levels are displaced by an order of $1/V$. This situation is discussed in detail and a relation between the density of energy levels and the phase shifts is given. This relation includes as a special case the Levinson's theorem[7),8)] between the number of discrete levels and the phase shifts. This kind of formulas can also be utilized to obtain a general relation which connects a self energy of closed system with phase shifts, which will be discussed in some other paper.

§ 1. Energy eigenstates and the variation of g

If any two hermitian operators, A_1 and A_2, which are defined as operators in a same Hilbert space have the same spectra, that is, if they have the sets of eigenvectors which are in one-to-one correspondence and the corresponding eigenvectors have the same eigenvalues, viz.,

$$A_1\psi_i^I=a_i\psi_i^I,\quad A_2\psi_i^{II}=a_i\psi_i^I, \tag{1·1}$$

then A_1 and A_2 are connected by a unitary operator:

$$A_2=UA_1U^\dagger,\quad \psi_i^{II}=U\psi_i^I. \tag{1·2}$$

Conversely, if A_1 and A_2 are connected by a unitary operator, then their energy spectra coincide. For instance, for the interacting fields which have no bound states and for which the self energies of the vacuum and of the particles can be neglected, the total and the free Hamiltonians H and H_0 are connected by the following relation[2),9)]:

$$H=U(0,\pm\infty)H_0U(\pm\infty,0). \tag{1·3}$$

When A has a parameter g and the above property holds for any pairs $A_1\equiv A(g_1)$ and $A_2\equiv A(g_2)$, the differentiation of A with respect to g gives

$$dA/dg=i[D_g, A], \tag{1·4}$$

where

$$D_g=-i(dU/dg)U^\dagger, \tag{1·5}$$

is the infinitesimal generating operator for the variation of g. Eq. (1·4) applied for (1·3) gives

$$dH/dg=i[D_g, H]. \tag{1·6}$$

When the spectrum of $A(g)$ changes with the variation of g, the relations (1·2) and (1·4) hold no longer. Let us consider, for example, a quantum mechanical model with a Hamiltonian which has a discrete level as a solution. Thus

$$H\psi_i=E_i\psi_i,\quad H\psi_d=E_d\psi_d, \tag{1·7}$$

where ψ_i and ψ_d are the eigensolutions of the continuous levels and of the discrete level respectively. From (1·7) H can be written as

$$H=\sum_i \psi_iE_i\psi_i^*+\psi_dE_d\psi_d^*. \tag{1·8}$$

Let the infinitesimal generating operator for these state vectors with respect to the variation of the coupling parameter g be D_g, then

$$d\psi_i/dg=iD_g\psi_i,\quad d\psi_d/dg=iD_g\psi_d, \tag{1·9}$$

where D_g must be hermitian to conserve the orthonormality of state vectors. We here confine ourselves to the region of g where the discrete level does not disap-

pear and a new discrete level does not appear; also let us suppose that we are considering a set of eigenstates in which the suffices i and d are independent of g. Now, differentiating (1·8) with respect to g, we get

$$dH/dg=i[D_g, H]+\psi_d(dE_d/dg)\psi_d^*. \qquad (1\cdot10)$$

In the case of quantum field theory, when ψ_i represents, for instance, the incoming plane plus outgoing scattered wave stationary states, E_i being the sum of energies of the incoming particles, the term arising from the self energies of the particles through dE_i/dg must be added to the second term of the right-hand side of (1·10). Comparing (1·10) with (1·6) it will be observed that the change of the energy spectrum is revealed in the second term of the right-hand side of (1·10).

Under the above preparatory consideration let us turn to a rigorous discussion of a general case of the quantum field theory. Let us introduce the following fundamental assumptions:

I) *There exists a Hilbert space* $\mathfrak{H}$ such that the field quantities and their arbitrary functions are represented by linear operators in this space.*

II) *The total Hamiltonian H contains one real parameter g and is differentiable with respect to it.*

III) *The complete set of the in-(out-) states at each g-value either coincides with $\mathfrak{H}$ or forms a sub-space in $\mathfrak{H}$.***

We fix this Hilbert space further by the condition that the basic field variables at time $t=0$ are represented by the operators independent of g. In some case we want to discuss the problem under a more restricted assumption that:

III′) *The complete set of the in-(out-) states at each g-value coincides with $\mathfrak{H}$.*

Writing

$$\mathcal{H}=dH/dg\equiv\mathcal{H}_g, \qquad (1\cdot11)$$

$$H_1=\int_0^g dg'\,\mathcal{H}_{g'}, \qquad (1\cdot12)$$

and

$$H_0=H]_{g=0}, \qquad (1\cdot13)$$

we have for the total Hamiltonian

$$H=H_0+H_1. \qquad (1\cdot14)$$

In usual theories in which H is dependent on g linearly, $\mathcal{H}$ is independent of g, and $H_1=g\mathcal{H}$. When g is varied, the energy spectrum of H changes according

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* We assume that $\mathfrak{H}$ has a positive definite metric.

** Therefore Eq. (2) is needed to hold only in this sub-space which will hereafter be denoted by $\mathfrak{H}_g$.

to possible occurrences of new bound states and to the variation of the energy eigenvalues of closed systems (the self energies of elementary particles and the binding energies of composite particles). Let us assume for simplicity that there is no bound state for $g<g_c$ and there is a single kind of composite particle for $g\geqq g_c$. Now introduce a complete set of stationary states $\{\Psi_I\}$ which are specified by g-independent indices I,

$$H\Psi_I=E_I\Psi_I. \tag{1·15}$$

In the region $g<g_c$ or $g\geqq g_c$ the sets $\{\Psi_I(g_1)\}$ and $\{\Psi_I(g_2)\}$ have one to one correspondence, the correspondence being represented by the index I. Now by the assumption II) and III) the difference of state vectors

$$\Psi_I(g+\delta g)-\Psi_I(g)$$

has a meaning, therefore we can introduce an infinitesimal generating operator D_g which is defined by

$$d\Psi_I/dg=iD_g\Psi_I, \tag{1·16}$$

where D_g must be hermitian* to conserve the orthonormality of $\{\Psi_I\}$. Taking the variation of the both sides of (1·15)

$$(H-E_I)\,d\Psi_I/dg=-(\mathcal{K}-dE_I/dg)\Psi_I. \tag{1·17}$$

Using (1·16), one is naturally led to the relation

$$-i[H, D_g]=\mathcal{K}-d\boldsymbol{E}/dg, \tag{1·18}$$

where $d\boldsymbol{E}/dg$ is defined as such an operator which is commuting with H and is diagonal in the representation spanned by $\{\Psi_I\}$ and for which the eigenvalue for Ψ_I is dE_I/dg:

$$[H, d\boldsymbol{E}/dg]=0,\quad (d\boldsymbol{E}/dg)\,\Psi_I=(dE_I/dg)\Psi_I. \tag{1·19}$$

(1·18) is rewritten as

$$dH/dg=i[D_g, H]+d\boldsymbol{E}/dg. \tag{1·18′}$$

(1·6) and (1·10) are special cases of (1·18′).

§ 2. Scattering problem

Conventionally the in-state is given by

$$\Psi^{in}=\Phi-(H_0-E-i\epsilon)^{-1}H_1\Psi^{in}. \tag{2·1}$$

In classical quantum mechanics the factor $(H_0-E-i\epsilon)^{-1}$ correctly expresses the required incoming plane plus outgoing scattered wave boundary condition, since the potential has a finite range and therefore there is no interaction at far distance

* This hermiticity is required only in $\mathfrak{H}_g$.

from the scattering center. But the situation is not so clear in quantized field theory because of the presence of the self fields, compound particles, the inelastic scattering, etc.

In a correct scheme, however, the following requirement must be satisfied:

"*The in-state must reduce to that defined by* (2·1), *when the scheme is applied to the case of classical quantum mechanics as well as to the case in which the interaction is infinitesimally small.*" (A)

Now let us introduce the infinitesimal variation operators D_g^{in} and D_g^{out} for the complete sets of in- and out-states respectively. From (1·18),

$$-i[H, D_g^{in(out)}]=\mathcal{K}-d\boldsymbol{E}^{in(out)}/dg, \tag{2·2}$$

where $d\boldsymbol{E}^{in}/dg$ is defined by

$$[d\boldsymbol{E}^{in}/dg, a^{in}(k)]=-(dk_0/dg)a^{in}(k), \text{ etc.} \tag{2·3}$$

For a discrete state Ψ the condition

$$\{D_g^{out}-D_g^{in}\}\Psi=0, \tag{2·4}$$

must be satisfied. The solution of (2·2) for D_g^{in} is indeterminate by a quantity commuting with H.

We regard (2·2) as a basic equation and assume, consequently, that

"D_g^{in} *is a function only of the operators* H, $\mathcal{K}$ *and* $d\boldsymbol{E}^{in}/dg$ *which constitute the equation.*" (B)

Therefore D_g^{in} is considered to be dependent on g only through H, $\mathcal{K}$ and $d\boldsymbol{E}^{in}/dg$. Equations (2·2) and (2·4), the requirements (A) and (B) and further the requirement of time reversibility[10)-12)] are sufficient to determine D_g^{in} and D_g^{out} uniquely (see Appendix III), the results being expressed by

$$\left.\begin{aligned} d\Psi_I^{in}/dg&=iD_g^{in}\Psi_I^{in}=-(H-E_I-i\epsilon)^{-1}(\mathcal{K}-dE_I/dg)\Psi_I^{in},\\ d\Psi_I^{out}/dg&=iD_I^{out}\Psi_I^{out}=-(H-E_I+i\epsilon)^{-1}(\mathcal{K}-dE_I/dg)\Psi_I^{out},\end{aligned}\right\} \tag{2·5}$$

(where I represents the set of the momenta of the incoming (outgoing) particles and $E_I=\sum_{i\in I}k_0{}^{(i)}$, $dE_I/dg=\sum_{i\in I}dk_0{}^{(i)}/dg$, $k_0=(\mu^2+k^2)^{1/2}$, etc., $dk_0/dg=(1/k_0)(d\mu/dg)$, etc.), which are looked upon as solutions of Eq. (1·17) and these expressions are nothing but to give the in- and the out-boundary conditions in a g-differential form. (2·5) satisfies the condition (2·4), since for $\Psi^{in}=\Psi^{out}=\Psi$,

$$i\{D_g^{out}-D_g^{in}\}\Psi=2\pi i\delta(H-E)(\mathcal{K}-dE/dg)\Psi,$$

and

$$(\Psi\mathcal{K}\Psi)=dE/dg.$$

(2·5) is important, because this form of the boundary condition facilitates the formulation of the scattering problem including composite particles, as will be seen

from the following discussion. In order that a stationary state for a scattering between elementary particles has a correct boundary condition, that is, the particles at the infinite past or future have correct observed masses, we had to transfer the mass counter term from the interaction Hamiltonian into the free Hamiltonian, so that

$$\Psi_I^{in}=\Phi_I-(H_0'-E_I-i\epsilon)^{-1}H_1'\Psi_I^{in}, \tag{2·6}$$

where

$$H_0'=H_0+\Delta H_0=H_0]_{\mu_0\to\mu,\,m_0\to m}, \quad H_1'=H_1-\Delta H_0, \tag{2·7}$$

and

$$H_0'\Phi_I=E_I\Phi_I. \tag{2·8}$$

There are some discussions[13),14)] that this formula can be applied to the case of a scattering including composite particles by modifying ΔH_0 so that the solutions of Eq. (2·8) include the bound states and that the corresponding eigenvalue E_I coincides with the binding energy. However, the term ΔH_0 for such cases is not an elementary quantity, nor it is sure that it can be found always. It will thus be not desirable to introduce such an ambiguous term for the purpose of constructing a formalism which has a sufficiently general applicability. It is shown, however, in Appendix III that the differentiation of (2·6) essentially reduces to Eq. (2·5) which is free from such an ambiguity.

The scattering problem is now formally to integrate the differential equation

$$d\,a^{in}(k)/dg=i[D_g^{in},\,a^{in}(k)],\ \text{etc.}, \tag{2·9}$$

under (2·5). The formal solution of it is represented by

$$a^{in}(k,\,g)=U_{gg_1}^{in}a^{in}(k,\,g_1)\,U_{gg_1}^{in\dagger}, \tag{2·10}$$

where

$$d\,U_{gg_1}^{in}/dg=iD_g^{in}U_{gg_1}^{in},\ U_{gg_1}^{in}]_{g=g_1}=1,\ U_{gg_1}^{in}1_{g_1}U_{gg_1}^{in\dagger}=1_g,\ U_{gg_1}^{in\dagger}1_gU_{gg_1}^{in}=1_{g_1}, \tag{2·11}$$

and 1_g is the projection operator for $\mathfrak{H}_g$. It is to be seen (see Appendix IV) that, if $a^{in}(k,\,g_1)$ satisfies (1) and (2), $a^{in}(k,\,g)$ satisfies (1) and (2) also.

In a renormalized theory of no bound state (subtraction of mass counter term is equivalent to regard the observed mass as g-independent: see Appendix II) $d\boldsymbol{E}^{in}/dg=0$, so that D_g^{in}, which is defined by (2·5), is rewritten as (α being infinitesimal and positive)

$$D_g^{in}=\int_0^{-\infty}d\tau\,\mathcal{H}(\tau)\,e^{-\alpha|\tau|}. \tag{2·12}*$$

The exponentially damping factor $e^{-\alpha|\tau|}$ in the integrand is necessitated for the convergence of the infinite time integral. This factor makes an explicit integra-

* This result would coincide with ref. 6), if α is put to be zero.

tion of D_g^{in} with respect to g difficult. But, if one assumes the adiabatic switching of the interaction:

$$\mathcal{H}(\tau) \to 0 \quad \text{for} \quad \tau \to \mp\infty, \tag{2·13}$$

this factor can be neglected ($\alpha=0$) and (2·12) can be easily integrated. The resulting S-matrix coincides with that of Feynman-Dyson and also the solution $a^{in}(k)$ of Eq. (2·9) under (2·12) coincides with that of Yang-Feldman (see Appendix II).

For the S-matrix

$$S_{I\,II}=(\Psi_I^{out}\Psi_{II}^{in}), \tag{2·14}$$

the following equation holds,

$$dS_{I\,II}/dg=i\sum_{III}S_{I\,III}D_{III\,II}^{in}-i\sum_{III}D_{I\,III}^{out}S_{III\,II}, \tag{2·15}$$

where

$$D_{I\,II}^{in}=(\Psi_I^{in}D_g^{in}\Psi_{II}^{in}),\quad D_{I\,II}^{out}=(\Psi_I^{out}D_g^{out}\Psi_{II}^{out}). \tag{2·16}$$

For no bound state renormalized theory,

$$dS_{I\,II}/dg=-2\pi i\sum_{III}S_{I\,III}\delta(E_{III}-E_{II})\overline{\mathcal{H}}_{III\,II}, \tag{2·17}$$

where

$$\overline{\mathcal{H}}_{I\,II}=(\Psi_I^{in}\mathcal{H}\Psi_{II}^{in}). \tag{2·18}$$

If the assumption (III′) holds, the following equation for $\overline{\mathcal{H}}_{I\,II}$ is obtained by differentiating (2·18),

$$d\overline{\mathcal{H}}_{I\,II}/dg=\overline{\mathcal{H}'}_{I\,II}+\sum_{III}\overline{\mathcal{H}}_{I\,III}\left\{\frac{1}{E_I-E_{III}-i\epsilon}-\frac{1}{E_{III}-E_{II}-i\epsilon}\right\}\overline{\mathcal{H}}_{III\,II}, \tag{2·19}$$

where

$$\overline{\mathcal{H}'}_{I\,II}=(\Psi_I^{in}d\mathcal{H}/dg\,\Psi_{II}^{in}). \tag{2·20}$$

§ 3. Example

Let us consider Lee's model as an illustration of the general scheme of the previous section. The detail of this model will be referred to the work of Pauli and Källen[15] (hereafter be referred as P&K). The total Hamiltonian is given by

$$H=H_0+H_1=H_0+\Delta H_0+H_1-\Delta H_0,$$

where

$$\left.\begin{aligned}
&H_0+\Delta H_0=m\sum_q\psi_V^\dagger(q)\psi_V(q)+m\sum_p\psi_N^\dagger(p)\psi_N(p)+\sum_k\omega a^\dagger(k)a(k),\\
&H_1=-gV^{-1/2}\sum_{q=p+k}(2\omega)^{-1/2}f(\omega)\{\psi_V^\dagger(q)\psi_N(p)a(k)+a^\dagger(k)\psi_N^\dagger(p)\psi_V(q)\},\\
&\Delta H_0=\delta m\sum_q\psi_V^\dagger(q)\psi_V(q),
\end{aligned}\right\} \tag{3·1}$$

m is the renormalized mass of (N- and) V-particle, g is the unrenormalized coupling constant and ψ_V is the unrenormalized operator. P&K's results are as follows:

$$\left.\begin{aligned} \Psi_V &= \Phi_V N + \sum_{k''} \Phi_{k''} g N F(\omega''), \\ \Psi_k &\equiv \Psi_k^{in} = \Phi_V \varphi(k) + \sum_{k''} \Phi_{k''} \varphi(k'', k), \end{aligned}\right\} \tag{3·2}$$

where $\boldsymbol{k}$ is the momentum of θ-particle in the free or in the interacting (N, θ) system, and

$$\left.\begin{aligned} &N^2 = (1+g^2/g_c^2)^{-1}, \quad 1/g_c^2 = \sum_{k''} F^2(\omega'')/\omega''^2, \\ &\delta m = -g^2 \sum_{k''} F^2(\omega'')/\omega'', \quad F(\omega) = (2V\omega)^{-1/2} f(\omega), \\ &\varphi(k) = \frac{-gF(\omega)/\omega}{1+g^2 \sum_{k''} (F^2(\omega'')/\omega'')(\omega''-\omega-i\epsilon)^{-1}}, \\ &\varphi(k', k) = \delta_{k'k} + gF(\omega')\varphi(k)(\omega'-\omega-i\epsilon)^{-1}. \end{aligned}\right\} \tag{3·3}$$

Since we are interested in this model only as an illustration, let us confine ourselves to the case where no ghost state appears: $1/g_c^2$ is finite and g is real. Introduce the following matrix notation,

$$\left.\begin{aligned} &\Phi_i = (\Phi_V, \Phi_k), \quad \Psi_i = (\Psi_V, \Psi_k) \quad \text{and} \quad \varphi_{ij} = \begin{pmatrix} N & \varphi(k') \\ gNF(\omega)/\omega & \varphi(k, k') \end{pmatrix}. \\ &\text{Then} \\ &\Psi_i = \sum_j \Phi_j \varphi_{ji}. \end{aligned}\right\} \tag{3·4}$$

Since m is considered to be independent of g, $d\boldsymbol{E}^{in}/dg = 0$. Therefore (2·5) becomes

$$d\Psi_i/dg = iD_g^{in}\Psi_i = -(H-E_i-i\epsilon)^{-1}\mathcal{H}\Psi_i, \tag{3·5}$$

where

$$E_i = (m, m+\omega) \quad \text{and} \quad \mathcal{H} = d'H_1/dg - d\,\Delta H_0/dg. \tag{3·6}$$

By taking the scalar product with Ψ_j, (3·5) becomes

$$\sum_l \varphi_{jl}^{\dagger}\, d\,\varphi_{li}/dg = -(E_i-E_j-i\epsilon)^{-1} \sum_{l,k} \varphi_{il}^{\dagger}\mathcal{H}_{lk}\varphi_{kj}, \tag{3·7}$$

where

$$\left.\begin{aligned} &\mathcal{H}_{ij} = (\Phi_i \mathcal{H} \Phi_j) = \begin{pmatrix} -\delta' m & -F(\omega') \\ -F(\omega) & 0 \end{pmatrix}, \\ &\delta' m = d\delta m/dg. \end{aligned}\right\} \tag{3·8}$$

It is easily shown that P&K's solution (3·3) satisfies (3·7). The Eq. (2·17) for the S-matrix becomes

$$dS_{kk'}/dg = -2\pi i\delta(\omega-\omega')\sum_{k''} S_{kk''}\overline{\mathcal{H}}_{k''k'}, \tag{3·9}$$

where

$$\overline{\mathcal{H}}_{ij} = (\Psi_i \mathcal{H} \Psi_j) = \sum_{l,k} \varphi_{il}^{\dagger} \mathcal{H}_{lk} \varphi_{kj}. \tag{3·10}$$

Decomposing into partial waves, (3·9) becomes

$$d\eta(\omega)/dg = -4\pi^2 \omega k \overline{\mathcal{H}}_{k,k}, \tag{3·11}$$

where $\eta(\omega)$ is the S-wave phase shift. Such formulas are suited for a theoretical discussion of some cases such as the discussion on properties which hold about phase shifts and energy spectrum (see § 5). For a practical evaluation, however, it is convenient to use the tool of the dispersion formula. The dispersion theoretical treatment for Lee's model has been given by Goldberger and Treiman.[16] While, if one uses (3) for the asymptotic condition, the S-matrix of this model is obtained exactly by the dispersion formula, we assume here that of Haag (4), partly for the purpose of illustrating the way how to treat the case of the scattering of a compound particle. Assuming

where

$$\left.\begin{aligned} a^{in}(k) &= \lim_{t\to-\infty} a'(k,t)\, e^{i\omega t}, \\ a'(k,t) &= e^{iHt} a'(k)\, e^{-iHt} \end{aligned}\right\} \tag{3·12}$$

is Haag's almost local operator, $a'(k)$ being unknown and restricted by the normalization and the steady condition:

$$(\Psi_0 a'(k) \Psi_{\theta(k')}) = \delta(\boldsymbol{k}-\boldsymbol{k}'), \quad (\Psi_0 H_{a'}(k) \Psi_\theta) = 0, \tag{3·13}$$

the S-matrix is given by

$$S_{kk'} = \delta_{kk'} + \delta(\omega-\omega') M(\omega), \tag{3·14}$$

where

$$M(\omega) - M_0(\omega) = \text{const}\times \int_0^{-\infty} dt\, e^{i\omega t} (\Psi_N [H_{a'}(k,t),\ H_{a'}(k)] \Psi_N), \tag{3·15}$$

$$M_0(\omega) = \text{const}\times (\Psi_N [H_{a'}(k),\ a'^{\dagger}(k)] \Psi_N) \tag{3·16}$$

is the equal time commutator, which vanishes for $a'(k) = a(k)$ but would not vanish for a compound particle, and

$$H_{a'}(k,t) = (\partial/\partial t + i\omega)\, a'(k,t). \tag{3·17}$$

Following the procedure of Goldberger and Treiman, one gets

$$M(\omega) - M_0(\omega) = (2\pi i G^2 F(\omega)/\omega)[1+\omega G^2 \sum_{k''} (F^2(\omega'')/\omega'')\,(\omega''-\omega-i\epsilon)^{-1}]^{-1}, \tag{3·18}$$

where

$$G^2=|(\Psi_N H_{a'}(k)\Psi_V)/F(\omega)|^2_{\omega=0}. \tag{3·19}$$

$a'(k)$ and, therefore, G is unknown. To determine G, differentiate the both sides of

$$(\Psi_N\{a^{out}(k)-a^{in}(k)\}\Psi_V)=2\pi\delta(\omega)(\Psi_N H_{a'}(k)\Psi_V),$$

then one obtains on account of (3·5) the relation

$$(d/dg)(\Psi_N H_{a'}(k)\Psi_V)]_{\omega=0}=(\Psi_k\mathcal{H}\Psi_V)]_{\omega=0}\equiv\overline{\mathcal{H}}_k]_{\omega=0}. \tag{3·20}$$

Assume that, while $\varphi(k)$ and $\varphi(k', k)$ are unknown, the eigenvalue problem for Ψ_V has been solved. Our task is to determine G from the basic equation (3·5). From (3·7) the following relations between $\varphi(k)$ and $\varphi(k', k)$ follow,

$$\left.\begin{aligned}&\varphi(k)+g\sum_{k''}(F(\omega'')/\omega'')\varphi(k'', k)=0,\\&(1/g+\delta' m/2\omega)\varphi(k)+(1/\omega)\sum_{k''}F(\omega'')\varphi(k'', k)=0.\end{aligned}\right\} \tag{3·21}$$

Substituting these relations into (3·10), we get

$$\overline{\mathcal{H}}_{ij}=\begin{pmatrix}0 & \overline{\mathcal{H}}_{k'}\\ \overline{\mathcal{H}}_{k}^{*} & \overline{\mathcal{H}}_{k,k'}\end{pmatrix}=\begin{pmatrix}0 & (N/g)\omega'\varphi(k')\\ (N/g)\omega\varphi^*(k) & (1/g)\varphi^*(k)(\omega+\omega')\varphi(k')\end{pmatrix}. \tag{3·22}$$

Further (2·19) reads

$$d\overline{\mathcal{H}}_{ij}/dg=\overline{\mathcal{H}}'_{ij}+\sum_l\overline{\mathcal{H}}_{il}\left\{\frac{1}{E_i-E_l-i\epsilon}-\frac{1}{E_l-E_j-i\epsilon}\right\}\overline{\mathcal{H}}_{lj}, \tag{3·23}$$

where

$$\overline{\mathcal{H}}'_{ij}=\begin{pmatrix}-N^2\delta'' m & -N\delta'' m\varphi(k')\\ -N\delta'' m\varphi^*(k) & -\delta'' m\varphi^*(k)\varphi(k')\end{pmatrix}. \tag{3·24}$$

(3·23) reduces to

$$\left.\begin{aligned}&N^2+\sum_{k''}|\varphi(k'')|^2=1,\\&d[(1/g)\varphi(k)]/dg=(2/g^2)\varphi(k)\sum_{k''}\omega''|\varphi(k'')|^2(\omega-\omega''-i\epsilon)^{-1}.\end{aligned}\right\} \tag{3·25}$$

Further, since $\overline{\mathcal{H}}_{ij}]_{g=0}=\mathcal{H}_{ij}$,

$$(1/g)\varphi(k)]_{g=0}=-F(\omega)/\omega. \tag{3·26}$$

From (3·25)

$$d\varphi_1/dg=-(2/g)\varphi_1(1-N^2)=-(2/g)\varphi_1(1/N)\,dN/dg, \tag{3·27}$$

where

$$\varphi_1=(1/g)\varphi(k)]_{\omega=0}. \tag{3·28}$$

The solution of (3·27) under (3·26) is

$$\varphi_1=-N^2F(\omega)/\omega]_{\omega=0}. \tag{3·29}$$

Compared with (3·22),

$$\overline{\mathcal{H}_k}]_{\omega=0}=-N^3F(0)=-F(0)\,d(gN)/dg. \tag{3·30}$$

Therefore from (3·20)

$$(\Psi_N H_{a'}(k)\Psi_V)]_{\omega=0}=-gNF(0), \tag{3·31}$$

and from (3·19)

$$G=gN=g_{re}, \tag{3·32}$$

which coincides with the renormalized coupling constant. The final result is, therefore, identical with Goldberger and Treiman.

§ 4. Bound state threshold of coupling constant

Let us consider the threshold $g=g_c$ of the coupling constant where the bound state appears.* To describe the incoming and the outgoing stationary states, we need the operators (the in- and out-notations are abbreviated) :

$$a_-(k),\ b_-(p) \quad \text{at} \quad g=g_c-\Delta g$$

and

$$a_+(k),\ b_+(p),\ c_+(q) \quad \text{at} \quad g=g_c+\Delta g$$

where Δg is infinitesimal and positive, and

$$[P_\mu, a_\mp(k)]=-k_\mu a_\mp(k),\ \ [P_\mu, b_\mp(p)]=-p_\mu b_\mp(p). \tag{4·1}$$

It seems contradictory that, while the sets of operators $(a_-(k), b_-(p))$ and $(a_+(k), b_+(p))$ satisfy the same commutation relations as well as Eq. (4·1), the former forms a complete set and the latter does not. This apparent contradiction will, however, be settled by the following consideration.

Let $P^j_{E_1E_2}$ be the projection operator such that

$$P^j_{E_1E_2}\Psi_{Ej'}=\begin{cases}\Psi_{Ej} & \text{for } j=j',\ E_1\leqq E<E_2\\ 0 & \text{for any other case,}\end{cases} \tag{4·2}**$$

where j is a set of eigenvalues of a mutually commuting set of constants of motion which are independent on g, and Ψ_{Ej} is any eigenstate belonging to j and to the energy E. $P^j_{E_1E_2}$ is a quantity determined only by the total Hamiltonian H, the energy values E_1 and E_2, and the symmetries corresponding to the operators which give the eigenvalues j. As H is assumed to be continuous with respect to g, $P^j_{E_1E_2}$ will also be continuous with respect to g, inasmuch as E_1 and E_2 do not coincide with the energies of the discrete levels. So we set the following assump-

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* Suppose that we are considering with the interacting meson and nucleon fields, and that there is no compound particle for $g<g_c$ and there is no compound particle except the deuteron, which is described by $c(\mathrm{q})$, for $g\geqq g_c$.

** Consider in the centre of mass system (0, 0, 0, i).

tion which seems reasonable on account of the assumptions II) & III) :

Assumption IV) *We assume that $P^{j}_{E_1E_2}$ is continuous with respect to g, in so far as E_1 and E_2 do not coincide with the energies of the discrete levels.*

If j equals the eigenvalue to which the deuteron belongs, and $E_1<2m$, $E_2=2m+\epsilon$ $(\epsilon>0)$, then $P^{j}_{E_1E_2}$ is continuous on g in the neighbourhood of g_c by the assumption, and in the limit $\epsilon\to 0$ it follows

$$c^{\dagger}(\boldsymbol{q})\,\Psi_0]_{q=0,\,g=g_c-\Delta g}=(b^{\dagger}(\boldsymbol{p})\,b^{\dagger}(\boldsymbol{p}))^{j}\,\Psi_0]_{p=0,\,g=g_c+\Delta g}\,, \tag{4·3}$$

where $(\quad)^{j}$ is some suitable combination which gives the eigenvalue j. Further, as will be made clear in § 5* the continuity of $P^{j}_{E_1E_2}$ for E_1, $E_2>2m$ leads to the continuity of the phase shifts as functions of g. Therefore there is no physical sense in the discrimination between $(a_{+}^{in,out}(\boldsymbol{k}),\ b_{+}^{in,out}(\boldsymbol{p}))$ from $(a_{-}^{in,out}(\boldsymbol{k}),\ b_{-}^{in,out}(\boldsymbol{p}))$. Hence we can put

$$a_{+}^{in,out}(\boldsymbol{k})=a_{-}^{in,out}(\boldsymbol{k}),\quad b_{+}^{in,out}(\boldsymbol{p})=b_{-}^{in,out}(\boldsymbol{p}). \tag{4·4}$$

Or $a^{in,out}(\boldsymbol{k})$ and $b^{in,out}(\boldsymbol{p})$ are continuous with respect to g. In order that this conclusion is consistent with completeness, we should understand the situation in such a way that the energy region belonging to the continuous levels at $g=g_c-\Delta g$ is different from that at $g=g_c+\Delta g$, which are restricted as follows:

The energy region for continuous levels belonging to j is

$$\begin{cases} E\geqq 2m & \text{at}\quad g=g_c-\Delta g, \\ E\geqq 2m+\epsilon & \text{at}\quad g=g_c+\Delta g. \end{cases} \tag{4·5}$$

Then the sets of operators $(a(\boldsymbol{k}),\ b(\boldsymbol{p}))_{-}$ and $(a(\boldsymbol{k}),\ b(\boldsymbol{p}),\ c(\boldsymbol{q}))_{+}$ coincide if we take, in addition to (4·5),

$$c(\boldsymbol{q})=(b(\boldsymbol{q}/2)\,b(\boldsymbol{q}/2))^{j}\quad\text{at}\quad g=g_c, \tag{4·6}**$$

which is consistent with (4·3).

The condition (4·5) is no restriction since the scattering process where the two of the incoming or outgoing particles have exactly the same momenta has a meaning as a limit only.

§ 5. Energy spectrum and phase shifts

As has been illustrated in (3·11), the matrix element of the interaction Hamiltonian on an energy shell is simply related to the phase shift. It represents the fact that the phase shift is proportional to the (infinitesimal) energy shift of a level in a continuous energy region. While this energy shift itself is infinitesimal, the density of the number of levels is infinite; so it leads to a finite modula-

* See (5.25), (5.27), (5.31) and (5.24).

** A similar relation is assumed in ref. 14).

tion in the level density. One can thus find a general connection between energy spectrum and phase shifts such as Levinson's theorem.

Consider the simultaneous eigenstates of H and S,

$$H\Psi_{nE}=E\Psi_{nE},\quad S\Psi_{nE}=e^{2i\eta_{nE}}\Psi_{nE}, \tag{5·1}$$

where

$$\Psi^{in}=S\Psi^{out},\quad SS^{\dagger}=S^{\dagger}S=1. \tag{5·2}$$

For a discrete level,

$$S\Psi=\Psi, \tag{5·3}$$

and it is clear that

$$d\Psi/dg=iD_g^{(1)}\Psi,\quad \bar{D}_g\Psi=0, \tag{5·4}$$

where

$$D_g^{(1)}=(D_g^{out}+D_g^{in})/2,\quad \bar{D}_g=D_g^{out}-D_g^{in}. \tag{5·5}$$

Ψ_{nE} can be expanded either by the in-states or by the out-states on the same energy shell which shall be denoted by $\Psi_{IE}^{in,out}$. It can be easily seen that they have the following form,

$$\Psi_{nE}=\sum_I \Psi_{IE}^{in}u_{In}e^{-i\eta_{nE}}=\sum_I \Psi_{IE}^{out}u_{In}e^{i\eta_{nE}}, \tag{5·6}$$

where u_{In} are the coefficients.

We first take into consideration the case in which the degeneration* of the S-matrix can be dissolved completely in terms of the constants of motion corresponding to certain symmetries, such as the magnitude and a component of the total momentum, charge number, particle number, etc. The energy region, where only the two body scattering is possible, belongs to this case. Then u_{In} are represented by the eigenfunctions for this constants of motion such as the spherical function $Y_l^m(\theta,\varphi)$, and are independent of g. Therefore differentiating (5·6) we get

$$d\Psi_{nE}/dg=iD_g^{in}\Psi_{nE}-i(d\eta_{nE}/dg)\Psi_{nE}=iD_g^{out}\Psi_{nE}+i(d\eta_{nE}/dg)\Psi_{nE}.$$

Averaging over these two expressions, one obtains

$$d\Psi_{nE}/dg=iD_g^{(1)}\Psi_{nE}. \tag{5·7}$$

Further starting from

$$d(\Psi_{Ik}^{out},\ \Psi_{IIk'}^{in})/dg=-i(\Psi_{Ik}^{out}\bar{D}_g\Psi_{IIk'}^{in})$$

one easily obtains the relation

$$d\eta_{nE}/dg=-[2k(E)\delta(0)]^{-1}(\Psi_{nE}\bar{D}_g\Psi_{nE}), \tag{5·8}$$

---

* We call here the S-matrix to have a degeneracy when the eigenvalue problem for it does not yet completely solved.

where we have normalized the state vectors by

$$(\Psi^{in}_{IE},\ \Psi^{in}_{II\,E'}) = k(E)\,\delta(E-E')\,\delta_{I\,II}, \tag{5·9}$$

I, II and $\delta_{I\,II}$ being independent of g.

Let us now turn to the general case for which the symmetries are not sufficient to resolve the degeneracy of S. Let j and P_j be the eigenvalue and the corresponding projection operator respectively for a commuting set of constants of motion which correspond to certain symmetries and are independent of g. Then the following relations hold:

$$d\,P_j/dg=0, \tag{5·10}$$

$$[H,\ P_j]=0, \tag{5·11}$$

$$[D_g^{in,out},\ P_j]=[D_g^{(1)},\ P_j]=[\bar{D}_g,\ P_j]=0, \tag{5·12}$$

and

$$[P_j,\ P_E]=0, \tag{5·13}$$

where P_E is the projection operator for the energy shell E. Since

$$P_E=\sum_I \Psi^{in}_{IE}\Psi^{in}_{IE}=\sum_I \Psi^{out}_{IE}\Psi^{out}_{IE}, \tag{5·14}$$

we have

$$\left.\begin{aligned} d\,P_E/dg&=i[D_g^{in},\ P_E]=i[D_g^{out},\ P_E],\\ \text{and hence}\qquad dP_E/dg&=i[D_g^{(1)},\ P_E],\end{aligned}\right\} \tag{5·15}$$

as well as

$$[\bar{D}_g,\ P_E]=0. \tag{5·16}$$

We find, as a generalization of (5·8), (see Appendix V)

$$\sum_a d\eta_{jaE}/dg=-[2\delta(0)]^{-1}\mathrm{Sp}[P_E P_j \bar{D}_g], \tag{5·17}$$

where we have put $n=(j,\ a)$ and have normalized the state vectors by

$$(\Psi^{in}_{IE},\ \Psi^{in}_{II\,E'})=\delta(E-E')\,\delta_{I\,II}, \tag{5·18}$$

I being dependent on g.

Now, while among the suffices $(n\,;\,E)=(j,\,a\,;\,E)$, $a=a(E)$ may depend on E in general, one can still give a certain one to one correspondence between Ψ_{jaE} and $\Psi_{jaE'}$ when E and E' are contained in a certain finite energy region. The energy region limited by two neighbouring energy thresholds, in which the possible channels for scattering is fixed, will be such one. We can then take the variables $a(E)$ in such a way that the two states, on energies E and E' respectively, are specified by the same values of $a(E)$ and $a(E')$ in the correspondence. Putting

$$Q_{j\,;\,E,E'}=\sum_a \Psi_{jaE'}\Psi_{jaE}, \tag{5·19}$$

and

$$\left.\begin{aligned} D^{in}_{j;E,E'} &= \mathrm{Sp}[Q_{j;E,E'} D^{in}_{g}], \\ D^{out}_{j;E,E'} &= \mathrm{Sp}[Q_{j;E,E'} D^{out}_{g}], \\ D^{(1)}_{j;E,E'} &= \mathrm{Sp}[Q_{j;E,E'} D^{(1)}_{g}], \\ \bar{D}_{j;E,E'} &= \mathrm{Sp}[Q_{j;E,E'} \bar{D}_{g}], \end{aligned}\right\} \tag{5·20}$$

one obtains (see AV·15)

$$\left.\begin{aligned} D^{in}_{j;E,E'} &= i(E-E'-i\epsilon)^{-1}\rho_{j;E,E'}, \\ D^{out}_{j;E,E'} &= i(E-E'+i\epsilon)^{-1}\rho_{j;E,E'}, \\ \bar{D}_{j;E,E'} &= 2\pi\delta(E-E')\rho_{j;E,E'}, \\ D^{(1)}_{j;E,E'} &= iP(E-E')^{-1}\rho_{j;E,E'}. \end{aligned}\right\} \tag{5·21}$$

As, on the energy shell $E'=E$,

$$Q_{j;E,E} = P_E P_j, \tag{5·22}$$

it follows

$$\rho_{j;E,E} = [2\pi\delta(0)]^{-1}\mathrm{Sp}[P_E P_j \bar{D}_g]. \tag{5·23}$$

Comparing with (5·8), one has

$$\sum_a d\eta_{jaE}/dg = -\pi\rho_{j;E,E}. \tag{5·24}$$

Since, as to be seen from (AV·15), the right-hand side represents the total energy variation in which the self energy part is subtracted, (5·8) or (5·24) reflects the fact that the phase shift is proportional to the energy shift.

The projection operator introduced in § 4 is expressed explicitly by

$$\begin{aligned} P^j_{E,E'} &= \int_E^{E'} dE'' \sum_a \Psi_{E''ja}\Psi_{E''ja} \\ &= \int_E^{E'} dE'' \sum_I P_j \Psi^{in}_{E''I}\Psi^{in}_{E''I} = \int_E^{E'} dE'' \sum_I P_j \Psi^{out}_{E''I}\Psi^{out}_{E''I}. \end{aligned}$$

Differentiation by g gives

$$d\,P^j_{E,E'}/dg = -i[P^j_{E,E'},\ D^{in}_g] = -i[P^j_{E,E'},\ D^{out}_g]. \tag{5·25}$$

Averaging

$$= -i[P^j_{E,E'},\ D^{(1)}_g], \tag{5·25′}$$

as well as

$$[P^j_{E,E'},\ \bar{D}_g] = 0. \tag{5·26}$$

Now

$$N^j_{E,E'} = \mathrm{Sp}[P^j_{E,E'}] \tag{5·27}$$

represents the number of levels belonging to the energy region $E\sim E'$ and to the eigenvalue j. $N^j_{E,E'}$ itself is infinite. But the difference

$$N^j_{E,E'}-N^{0j}_{E,E'}$$

will be finite, where

$$N^{0j}_{E,E'}=N^j_{E,E'}]_{g=0}. \tag{5·28}$$

In the following we confine ourselves to the case in which the assumption III′) is valid. Then we have the different complete sets of eigenstates corresponding to different g values, and therefore the spur of an operator A can be represented in various different ways:

$$\mathrm{Sp}[A]=\mathrm{Sp}_g[A]\equiv\sum_\alpha(\Psi_\alpha A\Psi_\alpha)+\int dE\sum_\alpha(\Psi_{nE}A\Psi_{nE}), \tag{5·29}$$

where the right-hand side is apparently dependent on g through Ψ's. Ψ_α are the discrete levels. But by the invariance of spur, $\mathrm{Sp}_g[A]$ is really independent of g except the dependence of A itself on g. Thus

$$d\,\mathrm{Sp}[A]/dg=d\,\mathrm{Sp}_g[A]/dg=\mathrm{Sp}_g[dA/dg]. \tag{5·30}$$

Differentiating $N^j_{E,E'}$ by g, one has from (5·27), (5·30) and (5·25′),

$$\begin{aligned}
dN^j_{E,E'}/dg&=-i\,\mathrm{Sp}[[P^j_{E,E'},\ D^{(1)}_g]]\\
&=-i\int_E^{E'}dE''\sum_n\int_E^{E'}dE'''\sum_{n'}\{(\Psi_{nE''}P^j_{E,E'}\Psi_{n'E'''})(\Psi_{n'E'''}D^{(1)}_g\Psi_{nE''})\\
&\qquad-(\Psi_{nE''}D^{(1)}_g\Psi_{n'E'''})(\Psi_{n'E'''}P^j_{E,E'}\Psi_{nE''})\}\\
&=-i\int_E^{E'}dE''\int_E^{E'}dE'''\sum_\alpha\{\delta(E''-E''')(\Psi_{jaE'''}D^{(1)}_g\Psi_{jaE''})\\
&\qquad-(\Psi_{jaE''}D^{(1)}_g\Psi_{jaE'''})\delta(E'''-E'')\}.
\end{aligned}$$

From (5·19) and (5·20), when E, E' belong to the continuous region,

$$=\int_E^{E'}dE''\int_E^{E'}dE'''\left\{\delta(E''-E''')\frac{P}{E'''-E''}\rho_{j;E''',E''}-\frac{P}{E''-E'''}\rho_{j;E'',E'''}\delta(E'''-E'')\right\}.$$

Making use of $\delta(x)P(1/x)=-\delta'(x)$

$$\begin{aligned}
dN^j_{E,E'}/dg&=-\int_E^{E'}dE''\{(\partial/\partial E'')\rho^j_{;E''',E''}+(\partial/\partial E'')\rho_{j;E'',E'''}\}_{E'''=E''}\\
&=-\int_E^{E'}dE''\,\partial\rho_{j;E'',E''}/\partial E''.
\end{aligned} \tag{5·31}*$$

* For the meaning of $\partial\eta/\partial E$ in the time-dependent standpoint, see ref. 17).

Therefore one finally obtains

$$dN^{j}_{E,E'}/dg = -\{\rho_{j;E',E'} - \rho_{j;E,E}\} - \sum_{i} \Delta\rho_{j;E_i,E_i}, \tag{5·32}$$

where $\Delta\rho_{j;E_i,E_i}$ is the discontinuity of $\rho_{j;E,E}$ at $E=E_i$, $E<E_i<E'$, occurring possibly at an energy threshold, etc. From (5·24), the above relation can also be written as

$$dN^{j}_{E,E'}/dg = \pi^{-1}\sum_{a}\{d\eta_{jaE'}/dg - d\eta_{jaE}/dg\} + \pi^{-1}\sum_{i}\sum_{a}\Delta(d\eta_{jaE_i}/dg). \tag{5·33}$$

Integrating on g,

$$N^{j}_{E,E'} - N^{0j}_{E,E'} = \pi^{-1}\sum_{a}\{\eta_{jaE'} - \eta_{jaE}\} + \pi^{-1}\sum_{i}\sum_{a}\Delta\eta_{ja}(i), \tag{5·34}$$

since $\eta=0$ at $g=0$ and there is no discontinuity in the g-dependence of η, as can be seen from (5·27) and (5·31) and the statement of § 4 about the continuity of $P^{j}_{E,E'}$.

From (5·34) we can infer the behaviour of the phase shift at the energy where resonance occurs. If the resonance occurs at $E=E_r$, then

$$N^{j}_{E_r-\epsilon,E_r+\epsilon} - N^{0j}_{E_r-\epsilon,E_r+\epsilon} = 1,$$

where ϵ is infinitesimal and positive. Therefore

$$\eta_{jaE_r+\epsilon} - \eta_{jaE_r-\epsilon} = \pi$$

from (5·34). Namely there is the jump of the phase shift by π at the resonance energy.

Also we can say that there is a virtual level at $E\approx E_v$ when there is a steep change in η_{jaE} with respect to E in the neighbourhood of E_v.

Let us take next the whole energy region. Putting

$$N^{j} = \mathrm{Sp}[P_j], \tag{5·35}$$

one has

$$dN^{j}/dg = \mathrm{Sp}[dP_j/dg] = 0. \tag{5·36}$$

Making use of (5·29),

$$dN^{j}/dg = (d/dg)\sum_{\alpha}(\Psi_\alpha\Psi_\alpha) + dN^{j}_{0,\infty}/dg = 0.$$

Taking into account the result (5·24) for the continuous region, one gets after the integration on g the result

$$m^{j} - m_0^{j} + \pi^{-1}\sum_{a}\{\eta_{ja}(\infty) - \eta_{ja}(0) + \sum_{i}\Delta\eta_{ja}(i)\} = 0, \tag{5·37}$$

where m^{j}, m_0^{j} are the numbers of the discrete levels belonging to j at g and at $g=0$ respectively.

If the inelastic processes are neglected, then only the two-body scattering is to be taken into account and (5·37) becomes

$$m^{j} - m_0^{j} + \pi^{-1}\{\eta_j(\infty) - \eta_j(0)\} = 0. \tag{5·38}$$

This is nothing but Levinson's theorem.

(5·36) is also written as

$$m^j - m_0^j + N_c^j - N_c^{0j} = 0, \tag{5·39}$$

where N_c^j is the total number of levels over the whole continuous energy region. We can see from this relation that, when a new bound state occurs, the total number of the continuous levels decreases by one.

The latter situation can be seen in more detail by investigating the neighbourhood of the energy threshold at the $g=g_c$. The projection operator

$$P^j_{2m-\epsilon,2m+\epsilon}$$

may be considered as continuous on g as stated in § 3, if we assume that the energy region permitted for the continuous level are different above and below the $g=g_c$, viz.,

$$E \geqq 2m \text{ at } g=g_c-\Delta g, \quad E \geqq 2m+\epsilon' \text{ at } g=g_c+\Delta g,$$

where ϵ' is small but finite. Therefore, taking $\epsilon < \epsilon'$,

$$N^j_{2m-\epsilon,2m+\epsilon} - N^{0j}_{2m-\epsilon,2m+\epsilon} = \pi^{-1}\{\eta_{j,2m+\epsilon} - \eta_{j,2m-\epsilon}\} \text{ at } g=g_c-\Delta g$$

and

$$=1 \qquad \text{at } g=g_c+\Delta g,$$

hence for the phase shift at $g=g_c-\Delta g$

$$\eta_{j,2m+\epsilon} - \eta_{j,2m} = \pi. \tag{5·40}$$

That is, when the coupling constant g is varied, the gradient of the phase shift at the neighbourhood of the energy threshold becomes steeper as g approaches g_c. The gradient reaches the discontinuity of π when $g=g_c$, and then the new bound state appears. The discontinuity produces an additional difference between $\eta(\infty)$ and $\eta(0)$ by π (see Figure 1).

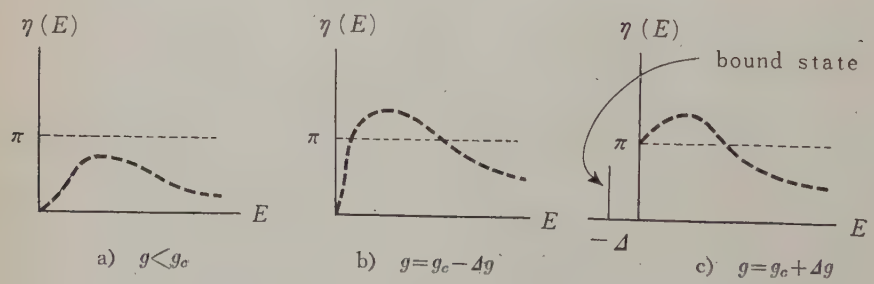

Fig. 1. Behaviour of the phase shift when a bound state appears.

§ 6. Discussions

The present scheme has the following features:

i) Scattering problem can be formulated whenever the total Hamiltonian is given, if it is assumed that the fields represent an assembly of particles at $t \sim \mp\infty$.

ii) It satisfies the usual in- (out-) boundary condition (2·1), when it is applied to (classical) quantum mechanics and also when the interaction is infinitesimally small.

iii) Scattering problem between compound particles can be formulated without ambiguity.

iv) It contains in principle a prescription to obtain all information from a given Hamiltonian.

In our scheme the scattering problem can be formulated whenever the algebraic property of basic variables (such as the commutation relations between local field quantities) and the form of the total Hamiltonian as a function of these variables are given, so that it does not depend on details of the given theory such as the canonical formalism, the local character, etc. Therefore the present scheme has a wider range of applicability than those so far.

We assumed that the theory has a parameter g and the solutions at any value of g have a certain meaning. However, the real world corresponds to the solutions at the observed value g_{obs}. Also the field theories at which we are conventionally working might be only mathematical artifices. Nevertheless, it may be hoped that some features of the present formalism can be maintained in a future theory (if, for instance, a future theory is given in a scheme of infinitesimal variations), since the operators $D_g^{in,out}$ have simple transformation properties and are quantities not of a local but of an overall character and the basic equations are independent of the details of the Hamiltonian formalism for local quantities.

Haag's almost local operator field $\phi'(x)$ for a compound particle will in some ways correspond to the structure of that particle. For example, the minimum value a_{min} of a:

$$[\phi'(\boldsymbol{x}, t), \phi'(\boldsymbol{x}', t)] \fallingdotseq 0 \quad \text{for} \quad |\boldsymbol{x}-\boldsymbol{x}'| < a,$$

will correspond to the radius of the extension of the particle. An elementary particle which has a structure, such as the nucleons which have the anomalous magnetic moment, has no essential difference from compound particle. To formulate a scattering between these particles, one must subtract the self fields, which give rise to the structure, from the interaction Hamiltonian. Though the mass renormalization technique subtracts the influence of the self field on the mass, it does not take the other effect into consideration, as is understood by observing that the Bethe-Salpeter wave function $(\Psi_0\psi(x)\Psi_p)$ for one nucleon under an external electromagnetic field satisfies the Dirac equation for a point particle and does not contain the Pauli term. It is related to the fact that in the usual treatments of the dispersion formula the microcausality condition is valid. If one is based on $\phi'(x)$, the effect of the structure will be revealed, for example, by the equal time commutator in the dispersion formula (such as $M_0(\omega)$ in (3·15)).

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Appendix I

Differentiation of various quantities

Consider an arbitrary Heisenberg operator which does not explicitly contain g:

$$O(t)=e^{iHt}Oe^{-iHt}. \tag{AI·1}$$

Although H is time independent, we introduce the following ordering for the exponential factor,

$$e^{iHt}=P_{0t}\exp\{i\int_0^t d\tau\, H(\tau)\}, \tag{AI·2}$$

where $H(\tau)=H$ and P_{0t} means the ordering that $H(\tau)$ nearer to t be put to the right in the products of series expansion of (AI·2), hence

$$d\,e^{iHt}/dg=\int_0^t d\tau\, P_{0t}\Big[i(dH(\tau)/dg)\exp\{i\int_0^t d\tau' H(\tau')\}\Big]$$

$$=i\int_0^t d\tau\, e^{iH\tau}\mathcal{K}\,e^{iH(t-\tau)}=i\int_0^t d\tau\,\mathcal{K}(\tau)\,e^{iHt}. \tag{AI·3}$$

Therefore we immediately get

$$d\,O(t)/dg=i[\int_0^t d\tau\,\mathcal{K}(\tau),\,O(t)], \tag{AI·4}$$

for Heisenberg operator $O(t)$ which does not explicitly depend on g.

For a discrete level

$$H\Psi=E\Psi, \tag{AI·5}$$

it is easily found that

$$[H,\,(1/2)\int_{-\infty}^{\infty} d\tau\,\epsilon(\tau)\mathcal{K}(\tau)]\Psi=i(\mathcal{K}-dE/dg)\Psi, \tag{AI·6}$$

as well as

$$d\,\Psi/dg=(i/2)\int_{-\infty}^{\infty} d\tau\,\epsilon(\tau)\,\mathcal{K}(\tau)\Psi, \tag{AI·7}$$

where

$$\epsilon(\tau)=\begin{cases}+1 \text{ for } \tau>0,\\ -1 \text{ for } \tau<0.\end{cases} \tag{AI·8}$$

For a Green's function we get from (AI·4) and (AI·7)

$$d\langle T[O(1)O(2)]\rangle_0/dg=-i\int_{-\infty}^{\infty} d\tau\langle T[\{\mathcal{K}(\tau)-dE_0/dg\}O(1)O(2)]\rangle_0, \tag{AI·9}$$

where $\langle \quad \rangle_0$ is the vacuum expectation value, E_0 is the vacuum energy and T is the time-ordering operator, and $O(1)$ and $O(2)$ are arbitrary Heisenberg operators at time t_1 and t_2 respectively. Analogous formulas hold for the expectation values of Heisenberg operators with respect to a discrete level Ψ:

$$d(\Psi T[O(1)O(2)]\Psi)/dg=-i\int_{-\infty}^{\infty}d\tau\,(\Psi T[\{\mathcal{H}(\tau)-dE/dg\}O(1)O(2)]\Psi). \tag{AI·10}$$

Let us next consider the transformation function of the interaction representation. The operator of the interaction representation is given by

$$\boldsymbol{O}(t)=e^{iH_0t}Oe^{-iH_0t}. \tag{AI·11}$$

It is clear that

$$d\boldsymbol{O}(t)/dg=0, \tag{AI·12}$$

for the operators not explicitly dependent on g, and that

$$d\boldsymbol{O}(t)/dg=e^{iH_0t}(dO/dg)\,e^{-iH_0t}, \tag{AI·13}$$

for the ones explicitly dependent on g.

Now for the transformation operator

$$U(t,t')=P_{tt'}\exp\{-i\int_{t'}^{t}\boldsymbol{H}_1(\tau)\,d\tau\}, \tag{AI·14}$$

which satisfies

$$O(t)=U^{\dagger}(t,0)\boldsymbol{O}(t)\,U(t,0), \tag{AI·15}$$

the differentiation gives

$$\begin{aligned} d\,U(t,t')/dg&=-i\int_{t'}^{t}d\tau\,P_{tt'}[\mathcal{H}^{int}(\tau)\,U(t,t')] \\ &=-i\int_{t'}^{t}d\tau\,U(t,\tau)\mathcal{H}^{int}(\tau)\,U(\tau,t')=-iU(t,0)\int_{t'}^{t}d\tau\,\mathcal{H}(\tau)\,U(0,t'). \end{aligned} \tag{AI·16}$$

Especially for t or $t'=0$,

$$\left.\begin{aligned} d\,U(0,t)/dg&=i\int_{0}^{t}d\tau\,\mathcal{H}(\tau)\,U(0,t), \\ d\,U(t,0)/dg&=-iU(t,0)\int_{0}^{t}d\tau\,\mathcal{H}(\tau), \end{aligned}\right\} \tag{AI·17}$$

which follow also from (AI·4) and (AI·15).

Appendix II

Connection with Feynman-Dyson theory

The Yang-Feldman type adiabatic assumption is written as

$$\varphi(x) \to \begin{cases} \varphi^{in}(x) & \text{for } t \to -\infty, \\ \varphi^{out}(x) & \text{for } t \to +\infty, \text{ etc.,} \end{cases} \tag{AII·1}$$

where $\varphi^{in(out)}(x)$, etc., satisfy the free field equation. The infinitesimal generating operator for $\varphi^{in(out)}(x)$ with respect to g-variation is immediately obtained by comparing the above condition with (AI·4), yielding

$$d\varphi^{in}(x)/dg = i[\int_0^{-\infty} d\tau \mathcal{H}(\tau), \varphi^{in}(x)], \quad d\varphi^{out}(x)/dg = i[\int_0^{\infty} d\tau \mathcal{H}(\tau), \varphi^{out}(x)], \tag{AII·2}$$

under an assumption on the convergence of the integration with respect to τ. The result (AII·2) is equivalent to choosing

$$D_g^{in(out)} = \int_0^{-\infty(+\infty)} d\tau \mathcal{H}(\tau),$$

except an additive term which commutes with H. The above form for $D_g^{in(out)}$, however, satisfies the following equation as can be seen at once,

$$-i[H, D_g^{in(out)}] = \mathcal{H} - \lim_{\tau \to -\infty(+\infty)} \mathcal{H}(\tau).$$

This form does not coincide with (2·2) when one assumes the usual adiabatic assumption for the interaction Hamiltonian,

$$\mathcal{H}(\tau) \to 0 \quad \text{for} \quad \tau \to \mp\infty,$$

but coincides, if one assumes the following condition, instead:

$$\mathcal{H}(\tau) \to d\boldsymbol{E}^{in(out)}/dg \quad \text{for} \quad \tau \to -\infty(+\infty). \tag{AII·3}$$

Now, assuming no bound states, let us consider the correspondence with the usual Feynman-Dyson theory in which the S-matrix is given in the interaction representation with real masses. Yet, if we can regard the observed mass to be dependent on g, we cannot apply Eq. (AI·17) to the interaction representation with real mass, because the transformation functions then have very complicated behaviours on g through μ and m. To avoid this difficulty, let us assume that the bare masses μ_0 and m_0 are dependent on g, whereas the real masses μ and m are independent of g. Thus,

where

$$\left.\begin{aligned} H &= H_0 + H_1 = H_0 + \Delta H_0 + H_1 - \Delta H_0 \equiv H_0' + H_1', \\ H_0' &= H_0 + \Delta H_0 = H_0]_{\mu_0 \to \mu,\, m_0 \to m} \end{aligned}\right\} \tag{AII·4}$$

is independent of g. Also, since μ and m, and therefore $E=\sum_i k_0^{(i)}$, are independent of g,

$$d\,\boldsymbol{E}^{in,out}/dg=0. \tag{AII·5}$$

Thus, our fundamental equation (2·2) becomes

$$i[H, D_g^{in,out}]=\mathcal{H}'\equiv d\,H_1'/dg=\mathcal{H}-d(\Delta H_0)/dg. \tag{AII·6}$$

Introducing the adiabatic assumption

$$\mathcal{H}'(\tau)=\mathcal{H}(\tau)-(d\Delta H_0/dg)(\tau)\to 0 \quad \text{for} \quad |\tau|\to 0, \tag{AII·7}$$

we get

$$D_g^{in,out}=\int_0^{\mp\infty} d\tau\,\mathcal{H}'(\tau),$$

$$U_g^{in,out}=U^{\dagger}(\mp\infty)=P(P^{-1})\exp\{i\int_0^{\mp\infty} d\tau\,\boldsymbol{H}_1'(\tau)\}, \tag{AII·8}$$

from (AI·17), as well as

$$S_{I\,II}=(\Phi_I'\,U(\infty,\,-\infty)\,\Phi_{II}'), \tag{AII·9}$$

where $\boldsymbol{H}_1'(\tau)$ is the interaction Hamiltonian in the interaction representation with real mass ($\boldsymbol{H}_1'(\tau)]_{\tau=0}=H_1'=H_1-\Delta H_0$ in which ΔH_0 is the mass counter term) and Φ_I' is the free particle state with real masses.

For quantum mechanics, (AII·7) and consequently (AII·9) hold rigorously; so that the requirement (A) of § 2 is really satisfied.*

The differential equation for $a^{in}(k)$ now becomes

$$d\,a^{in}(k)/dg=i[\int_0^{-\infty} d\tau\,\mathcal{H}(\tau),\, a^{in}(k)].$$

Comparing this with (AII·2), we see that $a^{in}(k)$ coincides with that given by Yang-Feldman.

Appendix III

In this Appendix it will be shown that the solution $D_g^{in,out}$ of Eq. (2·2) is uniquely determined under such additive requirements that $D_g^{in(out)}$ should be a function of H, $\mathcal{H}$ and $d\boldsymbol{E}^{in(out)}/dg$ only, that the steady condition (2·4) must be satisfied, and that D_g^{in} and D_g^{out} are transformed to each other under the time reversal.

Time reversal invariance[10)–12)] means that there is a unitary operator R such that

* Since the time-independent formalism is equivalent to that of Feynman-Dyson.[18)]

$$RH^*R^\dagger=H, \tag{AIII·1}$$

and also

$$R\Psi_I^{in*}=\Psi_{-I}^{out} \tag{AIII·2}$$

represents the time reversed state of Ψ_I^{in}, where $-I$ denotes the time reversed value of I (For example, momentum $k\to -k$, angular momentum $j\to -j$, charge $e\to e$, etc.).

We assume further that H_0 and $\mathcal{H}$ are separately invariant:

$$RH_0^*R^\dagger=H_0,\quad R\mathcal{H}^*R^\dagger=\mathcal{H}, \tag{AIII·3}$$

and that R is independent of g:

$$dR/dg=0. \tag{AIII·4}$$

Since

$$E_{-I}=E_I, \tag{AIII·5}$$

it is clear that

$$R(d\boldsymbol{E}^{in*}/dg)R^\dagger=d\boldsymbol{E}^{out}/dg. \tag{AIII·6}$$

Operating R on the complex conjugates of

$$d\Psi_I^{in}/dg=iD_g^{in}\Psi_I^{in},$$

one has

$$d\Psi_{-I}^{out}/dg=-iRD_g^{in*}R^\dagger\Psi_{-I}^{out}.$$

Therefore

$$RD_g^{in*}R^\dagger=-D_g^{out}. \tag{AIII·7}$$

Now by the assumption,

$$D_g^{in}=f\{H, \mathcal{H}, dE^{in}/dg\}. \tag{AIII·8}$$

From the hermiticity for D_g^{in},

$$f^\dagger=f, \tag{AIII·9}$$

where $f^\dagger$ means the function f in which the coefficients are transformed into their complex conjugates and the orders of the arguments are reversed. From (AIII·1), (AIII·3) and (AIII·6)

$$RD_g^{in*}R^\dagger=f^*\{H, \mathcal{H}, d\boldsymbol{E}^{out}/dg\},$$

where f^* means the function f in which the coefficients are transformed into their complex conjugates, hence

$$D_g^{out}=-f^*\{H, \mathcal{H}, d\boldsymbol{E}^{out}/dg\}. \tag{AIII·10}$$

From (AIII·9)

$$D_g^{out}=-\tilde{f}\{H, \mathcal{H}, d\boldsymbol{E}^{out}/dg\}, \tag{AIII·11}$$

where $\tilde{f}$ means the function f in which the orders of the arguments are reversed. The steady condition becomes

$$\{D_g^{in}-D_g^{out}\}\,\Psi=[f\{H,\,\mathcal{H},\,d\boldsymbol{E}^{in}/dg\}+\tilde{f}\{H,\,\mathcal{H},\,d\boldsymbol{E}^{out}/dg\}]\Psi=0, \qquad \text{(AIII·12)}$$

where Ψ is a discrete level.

The additive indeterminacy in the solution of (2·2) is a quantity which commutes with H and is by our assumption an arbitrary function of H and $d\boldsymbol{E}^{in}/dg$. Let it be

$$G_g^{in}=h\{H,\,d\boldsymbol{E}^{in}/dg\}, \qquad \text{(AIII·13)}$$

which satisfies similarly to the above,

$$h^{\dagger}=h,\quad G_g^{out}=-h^{*}\{H,\,d\boldsymbol{E}^{out}/dg\}=-\tilde{h}\{H,\,d\boldsymbol{E}^{out}/dg\}.$$

As H and $d\boldsymbol{E}^{in(out)}/dg$ commute, $\tilde{h}=h$ and therefore

$$G_g^{out}=-h\{H,\,d\boldsymbol{E}^{out}/dg\}. \qquad \text{(AIII·14)}$$

Therefore the steady condition reads

$$[h\{H,\,d\boldsymbol{E}^{in}/dg\}+h\{H,\,d\boldsymbol{E}^{out}/dg\}]\Psi=0,$$

or

$$h\{E_\alpha,\,dE_\alpha/dg\}=0. \qquad \text{(AIII·15)}$$

By our assumption f and h are not explicitly dependent on g. Therefore (AIII·15) represents, if it were not identically zero, an identity between E_α and dE_α/dg. This is a contradiction because this gives the same energy eigenvalue $E_\alpha=E_\alpha(g)$ for various discrete levels when this differential identity is solved. Therefore h must identically be zero:

$$h=0. \qquad \text{(AIII·16)}$$

Thus the theorem has been proved.

Next consider the matrix element of (2·2), then

$$\left.\begin{aligned}-i(E_I-E_{II})\,(\Psi_I^{in}D_g^{in}\Psi_{II}^{in})&=(\Psi_I^{in}(\mathcal{H}-d\boldsymbol{E}^{in}/dg)\,\Psi_{II}^{in}),\\ -i(E_I-E_{II})\,(\Psi_I^{out}D_g^{out}\Psi_{II}^{out})&=(\Psi_I^{out}(\mathcal{H}-d\boldsymbol{E}^{out}/dg)\Psi_{II}^{out}).\end{aligned}\right\} \qquad \text{(AIII·17)}$$

From (AIII·2), (AIII·3), (AIII·6) and (AIII·7)

$$\left.\begin{aligned}&(\Psi_I^{in}D_g^{in}\Psi_{II}^{in})^{*}=-(\Psi_I^{out}D_g^{out}\Psi_{II}^{out}),\quad (\Psi_I^{in}\mathcal{H}\Psi_{II}^{in})^{*}=(\Psi_{-I}^{out}\mathcal{H}\Psi_{-II}^{out}),\\ &(\Psi_I^{in}(d\boldsymbol{E}^{in}/dg)\,\Psi_{II}^{in})=(\Psi_I^{out}(d\boldsymbol{E}^{out}/dg)\,\Psi_{II}^{out})=dE_I/dg\cdot\delta_{I\,II}.\end{aligned}\right\} \qquad \text{(AIII·18)}$$

Now the general solution of the first equation of (AIII·17) is under our assumption

$$\begin{aligned}(\Psi_I^{in}D_g^{in}\Psi_{II}^{in})=&\{i(E_I-E_{II}-i\epsilon)^{-1}+c\delta(E_I-E_{II})\}\,(\Psi_I^{in}(\mathcal{H}-d\boldsymbol{E}^{in}/dg)\,\Psi_{II}^{in})\\ &+\delta(E_I-E_{II})\,(\Psi_I^{in}h\{\mathcal{H},\,d\boldsymbol{E}^{in}/dg\}\,\Psi_{II}^{in}),\end{aligned} \qquad \text{(AIII·19)}$$

where c is an arbitrary real constant independent of g and h is an arbitrary function which satisfies $h=h^{\dagger}=h^{*}=\tilde{h}$. Taking complex conjugation and let $I\to -I$, $II\to -II$, we have

$$(\Psi_I^{out} D_g^{out} \Psi_{II}^{out}) = \{i(E_I-E_{II}+i\epsilon)^{-1} - c\delta(E_I-E_{II})\}(\Psi_I^{out}(\mathcal{H}-d\boldsymbol{E}^{out}/dg)\Psi_{II}^{out}) - \delta(E_I-E_{II})(\Psi_I^{out} h\{\mathcal{H}, d\boldsymbol{E}^{out}/dg\}\Psi_{II}^{out}). \quad \text{(AIII·20)}$$

Taking into account the relation

$$(\Psi\mathcal{H}\Psi)=dE/dg,\ (\Psi(\mathcal{H}-d\boldsymbol{E}^{in,out}/dg)\Psi)=0,\ \text{for a discrete level,}$$

it follows from the steady condition similarly to the previous proof that h must identically vanish. To decide further the indeterminate constant c, let us consider $g=\Delta g$ to be infinitesimally small for which the stationary state is given by

$$\Psi_I^{in}=\Phi_I+iD_{g=0}^{in}\Phi_I\Delta g. \quad \text{(AIII·21)}$$

Accordingly, the wave matrix is

$$(\Phi_{II}\Psi_I^{in})=\delta_{II\,I}-\{(E_{II}-E_I-i\epsilon)^{-1}+ic\delta(E_{II}-E_I)\}(\Phi_{II}(\mathcal{H}-d\boldsymbol{E}^{in}/dg)\Phi_I)\Delta g,$$

in which the interaction Hamiltonian and the self energy is infinitesimally small. Then, by the requirement (A) of § 2,

$$c=0. \quad \text{(AIII·22)}$$

Thus one finally obtains the results:

$$\left.\begin{aligned}(\Psi_I^{in} D_g^{in}\Psi_{II}^{in}) &= i(E_I-E_{II}-i\epsilon)^{-1}(\Psi_I^{in}(\mathcal{H}-d\boldsymbol{E}^{in}/dg)\Psi_{II}^{in}),\\ (\Psi_I^{out} D_g^{out}\Psi_{II}^{out}) &= i(E_I-E_{II}+i\epsilon)^{-1}(\Psi_I^{out}(\mathcal{H}-d\boldsymbol{E}^{out}/dg)\Psi_{II}^{out}).\end{aligned}\right\} \quad \text{(AIII·23)}$$

These formulas are rewritten as follows,

$$\left.\begin{aligned}d\Psi_I^{in}/dg &= iD_g^{in}\Psi_I^{in} = -(H-E_I-i\epsilon)^{-1}(\mathcal{H}-dE_I/dg)\Psi_I^{in},\\ d\Psi_I^{out}/dg &= iD_g^{out}\Psi_I^{out} = -(H-E_I+i\epsilon)^{-1}(\mathcal{H}-dE_I/dg)\Psi_I^{out}.\end{aligned}\right\} \quad \text{(AIII·24)}$$

It will be interesting to compare this result with a result which is obtained when the both sides of (2·6) are directly differentiated with respect to g. By differentiating the both sides of (2·6) and (2·8), we have

$$\{1+(H_0'-E_I-i\epsilon)^{-1}H_1'\}d\Psi_I^{in}/dg = d\Phi_I/dg - \{(d/dg)(H_0'-E_I-i\epsilon)^{-1}H_1' + (H_0'-E_I-i\epsilon)^{-1}dH_1'/dg\}\Psi_I^{in}, \quad \text{(AIII·25)}$$

and

$$(H_0'-E_I)d\Phi_I/dg = -(dH_0'/dg-dE_I/dg)\Phi_I,$$

or

$$d\Phi_I/dg = -\{P(H_0'-E_I)^{-1}+ic\delta(H_0'-E_I)\}(dH_0'/dg-dE_I/dg)\Phi_I. \quad \text{(AIII·26)}$$

Substituting (AIII·26) and

$$d(H_0'-E_I-i\epsilon)^{-1}/dg=-(H_0'-E_I-i\epsilon)^{-1}(d\,H_0'/dg-d\,E_I/dg)(H_0'-E_I-i\epsilon)^{-1}$$

into (AIII·25), we have

$$\{1+(H_0'-E_I-i\epsilon)^{-1}H_1'\}\,d\,\Psi_I^{in}/dg=ic'\delta(H_0'-E_I)(d\,H_0'/dg-dE_I/dg)\,\Phi_I$$
$$-(H_0'-E_I-i\epsilon)^{-1}(d\,H_0'/dg-d\,E_I/dg)\{\Phi_I-(H_0'-E_I-i\epsilon)^{-1}H_1'\,\Psi_I^{in}\}$$
$$-(H_0'-E_I-i\epsilon)^{-1}(dH_1'/dg)\,\Psi_I^{in}.$$

Evidently the first term of the right-hand side of this equation vanishes. Substituting Eq. (2·6) into the second term and taking into account that

$$d\,H_0'/dg+d\,H_1'/dg=\mathcal{H},$$

it follows that

$$\text{the right-hand side}=-(H_0'-E_I-i\epsilon)^{-1}(\mathcal{H}-d\,E_I/dg)\,\Psi_I^{in}.$$

Further, on account of the relation

$$\{1+(H_0'-E_I-i\epsilon)^{-1}H_1'\}^{-1}=(H-E_I-i\epsilon)^{-1},$$

which follows from the identity

$$\frac{1}{A-\Delta}=\frac{1}{A}+\frac{1}{A}\Delta\frac{1}{A}+\frac{1}{A}\Delta\frac{1}{A}\Delta\frac{1}{A}+\cdots,$$

we have the result

$$d\,\Psi_I^{in}/dg=c''\,\Psi_I^{in}-(H-E_I-i\epsilon)^{-1}(\mathcal{H}-dE_I/dg)\,\Psi_I^{in},\qquad\text{(AIII·27)}$$

where the constant c'' is indeterminate.

Appendix IV

If $a^{in}(k)$, etc., satisfy (1) and (2) at some g-value $g=g_1$, and satisfy (5) at an arbitrary value of g, then $a^{in}(k)$, etc., satisfy (1) and (2) at an arbitrary g.

Proof: Setting

$$A_g=[P_\mu,\,a^{in}(k)]+k_\mu a^{in}(k),$$

we consider the differentiation

$$d\,A/dg=[dP_\mu/dg,\,a^{in}(k)]+[P_\mu,\,d\,a^{in}(k)/dg]+(dk_\mu/dg)\,a^{in}(k)+k_\mu d\,a^{in}(k)/dg.$$

From (5) and (2·2), $(\partial P_0/\partial g\equiv d\boldsymbol{E}^{in}/dg)$

$$dA/dg=i[D_g^{in},[P_\mu,\,a^{in}(k)]+k_\mu a^{in}(k)]+[\partial P_\mu^{in}/\partial g,\,a^{in}(k)]+(dk_\mu/dg)\,a^{in}(k),$$

where the second and the third terms of the right-hand side cancel out on account of (2·3), hence

$$d\,A/dg=i[D_g^{in},\,A].$$

In terms of

$$U^{in}_{gg_1}=P_{gg_1}\exp\{i\int_{g_1}^{g}dg' D^{in}_{g'}\},$$

A_g is integrated, yielding

$$A_g=U^{in}_{gg_1}A_{g_1}U^{in\dagger}_{gg_1}.$$

However, $A_{g_1}=0$ by our condition. Therefore

$$A_g=0,$$

or

$$[P_\mu, a^{in}(k)]=-k_\mu a^{in}(k),$$

for an arbitrary g. Further, it is clear that the commutation relations are conserved because of the hermiticity of D^{in}_g in Eq. (5).

Appendix V

Since we have normalized the state vectors by Eq. (5·18), where (IE) and (IIE') are the indices which specify the incoming particles,

$$d\{\delta(E-E')\delta_{I\,II}\}/dg=0, \tag{AV·1}$$

and

$$\sum_I\int dE\,\Psi^{in}_{IE}\Psi^{in}_{IE}=1. \tag{AV·2}$$

Further put

$$(\Psi^{in}_{IE}P_j\Psi^{in}_{IIE'})=\delta(E-E')\delta^j_{I\,II}, \tag{AV·3}$$

then clearly

$$d\{\delta(E-E')\delta^j_{I\,II}\}/dg=0. \tag{AV·4}$$

Define the S-matrix on an energy shell by

$$(\Psi^{out}_{IE}\Psi^{in}_{IIE'})=\delta(E-E')S_{I\,II}(E), \tag{AV·5}$$

then the unitarity reads

$$\sum_{III}S^\dagger_{I\,III}(E)S_{III\,II}(E)=\delta_{I\,II}. \tag{AV·6}$$

Further, let the eigenstate of S be specified by $n=(j, a)$, then the normalization and the completeness for the amplitudes, which are now dependent on g, are

$$(u_{In}, u_{In'})=\delta_{nn'},\ \sum_n u_{In}u^\dagger_{nII}=\delta_{I\,II}\ \text{and}\ \sum_a u_{Ija}u^\dagger_{jaII}=\delta^j_{I\,II}. \tag{AV·7}$$

Accordingly, the S-matrix is represented explicitly by

$$S_{I\,II}(E)=\sum_n u_{In}e^{2i\eta_{nE}}u^\dagger_{nII}. \tag{AV·8}$$

Also

$$(\Psi^{out}_{IE}\, P_j \Psi^{in}_{IIE'}) = \delta(E-E')\, S^j_{I\,II}(E), \qquad \text{(AV·9)}$$

where

$$S^j_{I\,II}(E) = \sum_a u_{Ija} e^{2i\eta_{jaE}} u^\dagger_{jaE}. \qquad \text{(AV·10)}$$

Now

$$-i\mathrm{Sp}[P_E P_j \bar{D}_g] = -i \sum_I (\Psi^{in}_{IE} P_j \bar{D}_g \Psi^{in}_{IE})$$

$$= -i \sum_I \sum_{II} \int dE' (\Psi^{in}_{IE} P_j \Psi^{out}_{IIE'}) (\Psi^{out}_{IIE'} P_j \bar{D}_g \Psi^{in}_{IE}),$$

making use of (AV·8), it becomes

$$= \sum_I \sum_{II} \sum_a \sum_{a'} u_{Ija'} e^{-2i\eta_{ja'E}} u^\dagger_{ja'II} [d\{\delta(E-E') u_{IIja} e^{2i\eta_{jaE}} u^\dagger_{jaI}\}/dg]_{E'=E}$$

$$= 2i\delta(0) \sum_a d\eta_{jaE}/dg + [d\,\delta(E-E')/dg]_{E=E'} \sum_I \delta_{II}{}^j$$

$$+ \delta(0) \sum_a \{(u_{ja}, du_{ja}/dg) + (du_{ja}/dg, u_{ja})\}$$

$$= 2i\delta(0) \sum_a d\eta_{jaE}/dg + \sum_I [d/dg\{\delta(E-E')\delta^j_{II}\}]_{E=E'}.$$

By virtue of (AV·4) the second term vanishes. One therefore finally obtains the following result:

$$\sum_a d\eta_{jaE}/dg = -[2\delta(0)]^{-1} \mathrm{Sp}[P_E P_j \bar{D}_g]. \qquad \text{(AV·11)}$$

From (5·6) and (5·19),

$$Q_{j;\,E,E'} = \sum_I \sum_{II} \Psi^{in}_{IE} v_{IE,\,IIE'} \Psi^{in}_{IIE'} = \sum_I \sum_{II} \Psi^{out}_{IE} v^*_{IE,\,IIE'} \Psi^{out}_{IIE'}, \qquad \text{(AV·12)}$$

where

$$v_{IE,\,IIE'} = \sum_a u_{Ija} e^{-i(\eta_{jaE} - \eta_{jaE'})} u^\dagger_{jaII}. \qquad \text{(AV·13)}$$

One easily finds that

$$\mathrm{Sp}[Q_{j;\,E,E'}\, d\boldsymbol{E}^{in}/dg] = \mathrm{Sp}[Q_{j;\,E,E'}\, d\boldsymbol{E}^{out}/dg]. \qquad \text{(AV·14)}$$

Therefore

$$\rho_{j;\,E,E'} \equiv \mathrm{Sp}[Q_{j;\,E,E'}(\mathcal{H} - d\boldsymbol{E}^{in}/dg)] = \mathrm{Sp}[Q_{j;\,E,E'}(\mathcal{H} - d\boldsymbol{E}^{out}/dg)]$$

$$= \sum_a (\Psi_{jaE}(\mathcal{H} - d\boldsymbol{E}^{in(out)}/dg)\, \Psi_{jaE'})$$

$$= \sum_I \sum_{II} (\Psi^{in}_{IE}(\mathcal{H} - d\boldsymbol{E}^{in}/dg)\, \Psi^{in}_{IIE'})\, v_{IIE',\,IE}$$

$$= \sum_I \sum_{II} (\Psi^{out}_{IE}(\mathcal{H} - d\boldsymbol{E}^{out}/dg)\, \Psi^{out}_{IIE'})\, v^*_{IIE',\,IE}. \qquad \text{(AV·15)}$$

References

1) C. N. Yang and G. Feldman, Phys. Rev. **79** (1950), 973.
2) M. Gell-Mann and F. Low, Phys. Rev. **84** (1951), 350.
3) J. Schwinger, Proc. Nat. Acad. Sci. **37** (1951), 452, 455.
4) A. Klein, Phys. Rev. **99** (1955), 1000.
5) R. Haag, Phys. Rev. **112** (1958), 669.
6) T. Imamura, S. Sunakawa and R. Utiyama, Prog. Theor. Phys. **11** (1954), 291.
7) N. Levinson, Kgl. Danske Vid. Selsk. Mat.-fys. Medd. **25** (1949), No. 9.
8) J. M. Jauch, Helv. Phys. Acta **30** (1957), 143.
9) F. Low, Phys. Rev. **97** (1955), 1392.
10) E. P. Wigner, Göttinger Nachrichten (1932), 546.
11) J. Schwinger, Phys. Rev. **82** (1951), 914.
12) G. Morpurgo, L. A. Radicati and B. F. Touscheck, Nuovo Cimento **12** (1954), 677.
13) Bogoliubov, Medvedev and Polivanov, lecture note (1957).
14) T. Imamura, Prog. Theor. Phys. **18** (1957), 559.
15) G. Källen and W. Pauli, Kgl. Danske Vid. Selsk. Mat.-fys. Medd. **30** (1955), No. 7.
16) M. L. Goldberger and S. B. Treiman, Phys. Rev. **113** (1958), 1663.
17) E. P. Wigner, Phys. Rev. **98** (1955), 145.
18) S. T. Epstein, Phys. Rev. **98** (1955), 196.

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Flow over a Wavy Wall in Magneto-Gas-Dynamics. Part I

——*Equilibrium Flow*——

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The hydromagnetic plane steady flow of an infinitely conducting gas over a thin infinite wave-shaped wall is discussed under the condition that the directions of the flow velocity and magnetic field are in the same plane. When the external magnetic field is parallel to the undisturbed stream, the flow is shown to be current-free and irrotational except for some surface currents. For the case of external magnetic field perpendicular to the undisturbed stream, the hydromagnetic effects consist of the waves of electric currents and vorticity extending from the body. The drag is determined both for subsonic and supersonic cases.

§ 1. Introduction

In a paper of fundamental significance, Sears and Resler[1] concluded that the magnetic interaction of a flying body with the surrounding plasma may have very important applications in aerodynamics, especially if the ' seeding ' of easily ionizable substances is used in order to enhance the conductivity of the air. Further, they studied in another paper[2] the motion of thin airfoils in fluids of high electrical conductivity. Their analysis, which was confined to inviscid, incompressible, non-heat-conducting fluids, contained some interesting results and gave a clue for predicting analogous results in more complicated phenomena arising from the compressibility of the air which may be either in thermodynamical equilibrium or chemically non-equilibrium state. In the present paper we investigate the flow of an electrically conducting fluid over an infinite wavy wall, taking the compressibility of the fluid into consideration. The fluid is assumed to be in thermal equilibrium. The case of non-equilibrium flow will constitute the subject of a future paper.

§ 2. Fundamental equations

Let $\bar{\rho}$ be the density of an inviscid, electrically conducting gas moving with the velocity $\boldsymbol{q}$. Suppose that its electrical conductivity is infinite and denote the magnetic field, electric field and electric current in it by $\boldsymbol{H}$, $\boldsymbol{E}$ and $\boldsymbol{J}$ respectively. These quantities will be measured in electromagnetic units. The permeability of the fluid is assumed to be unity. Under the usual hydromagnetic approximations, the equations governing the steady flow are given by[7]

$$\text{curl } \boldsymbol{E}=0, \tag{1}$$

$$\text{curl } \boldsymbol{H}=4\pi\boldsymbol{J}, \tag{2}$$

$$\text{div } \boldsymbol{H}=0, \tag{3}$$

$$\boldsymbol{E}+\boldsymbol{q}\times\boldsymbol{H}=0, \tag{4}$$

$$\text{div}(\bar{\rho}\boldsymbol{q})=0, \tag{5}$$

$$\boldsymbol{q}\cdot\nabla\boldsymbol{q}=-\frac{1}{\bar{\rho}}\,\text{grad}\,\bar{p}+\frac{1}{\bar{\rho}}\boldsymbol{J}\times\boldsymbol{H}, \tag{6}$$

where $\bar{p}$ is the pressure in the fluid. Elimination of $\boldsymbol{E}$ between (1) and (4) gives

$$\text{curl}\,(\boldsymbol{q}\times\boldsymbol{H})=0. \tag{7}$$

Substituting (2) into (6), we get

$$\boldsymbol{q}\cdot\nabla\boldsymbol{q}=-\frac{1}{\bar{\rho}}\,\text{grad}\,\bar{p}+\frac{1}{4\pi\bar{\rho}}\,\text{curl}\,\boldsymbol{H}\times\boldsymbol{H}. \tag{8}$$

For the plane flows, Eqs. (3), (5), (7) and (8) give five equations for six unknowns q_x, q_y, h_x, h_y, $\bar{p}$, $\bar{\rho}$. In order to have a deterministic set of equations, we have to assume a relationship between the pressure and the density. Here we assume that the gas obeys the adiabatic law

$$\bar{p}=k(\bar{\rho})^{\gamma}. \tag{9}$$

§ 3. Small perturbation flow *Case I*

We shall first consider the flow over a wave-shaped wall, taking the undisturbed flow velocity U_∞ and the applied magnetic field H_∞ at infinite distance from the wall to be parallel with each other. If the amplitude of the wave of the wall is small, we may safely assume that the velocity $\boldsymbol{q}$, the magnetic field $\boldsymbol{H}$, the pressure $\bar{p}$ and the density $\bar{\rho}$ are only slightly different from their undisturbed values, that is,

$$\left.\begin{aligned}
&\boldsymbol{q}=(U_\infty+u,\ v,\ 0)\ \text{ where } u,\ v\ll U_\infty,\\
&\boldsymbol{H}=(H_\infty+h_x,\ h_y,\ 0)\ \text{ where } h_x,\ h_y\ll H_\infty,\\
&\bar{\rho}=\rho_\infty+\rho,\ \text{where } \rho\ll\rho_\infty,\\
&\bar{p}=p_\infty+p,\ \text{where } p\ll p_\infty.
\end{aligned}\right\} \tag{10}$$

If we neglect the squares and the products of the perturbed quantities in Eqs. (5), (7) and (8), they reduce to

$$\rho_\infty\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+U_\infty\frac{\partial\rho}{\partial x}=0, \tag{11}$$

$$U_\infty h_y-vH_\infty=0, \tag{12}$$

$$U_\infty \frac{\partial u}{\partial x} = (-1/\rho_\infty) \frac{\partial p}{\partial x}, \tag{13}$$

$$U_\infty \frac{\partial v}{\partial x} = (-1/\rho_\infty) \frac{\partial p}{\partial y} + \frac{H_\infty}{4\pi\rho_\infty}\left[\frac{\partial h_y}{\partial x} - \frac{\partial h_x}{\partial y}\right], \tag{14}$$

$$\frac{dp}{d\rho} = a_0^2 = (\text{square of sound velocity}). \tag{15}$$

Eliminating v and ρ from (11) by making use of (3), (9) and (12), we have

$$\frac{1}{U_\infty}\frac{\partial u}{\partial x} - \frac{1}{H_\infty}\frac{\partial h_x}{\partial x} + \frac{1}{\rho_\infty a_0^2}\frac{\partial p}{\partial x} = 0. \tag{16}$$

Integration of (16) gives

$$\frac{u}{U_\infty} - \frac{h_x}{H_\infty} + \frac{p}{\rho_\infty a_0^2} = \text{constant.}^* \tag{17}$$

Further, integration of (13) gives (see footnote* on this page)**

$$\rho_\infty U_\infty u + p = 0. \tag{18}$$

Elimination of p between (17) and (18) gives

$$\frac{u}{U_\infty} - \frac{h_x}{H_\infty} - \frac{U_\infty u}{a_0^2} = \text{constant} = C_1, \text{ (say)}. \tag{19}$$

Recalling that u, h_x vanish at infinity, we have

$$C_1 = 0.$$

Then Eq. (19) reduces to

$$h_x = \frac{1}{U_\infty}(1 - M^2) H_\infty u, \tag{20}$$

where M is the Mach number defined by $M = U_\infty / a_0$. Eliminating h_x and h_y from (14) with the help of (12) and (20), we have

$$(M^2 - m_0^2)\frac{\partial v}{\partial x} = [M^2 - m_0^2(1 - M^2)]\frac{\partial u}{\partial y}, \tag{21}$$

where

$$m_0^2 = H_\infty^2 / 4\pi\rho_\infty a_0^2 = A_e^2 / a_0^2 = \left[\frac{\text{Alfven's velocity}}{\text{sound velocity}}\right]^2.$$

Eliminating $\partial\rho/\partial x$ from (11) by making use of (18), we obtain

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* The R. H. S. of (17) cannot be a function of y on account of the boundary conditions discussed in page 725.

** Since all the perturbed quantities vanish at infinity.

$$(1-M^2)\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \tag{22}$$

Thus the problem has been reduced to the boundary value problem of Eqs. (21) and (22). From (21), we may introduce a potential function ϕ and express u, v in the form

$$u=\frac{\partial \phi}{\partial x}, \quad v=\frac{M^2-m_0^2(1-M^2)}{M^2-m_0^2}\frac{\partial \phi}{\partial y}. \tag{23}$$

Substituting (23) into (22), we obtain

$$\frac{(1-M^2)(M^2-m_0^2)}{M^2-m_0^2(1-M^2)}\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}=0. \tag{24}$$

The vorticity Ω is given from (23) by

$$\Omega=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=(A_e^2/U_\infty^2)\Omega+(A_e^2/U_\infty^2)M^2\frac{\partial u}{\partial y}. \tag{25}$$

If Alfven's velocity is very much smaller than the velocity of sound, as frequently is the case, Eq. (25) reduces to $\Omega=0$, which shows that the flow is irrotational except for the particular case in which the stream speed U_∞ is equal to Alfven's velocity. For small values of Alfven's velocity, we find that the current density j_z, which is given by

$$\frac{\partial h_y}{\partial x}-\frac{\partial h_x}{\partial y}=4\pi j_z, \tag{26}$$

is also zero to the first order.

In the special case stated above, in which the disturbance produced by the body is propagated in the form of an Alfven wave, with the speed of the body relative to the undisturbed stream, the body moves in a 'cavity' of its own shape and exerts no force on the fluid. At any other flow speed, Eq. (24) shows that the velocity field about the body can be reduced to that of a non-conducting fluid by changing the scale of x or y, so that it is determined by the boundary conditions and the required cyclic constants. Therefore we may derive a number of solutions of this case from the familiar first-order solutions of plane irrotational compressible flow in non-conducting fluid that may be furnished by, say, the thin airfoil theory.

Since, however, there are surface currents at the interface between the body and the fluid, the formula of the force on the body must be modified. The behaviour of this force may be seen in an example given in § 4.

Boundary Conditions

As remarked above, for all stream speeds different from that of Alfven's speed, the flow field about the body is essentially the same as in the non-magnetic case and is determined by the following boundary conditions:

(i) The direction of the flow must be tangential to the surface of the body.

(ii) At infinity, the perturbation velocities must tend to zero or, at least, finite (depending on the problem). In our case they must tend to zero.

Mathematically speaking, the first condition may be expressed as

$$\boldsymbol{q}\cdot\nabla f=0,$$

where $f(x, y)=0$ gives the equation of the surface. In terms of the perturbation velocities, this may be written as

$$v/U_\infty=-\left[\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y}\right]=\frac{dy}{dx}.$$

In the case of thin body, y differs only slightly from zero, and we may expand $v(x, y)$ in power series of y:

$$v(x,\ y)=v(x,\ 0)+y\left(\frac{\partial v}{\partial y}\right)_{y=0}+\cdots.$$

Retaining only the first term, we get

$$v(x,\ 0)=U_\infty\left(\frac{dy}{dx}\right)_{\text{body}}. \tag{27}$$

Although the boundary condition (27) is the same for the subsonic and supersonic flows, Eq. (24) governing both the flow field may change its type according to the subsonic or supersonic cases (it always changes its type in ordinary hydrodynamical case),[5] and each cases will be dealt with separately in the follwing section.

§ 4. Two-dimensional flow over a wave-shaped wall (insulator)

Case 1. Subsonic Flow

As remarked by some authors,[3),4),8)] the equation for the steady hydromagnetic flow can be hyperbolic even for the subsonic flow velocity if the magnetic field is of moderate strength. But we shall restrict ourselves to the case in which Eq. (24) is elliptic, i.e. $1-M^2>0$ and $M>m_0$ so that $\lambda^2=(1-M^2)(M^2-m_0^2)/\{M^2-m_0^2(1-M^2)\}>0$ in Eq. (24). The general solution of Eq. (24) is given by

$$\phi(x,\ y)=X(x)\,Y(y), \tag{28}$$

where

$$X(x)=A_1\sin kx+A_2\cos kx,$$
$$Y(y)=A_3\,e^{-k\lambda y}+A_4e^{k\lambda y}.$$

The boundary condition at infinity requires that $A_4=0$. At the wall

$$(v)_{y=0}=v_0\left(\frac{\partial\phi}{\partial y}\right)_{y=0}=v_0X(x)\left(\frac{dY}{dy}\right)_{y=0}, \tag{29}$$

where

$$v_0=\frac{M^2-m_0^2(1-M^2)}{M^2-m_0^2}. \tag{30}$$

Now if the equation of the boundary is given by $y=\epsilon \sin Lx$ where ϵ denotes the amplitude of the wavy wall and $l=2\pi/L$ the wave length, it follows from Eq. (27), that

$$(v)_{y=0}=U_\infty \epsilon L \cos Lx. \tag{31}$$

Equating (29) and (31), we get

$$-A_3 k v_0 \lambda (A_1 \sin kx+A_2 \cos kx)=U_\infty \epsilon L \cos Lx. \tag{32}$$

From Eq. (32) follows

$$A_1=0,\ k=L,\ -A_3 A_2 \lambda v_0=U_\infty \epsilon.$$

Hence the solution is obtained as

$$\phi(x, y)=-U_\infty \epsilon v_0^{-1} e^{-L\lambda y} \cos Lx, \tag{33}$$

and

$$\left.\begin{aligned} u&=U_\infty \epsilon L \lambda^{-1} v_0^{-1} e^{-L\lambda y} \sin Lx,\\ v&=U_\infty \epsilon L e^{-L\lambda y} \cos Lx. \end{aligned}\right\} \tag{34}$$

Using Eqs. (12) and (20), we get

$$\left.\begin{aligned} h_x&=H_\infty \epsilon L \lambda^{-1} v_0^{-1}(1-M^2)\, e^{-L\lambda y} \sin Lx,\\ h_y&=\epsilon L H_\infty e^{-L\lambda y} \cos Lx. \end{aligned}\right\} \tag{35}$$

The pressure coefficient C_p at the boundary is given by

$$(C_p)_{\text{boundary}}=-2[u/U_\infty]_{y=0}=-2\epsilon L \lambda^{-1} v_0^{-1} \sin Lx.$$

If the surface current per unit length flowing in the z direction is denoted by J_s, then J_s is given from (2) by

$$4\pi J_s=-h_x(x, +0)+h_x(x, -0).$$

The force exerted on the wall due to the surface currents is $H_\infty J_s$ per unit area. Therefore the net force on the wall is given by the difference between the local static pressure and the force $H_\infty J_s$, and denoting it by $p_n(x)$, we have

$$\begin{aligned} p_n(x)&=\bar{p}(x, 0)-H_\infty J_s,\\ &=p_\infty-\rho_\infty U_\infty u(x, 0)-\frac{H_\infty}{4\pi}\{h_x(x, -0)-h_x(x, +0)\},\\ &=p_\infty-\rho_\infty U_\infty^2 \epsilon L \lambda^{-1} v_0^{-1}\left\{1-\frac{2}{U_\infty^2}(1-M^2)A_e^2\right\} \sin Lx. \end{aligned} \tag{36}$$

For $M\to 0$, (36) tends to the corresponding expression in incompressible case[2)]

(where the equation of the wall is $y=\epsilon e^{i\lambda x}$).

Discussion of the result

If we compare the above result with the results of subsonic non-magnetic flow, it is seen that the pressure coefficient is reduced by the factor $\{1-2(1-M^2)A_e^2U_\infty^{-2}\}$, the amplitude of the net pressure disturbance on the wall is also reduced, but the drag remains unchanged.

Case 2. Supersonic Flow

For the supersonic flow, $M^2-1>0$, we consider only the case in which $M>m_0$, $M>1$. With this restriction on M and m_0 we find that

$$\lambda^2=-(M^2-1)(M^2-m_0^2)/\{M^2+m_0^2(M^2-1)\}=-k^2<0.$$

The general solution of Eq. (24) is the sum of two arbitrary functions $f_1(x-ky)$ and $f_2(x+ky)$. The boundary condition at infinity requires that f_2 must be zero. Eq. (31) requires that

$$-kv_0f_1'(x)=U_\infty\epsilon L\cos Lx,$$

where f_1' denotes the derivative of f_1 with respect to its argument. Therefore we have

$$f_1(x)=-U_\infty\epsilon k^{-1}v_0^{-1}\sin Lx.$$

Hence

$$\phi(x,y)=f_1(x-ky)=-U_\infty\epsilon k^{-1}v_0^{-1}\sin L(x-ky).$$

Substituting this expression for ϕ into (12), (20), (23) and (36), we obtain the following results:

$$\left.\begin{aligned}&u=-U_\infty\epsilon k^{-1}v_0^{-1}L\cos L(x-ky)\ ;\ v=U_\infty\epsilon L\cos L(x-ky),\\&h_x=H_\infty\epsilon Lk^{-1}v_0^{-1}(M^2-1)\cos L(x-ky)\ ;\ h_y=H_\infty\epsilon L\cos L(x-ky),\\&(C_p)_{\text{boun 'ary}}=2\epsilon Lk^{-1}v_0^{-1}\cos Lx.\end{aligned}\right\}\qquad(37)$$

$$C_D=\text{Drag coefficient}=(2/k)\left(\frac{\overline{dy}}{dx}\right)^2,$$

where

$$\left(\frac{\overline{dy}}{dx}\right)^2=\frac{1}{l}\int_0^l\left(\frac{dy}{dx}\right)^2dx.$$

Discussion of the supersonic solution

(1) It is seen from (37) that the disturbance has the same value along the straight lines:

$$x-\left[\frac{(M^2-1)(M^2-m_0^2)}{M^2+m_0^2(M^2-1)}\right]^{1/2}y=\text{constant}.\qquad(38)$$

(2) As in the subsonic case, the pressure coefficient is reduced.
(3) The magnitude of the drag coefficient per unit wavelength is also reduced.

§ 5. Small perturbation flow *Case II*

Next we consider the plane steady flows in which the undisturbed stream velocity and the magnetic field are uniform and perpendicular with each other. In this case we may put

$$\boldsymbol{q}=(U_\infty+u,\ v,\ 0) \text{ where } u,\ v\ll U_\infty, \tag{39}$$

and

$$\boldsymbol{H}=(h_x,\ H_\infty+h_y,\ 0) \text{ where } h_x,\ h_y\ll H_\infty. \tag{40}$$

Assuming that the ratios of the perturbation to the undisturbed values are of the same order of magnitude both for the velocity and magnetic fields, the equations of momentum (8) become, to the first order,

$$\rho_\infty U_\infty\frac{\partial u}{\partial x}=-\frac{\partial p}{\partial x}+\frac{H_\infty}{4\pi}\left(\frac{\partial h_x}{\partial y}-\frac{\partial h_y}{\partial x}\right), \tag{41}$$

$$\rho_\infty U_\infty\frac{\partial v}{\partial x}=-\frac{\partial p}{\partial y}. \tag{42}$$

Equation of continuity (15) reduces to

$$U_\infty\frac{\partial p}{\partial x}+\rho_\infty a_0^2\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0. \tag{43}$$

Conservation of magnetic field gives

$$\frac{\partial h_x}{\partial x}+\frac{\partial h_y}{\partial y}=0. \tag{44}$$

Ohm's law gives

$$H_\infty\frac{\partial u}{\partial y}-U_\infty\frac{\partial h_x}{\partial x}=-\frac{1}{4\pi\sigma}\nabla^2 h_x, \tag{45}$$

$$H_\infty\frac{\partial u}{\partial x}+U_\infty\frac{\partial h_y}{\partial x}=\frac{1}{4\pi\sigma}\nabla^2 h_y. \tag{46}$$

Eliminating $\partial p/\partial x$ between (41) and (43), we get

$$(1-M^2)\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=\frac{U_\infty H_\infty}{4\pi\rho_\infty a_0^2}\xi, \tag{47}$$

where

$$\xi=\frac{\partial h_y}{\partial x}-\frac{\partial h_x}{\partial y}. \tag{48}$$

From (45) and (46) we get

$$H_\infty \nabla^2 u = \left\{ \frac{1}{4\pi\sigma} \nabla^2 - U_\infty \frac{\partial}{\partial x} \right\} \xi, \tag{49}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Eliminating u between (47) and (49), we obtain

$$\nabla^2 \frac{\partial v}{\partial y} = \left[\frac{H_\infty U_\infty}{4\pi \rho_\infty a_0^2} \nabla^2 - \frac{(1-M^2)}{H_\infty} \left(\frac{1}{4\pi\sigma} \nabla^2 \frac{\partial}{\partial x} - U_\infty \frac{\partial^2}{\partial x^2} \right) \right] \xi. \tag{50}$$

From (49) and (50) we have

$$\nabla^2 \frac{\partial \Omega}{\partial y} = \left[\left(\frac{H_\infty U_\infty}{4\pi \rho_\infty a_0^2} + \frac{U_\infty}{H_\infty} \right) \nabla^2 \xi_x - \frac{1}{4\pi\sigma H_\infty} \nabla^4 \xi + M^2 \left(\frac{1}{4\pi\sigma H_\infty} \nabla^2 - \frac{U_\infty}{H_\infty} \frac{\partial}{\partial x} \right) \xi_{xx} \right]. \tag{51}$$

By cross-differentiation of (41) and (42) we get

$$\frac{\partial \Omega}{\partial x} = \frac{H_\infty}{4\pi \rho_\infty U_\infty} \frac{\partial \xi}{\partial y}. \tag{52}$$

From (51) and (52), we find that Ω and ξ satisfy the same equation, that is,

$$\left(\frac{H_\infty U_\infty}{4\pi \rho_\infty a_0^2} + \frac{U_\infty}{H_\infty} \right) \nabla^2 \begin{pmatrix} \xi \\ \Omega \end{pmatrix}_{xx} - \frac{1}{4\pi\sigma H_\infty} \left(\nabla^4 - M^2 \nabla^2 \frac{\partial^2}{\partial x^2} \right) \begin{pmatrix} \xi \\ \Omega \end{pmatrix}_x$$
$$- \frac{U_\infty}{H_\infty} M^2 \begin{pmatrix} \xi \\ \Omega \end{pmatrix}_{xxxx} = \frac{H_\infty}{4\pi \rho_\infty U_\infty} \nabla^2 \begin{pmatrix} \xi \\ \Omega \end{pmatrix}_{yy}. \tag{53}$$

For the case of infinite conductivity we find that Eq. (53) reduces to

$$\left[\{M^2 + m^2(1-M^2)\} \frac{\partial^4}{\partial x^4} + (M^2 + m^2 - 1) \frac{\partial^4}{\partial x^2 \partial y^2} - \frac{\partial^4}{\partial y^4} \right] \begin{pmatrix} \xi \\ \Omega \end{pmatrix} = 0, \tag{54}$$

where m is the ratio of the stream speed to Alfven's velocity, that is,

$$m = U_\infty / H_\infty (4\pi\rho_\infty)^{-1/2} = 1/m_0. \tag{55}$$

Since the discriminant of Eq. (54) is always positive, the solution of (54) is given by

$$\xi = f_1'(x+ik_1 y) + f_2'(x-ik_1 y) + g_1'(x+ik_2 y) + g_2'(x-ik_2 y), \tag{56}$$

$$\Omega = f_3'(x+ik_1 y) + f_4'(x-ik_1 y) + g_3'(x+ik_2 y) + g_4'(x-ik_2 y), \tag{57}$$

where

$$k_{1,2}^2 = -\left(\frac{M^2+m^2-1}{2} \right) \pm \left\{ \left(\frac{M^2+m^2+1}{2} \right)^2 - M^2 m^2 \right\}^{1/2}. \tag{58}$$

The components of the velocity and the magnetic field, as obtained from (44), (47), (48), (56) and (57), are as follows;

$$u=\frac{1}{1-M^2-k_1^2}\left\{\frac{U_\infty M^2}{H_\infty m^2}(f_1+f_2)-ik_1(f_3-f_4)\right\}$$
$$+\frac{1}{1-M^2-k_2^2}\left\{\frac{U_\infty M^2}{H_\infty m^2}(g_1+g_2)-ik_2(g_3-g_4)\right\}+\frac{\partial\phi_1}{\partial x}, \tag{59}$$

$$v=\frac{1}{1-M^2-k_1^2}\left\{\frac{U_\infty M^2}{H_\infty m^2}ik_1(f_1-f_2)+(1-M^2)(f_3+f_4)\right\}$$
$$+\frac{1}{1-M^2-k_2^2}\left\{\frac{U_\infty M^2}{H_\infty m^2}ik_2(g_1-g_2)+(1-M^2)(g_3+g_4)\right\}+\frac{\partial\phi_1}{\partial y}, \tag{60}$$

$$h_x=\frac{ik_1}{1-k_1^2}\{f_2-f_1\}+\frac{ik_2}{1-k_2^2}\{g_2-g_1\}+\frac{\partial\phi_2}{\partial x}, \tag{61}$$

$$h_y=\frac{1}{1-k_1^2}\{f_1+f_2\}+\frac{1}{1-k_2^2}\{g_1+g_2\}+\frac{\partial\phi_2}{\partial y}, \tag{62}$$

where ϕ_1 and ϕ_2 are solutions of the following equations respectively:

$$(1-M^2)\frac{\partial^2\phi_1}{\partial x^2}+\frac{\partial^2\phi_1}{\partial y^2}=0, \quad \nabla^2\phi_2=0. \tag{63}$$

Substituting the result (59)–(62) into Eq. (41) and integrating it with respect to x, we get

$$\rho_\infty U_\infty u+p=-\frac{H_\infty}{4\pi}[f_1+f_2+g_1+g_2]+\text{constant}. \tag{64}$$

Dealing with Eq. (42) in a similar manner and integrating it with respect to y, we have

$$\rho_\infty U_\infty u+p=-\frac{\rho_\infty U_\infty}{ik_1}(f_1-f_4)-\frac{\rho_\infty U_\infty}{ik_2}(g_3-g_4)+\text{constant}. \tag{65}$$

Comparing (64) and (65), we find that

$$\left.\begin{aligned} f_3&=\frac{ik_1}{m^2}\frac{U_\infty}{H_\infty}f_1, \quad f_4=-\frac{ik_1}{m^2}\frac{U_\infty}{H_\infty}f_2,\\ g_3&=\frac{ik_2}{m^2}\frac{U_\infty}{H_\infty}g_1, \quad g_4=-\frac{ik_2}{m^2}\frac{U_\infty}{H_\infty}g_2. \end{aligned}\right\} \tag{66}$$

Similarly substituting (59)–(62) into Eq. (46) with the right-hand side put equal to zero, we get

$$H_\infty\frac{\partial^2\phi_1}{\partial x^2}=-U_\infty\frac{\partial^2\phi_2}{\partial x\partial y}. \tag{67}$$

The compatibility of Eq. (67) with (63) requires that the functions $\phi_1(x, y)$ and $\phi_2(x, y)$ must be identically zero for $M\neq 0$.

Now we confine our consideration to the case in which $M^2+m^2>1$. In this

case we find that $k_2^2<0$, and for k_1^2 the following three possibilities arise:*

$$(1)\ k_1^2=0 \quad (2)\ k_1^2>0 \quad (3)\ k_1^2<0.$$

§ 6. Two-dimensional flow of an electrically conducting fluid over an infinite wave-shaped wall (insulator)

If the shape of the wall is given by the equation $y=\epsilon e^{iLx}$, the boundary condition at the wall gives to the first order

$$v(x, 0)=U_\infty y'(x)=iU_\infty \epsilon L e^{iLx}. \tag{68}$$

Now because of the sinusoidal shape of the wall, all the physical quantities may be assumed to be sinusoidal, so that the functions f_1, f_2, g_1, g_2, etc., may be expressed in the form: (constant) $\times \exp[iL(x\pm ik_{1,2}y)]$. Then assuming that no perturbations travel towards the wall from the undisturbed stream, we do not take the solutions of the type $f_2(x-ik_1y)$ and $g_2(x-ik_1y)$ for $k_{1,2}^2<0$. With this requirement the case (ii) mentioned above becomes indeterminate and the case (i) is not satisfied, so that the only possible case is the case (iii) where $k_{1,2}^2<0$, i.e. $k_1=iL_0$, $k_2=i\beta_0$.** With these restrictions on k_1^2 and k_2^2 and using (66) in Eqs. (59)–(62), we have

$$u=\frac{U_\infty}{H_\infty m^2}\frac{(M^2-L_0^2)}{(1-M^2+L_0^2)}f_1(x-L_0y)+\frac{U_\infty}{H_\infty m^2}\frac{(M^2-\beta_0^2)}{(1-M^2+\beta_0^2)}g_1(x-\beta_0y), \tag{69}$$

$$v=-\frac{U_\infty}{H_\infty m^2}\frac{L_0}{(1-M^2+L_0^2)}f_1(x-L_0y)-\frac{U_\infty}{H_\infty m^2}\frac{\beta_0}{(1-M^2+\beta_0^2)}g_1(x-\beta_0y), \tag{70}$$

$$h_x=\frac{L_0}{1+L_0^2}f_1(x-L_0y)+\frac{\beta_0}{1+\beta_0^2}g_1(x-\beta_0y), \tag{71}$$

$$h_y=\frac{1}{1+L_0^2}f_1(x-L_0y)+\frac{1}{1+\beta_0^2}g_1(x-\beta_0y). \tag{72}$$

As remarked earlier, all the field quantities are assumed to be sinusoidal, so that f_1 and g_1 are expressed as

$$f_1(x-L_0y)=Fe^{iL(x-L_0y)}, \tag{73}$$

$$g_1(x-\beta_0y)=Ge^{iL(x-\beta_0y)}. \tag{74}$$

As the wall is an insulator, no current flows in it, that is, the magnetic field for $y<0$ is expressed in terms of a potential function, say $\bar{\phi}(x, y)$. Thus, for $y<0$,

---

* $k_1^2 \gtreqless 0$ corresponds to $M^2 \gtreqless m^2(M^2-1)$, and the cases in which $M\leqq 1$ (sonic and subsonic flows) are inadmissible may be distinguished readily.

** $k_1=-iL_0$, $k_2=-i\beta_0$ give no new result.

$$h_x=\frac{\partial\bar{\phi}}{\partial x}=iL\bar{H}e^{iL(x-yt)}=i\frac{\partial\bar{\phi}}{\partial y}=ih_y, \tag{75}$$

where $\bar{H}=$ constant. In order to satisfy (44), we require the continuity of the component h_y at the wall, and therefore we get from (72) and (75)

$$L\bar{H}=\frac{F}{1+L_0^2}+\frac{G}{1+\beta_0^2}. \tag{76}$$

Eq. (68) gives

$$iU_\infty\epsilon L=-\frac{L_0U_\infty}{H_\infty m^2(1-M^2+L_0^2)}F-\frac{\beta_0U_\infty}{H_\infty m^2(1-M^2+\beta_0^2)}G. \tag{77}$$

In the present problem, we shall also require the component h_x to be continuous at the solid fluid interface, because a first-order surface current would produce first-order force components in the tangential direction and there is no mechanism here to resist such force components. Thus, using Eqs. (71) and (75), we get

$$iL\bar{H}=\frac{L_0}{1+L_0^2}F+\frac{\beta_0}{1+\beta_0^2}G. \tag{78}$$

The solutions of simultaneous equations (76)–(78) are given by

$$F=\frac{i(i-\beta_0)L\epsilon H_\infty m^2(1+L_0^2)(1-M^2+L_0^2)(1-M^2+\beta_0^2)}{\beta_0(i-L_0)(1+\beta_0^2)(1-M^2+L_0^2)-(i-\beta_0)L_0(1+L_0^2)(1-M^2+\beta_0^2)}, \tag{79}$$

$$G=-\frac{(i-L_0)(1+\beta_0^2)}{(i-\beta_0)(1+L_0^2)}F, \tag{80}$$

$$\bar{H}=\frac{H_\infty m^2\epsilon(L_0-\beta_0)(1-M^2+L_0^2)(1-M^2+\beta_0^2)}{(\beta_0-L_0)(1+M^2-L_0\beta_0)+(L_0+\beta_0)\{(1-M^2)(L_0^2+\beta_0^2-L_0\beta_0)+L_0^2\beta_0^2\}}. \tag{81}$$

Substituting the expressions (66), (73), (74), (79) and (80) into (65), we find that the pressure at the wall is given by

$$p=p_\infty+\rho_\infty U_\infty^2 iL\epsilon\left\{\frac{(i-L_0)(1+\beta_0^2)(1-M^2+L_0^2)-(i-\beta_0)(1+L_0^2)(1-M^2+\beta_0^2)}{\beta_0(i-L_0)(1+\beta_0^2)(1-M^2+L_0^2)-(i-\beta_0)L_0(1+L_0^2)(1-M^2+\beta_0^2)}\right\}. \tag{82}$$

The drag may be calculated with this knowledge of the pressure distribution.

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References

1) E. L. Resler, Jr. and W. R. Sears, Jour. Aero Space Sciences, **25** (1958), 235.
2) E. L. Resler, Jr. and W. R. Sears, Jour. Fluid Mech. **5** (1959), 257.
3) W. R. Sears, Lecture at the Eighth Japan National Congress of Applied Mechanics, Tokyo, 1958.
4) T. Taniuti, Prog. Theor. Phys. **19** (1958), 749.
5) H. W. Liepmann and A. Roshko, *Elements of Gas Dynamics*, 1957.
6) References 3) and 4) have been obtained from the bibliography of the paper by Y. Kato and T. Taniuti, Prog. Theor. Phys. **21** (1959), 606.
7) T. G. Cowling, *Magnetohydrodynamics* (2nd Edition) 1958.
8) K. O. Friedrichs and H. Kranzer, *Non-linear wave motion in magneto-hydrodynamics*, New York University, Inst. Math. Sci. Report MH-8, 1958 (unpublished).

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On the Principle of Invariance in a Semi-Infinite Inhomogeneous Atmosphere

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With the aid of the extension of Chandrasekhar's invariance method and the principle of invariant imbedding stated by Bellman and Kalaba, respectively, we obtain the emergent intensity of diffusely reflected radiation in a semi-infinite atmosphere of arbitrary stratification. The result reduces to that given by Sobolev and later by us.

§ 1. Introduction

The problem of diffuse reflection in semi-infinite homogeneous media has been first attacked by Ambarzumian[1] with the aid of the physical method based on the principle of invariance. Then, extensively extending the principle, Chandrasekhar,[2] has formulated firstly a complete set of the principles of invariance. The problem has recently been reconsidered by Busbridge,[3] Sobolev,[4] and Ueno.[5]

In the majority of transfer problems met in practice the optical properties of the atmosphere change in traversing the layer. Then in the present paper we consider the diffuse reflection in a semi-infinite inhomogeneous atmosphere scattering radiation isotropically.

With the aid of the extension of Chandrasekhar's invariance method in R. T. and the principle of invariant imbedding stated by Bellman and Kalaba,[6] respectively, we obtain the angular distribution of the diffusely reflected radiation under consideration. The result is equal to that given by Sobolev[7] and Ueno,[8] respectively. The scattering function can also be deduced from the integral equation for the backward scattering function given by Bellman and Kalaba and Ueno,[8] respectively.

§ 2. The invariance of the law of diffuse reflection

We consider the problem of diffuse reflection by a semi-infinite plane-parallel atmosphere scattering radiation isotropically with an albedo, $0<\omega(\tau)<1$. Let parallel rays of a net flux πF in the direction $-\mu_0$ be incident on the outer boundary of the atmosphere under consideration.

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Following the notation of Chandrasekhar in R. T., the equation of transfer appropriate to the present case is of the form

$$\mu \frac{dI(\tau, \mu)}{d\tau} = I(\tau, \mu) - \mathfrak{J}(\tau), \tag{1}$$

where the source function $\mathfrak{J}(\tau)$ is

$$\mathfrak{J}(\tau) = \frac{F}{4}\omega(\tau)\, e^{-\tau/\mu_0} + \frac{1}{2}\omega(\tau)\int_{-1}^{+1} I(\tau, \mu')\, d\mu'. \tag{2}$$

Eq. (1) should be solved subject to the boundary condition

$$I(0, -\mu) = 0 \quad (0 < \mu \leq 1). \tag{3}$$

Let $I(\tau, +\mu)$ be the intensity of the radiation in the upward direction and $I(\tau, -\mu)$ be that in the downward direction. The emergent intensity $I(0, \mu)$ diffusely reflected in the direction μ can be expressed in terms of a scattering function $S(\mu, \mu_0)$ (cf. R. T., Chap. I, Eq. (125))

$$I(0, \mu) = \frac{F}{4\mu} S(\mu, \mu_0). \tag{4}$$

In a manner similar to that used by Chandrasekhar (cf. R. T., Chap. IV), we can formulate the invariance of the law of diffuse reflection under consideration. Consider a semi-infinite inhomogeneous atmosphere bounded by the surface $\tau = t$. On decomposing the non-classical field into the reduced incident radiation and the diffuse radiation field, an outward intensity in the direction $+\mu$ becomes

$$I(\tau, +\mu) = \frac{F}{4\mu} e^{-(\tau - t)/\mu_0} S(\tau\,;\, \mu, \mu_0) + \frac{1}{2\mu}\int_0^1 S(\tau\,;\, \mu, \mu') I(\tau, -\mu')\, d\mu'. \tag{5}$$

Eq. (5) represents the invariance of the law of diffuse reflection. In the present case we shall use the scattering function $S(\tau\,;\, \mu, \mu_0)$ in place of $S(\mu, \mu_0)$.

The equation of transfer appropriate to this case is Eq. (1) whose source function $\mathfrak{J}(\tau)$ is

$$\mathfrak{J}(\tau) = \frac{F}{4}\omega(\tau) e^{-(\tau - t)/\mu_0} + \frac{1}{2}\omega(\tau)\int_{-1}^{+1} I(\tau, \mu')\, d\mu', \tag{6}$$

together with the boundary condition

$$I(t, -\mu) = 0 \quad (0 < \mu \leq 1). \tag{7}$$

§ 3. The integral equation for the scattering function

In a manner similar to that used by Chandrasekhar in R. T., differentiating Eq. (5) with respect to τ, using Eqs. (1) and (6), and then passing to the

limit $\tau=t$, we obtain the integral equation for the scattering function.

In this manner, recalling the boundary condition (7), we have

$$\left[\frac{dI(\tau,+\mu)}{d\tau}\right]_{\tau=t}=\frac{F}{4\mu}\left[-\frac{1}{\mu_0}S(t;\mu,\mu_0)+\frac{\partial S(t;\mu,\mu_0)}{\partial t}\right]$$
$$+\frac{1}{2\mu}\int_0^1 S(t;\mu,\mu')\left[\frac{dI(\tau,-\mu')}{d\tau}\right]_{\tau=t}d\mu'. \tag{8}$$

The derivatives appearing in Eq. (8) can be obtained from the transfer equation (1) with the source function (6). Then

$$\left[\frac{dI(\tau,+\mu)}{d\tau}\right]_{\tau=t}=\frac{1}{\mu}\left[\frac{F}{4\mu}S(t;\mu,\mu_0)-\mathfrak{T}(t)\right], \tag{9}$$

$$\left[\frac{dI(\tau,-\mu')}{d\tau}\right]_{\tau=t}=\frac{1}{\mu'}\mathfrak{T}(t), \tag{10}$$

where

$$\mathfrak{T}(t)=\frac{F}{4}\omega(t)\left[1+\frac{1}{2}\int_0^1 S(t;\mu',\mu_0)\frac{d\mu'}{\mu'}\right]. \tag{11}$$

Making use of Eqs. (9) and (10), after some minor rearrangement of the terms, we get from Eq. (8)

$$\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)S(t;\mu,\mu_0)-\frac{\partial S(t;\mu,\mu_0)}{\partial t}$$
$$=\omega(t)\left[1+\frac{1}{2}\int_0^1 S(t;\mu'',\mu_0)\frac{d\mu''}{\mu''}+\frac{1}{2}\int_0^1 S(t;\mu,\mu')\frac{d\mu'}{\mu'}\right.$$
$$\left.+\frac{1}{4}\int_0^1 S(t;\mu'',\mu_0)\frac{d\mu''}{\mu''}\int_0^1 S(t;\mu,\mu')\frac{d\mu'}{\mu'}\right]. \tag{12}$$

Now we shall have to take the reciprocity principle into consideration. Let the stochastic integral equation for the probability distribution of emission of photon $p(\mu;\tau)$ under consideration (cf. Ueno[9]) be

$$p(\mu;\tau)=\omega(\tau)\,e^{-(\tau-t)/\mu}+\frac{1}{2}\omega(\tau)\bar{\Lambda}_\tau\{p(\mu;\tau')\}, \tag{13}$$

where the truncated Hopf operator $\bar{\Lambda}$ is

$$\bar{\Lambda}_\tau\{f(\tau')\}=\frac{1}{2}\int_t^\infty f(\tau')E_1(|\tau'-\tau|)\,d\tau', \tag{14}$$

and E_1 is the first exponential integral

$$E_1(\tau)=\int_0^1 e^{-\tau/\mu}\frac{d\mu}{\mu}\,. \tag{15}$$

Multiply Eq. (13) by $p(\mu_0;\tau)/\omega(\tau)$ and integrate with respect to τ over (t,∞). Writing

$$S(t;\mu,\mu_0)=\frac{1}{2}\int_t^\infty p(\mu_0;\tau)\,e^{-(\tau-t)/\mu}\,d\tau, \tag{16}$$

we have

$$\int_t^\infty p(\mu_0;\tau)p(\mu;\tau)\frac{d\tau}{\omega(\tau)}=\int_t^\infty p(\mu;\tau')p(\mu_0;\tau')\frac{d\tau'}{\omega(\tau')}-S(t;\mu_0,\mu)+S(t;\mu,\mu_0). \tag{17}$$

Then

$$S(t;\mu,\mu_0)=S(t;\mu_0,\mu). \tag{18}$$

Writing

$$H(\mu,t)=1+\frac{1}{2}\int_0^1 S(t;\mu,\mu')\frac{d\mu'}{\mu'}, \tag{19}$$

using the principle of reciprocity (18), Eq.(12) becomes

$$\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)S(t;\mu,\mu_0)-\frac{\partial S(t;\mu,\mu_0)}{\partial t}=\omega(t)H(\mu,t)H(\mu_0,t). \tag{20}$$

From Eq. (20), we get

$$S(t;\mu,\mu_0)=\int_t^\infty \omega(\tau)H(\mu,\tau)H(\mu_0,\tau)\,e^{-(\tau-t)(1/\mu+1/\mu_0)}\,d\tau, \tag{21}$$

where

$$H(\mu,\tau)=1+\frac{1}{2}\int_0^1\frac{d\mu'}{\mu'}\int_\tau^\infty \omega(\tau')\,H(\mu,\tau')H(\mu',\tau')\,e^{-(\tau'-\tau)(1/\mu+1/\mu')}\,d\tau'. \tag{22}$$

Eqs. (21) and (22) are equal to those given by Sobolev[7] and Ueno[8], respectively.

Then the required law of diffuse reflection in a semi-infinite atmosphere of arbitrary stratification is of the form

$$I(0,+\mu)=\frac{F}{4\mu}[S(t;\mu,\mu_0)]_{t=0}. \tag{23}$$

§ 4. The principle of invariant imbedding

The straightforward application of Ambarzumian's invariance method to the

arbitrary stratification does not hold true, because it is based on the assumption that an albedo in the added or subtracted layer is the same as in the original medium. Then, in order to extend his procedure to the present case, we have to allow for a modification due to the arbitrary stratification. Based on the principle of invariant imbedding stated recently by Bellman and Kalaba, we attack the problem.

The principle, as applied to the problem in question, consists in the following: The diffuse reflecting power of semi-infinite plane-parallel atmosphere is changed if we add (or, subtract) to (or, from) the outer boundary a flat layer of infinitely small optical thickness Δt whose albedo in the added (or, subtracted) layer is different from that of the original medium. Computing each contributions to the change, we obtain the functional equation governing the stochastic process of multiple scattering of photon. In such a way we get an invariant process, by imbedding the original process within the new set of processes. We shall take Δt so small that its square can be neglected in comparison with Δt itself.

Following Ambarzumian, after tracing all processes in the passage of the radiation through the additional layer of optical thickness Δt, we have

$$I(t-\Delta t;\mu,\mu_0)=I(t;\mu,\mu_0)\left(1-\frac{\Delta t}{\mu}-\frac{\Delta t}{\mu_0}\right)+\frac{F}{4\mu}\omega(t)\Delta t$$
$$+\frac{1}{2}\omega(t)\Delta t\int_0^1 I(t;\mu,\mu')\frac{d\mu'}{\mu'}+\frac{1}{2}\frac{\omega(t)}{\mu}\Delta t\int_0^1 I(t;\mu'',\mu_0)d\mu''$$
$$+\frac{\omega(t)}{F}\Delta t\int_0^1 I(t;\mu,\mu')\frac{d\mu'}{\mu'}\int_0^1 I(t;\mu'',\mu_0)d\mu'', \tag{24}$$

where

$$I(t;\mu,\mu_0)=\frac{F}{4\mu}S(t;\mu,\mu_0). \tag{25}$$

In the limit, as $\Delta t\to 0$, this provides Eq. (12).

Recalling the principle of reciprocity, we get the law of diffuse reflection in terms of the scattering function thus obtained.

We consider some special cases.

1. When an albedo is constant throughout the atmosphere and t is put to be zero, Eqs. (21) and (22) reduce to those given in R. T., Chap. IV, Eqs. (41) and (42).

2. Let $\omega=\omega(\tau)$ for $0\leq t\leq\tau\leq\tau_1$ and $\omega=\omega_1$ for $\tau_1\leq\tau<\infty$. From Eqs. (21) and (22), we get

$$S(t,\tau_1;\mu,\mu_0)=\int_t^{\tau_1}\omega(\tau)H(\mu,\tau)H(\mu_0,\tau)\,e^{-(\tau-t)(1/\mu+1/\mu_0)}d\tau+S(\mu,\mu_0)\,e^{-(\tau_1-t)(1/\mu+1/\mu_0)}, \tag{26}$$

where

$$H(\mu, \tau)=1+\frac{1}{2}\int_0^1 \frac{d\mu'}{\mu'}\int_\tau^{\tau_1}\omega(\tau')H(\mu, \tau')H(\mu', \tau')\,e^{-(\tau'-\tau)(1/\mu+1/\mu')}\,d\tau'$$

$$+\frac{\omega_1}{2}\mu H(\mu)\int_0^1 \frac{H(\mu')}{\mu+\mu'}\,e^{-(\tau_1-\tau)(1/\mu+1/\mu')}\,d\mu'. \qquad (27)$$

In Eq. (27) $S(\mu, \mu_0)$ and $H(\mu)$ are respectively equal to those given by Chandrasekhar in R. T. Assuming $\omega_1=0$, and writing

$$X(\mu, \tau, \tau_1)=H(\mu, \tau), \qquad (28)$$

Eq. (26) becomes

$$S(t, \tau_1; \mu, \mu_0)=\int_t^{\tau_1}\omega(\tau)X(\mu, \tau, \tau_1)X(\mu_0, \tau, \tau_1)\,e^{-(\tau-t)(1/\mu+1/\mu_0)}\,d\tau, \qquad (29)$$

where

$$X(\mu, \tau, \tau_1)=1+\frac{1}{2}\int_0^1 \frac{d\mu'}{\mu'}\int_\tau^{\tau_1}\omega(\tau')X(\mu, \tau', \tau_1)X(\mu', \tau', \tau_1)\,e^{-(\tau'-\tau)(1/\mu+1/\mu')}\,d\tau'. \qquad (30)$$

Eq. (29) is the expression for the forward scattering function in a finite inhomogeneous atmosphere, i.e. the scattering function in the case of monodirectional illumination of the surface $\tau=t$. So far as we know, Eqs. (29) and (30) are new. The integral equation governing Eq. (29) corresponds to one of the functional relations for the reflectance operators given by Preisendorfer.[10] Eq. (26) with $\omega(\tau)=\omega_0$ reduces to that given by Sobolev.[7] The derivation of the backward scattering function from the forward ones should be carried out based on a full knowledge of the functional relations between the forward and the backward stochastic processes of multiple scattering of photons. Then the problem will be reconsidered in our later paper.

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Note added in proof : Our discussion concerning the application of invariant imbedding technique to the derivation of the stochastic equation for the probability distribution of emission in arbitrary stratification will appear in Journal of Mathematical Analysis and Applications (1960).

References

1) V. A. Ambarzumian, Astron. J. (USSR) **19** (1942), 1.
2) S. Chandrasekhar, Ap. J. **105** (1947), 151, 441; *Radiative Transfer* (London, Oxford University Press, 1950) : hereafter referred to as "R. T.".
3) I. W. Busbridge, M. N. **115** (1955), 391; *The Mathematics of Radiative Transfer, Cambridge Tract*, No. 50 (London, Cambridge Uneversity Press, 1960).
4) V. V. Sobolev, *The Transport of Radiant Energy in the Atmosphers of the Stars and Planets* (in Russian), (Gostekhizdat, 1956); Astron. J. (USSR) **34** (1957), 336.
5) S. Ueno, Ann. d'Ap. **22** (1959), 468.
6) R. Bellman and R. Kalaba, Proc. Nat. Acad. Sci. **42** (1956), 629.
7) V. V. Sobolev, C. R. (Doklady), Acad. Sci. U.R.S.S. **111** (1956), 1000.
8) S. Ueno, to appear in Ap. J. **132** (1960).
9) S. Ueno, Ap. J. **126** (1957), 413.
10) R. W. Preisendorfer, Proc. Nat. Acad. Sci. **44** (1958), 323.

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Local Properties in Non-Equilibrium Systems

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The development of a satisfactory statistical mechanics for non-equilibrium states has been retarded by the difficulty of making a satisfactory definition of "local temperature", that is, a definition which does not depend on the very restrictive assumption of local equilibrium. We first define, for any subvolume ΔV of the system, the "local statistical operator" $W(\Delta V)$, which can be deduced directly from the statistical operator (density matrix) U of the system as a whole. Next, we define a local entropy $S(\Delta V)$ and a local energy $E(\Delta V)$. Finally, we introduce a new requirement for the macroscopic independent variables which define the macroscopic state of the system: such variables must be directly deducible from a knowledge of the present state (density matrix) of the system, without reference to its future time development, i. e., without reference to its Hamiltonian H. Of any pair of thermodynamically conjugate variables, at most one is permissible in the light of this requirement. In particular, volume, particle numbers, and entropy are permissible macroscopic independent variables; pressure, chemical potential, and temperature are not. These latter are permissible as *dependent* variables, only, unless the system happens to be in equilibrium.

§ 1. Introduction

In equilibrium statistical mechanics, entropy and internal energy are defined by

$$S=-k\,\mathrm{Tr}(U\ln U), \tag{1·1}$$

$$E=\mathrm{Tr}(HU), \tag{1·2}$$

where U is the statistical matrix for the ensemble, normalized to

$$\mathrm{Tr}\,U=1, \tag{1·3}$$

and H is the Hamiltonian of the system. The temperature T is then defined thermodynamically through

$$1/T=(\partial S/\partial E)_V. \tag{1·4}$$

If the statistical matrix is the one appropriate for a canonical ensemble (isothermal system), i.e.,

$$U=\exp[\beta(F-H)] \tag{1·5}$$

then it follows that T and β are related through

$$\beta=1/kT. \tag{1·6}$$

Recently, considerable progress has been made in non-equilibrium statistical mechanics,[1] particularly in relation to the "linear response" of systems to applied external fields. An example of such a linear response is the electrical current produced by an applied electric field, the response function in question being the electrical conductivity. However, when one considers such linear processes as heat conduction, or such non-linear effects as the production of Joule heating, difficulties of principle arise. These difficulties are connected with the concept of "local temperature" in a specimen which is not in thermal equilibrium. Mathematically speaking, we can find the statistical matrix $U(t)$ as a function of time, at least in principle, but we do not know how to deduce from this knowledge what a local thermometer would measure.

The conventional way out of this difficulty is to assume "local equilibrium" to exist. Formally, the Hamiltonian H of the system is written, by the method of second quantization, as

$$H=\int H(x)d^3x \tag{1·7}$$

and the statistical matrix $U(t)$ is assumed to be of the form:

$$U(t)=\exp\left\{\int d^3x\beta(x,t)[F(t)-H(x)]\right\} \tag{1·8}$$

where $F(t)$ is a time-dependent constant necessary for normalization according to (1·3). From the function $\beta(x, t)$ we then deduce the local temperature $T(x, t)$ according to (1·6). If $\beta(x, t)$ is very nearly equal to a constant, then we have small deviations from full thermal equilibrium.

The concept of "local equilibrium" can be generalized somewhat by allowing for a local stream velocity, for example in a flowing gas. In the neighbourhood of any given point in the material, one must make a transformation to moving coordinates appropriate to the local mean velocity $\bar{v}(x, t)$, and in these transformed coordinates the local statistical matrix looks like (1·8). This transformation shall not concern us in what follows.

There are objections in principle to this procedure. The assumption of local equilibrium is a very strong and restrictive condition, and it should not be necessary at all. It should be possible to define "local" properties for any system in which the properties in question vary slowly with distance. This condition is much weaker than the assumption of local equilibrium at all times.

Indeed, even if we start at time $t=0$ with a statistical matrix appropriate to local equilibrium, of form (1·8), the Kubo formalism[1] necessarily leads to a time-dependent operator $U(t)$ *not* of the form (1·8), at times $t>0$. *The assumption* (1·8) *is inconsistent with the equations of motion of the system.*

As a simple example of slow distance variation but complete absence of local equilibrium consider a glass box exposed to sunlight. The photon gas inside the box presents conditions which are essentially constant throughout the box (or slowly varying if the box is of astronomical dimensions) but which are definitely very far from local equilibrium (i.e., local black-body radiation). All the photons move in one direction away from the Sun, at the speed of light. No matter what moving frame of reference we transform to, it is impossible to make any photons appear to move toward the Sun. Hence there exists no frame of reference in which the local situation looks even approximately like thermal equilibrium, black-body radiation. Yet, local properties should exist.

In statistical mechanics, the entropy S is a simpler concept than the temperature T. The definition of entropy, (1·1), can be used directly for non-equilibrium systems, since no assumption is made about the statistical matrix U.[2] Similarly, the internal energy E is defined by (1·2) even for non-equilibrium states. However, these quantities refer to the system as a whole, whereas in heat conduction, for example, we are interested in variations in the *local* entropy, energy, and other quantities.

The first purpose of this paper is to suggest a definition of *localized entropy* which is applicable to any system with slow distance variations, whether or not it is in local equilibrium; i.e., no use will be made of the assumption (1·8). The second purpose is to discuss, in the light of a new requirement, the nature of the quasi-thermodynamic independent variables which can be employed in non-equilibrium situations. We shall see, in particular, that a "local temperature" is *not* a suitable independent variable, whereas local entropy is permissible.

§ 2. The localized statistical operator

Consider a small volume ΔV inside the box of volume V. Assume for simplicity that the number N of particles inside the box is given. The statistical operator U at any given time can then be expressed in configuration space, in terms of matrix elements:

$$\langle x_1 x_2 \cdots x_N | U | x_1' x_2' \cdots x_N' \rangle \tag{2·1}$$

and the normalization condition is, for identical particles,

$$\mathrm{Tr}\, U = \frac{1}{N!} \int d^3x_1 d^3x_2 \cdots d^3x_N \langle x_1 x_2 \cdots x_N | U | x_1 x_2 \cdots x_N \rangle = 1. \tag{2·2}$$

(2·1) is of course symmetric or antisymmetric under the exchange of the coordinates (both primed and unprimed) of any two particles, depending upon whether the particles obey Bose-Einstein or Fermi-Dirac statistics.

We now define, starting from (2·1), a Fock space operator W as follows: First of all, W commutes with the number operator $\mathcal{N}$. In the subspace $\mathcal{N}=0$, we define

$$W_0=\frac{1}{N!}\int_G d^3x_1 d^3x_2\cdots d^3x_N\langle x_1x_2\cdots x_N|U|x_1x_2\cdots x_N\rangle \qquad (2\cdot3)$$

where the region G of integration is defined by the condition that all coordinates lie outside of the small volume ΔV but inside the box V. That is, W_0 is simply the probability of finding all N particles simultaneously outside the small volume ΔV.

In the space $\mathcal{N}=1$, we define the operator W_1 by the matrix elements:

$$\langle x_1|W_1|x_1'\rangle=\frac{1}{(N-1)!}\int_{R_{N-1}} d^3x_2 d^3x_3\cdots d^3x_N\langle x_1x_2\cdots x_N|U|x_1'x_2\cdots x_N\rangle \qquad (2\cdot4)$$

where the region of integration R_{N-1} is such that all of $x_2, x_3, \cdots, x_N$ lie inside V but outside ΔV; x_1 and x_1', on the other hand, are restricted to lie inside ΔV. The probability P_1 of finding exactly one particle inside ΔV, all the other $N-1$ particles outside ΔV, is then

$$P_1=\mathrm{Tr}\,W_1=\frac{1}{1!}\int_{\Delta V} d^3x_1\langle x_1|W_1|x_1\rangle. \qquad (2\cdot5)$$

Next we consider the two-particle subspace $\mathcal{N}=2$, and define the operator W_2 by the matrix elements:

$$\langle x_1x_2|W_2|x_1'x_2'\rangle=\frac{1}{(N-2)!}\int_{R_{N-2}} d^3x_3 d^3x_4\cdots d^3x_N\langle x_1x_2x_3\cdots x_N|U|x_1'x_2'x_3\cdots x_N\rangle \qquad (2\cdot6)$$

where R_{N-2} is a region such that all of $x_3, x_4, \cdots, x_N$ lie outside ΔV, and the free coordinates x_1, x_2, x_1', and x_2' all lie inside ΔV. The probability of finding exactly 2 particles in ΔV is then

$$P_2=\mathrm{Tr}\,W_2=\frac{1}{2!}\int_{\Delta V} d^3x_1\int_{\Delta V} d^3x_2\langle x_1x_2|W_2|x_1x_2\rangle. \qquad (2\cdot7)$$

The higher-order operators W_k are defined in analogy to (2·4) and (2·6), and describe the statistical situation under conditions in which exactly k particles are inside the volume ΔV. The final one, W_N, is simply equal to U, (2·1), but with the restriction that all of $x_1, x_2, \cdots, x_N, x_1', x_2', \cdots, x_N'$ must lie inside ΔV. This is of course a highly unlikely situation, and may be impossible if the particles have non-zero size (hard core interaction).

The Fock space operator W is defined by the sequence of operators W_k, i.e.,

$$W=\begin{pmatrix} W_0 \\ \langle x_1|W_1|x_1'\rangle \\ \langle x_1x_2|W_2|x_1'x_2'\rangle \\ \cdots\cdots \\ \langle x_1x_2\cdots x_N|W_N|x_1'x_2'\cdots x_N'\rangle \end{pmatrix}. \qquad (2\cdot8)$$

The operator (2·8) contains *all* the physical information about the ensemble obtainable from measurements carried on entirely within the volume ΔV. W is therefore the sum total of the "local information". It is clear from the definitions so far that W is normalized to unit trace:

$$\mathrm{Tr}\, W=\mathrm{Tr}\, W_0+\mathrm{Tr}\, W_1+\mathrm{Tr}\, W_2+\cdots+\mathrm{Tr}\, W_N=P_0+P_1+P_2+\cdots+P_N=1. \quad (2\cdot9)$$

It should be noted that all the free coordinates which appear in (2·8) are restricted to lie within ΔV. That is, the matrix element

$$\langle x_1 x_2 \cdots x_k | W_k | x_1' x_2' \cdots x_k' \rangle$$

is defined to be zero whenever at least one of $x_1, \cdots, x_k'$ lies outside ΔV.

Although this definition of the localized statistical operator W is simple and intuitive, it is not always the easiest form for practical purposes. We now proceed to show how the operator W can be written in second quantization language, given any operator U for the whole system.

We start by constructing two orthonormal sets of functions: The set $\varphi_n(x)$ is complete in the volume ΔV, and zero outside. The set $\chi_m(x)$ is complete in the volume $V-\Delta V$, and each function is zero inside ΔV. The sum of these two orthonormal sets forms a complete orthonormal set for the whole volume V.

We associate creation and destruction operators a_n^+, a_n with the states $\varphi_n(x)$ in the usual way, depending on the statistics of the particles. Similarly, we associate creation and destruction operators b_m^+, b_m with the states $\chi_m(x)$. We define the occupation number operators:

$$N_n=a_n^+ a_n \quad (2\cdot10)$$

and

$$M_m=b_m^+ b_m \quad (2\cdot11)$$

and we use simultaneous eigenstates of all these operators as our basic vectors in Hilbert space. The number of particles in ΔV is the eigenvalue of the operator

$$N(\Delta V)=\sum_n N_n. \quad (2\cdot12)$$

The original statistical operator U for the whole volume has matrix elements*

$$\langle N_1 N_2 \cdots ; M_1 M_2 \cdots | U | N_1' N_2' \cdots ; M_1' M_2' \cdots \rangle. \quad (2\cdot13)$$

The matrix elements of the localized statistical operator W are then defined as follows: The matrix element of W vanishes unless all the M_m and M_m' are zero. In the latter case,

$$\langle N_1 N_2 \cdots ; 0, 0, \cdots | W | N_1' N_2' \cdots ; 0, 0, \cdots \rangle = \sum_{M_1, M_2 \cdots} \langle N_1 N_2 \cdots ; M_1 M_2 \cdots | U | N_1' N_2' \cdots ; M_1 M_2 \cdots \rangle. \quad (2\cdot14)$$

* Here we need no longer suppose that the total number of particles in V is given.

That is, W is the partial trace of U over the Hilbert space associated with the M_m operators.

It is possible to express (2·14) as an operator relation between W and U, without writing down the matrix elements explicitly. To do this, we introduce a second, formal set of creation and destruction operators, called B_m^+ and B_m, and obeying the same commutation (or anticommutation) rules as the b_m^+, b_m. We define the operator Q by

$$Q = \sum_m b_m B_m \tag{2·15}$$

and we note that the terms of this sum commute with each other, with either statistics. Furthermore, let $\mathcal{M}$ be the linear manifold of all wave functions ψ which satisfy

$$b_m^+ b_m \psi = 0, \quad B_m^+ B_m \psi = 0 \qquad \text{(all } m\text{)} \tag{2·16}$$

and let P be the projection operator onto this manifold $\mathcal{M}$. We then assert that there exists the operator identity:

$$W = P \exp(Q)\, U \exp(Q^+)\, P. \tag{2·17}$$

The proof of this assertion is straightforward: we expand the exponentials and make use of the identity

$$P(B_m)^{M_m}(B_m^+)^{M_{m'}} P = P \delta_{M_m M_{m'}} M_m !. \tag{2·18}$$

This leads to

$$W = \sum_{M_1 M_2 \cdots}{}' P A(M_1 M_2 \cdots)\, U A^+(M_1 M_2 \cdots)\, P \tag{2·19}$$

where

$$A^+(M_1 M_2 \cdots) = \prod_m \frac{(b_m^+)^{M_m}}{(M_m !)^{1/2}}. \tag{2·20}$$

This operator, however, is just the one which, operating on the vacuum state, produces a configuration with occupation numbers $M_1, M_2, \cdots$.* Thus all matrix elements of (2·19) are identical with the results obtained from the explicit formula (2·14).

The formal, labelling operators B_m do not occur in the original U, and the projection operators P on either side of (2·17) ensure that only the vacuum state for the $B_m^+ B_m$ need ever be considered.

Equation (2·17) is our general definition of the localized statistical operator.

Although we have discussed the projection operation $U \to W$ from the point of view of localization, it should be noted that it is actually more general, and

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* In the case of Fermi-Dirac statistics, the order in which the states m are written in (2·20) matters to the extent of a sign. No ambiguity exists in (2·19) since the Hermitean conjugate operator A also occurs in (2·19).

may correspond to quite different ways of dividing the system into component parts. For example, consider an ionized plasma. It is conventional to distinguish between the "electron temperature" and the "ion temperature" of the plasma; the two can be very different in practice. If we let the operators a_n refer to electrons, the operators b_m and B_m to ions, then equation (2·17) yields the projected statistical operator for electrons only. Note that the projected statistical operator can *always* be defined, even if the electron-ion interaction is of major importance. The projected statistical operator may however be of limited usefulness, under such circumstances (see later).

§ 3. Definition of local entropy and local energy

We are now in a position to define the entropy contained in the volume ΔV. The local statistical operator W, (2·17) contains within itself all the information obtainable from measurements carried out entirely within ΔV. The entropy within ΔV, $S(\Delta V)$, can thus be defined in analogy to the entropy S of the whole system, equation (1·1), by

$$S(\Delta V) = -k \operatorname{Tr}(W \ln W) = -k \sum_{n=0}^{N} \operatorname{Tr}(W_n \ln W_n). \qquad (3\cdot1)$$

(3·1) measures the amount of disorder (neginformation) observable from measurements within ΔV, and we hereby propose this quantity as the definition of the local entropy.[3]

In the limiting case in which we take ΔV as the full volume V, we get $W=U$ and $S(\Delta V)=S(V)=S$. On the other hand, in general it will *not* be true that the overall entropy, (1·1), is given by the sum of the local entropies $S(\Delta V)$ over all the cells ΔV into which the volume of the box is divided. The extent to which S departs from the sum of the $S(\Delta V)$ gives a measure of the localizability of the entropy in the ensemble under consideration. If the cells ΔV are chosen so that each one contains many particles on the average, but is nevertheless small compared with dimensions over which the local properties change appreciably, then we expect the sum of the $S(\Delta V)$ to lie close to S. But no such choice of cells may be possible; i.e., regions small enough to be locally uniform may contain too few particles on the average to make a statistical treatment meaningful, or else there may exist long range correlations. In such a case, the sum of the $S(\Delta V)$ deviates appreciably from the total S.

The definition (3·1) does not require any special assumptions about the original statistical matrix U. In particular, it is *not* necessary to have local equilibrium in order to define a local entropy.

The local statistical operator W can be used to define not only the local entropy, but other local properties of the system. For example, the local energy $E(\Delta V)$ contained in ΔV is defined by

$$E(\Delta V) = \mathrm{Tr}(HW) \tag{3·2}$$

where H is the Hamiltonian of the system.

There are some delicate points in connection with this last definition. First, if H contains a sum of potential energies of pairs of particles, $v(r_{ij})$, then it may happen that particle i is in ΔV whereas particle j is outside ΔV. Such interactions are not counted in $E(\Delta V)$, and to this extent the sum of the $E(\Delta V)$ differs from $E=\mathrm{Tr}(HU)$. Unless $v(r_{ij})$ is longranged, however, this is a surface effect which can be made small by increasing the cell-size ΔV. If $v(r_{ij})$ is the long-ranged Coulomb interaction, the energy can be localized only by explicit introduction of the electric potential.

The second difficulty arises in connection with the kinetic energy. The complete set $\varphi_n(x)$ for the volume ΔV satisfies some boundary conditions on the boundary of ΔV; for example, we may choose $\varphi_n(x)=0$ on the boundary of ΔV. Similarly, the complete set $\chi_m(x)$ for the remainder of the volume, $V-\Delta V$, satisfies some boundary condition on the boundary of ΔV, and let us suppose for the sake of illustration that $\chi_m(x)=0$, also, on this boundary. Now a typical eigenfunction of H in the volume V does not vanish on the boundary of ΔV. It can be expanded in terms of the combined set of functions $\varphi_n(x)$ and $\chi_m(x)$, but the expansion cannot be differentiated term by term, and difficulties arise with the second derivatives in the kinetic energy operator. However, the errors which are made by ignoring these problems again amount to surface corrections only, *unless* a particular single-particle (or effective single-particle) quantum state is occupied by a macroscopic number of particles; this happens in Bose-Einstein condensation. Thus local energies cannot be defined sensibly for a condensed Bose-Einstein gas.

The projection $U\to W$ need not refer to localization but may correspond to some other division of the system into parts, as discussed at the end of section 2. We define $U\to\widetilde{W}$ to be the "complementary" projection, e.g., the projection onto the volume $V-\Delta V$ for the localization case, the projection onto the ions when $U\to W$ is a projection onto electrons, etc. We can then state a necessary condition that this division of the system into parts be of immediate physical usefulness: The sum of the partial entropies must be nearly equal to the full entropy:

$$S(U) \cong S(W) + S(\widetilde{W}). \tag{3·3}$$

This condition takes the place of the usual, more vague, requirement that the interaction between the two parts of the system be weak enough to "allow the definition of separate temperatures". As we shall see in the next section, partial temperatures can not be defined in a satisfactory fashion.

Considering the entropy, as we do here, as a measure of disorder or "neginformation", condition (3·3) says that the separation of a system into parts is immediately useful if the information lost by ignoring the statistical correlations between the parts does not increase the disorder (neginformation) significantly.

Although (3·3) is a necessary condition to have the division into parts immediately useful, we do *not* consider it a necessary condition on the concept of partial entropies. If the two sides of (3·3) differ, then the difference is to be interpreted as an "entropy of interaction" between the parts; it is the disorder arising from the interaction between the two parts of the system. The situation is quite analogous to that of partial energies: We do not reject the concept of

the energy of a subsystem, merely because there are also interaction energies between subsystems.

Condition (3·3) is a condition on the density matrix, not on the Hamiltonian. However, if we require that (3·3) be satisfied not only at time $t=0$ but also for some significant (compared to relaxation times) time interval thereafter, then (3·3) becomes an implicit condition on the interaction Hamiltonian. In the Appendix, we calculate a specific example, and mention two others, in which (3·3) holds even though particles can pass freely between the two parts into which the system is divided. Thus, although a weak interaction between the two parts implies (3·3), the converse is not true.

§ 4. Non-equilibrium states

In equilibrium thermodynamics, the macroscopic state of a system can be described in a number of different, equivalent ways. For example, we may specify the volume V *or* the pressure p, the temperature T *or* the entropy S, the number of particles N *or* the chemical potential μ. This choice is purely a matter of convenience. The transformation from one set of variables to another is effected by means of the thermodynamic potentials. Let x and y be two thermodynamically conjugate variables, such as p and V, and let Φ be the appropriate thermodynamic potential for use with the variable x. Then y is defined by

$$y=\frac{\partial \Phi}{\partial x} \tag{4·1}$$

where the other independent variables are kept constant during the differentiation. A suitable thermodynamic potential exists for every reasonable choice of independent variables. For example, the free energy $F=F(V, T, N)$ and the internal energy $E=E(V, S, N)$ are the appropriate thermodynamic potentials for the independent variables V, T, N and V, S, N, respectively; T is defined in terms of $E(V, S, N)$ by

$$T=\left(\frac{\partial E}{\partial S}\right)_{V,N} \tag{4·2}$$

in accordance with the general rule (4·1), and in agreement with (1·4).

The situation is completely different, however, in the absence of full equilibrium. It seems reasonable to require that *the macroscopic state of a system must be deducible from its microscopic state, i.e., from its statistical matrix U at time t, without reference to the time-development of the system, i.e., without reference to the Hamiltonian H.* For example, if the system is in a definite quantum state ψ at time t, specification of the wave function ψ specifies the state of the system; two systems with completely different Hamiltonians can nevertheless be in the same state ψ at the particular time t. Of course, they will not remain in the same state thereafter, but the state at time t is the same. If

this point of view is extended to statistical systems, which are described by a density matrix U rather than by a single wave function ψ, our postulate follows directly. We shall call a macroscopic variable *state-determined* if its value at time t can be deduced directly from the density matrix $U(t)$, without knowledge of the Hamiltonian H of the system.

Now any thermodynamic potential involves knowledge of the Hamiltonian. The internal energy E is the statistical average of H, see (1·2), and all other thermodynamic potentials contain E, for example

$$F=E-TS. \tag{4·3}$$

We conclude from the general relation (4·1) that *if the macroscopic variable x is state-determined, its thermodynamic conjugate variable y is not state-determined.*

Since we must restrict ourselves to state-determined variables, *for non-equilibrium systems, the choice of macroscopic variables is no longer free.* To be precise, of the pairs (V, p), (S, T), and (N, μ) we *must* choose V, S, N as our independent macroscopic variables. The other ones violate our basic requirement.

Actually, this is quite well-known for the pair of variables V, p. In non-equilibrium states, the pressure is not a suitable variable, since more complicated stress systems are possible even in a system which is homogeneous and isotropic under equilibrium conditions, such as a gas. Noone suggests using the gas pressure as a suitable independent variable away from equilibrium. The "state-determined" feature is also quite obvious: Knowing the distribution function of the gas molecules in phase space at time t is simply not enough to determine the pressure which the gas exerts on the walls of the container. This knowledge must be supplemented by knowledge of the forces with which the gas molecules attract or repel each other, i.e., by knowledge of the Hamiltonian of the system. On the other hand, the volume is clearly defined, no matter what the Hamiltonian of the particles within that volume.

Since the entropy of the whole system, S, and the entropy of a subsystem, $S(\Delta V)$, are obviously state-determined macroscopic variables, it follows that the temperature, overall or partial, is not state-determined, and thus local temperature is *in principle* an unsuitable macroscopic variable. This explains why it has proved so difficult to define local temperature: it just cannot be done!

There exist a number of variation principles for the equilibrium state. For example, the Gibbs distribution (1·5) is the one which gives maximum entropy S, subject to given internal energy E, volume V, and number of particles N.[4] From our present point of view, it would be preferable to have a variation principle in which the "given" quantities are state-determined, and the quantity to be varied depends on the Hamiltonian. This is easy to find; we merely observe that $E=E(V, S, N)$ is a thermodynamic potential, and hence assumes its minimum value in equilibrium. The variation principle therefore reads

$$E=\mathrm{Tr}(HU)=\text{minimum} \tag{4·4a}$$

subject to

$$-k\,\mathrm{Tr}(U\ln U)=S, \tag{4·4b}$$

$$\mathrm{Tr}(\mathcal{N}(U)=N, \tag{4·4c}$$

$$\mathrm{Tr}(U)=1. \tag{4·4d}$$

If the number of particles is known, condition (4·4c) is replaced by the stronger condition that $U=0$ unless the number of particles equals the given value N.

It is only with a "state-determined variation principle" that one gets a simple limit for the absolute zero. In our present context, the absolute zero is defined by the condition $S=0$. It is well known that this condition, when substituted into (4·4b), implies that U must be a projection operator onto some one definite quantum state ψ. (4·4d) becomes the normalization condition for the wave function ψ, and (4·4a) simply reduces to

$$E=(\psi, H\psi)=\text{minimum} \quad (\text{for } S=0) \tag{4·5}$$

that is, the ordinary Rayleigh-Ritz variation principle of quantum mechanics.

By contrast, the variation principle of maximum entropy for given E, cannot be used at all in the limit of the absolute zero. For we would have to *specify* the ground state energy E, which is a quantity we don't know for a complicated system, and *deduce* the zero value of the entropy, which is really what we start from.

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Appendix

Local Entropy in an Independent Particle System

As an illustration, we now determine the local entropy, contained in a volume ΔV, of a classical independent particle system. Although the *particles* are independent, the *volumes* ΔV and $V-\Delta V$ can hardly be said to be in weak interaction; particles continually cross the boundary between these two volumes, in both directions. Furthermore, we do *not* restrict the initial distribution function in any way, beyond requiring no correlation between particles at time $t=0$. For example, the situation in which all particles are in one half of the box at time $t=0$, and thereafter diffuse freely, is included in this calculation.

In the classical limit[3] the density matrix U becomes a density function in phase space, which we write in the form:

$$\rho(p_1q_1, \cdots, p_N q_N, t) = N! \prod_{i=1}^{N} f(p_i, q_i, t). \tag{A·1}$$

This product form corresponds to the independent particle assumption; the function $f(p, q, t)$ can be specified freely at time $t=0$. The trace goes over into the integral:

$$\mathrm{Tr}(U) \to \int \frac{d^{3N}p\, d^{3N}q}{N!\,(h)^{3N}} \rho(p_1q_1\cdots) = \left[(1/h^3) \int d^3p\, d^3q f(p, q, t) \right]^N = 1 \tag{A·2}$$

so that the normalization condition on f is

$$h^{-3} \int d^3p\, d^3q f(p, q, t) = 1. \tag{A·3}$$

The total entropy in the volume V is given by the classical limit of (1·1), namely

$$S = -k \int \frac{d^{3N}p\, d^{3N}q}{N!\,(h)^{3N}} \rho \ln \rho = -k \ln (N!) - kNh^{-3} \int d^3p\, d^3q f \ln f. \tag{A·4}$$

We use the definition

$$L(V) = h^{-3} \int d^3p\, d^3q f \ln f \tag{A·5}$$

and the Stirling approximation for $\ln(N!)$, to get

$$S(V) \doteq -kN[\ln N - 1 + L(V)]. \tag{A·6}$$

The notation $S(V)$ serves to indicate that this is the entropy for the entire volume. In spite of the term $N \ln N$, $S(V)$ has the correct dependence on volume, since $L(V)$ contains a contribution $-\ln V$. The Liouville equation ensures that (A·6) is independent of time.

We now project onto the subvolume ΔV. The probability of finding a single particle in ΔV, assuming that there are no other particles (i. e., $N=1$), is given by

$$w_1 = h^{-3} \int_{\Delta V} d^3q \int d^3p\, f(p, q, t) \tag{A·7}$$

and we shall also use the notation

$$w_0 = 1 - w_1 = h^{-3} \int_{V-\Delta V} d^3q \int d^3p\, f(p, q, t). \tag{A·8}$$

The classical analogue of the projected density matrix W_n of section 2 is

$$\rho_n(p_1 q_1, \cdots, p_n q_n, t) = \int_{V-\Delta V} \frac{d^{3(N-n)}q\, d^{3(N-n)}p}{(N-n)!\,(h)^{3(N-n)}} \rho(p_1q_1, \cdots)$$

$$= \frac{N!}{(N-n)!} w_0^{N-n} \prod_{i=1}^{n}{}' f(p_i, q_i, t) \tag{A·9}$$

where the prime on the product indicates that all the coordinates q_i must lie within ΔV. Integration over the phase space of the n particles then gives P_n, the probability of finding exactly n particles in ΔV, as:

$$P_n = \frac{N!}{n!(N-n)!} w_1^n \, w_0^{N-n}. \tag{A·10}$$

The sum of all the P_n is unity, as it must be.

The $\mathrm{Tr}(W_n \ln W_n)$ which appears in the quantum mechanical expression for the entropy, (3·1), has the classical limit:

$$\int_{\Delta V} \frac{d^{3n}p \; d^{3n}q}{n!\,(h)^{3n}} \rho_n \ln \rho_n = P_n \left[\ln \left(\frac{N!\, w_0^{N-n}}{(N-n)!} \right) + nL(\Delta V) \right] \tag{A·11}$$

where

$$L(\Delta V) = \frac{\int_{\Delta V} d^3p \; d^3q f \ln f}{\int_{\Delta V} d^3p \; d^3q f} = (w_1 h^3)^{-1} \int_{\Delta V} d^3p \; d^3q f \ln f. \tag{A·12}$$

We must evaluate the sum of (A·11) over all n. As a result of the factor P_n in front, there is a sharp peak at the expected value of n,

$$\bar{n}(\Delta V) = w_1 N. \tag{A·13}$$

Using Stirling's approximation, and replacing $w_0 = (N-\bar{n})/N$ by $(N-n)N$, we get

$$\ln \frac{N!\, w_0^{N-n}}{(N-n)!} \cong n(\ln N - 1) \tag{A·14}$$

A detailed analysis shows that the error terms in (A·14) are negligible after the summation over n. The result for the local entropy therefore is:

$$S(\Delta V) = -k\bar{n}(\Delta V)[\ln N - 1 + L(\Delta V)] + \mathrm{Order}(1). \tag{A·15}$$

Unlike the total entropy (A·6), $S(\Delta V)$ is time-dependent, both through the factor $\bar{n}(\Delta V)$ in front, and through the quantity $L(\Delta V)$.

By interchanging the roles of ΔV and $V - \Delta V$, we obtain for the entropy of the complementary volume:

$$S(V-\Delta V) = -k\bar{n}(V-\Delta V)[\ln N - 1 + L(V-\Delta V)] + \mathrm{Order}(1) \tag{A·16}$$

where the notation is obvious.

Although both $S(\Delta V)$ and $S(V-\Delta V)$ are time-dependent, it follows from (A·15), (A·16), (A·13), and (A·12) that their sum is time-independent and equal to the total entropy computed directly:

$$S(\Delta V) + S(V-\Delta V) = S(V) + \mathrm{Order}\ (1). \tag{A·17}$$

This result may appear quite surprising at first sight. For example, if the par-

ticles are initially all in one half of the box, and ΔV and $V-\Delta V$ denote the two halves of the box, (A·17) shows that we do *not* get the extra term $kN \ln 2$ in the entropy, expected in final equilibrium.

The derivation of (A·17) depends on the independent-particle, and classical character of the system. We do not know the general conditions under which (A·17) holds. However, we know of two other systems for which entropy is additive.

(1) A classical gas of repulsive, infinitesimally small spheres, constrained to move along a line. Since a collision between two such mass-points merely exchanges their identity, this system is equivalent, in statistical mechanics, to a one-dimensional ideal gas.

(2) A quantum-mechanical, fully condensed ideal Bose-Einstein gas; all particles are assumed to occupy the same single-particle quantum state, but the state in question is not assumed to be an eigenstate of the Hamiltonian, so that the system changes its properties as time goes on. The total entropy $S(V)$ is of course zero, and the sum of the partial entropies is non-zero. But it can be shown that this sum is only of order $\ln N$, and thus of no consequence in statistical mechanics.

The division of the volume V into cells is a "coarse-graining" process. From the point of view of coarse-graining, then, the equality (A·17) shows that, for these three systems, coarse-graining in x-space alone is not enough to hide the memory of the initial distribution function.

From our present point of view, however, it is more interesting to observe that *(A·17), and thus its equivalent (3·3), holds for several systems in which the different cells are by no means isolated from each other, even approximately.* Particles pass freely between cells, yet the entropy is additive except for negligible terms. We coclude that, although a weak interaction between cells implies additivity, the converse is not true.

References

1) R. Kubo, review article to be published in Prog. Theor. Phys. Supplements. "*Some Aspects of the Statistical-Mechanical Theory of Irreversible Processes*," *Lectures in Theoretical Physics* (Edited by W. E. Britten and L. G. Dunham) Vol. 1 (Interscience Publishers, INC., New York, 1959).
2) This is the "fine-grained" entropy, which fails to increase with time in an isolated system if the latter is idealized to be enclosed in perfectly reflecting mirror walls. Compare: J. M. Blatt, Prog. Theor. Phys. **22** (1959), 745.
3) In the classical limit, it can be shown that (3·1) reduces to the definition suggested by R. L. Stratonovich, Jour. Exp. Theor. Phys. U. S. S. R. **28** (1955), 409; Soviet Physics—JETP **1** (1955), 254.
4) Landau and Lifschitz, *Statistical Physics* (Pergamon Press, 1959).

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Nuclear Potential in Many-Body Problems

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We investigate whether the nuclear potential in the many-body system is the same with that in the two-body system or not. At first sight, it seems that the nuclear potentials are different in these two cases, because the momenta of mediating mesons are limited due to the Pauli principle in the case of many-body systems. It is concluded, however, that the energy of a nucleon gas can be calculated correctly by using the T.M.O. potential which is constructed under the assumption of the existence of only two nucleons, and that extra many-body forces must not be introduced into the system in addition to the above two-body potential. For simplicity, we consider an idealized nucleon gas which interacts with a symmetrical scalar meson field so weakly that the perturbation-theoretic treatment is justified.

§ 1. Introduction

Various theories have been proposed up to the present about the meson-theoretical construction of the nuclear potential between two nucleons.[1)–3)] For example, Brueckner and Watson[1)] constructed it by calculating the Feynman diagrams shown in Fig. 1. It is, however, very doubtful to apply such a potential which is derived in the presence of only two nucleons to the study of many-body systems, because the momenta of mediating mesons are limited due to the Pauli principle. (The virtual nucleons represented by bold lines in Fig. 1, for example, must be in unoccupied states, consequently the momenta of the mesons in those figures cannot be arbitrary.) It is the purpose of this paper to resolve this doubt.

In § 2, we calculate the energy of a nucleon gas without introducing any nuclear potential, that is, by considering the meson field explicitly. For simplicity, we here consider the nucleon gas which interacts so weakly with a symmetrical scalar meson field that the perturbation-theoretic treatment is meaningful. (Such a system, of course, collapses. But it is not the purpose of this paper to discuss the saturation of the nucleon gas. We assume that the density of the nucleon gas is kept at an appropriate value by the origin other than the particular properties of the interaction, for example, by confining the entire system in a completely adiabatic box.) In § 3, we calculate the formal expression of the energy of the

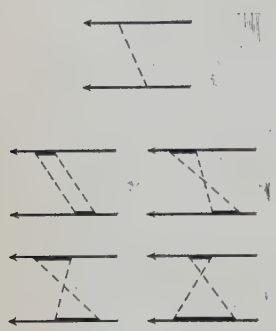

Fig. 1. Feynman diagrams used by Brueckner and Watson. In this paper, the time progresses from the right to the left.

nucleon gas without using any meson field explicitly, that is, by introducing a nuclear potential. Comparing this expression with the energy in the previous section, we decide the nuclear potential which gives the energy of a nucleon gas correctly. As the results, it is shown that the T.M.O. potential which is derived from the two-body problem gives correctly the energy of a nucleon gas, if we use the renormalized value for the coupling constant, and that the introduction of extra many-body forces is prohibited within the present approximation.

§ 2. Energies of a nucleon gas

The Hamiltonian of our system is

$$H=H_0+H_1, \tag{1}$$

$$H_0=\sum E_p a^+_{p,t} a_{p,t}+\sum \omega_q b^+_{q,i} b_{q,i}, \tag{2}$$

$$H_1=\sum \frac{g_0}{\sqrt{2\Omega\omega_q}}[(t_1|\tau_i|t_2)\, a^+_{p+q,t_1} a_{p,t_2} b_{q,i}+\text{c.c.}], \tag{3}$$

where $a^+_{p,t}$ and $a_{p,t}$ are the usual creation and annihilation operators for nucleons and $b^+_{q,i}$ and $b_{q,i}$ are those for symmetrical scalar mesons, $\boldsymbol{p}$ and $\boldsymbol{q}$ are wave vectors, t stands for the isotopic spin state of a nucleon, i for the charge state of a meson, $\boldsymbol{\tau}$ is an isotopic spin operator, $E_p=\boldsymbol{p}^2/2M$, $\omega_q=\sqrt{\boldsymbol{q}^2+\mu^2}$, M and μ are, respectively, the mass of a nucleon and a meson and g_0 is the unrenormalized coupling constant. We take $\hbar=c=1$. We confine the entire system in a cube with a volume Ω and require a periodicity condition on the surface. For simplicity, we do not consider Pauli spins but isotopic spins only. So, two nucleons can occupy an energy level.

Regarding H_0 as the unperturbed Hamiltonian and H_1 as a perturbation, we calculate the ground-state energy of the nucleon gas. In order to avoid the difficulty of divergences, we concern ourselves only with the difference between the energy at the density in question and that at the low density limit. It is this difference to be observed as the binding energy.

There are two processes in the second-order as shown in Fig. 2. The value of (a), however, is zero because of $\boldsymbol{\tau}$-spin. The value of (b) is

$$E^{(2)}=\frac{3g_0^2}{\Omega}\sum_{p_1 p_2}(|\boldsymbol{p}_1|<p_F,\ |\boldsymbol{p}_2|<p_F)\frac{1}{\omega^2_{p_1-p_2}}, \tag{4}$$

where we have already subtracted a constant which is independent of the nucleon density and neglected the terms of $O(\mu^2/M^2)$. In Eq. (4), p_F is the momentum of a nucleon at the Fermi surface.

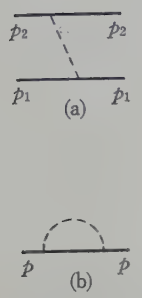

Fig. 2. Second-order diagrams.

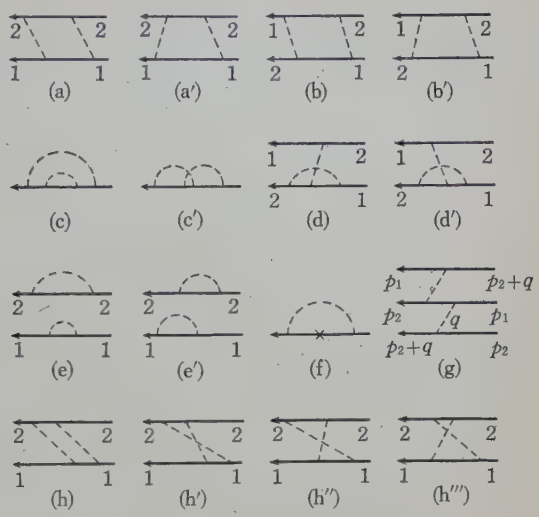

Fig. 3. Fourth-order diagrams. (f) means the expansion of the energy denominator in the second-order self-energy.

The fourth-order processes are drawn in Fig. 3. The diagrams of which the values are zero are not drawn there.

As an example of calculations, we first consider diagrams (a) and (a'). Expanding them with respect to μ/M, we have

$$E_a^{(4)}+E_{a'}^{(4)}=(E_a^{(4)}+E_{a'}^{(4)})_{Ad}+(E_a^{(4)}+E_{a'}^{(4)})_{Non\text{-}ad}+O(\mu/M), \tag{5}$$

$$(E_a^{(4)}+E_{a'}^{(4)})_{Ad}=-\frac{6Mg_0^4}{\Omega^2}\sum_{\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{q}}(|\boldsymbol{p}_1|<p_F,\ |\boldsymbol{p}_2|<p_F,\ |\boldsymbol{p}_1+\boldsymbol{q}|>p_F,\ |\boldsymbol{p}_2+\boldsymbol{q}|>p_F)$$

$$\times\frac{1}{\omega_q^4[\boldsymbol{q}\cdot(\boldsymbol{q}+\boldsymbol{p}_1+\boldsymbol{p}_2)]}, \tag{6}$$

$$(E_a^{(4)}+E_{a'}^{(4)})_{Non\text{-}ad}=\frac{6g_0^4}{\Omega^2}\sum_{\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{q}}(|\boldsymbol{p}_1|<p_F,\ |\boldsymbol{p}_2|<p_F,\ |\boldsymbol{p}_1+\boldsymbol{q}|>p_F,\ |\boldsymbol{p}_2+\boldsymbol{q}|>p_F)$$

$$\times\frac{1}{\omega_q^5}. \tag{7}$$

Eq. (6) corresponds to the repetition of the second-order adiabatic potential [c.f. Eq. (15)] and Eq. (7) is a nonadiabatic effect.

As another example, we next consider the fourth-order self-energy (diagram (c)). This is

$$E_c^{(4)}=-\frac{9}{2}\frac{g_0^4}{\Omega^2}\sum_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{q}'}(|\boldsymbol{p}|<p_F,\ |\boldsymbol{p}+\boldsymbol{q}|>p_F,\ |\boldsymbol{p}+\boldsymbol{q}+\boldsymbol{q}'|>p_F)$$

$$\times\frac{1}{\omega_q^3\omega_{q'}(\omega_q+\omega_{q'})}. \tag{8}$$

This energy contains a constant part which does not vanish at the low density limit. After subtracting this constant, we have

$$E_c^{(4)} \to \frac{9}{2}\frac{g_0^4}{\Omega^2}\sum_{p_1,p_2,q}(|\boldsymbol{p}_1|<p_F,\ |\boldsymbol{p}_2|<p_F)\left\{\frac{1}{\omega_q^{\,2}\,\omega_{p_1-p_2}^3}-\frac{1}{\omega_q^{\,3}\,\omega_{p_1-p_2}^2}\right.$$
$$\left.+\frac{1}{\omega_q^{\,3}\,\omega_{p_1-p_2}\,(\omega_q+\omega_{p_1-p_2})}\right\} \tag{9}$$
$$+\frac{9}{2}\frac{g_0^4}{\Omega^2}\sum_{p_1,p_2,q}(|\boldsymbol{p}_1|<p_F,\ |\boldsymbol{p}_2|<p_F,\ |\boldsymbol{p}_1+\boldsymbol{q}|>p_F)\frac{1}{\omega_q^{\,3}\omega_{q+p_1+p_2}(\omega_q+\omega_{q+p_1+p_2})}.$$

We do not write out the values of the other diagrams, because we can calculate them very easily. In all of them, the meson momenta are limited as in Eqs. (6), (7) and (9), but the sum of all fourth-order diagrams is

$$E^{(4)}=-\frac{3Mg_0^4}{\Omega^2}\sum_{p_1,p_2,q}(|\boldsymbol{p}_1|<p_F,\ |\boldsymbol{p}_2|<p_F,\ |\boldsymbol{p}_1+\boldsymbol{q}|>p_F,\ |\boldsymbol{p}_2+\boldsymbol{q}|>p_F)$$
$$\times\left\{\frac{2}{\omega_q^{\,4}[\boldsymbol{q}\cdot(\boldsymbol{q}+\boldsymbol{p}_1+\boldsymbol{p}_2)]}+\frac{1}{\omega_q^{\,2}\,\omega_{q+p_1+p_2}^2[\boldsymbol{q}\cdot(\boldsymbol{q}+\boldsymbol{p}_1+\boldsymbol{p}_2)]}\right\}$$
$$+\frac{6g_0^4}{\Omega^2}\sum_{p_1,p_2,q}(|\boldsymbol{p}_1|<p_F,\ |\boldsymbol{p}_2|<p_F)$$
$$\times\left\{-\frac{2}{\omega_q^{\,3}\omega_{p_1-p_2}^2}+\frac{1}{\omega_q^{\,3}\omega_{q+p_1+p_2}^2}+\frac{1}{\omega_q^{\,3}\omega_{q+p_1+p_2}(\omega_q+\omega_{p_1+p_2+q})}\right\}, \tag{10}$$

and the meson momentum is not limited at all in the term which is independent of M.

The first term in the curly bracket in Eq. (10) is divergent, but this term is proportional to $E^{(2)}$. Performing the charge renormalization, that is, substituting the renormalized value g for g_0, we may drop this term.

§ 3. Nuclear Potential

The above system is described alternatively with the Hamiltonian

$$H=K+U,\quad K=\sum\frac{\boldsymbol{p}_n^{\,2}}{2M}, \tag{11}$$

where U is a nuclear potential with the force range$\sim\mu^{-1}$. Regarding U as a perturbation, we calculate the ground-state energy of the system.

If U is expressed in terms of a coupling strength $g^2/4\pi$ and of a ratio μ/M as follows,

$$U=\frac{g^2}{4\pi}\left\{v+\frac{\mu}{M}v'+\left(\frac{\mu}{M}\right)^2v''+\cdots\right\}$$
$$+\left(\frac{g^2}{4\pi}\right)^2\left\{u+\frac{\mu}{M}u'+\left(\frac{\mu}{M}\right)^2u''+\cdots\right\}+\cdots, \tag{12}$$

the energy of the system is

$$E=\langle K\rangle+\frac{g^2}{4\pi}\left\{\langle v\rangle+\frac{\mu}{M}\langle v'\rangle+\left(\frac{\mu}{M}\right)^2\langle v''\rangle+\cdots\right\}$$
$$+\left(\frac{g^2}{4\pi}\right)^2\left\{\frac{M}{\mu}\langle v(1/\alpha)v\rangle+\langle u+v(1/\alpha)v'+v'(1/\alpha)v\rangle\right.$$
$$\left.+\frac{\mu}{M}\langle u'+v(1/\alpha)v''+v''(1/\alpha)v+v'(1/\alpha)v'\rangle+\cdots\right\}+\cdots, \qquad (13)$$

where $\langle A\rangle$ stands for the expectation value of A in the ground state of H_0 and

$$1/\alpha=(\mu/M)[\langle K\rangle-K]^{-1} \qquad (14)$$

which is an operator of the order of unity owing to the factor μ/M.

By comparing Eq. (13) with the energy calculated in the previous section, we can decide the nuclear potential uniquely. In the first place, by comparing the $g^2/4\pi$-term in Eq. (13) with $E^{(2)}$ [Eq. (4)], we have

$$\frac{g^2}{4\pi}v=-\frac{1}{2}\sum_{n\neq m}\frac{g^2}{\Omega}\boldsymbol{\tau}^{(n)}\cdot\boldsymbol{\tau}^{(m)}\sum_k\frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{r}_n-\boldsymbol{r}_m)}}{\omega_k^2}$$
$$=\frac{1}{2}\sum_{n\neq m}\left(-\frac{g^2}{4\pi}\right)\boldsymbol{\tau}^{(n)}\cdot\boldsymbol{\tau}^{(m)}\frac{e^{-\mu R_{nm}}}{R_{nm}}, \qquad (15)$$

where $R_{nm}=|\boldsymbol{r}_n-\boldsymbol{r}_m|$.

Then, comparing the $(g^2/4\pi)^2$-term in Eq. (13) with $E^{(4)}$ [Eq. (10)], we have

$$\left(\frac{g^2}{4\pi}\right)^2u=-\sum_{n\neq m}\frac{g^4}{\Omega^2}\boldsymbol{\tau}^{(n)}\cdot\boldsymbol{\tau}^{(m)}\sum_{k_1,k_2}\left[\frac{1}{\omega_{k_1}^3\omega_{k_2}^2}+\frac{1}{\omega_{k_1}^3\omega_{k_2}(\omega_{k_1}+\omega_{k_2})}\right]e^{i(\boldsymbol{k}_1+\boldsymbol{k}_2)\cdot(\boldsymbol{r}_n-\boldsymbol{r}_m)}$$
$$=\frac{1}{2}\sum_{n\neq m}\left[-\left(\frac{g^2}{4\pi}\right)^2\right]\boldsymbol{\tau}^{(n)}\cdot\boldsymbol{\tau}^{(m)}\frac{8}{\pi}\frac{K_0(2\mu R_{nm})}{R_{nm}}, \qquad (16)$$

where K_0 is a modified imaginary Bessel function.

Thus, the two-body potential which gives the energy of the nucleon gas correctly to the fourth-order is

$$U_{1,2}=\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)}\left[-\frac{g^2}{4\pi}\frac{e^{-\mu R_{12}}}{R_{12}}-\left(\frac{g^2}{4\pi}\right)^2\frac{8}{\pi}\frac{K_0(2\mu R_{12})}{R_{12}}\right]. \qquad (17)$$

This is just the nuclear potential of Nishijima-T.M.O. type, but not that of B. W. type. By the way, the B. W. potential for the present simple model is

$$U_{1,2}^{B.W.}=\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)}\left[-\frac{g^2}{4\pi}\frac{e^{-\mu R_{12}}}{R_{12}}+\left(\frac{g^2}{4\pi}\right)^2\frac{4}{\pi}\frac{e^{-\mu R_{12}}}{R_{12}}K_0(\mu R_{12})\right.$$
$$\left.-\left(\frac{g^2}{4\pi}\right)^2\frac{8}{\pi}\frac{K_0(2\mu R_{12})}{R_{12}}\right]$$
$$+\left[-\left(\frac{g^2}{4\pi}\right)^2\frac{6}{\pi}\frac{e^{-\mu R_{12}}}{R_{12}}K_0(\mu R_{12})\right]. \qquad (18)$$

§ 4. Conclusion

We here have analized perturbation-theoretically the energy of a nucleon gas which interacts with a symmetrical scalar meson field and have obtained the conclusion that the energy can be calculated correctly with the T.M.O. two-body potential.

In many-body systems, the momenta of mediating mesons are limited on account of Pauli principle. So the interaction in its proper sense is modified. But, the mass and the mesonic charge of a nucleon are changed at the same time. Moreover, many-body forces arise [(g) in Fig. 3]. These shifts are canceled by each other as a whole. The net energy can be calculated with the T.M.O. two-body potential.

Thus, in many-body problems, we must be careful to introduce any extra many-body forces, if we use the ordinary two-body potential. At least, it is a mistake to introduce the three-body force of the g^4-order into the present simple system.

References

1) Brueckner and Watson, Phys. Rev. **92** (1953), 1023.
(We call the potential constructed according to the method of this paper the B.W. potential.)
2) Nishijima, Prog. Theor. Phys. **6** (1951), 815, 911.
Taketani, Machida, and Onuma, Prog. Theor. Phys. **7** (1952), 45. (We call the potential constructed according to the method of these papers the T.M.O. potential.)
3) Fukuda, Sawada and Taketani, Prog. Theor. Phys. **12** (1954), 156.

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Internal Quantum States of Hyperspherical (Nakano) Relativistic Rotators

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If one quantizes with conventional methods the classical internal motion of the hyperspherical (Nakano) relativistic rotator, the relativistic Hamiltonian splits into the sum of two complex conjugate spherical complex three-dimentional rotators. The corresponding excited "levels" also split into fine structure states, each "level" being naturally associated to a vector space irreducible under complex transformations isomorphic to the full Lorentz group. It finally turns out that every state is characterized by a set of quantum numbers with which we can classify typical families of rotator levels.

Introduction

In a recent paper[1] in collaboration with T. Takabayasi we have developed the classical theory of relativistic rotators on the basis of Lagrangian and Hamiltonian formalism and stressed its connection with bilocal theory. This raises immediately a new problem: What happens when we substitute relativistic rotators for point particles as possible starting point of quantum mechanics? As we shall see it turns out that the exploration of this problem raises many difficult questions, but may also open far-reaching perspectives on the nature and physical interpretation of quantum mechanics itself.

However, in order not to confuse the picture, let us make a few preliminary remarks.

The first is that we consider this paper as a very simple first step towards the complete understanding of the preceding problem.

The second remark is that we shall now voluntarily and systematically leave aside all problems of physical interpretation (which shall be treated in subsequent papers*). The reader will find here only some mathematical consequences of the formal application of conventional methods of quantization. We only wish to say at this stage:

a. that we shall connect these rotators with real physical properties of the "subquantum level of matter" recently introduced[2] as a necessary consequence of the causal interpretation of quantum mechanics;

* To be submitted shortly to the Physical Review.

b. that we aim to justify later this conventional procedure on the basis of material properties of this subquantum level (such as continuity and pulse-like motions) so that its utilization is not necessarily connected with the usual difficulties of present field theories.

Now let us return to our problem. As one knows the passage from a line classical Hamiltonian $H(\Pi_\alpha^\xi, q_\mu^\xi)$ for the corresponding operator (acting on the wave function Φ) is realized by substituting operators $-j\hbar\cdot\partial/\partial q_\mu^\xi$ for the conjugate momenta $\Pi_\mu^\xi=(\partial/\partial\dot{q}_\mu^\xi)L(q_\alpha^\xi, \dot{q}_\alpha^\xi)$ associated with the classical Lagrangian $L(q_\alpha^\xi, \dot{q}_\alpha^\xi)$ or alternatively by replacing Poisson brackets $[\Pi_\alpha^\xi, q_\alpha^\xi]=-1$ by commutators:

$$[\Pi_\alpha^\xi, q_\alpha^\xi]=-j\hbar.$$

One must remark here, as Møller and Synge[3] have shown, that when one operates in Minkowski space a distinction must be made between the imaginary quantity $j(j^2=-1)$ utilized in the quantization procedure and the usual $i(i^2=-1)$ which is naturally introduced in the theory by the very nature of four-dimensional space-time. Indeed, when we introduce relativistic Euler angles the theory will be closely bound to the quaternion group (with complex coefficients) and we shall show that i and j can be identified with two of the three classical quaternionic units.

To illustrate this procedure we can apply this method of quantization to the classical point particle. As we have shown[1] the line Lagrangian associated with its motion can be written:

$$L=\frac{1}{2}m\dot{x}_\mu\dot{x}_\mu.$$

If we therefore substitute $-j\hbar\cdot\partial/\partial x_\mu$ for $G_\mu=\partial L/\partial\dot{x}_\mu$ in the relativistic Hamiltonian,

$$H=\frac{1}{2m}G_\mu G_\mu=-\frac{1}{2}mc^2,$$

we get

$$H=-\frac{\hbar^2}{2m}\Box,$$

where $\Box$ is the dalembertian and the corresponding wave equation just reduces to

$$(\Box-\chi^2)\Phi=0$$

with $\chi^2=(m^2c^2/\hbar^2)$, that is, the usual Klein-Gordon equation.

If we start now from a Nakano rotator* one can show that its internal Lagrangian with the use of a moving tetrad b_μ^ξ:

* Various types of Lagrangians implying a constant rest mass are possible. One of these proposed by one of us (J.P.V.) takes the simple form $L_{(total)}=L_E+L$ with $L_E=\alpha^r b_\mu{}^r\dot{x}_\mu+(\lambda/2)(\dot{x}_\mu\dot{x}_\mu+c^2)+\rho\lambda+\mu\rho^2$ where $\alpha^r(r=1, 2, 3)$, λ, ρ, μ (and $\lambda_{\mu\nu}$) are Lagrange multipliers.

$$L=-\frac{1}{4}I\omega_{\alpha\beta}\omega_{\alpha\beta}+\lambda_{\mu\nu}(b_\mu{}^\xi b_\nu{}^\xi-\delta_{\mu\nu}),$$

$$(\omega_{\alpha\beta}=b_\alpha{}^\xi \dot{b}_\beta{}^\xi, \quad I: \text{ moment of inertia})$$

discussed in our first paper leads,[1] when we introduce as new variables relativistic Euler angles to the Hamiltonian:

$$H=\frac{1}{2I}(S_k^+ S_k^+ + S_k^- S_k^-), \quad k\sim 1, 2, 3$$

where the $S_k^\pm$ are defined as projections of the angular momentum $I\omega_{\alpha\beta}$ (expressed in terms of these new variables) on self-dual tensors associated by Einstein and Meyer to the moving tetrad. As we have seen, this Hamiltonian splits into two complex conjugate parts associated with complex conjugate three-dimensional complex rotations. According to our program, our present objective is just to analyze some mathematical aspects of the conventional quantization of this Hamiltonian using these new basic variables. This is an important point since the result of any quantization procedure may vary according to the system of variables chosen. In the point particle case we used x_μ, thus obtaining a wave in space-time. With relativistic Euler angles we quantize the tetrad orientation directly.* The quantum levels obtained in this way evidently correspond to what we can call "internal" states to distinguish them from the quantum "external" states associated with the motion of the tetrad's orign x_μ. This latter question or more precisely the problem of the quantization of the motion of x_μ associated with given "internal" states will be discussed in a subsequent paper.

This quantization will be done in three steps. In the first section we shall briefly discuss the new variables, the so-called relativistic Euler angles, which shall greatly simplify our quantization procedure and give the explicit form of the new Hamiltonian operator. As we shall see the theory will be invariant under the full group of complex conjugate three-dimensional rotations isomorphic to the full Lorentz group.

In § 2, we shall study the corresponding "eigenfunctions" and summarize certain mathematical results established by two of us (P. H. and J. P. V.)[4]

In § 3, we shall study the connection between the new "internal" quantum rotator levels and irreducible representations of the complex threedimensional rotation group, connection which leads to a classification of these excited levels.

§ 1

According to this program, let us first transform the Lagrangian with the help of the relativistic Euler angles:

* This quantization can only have a clear physical meaning if this orientation is quantized with respect to a kinematical frame $a_\mu{}^\xi$ attached to the moving body. Such a frame can be determined for example by the principal axis of dilatation introduced by Fukutome. In the rest of this paper however we only assumed that the $a_\mu{}^\xi$ correspond to a well defined frame in which $L_{(total)}$ is given.

$$\omega^+ = \{\theta^+, \varphi^+, \psi^+\},$$
$$\omega^- = \{\theta^-, \varphi^-, \psi^-\},$$

with

$$\begin{cases} \varphi^\pm = \varphi_1 \pm i\varphi_2, \\ \theta^\pm = \theta_1 \pm i\theta_2, \\ \psi^\pm = \psi_1 \pm i\psi_2. \end{cases}$$

The real quantities $\varphi_1, \theta_1, \psi_1$, represent the usual space Euler angles, while $i\varphi_2, i\theta_2, i\psi_2$ represent hyperbolic angles (varying from $-\infty$ to $+\infty$) expressing pure Lorentz transformations. The physical interpretation of these six angles is clear. If one starts from a fixed tetrad a_μ^ξ we can determine the orientation of the moving tetrad b_μ^ξ which defines the relativistic rotator by a homogeneous Lorentz transform:

$$b_\mu{}^\xi = \Lambda_\eta^\xi a_\mu{}^\eta, \quad \mu, \xi \sim 1, 2, 3, 4.$$

where the Λ_η^ξ are functions of ω^+ and ω^- which we have given explicitly in a preceding paper.[5] More precisely, if we associate to the tetrads a_μ^ξ and b_μ^ξ complex self-dual antisymmetrical tensors:

$$\Gamma_{\mu\nu}^{r\pm} = \varepsilon^{rst} b_\mu{}^s b_\nu{}^t \pm (b_\mu{}^r b_\nu{}^4 - b_\nu{}^r b_\mu{}^4),$$

and

$$\Lambda_{\mu\nu}^{r\pm} = \varepsilon^{rst} a_\mu{}^s a_\nu{}^t \pm (a_\mu{}^r a_\nu{}^4 - a_\nu{}^r a_\mu{}^4), \quad r, s, t \sim 1, 2, 3$$

we can show that their space-like parts:

$$B_k^{r\pm} = b_k{}^r b_4{}^4 - b_4{}^r b_k{}^4 \pm \varepsilon_{ijk} b_i{}^r b_j{}^4,$$
$$A_k^{r\pm} = a_k{}^r a_4{}^4 - a_4{}^r a_k{}^4 \pm \varepsilon_{ijk} a_i{}^r a_j{}^4 \quad r \sim 1, 2, 3 \quad i, j, k \sim 1, 2, 3$$

behave exactly like three orthogonal unitary complex vectors $\boldsymbol{B}^r$ and $\boldsymbol{A}^r$ belonging to two complex conjugate three-dimensional spaces $E^\pm$.

Indeed, one can show[5] that under a homogeneous Lorentz transform defined by the six Euler angles, the A^{r+} transform into the B^{r+} in E^+, according to the complex three-dimensional rotation defined by the Euler angles ω^+; while A^{r-} transform into B^{r-} in E^- under the rotation determined by ω^-. We note here that in the non-relativistic limit $B_k^{r\pm} \to b_k^r$ and $A_k^{r\pm} \to a_k^r$ so that the classical limit of our Nakano rotator is just the classical spherical rigid body.

If we consider an observer in the frame B_k^{r+} or B_k^{r-} he will see $B_k^{r+} \equiv B_k^{r-} \equiv b_k^r$, while an observer in the frames $A_k^{r\pm}$ will see $A_k^{r+} \equiv A_k^{r-} \equiv a_k^r$; so that the preceding formulas just correspond to the relativistic transformation of orthogonal three frames. Mathematically, this implies that if we define in ordinary three-dimensional space three fixed unitary vectors a_k^r and three moving unitary and orthogonal vectors b_k^r we can determine matrix elements Λ_k^r as functions of the three real Euler angles $\omega = \{\theta, \varphi, \psi\}$ by the relation

$$b_k^r = \Lambda_s^r a_k^s \quad r, s \sim 1, 2, 3 \quad k \sim 1, 2, 3$$

(with $\Lambda_j^r \equiv \Lambda_j^r(\omega)$), we have

$$\begin{cases} B_k^{r+} = \Lambda_s^r(\omega^+) A_k^{s+} \\ B_k^{r-} = \Lambda_s^r(\omega^-) A_k^{s-}. \end{cases}$$

We are now in a position to quantize H. But before we do that, in order to clarify our procedure, let us briefly recall certain well-known results of the quantization of the three-dimensional rigid spherical rotator, or rather the quantization of its rotational motion, when we leave aside its radial behaviour.

If we characterize this rotator by a moving frame of three orthogonal unitary vectors b_k^r and depict their orientation with respect to a fixed frame a_k^r by three Euler angles θ, φ, ψ, the classical rotation Lagrangian associated to the spherical case becomes (I=moment of inertia)

$$L = \frac{1}{4} I \omega_{ij} \omega_{ij} + \lambda_{ij}(b_i^r b_j^r - \delta_{ij}),$$

where $$\omega_{ij} = \dot{b}_i^r b_j^r,$$

or $$L = \frac{1}{2} I \dot{b}_i^r \dot{b}_i^r + \lambda_{ij}(b_i^r b_j^r - \delta_{ij}),$$

$$L = \frac{I}{2}(\dot{\varphi}^2 + \dot{\theta}^2 + \dot{\psi}^2 + 2\dot{\varphi}\dot{\psi}\cos\theta),$$

with $\dot{G} = dG/dt$, when expressed in terms of Euler angles. Introducing the canonical momenta $p_\theta = \partial L/\partial\dot{\theta}$, $p_\phi = \partial L/\partial\dot{\varphi}$ and $p_\psi = \partial L/\partial\dot{\psi}$ we obtain for the projections of the angular momentum $\boldsymbol{S}$ on both frames certain functions of θ, φ, ψ, p_θ, p_ϕ, and p_ψ. Let us call

S_k the projection of $\boldsymbol{S}$ on a_k^r,

S_k' the projections of $\boldsymbol{S}$ on b_k^r,

where $$S_i = I\omega_i = \frac{I}{2}\varepsilon_{ijk}\omega_{jk}.$$

We discover after a simple calculation that the Hamiltonian becomes

$$H = \frac{1}{2I} S_k S_k = \frac{1}{2I} S_k' S_k'.$$

The quantization is performed as usual. Replacing p_θ, p_ϕ, p_ψ by $-j\hbar\cdot\partial/\partial\theta$, $-j\hbar\cdot\partial/\partial\varphi$ and $-j\hbar\cdot\partial/\partial\psi$, we get for the projections of $\boldsymbol{S}$ on a_k^r the operators J_κ defined by the expressions

$$\left.\begin{aligned} J_1 &= -j\hbar\left(-\sin\varphi\frac{\partial}{\partial\theta}-\cos\varphi\cot\theta\frac{\partial}{\partial\varphi}+\frac{\cos\varphi}{\sin\theta}\frac{\partial}{\partial\psi}\right), \\ J_2 &= -j\hbar\left(\cos\varphi\frac{\partial}{\partial\theta}-\sin\varphi\cot\theta\frac{\partial}{\partial\varphi}+\frac{\sin\varphi}{\sin\theta}\frac{\partial}{\partial\psi}\right), \\ J_3 &= -j\hbar\frac{\partial}{\partial\varphi}, \end{aligned}\right\} \tag{1a}$$

and on b_k^r the operators J_k' with

$$\left.\begin{aligned} J_1' &= -j\hbar\left(\sin\psi\frac{\partial}{\partial\theta}+\cos\psi\cot\theta\frac{\partial}{\partial\psi}-\frac{\cos\psi}{\sin\theta}\frac{\partial}{\partial\varphi}\right), \\ J_2' &= -j\hbar\left(\cos\psi\frac{\partial}{\partial\theta}-\sin\psi\cot\theta\frac{\partial}{\partial\psi}+\frac{\sin\psi}{\sin\theta}\frac{\partial}{\partial\varphi}\right), \\ J_3' &= -j\hbar\frac{\partial}{\partial\psi}. \end{aligned}\right\} \tag{1b}$$

These operators are exactly proportional to the infinitesimal rotation operators associated with the directions a_k^r and b_k^r. We see also that the quantized Hamiltonian becomes

$$H=\frac{1}{2I}(J)^2,$$

with $$(J)^2=J_kJ_k=J_k'J_k'.$$

We know also that these operators satisfy the commutation relations of the three-dimensional infinitesimal rotation group:

$$\begin{cases} [J_i,\ J_j]=-j\hbar J_k, \\ [J_i',\ J_j']=-j\hbar J_k', \\ [J_i,\ J_j']=0, \end{cases}$$

besides $$[H,\ J_3]=[H,\ J_3']=0,$$

and then we obtain the angular part of all eigenstates by the simultaneous eigenfunctions of

$$(J)^2,\ J_3 \text{ and } J_3';$$

that is, the set $Y_l^{m,m'}(\omega)$ satisfying

$$\begin{cases} J^2\,Y_l^{m,m'}(\omega)=l(l+1)\,Y_l^{m,m'}(\omega), \\ J_3\,Y_l^{m,m'}(\omega)=m\,Y_l^{m,m'}(\omega), \\ J_3'\,Y_l^{m,m'}(\omega)=m'\,Y_l^{m,m'}(\omega), \end{cases}$$

where we put $\hbar=1$,

with $l=0, 1/2, 1, 3/2, \cdots; \quad m, m'=-l, -l+1, \cdots, l-1, l,$

and

$$Y_l^{m,m'}(\omega)=\left(\sin\frac{\theta}{2}\right)^{-m+m'}\left(\cos\frac{\theta}{2}\right)^{-m-m'}\frac{d^{l-m}}{d\left(\sin^2\frac{\theta}{2}\right)^{l-m}}$$

$$\times\left[\left(\sin^2\frac{\theta}{2}\right)^{l-m'}\left(\cos^2\frac{\theta}{2}\right)^{l+m'}\right]e^{i(m\varphi+m'\psi)}, \tag{2}$$

(corresponding to the well-known energy levels $l(l+1)$.) We know moreover that when we fix l and m', the set $Y_l^{m,m'}(\omega)$ determines a subspace of the classical Hilbert space which transforms into itself under an irreducible representation $\mathfrak{D}(l)$ of the three-dimensional rotation group.

We are now in a position to quantize H. As we have seen in a preceding paper[1] if we replace d/dt by $d/d\tau$ the scalar relativistic Hamiltonian H splits into the sum of two complex conjugate Hamiltonians,

$$H=\frac{1}{2I}\left(M_{\alpha\beta}^{+}M_{\alpha\beta}^{+}+M_{\alpha\beta}^{-}M_{\alpha\beta}^{-}\right)$$

$$=\frac{1}{2I}\left(\dot{B}_k^{r+}\dot{B}_k^{r+}+\dot{B}_k^{r-}\dot{B}_k^{r-}\right)$$

$$=H^{+}+H^{-}.$$

That is exactly the sum of the Hamiltonians of two complex conjugate three-dimensional rigid rotators (with $d/d\tau$ replacing d/dt). We have shown also that if we call $S_k^{\pm}$ the projections of the total angular momentum $M_{\alpha\beta}$ on $\Lambda_{\mu\nu}^{r\pm}$, that is,

$$S_k^{\pm}=M_{\alpha\beta}\Lambda_{\alpha\beta}^{k\pm},$$

and similarly the projections on the moving frame

$$S_k'^{\pm}=M_{\alpha\beta}\Gamma_{\alpha\beta}^{k\pm},$$

we have

$$H=\frac{1}{2I}\left(S_k^{+}S_k^{+}+S_k^{-}S_k^{-}\right)$$

$$=\frac{1}{2I}\left(S_k'^{+}S_k'^{+}+S_k'^{-}S_k'^{-}\right),$$

where $S_k^{\pm}$ and $S_l'^{\pm}$ are exactly given by the three-dimentional usual relations when we replace ω by $\omega^{\pm}$. This is remarkable since S_k^{+} and $S_k'^{+}$ depend only on ω^{+}, while S_k^{-} and $S_k'^{-}$ are the same functions with ω^{-} replacing ω^{+}.

As stated before, we perform the quantization by substituting the operators $-j\hbar\cdot\partial/\partial\omega^{+}$ for the quantities $p_{\omega}{}^{\pm}=\partial L/\partial\dot{\omega}^{\pm}$. This replaces the quantities $S_k^{\pm}$ by the set of operators:

$$\left.\begin{aligned} J_1^{\pm} &= -j\hbar\left(-\sin\varphi^{\pm}\,\partial/\partial\theta^{\pm} - \cos\varphi^{\pm}\cot\theta^{\pm}\,\partial/\partial\varphi^{\pm} + \frac{\cos\varphi^{\pm}}{\sin\theta^{\pm}}\,\partial/\partial\psi^{\pm}\right),\\ J_2^{\pm} &= -j\hbar\left(\cos\varphi^{\pm}\,\partial/\partial\theta^{\pm} - \sin\varphi^{\pm}\cot\theta^{\pm}\,\partial/\partial\varphi^{\pm} + \frac{\sin\varphi^{\pm}}{\sin\theta^{\pm}}\,\partial/\partial\psi^{\pm}\right),\\ J_3^{\pm} &= -j\hbar\,\partial/\partial\varphi^{\pm}, \end{aligned}\right\} \tag{3}$$

which are proportional to infinitesimal transformations conserving bivectors $I_{\mu\nu}^{r\pm}$ in space-time (or infinitesimal three-dimensional complex rotations around $A_k^{r\pm}$ in $E^{\pm}$), while $S_k^{\prime\pm}$ become

$$\left.\begin{aligned} J_1'^{\pm} &= -j\hbar\left(\sin\psi^{\pm}\,\partial/\partial\theta^{\pm} + \cos\psi^{\pm}\cot\theta^{\pm}\,\partial/\partial\psi^{\pm} - \frac{\cos\psi^{\pm}}{\sin\theta^{\pm}}\,\partial/\partial\varphi^{\pm}\right),\\ J_2'^{\pm} &= -j\hbar\left(\cos\psi^{\pm}\,\partial/\partial\theta^{\pm} - \sin\psi^{\pm}\cot\theta^{\pm}\,\partial/\partial\psi^{\pm} + \frac{\sin\psi^{\pm}}{\sin\theta^{\pm}}\,\partial/\partial\varphi^{\pm}\right),\\ J_3'^{\pm} &= -j\hbar\,\partial/\partial\psi^{\pm}, \end{aligned}\right\} \tag{4}$$

having the same signification with respect to $I'^{r\pm}_{\mu\nu}$ and $B_k^{r\pm}$. These operators satisfy the commutation relations of the three-dimensional infinitesimal rotation group, that is,

$$\left.\begin{aligned} [J_i^{\pm},\ J_j^{\pm}] &= -j\hbar\,J_k^{\pm},\\ [J_i^{\pm},\ J_j'^{\pm}] &= -j\hbar\,J_k'^{\pm}. \end{aligned}\right\} \quad i, j, k \sim 1, 2, 3$$

This is not astonishing since they are associated with complex rotations in $E^{\pm}$ isomorphic to Lorentz transformations in Minkowski space-time. We find also

$$[J_k^{+},\ J_j^{-}] = [J_k^{+},\ J_j'^{-}] = [J_k'^{+},\ J_j'^{-}] = 0.$$

Besides

$$[H,\ J_3^{+}] = [H,\ J_3'^{+}] = 0,$$
$$[H,\ J_3^{-}] = [H,\ J_3'^{-}] = 0.$$

Introducing further the operators

$$(J^{+})^2 = (J'^{+})^2 = J_k^{+}J_k^{+} = J_k'^{+}J_k'^{+},$$
$$(J^{-})^2 = (J'^{-})^2 = J_k^{-}J_k^{-} = J_k'^{-}J_k'^{-},$$

satisfying the commutation relations

$$[(J^{+})^2,\ J_3^{+}] = [(J^{+})^2,\ J_3'^{+}] = 0,$$
$$[(J^{-})^2,\ J_3^{-}] = [(J^{-})^2,\ J_3'^{-}] = 0,$$
$$[(J^{+})^2,\ (J^{-})^2] = 0,$$

we see that the quantized Hamiltonian becomes

$$H = H^{+} + H^{-}$$
$$= \frac{1}{2I}\,(J_k^{+}J_k^{+} + J_k^{-}J_k^{-}),$$

H^+ commuting with H^-. Naturally this implies, if we generalize the usual three-dimensional procedure, that the "levels" of the hyperspherical rotator are products of the eigenfunctions $Y_{l^+}^{m^+,m'^+}(\omega^+)$ of the three operators $J_k^+ J_k^+$, J_3^+, $J_3'^+$, by the eigenfunctions $Y_{l^-}^{m^-,m'^-}(\omega^-)$ of $J_k^- J_k^-$, J_3^- and $J_3'^-$ with

$$\left.\begin{aligned} &J_k^{\pm} J_k^{\pm}\, Y_{l^{\pm}}^{m^{\pm},m'^{\pm}}(\omega^{\pm}) = [l^{\pm}(l^{\pm}+1)]\, Y_{l^{\pm}}^{m^{\pm},m'^{\pm}}(\omega^{\pm}),\\ &J_3^{\pm}\, Y_{l^{\pm}}^{m^{\pm},m'^{\pm}}(\omega^{\pm}) = m^{\pm}\, Y_{l^{\pm}}^{m^{\pm},m'^{\pm}}(\omega^{\pm}),\\ &J_3'^{\pm}\, Y_{l^{\pm}}^{m^{\pm},m'^{\pm}}(\omega^{\pm}) = m'^{\pm}\, Y_{l^{\pm}}^{m^{\pm},m'^{\pm}}(\omega^{\pm}), \qquad \text{(we put here } \hbar=1)\end{aligned}\right\} \quad (5)$$

$$l^+,\ l^-=0,\ 1/2,\ 1,\ 3/2\cdots\cdots,$$

$$m^+,\ m^-=-l^{\pm},-l^{\pm}+1,\ \cdots,\ l^{\pm}-1,\ l^{\pm},$$

$$m'^+,\ m'^-=-l^{\pm},\ -l^{\pm}+1,\ \cdots,\ l^{\pm}-1,\ l^{\pm}.$$

However, if we impose on H the physical condition that it is a scalar under transformations of the full complex rotation group (that is, including reflections through the origin of the moving frame) we are led to associate the "angular" part of the eigenfunctions with simultaneous eigenfunctions of the six commuting operators

$$J_k^+ J_k^+,\ J_3^+,\ J_k^- J_k^-,\ J_3^-,\ S_k' S_k',\ S_3'$$

with $\qquad S_k' = J_k'^+ + J_k'^-.$

As we shall see in the next section these are linear products of the $Y_{l^+}^{m^+,m'^+}(\omega)$ and $Y_{l^-}^{m^-,m'^-}(\omega^-)$ multiplied by suitable Clebsh-Gordan coefficients. These eigenfunctions which we shall denote $Z_{l^+,l^-,s'}^{m^+,m^-,m'}(\omega^+,\omega^-)$ constitute the four-dimensional generalization of the levels of the three-dimensional rotator. They satisfy transformations of the full complex rotation group and will now enable us to study the internal level classification of our relativistic rotator.

§ 2

According to our program we shall now recall certain mathematical results concerning the eigenfunctions of the operators defined in the preceding section. No detailed mathematical proofs shall be given here since they have already been published in specialized papers which will be quoted as references and can be consulted if necessary. Such a summary is however useful as starting point for the quantum theory of relativistic rotators. The following theorems can be established:

Theorem I: If one considers the set of operators J_3^+, $J_3'^+$, $J_k^+ J_k^+$, it is possible to find a denumerable set of eigenfunctions $Y_{l^+}^{m^+,m'^+}(\omega^+)$ satisfying relations (5) where the eigenvalues l^+ may take all real integer or half-integer values: 0, 1/2, 1, 3/2, $\cdots$; the associated values of m^+ and m'^+ being distributed over the set:

$$-l^+,\ -l^+ +1,\ \cdots,\ l^+ -1,\ l^+.$$

We shall now make three remarks.

This theorem is evidently the generalization to complex three-dimensional rotations of the usual three-dimensional theory.

As we have shown in the preceding section, any Lorentz transform on the b_μ^ξ frame induces two complex conjugate rotations of the $B_k^{r\pm}$ frames (defined by $\omega^\pm$). Both groups (Lorentz transformations and complex three-dimensional rotations) are locally isomorphic.

The explicit form of the eigenfunctions has been given by C. Van Winter.[6] We get

$$Y_{l^+}^{m^+,m'^+}(\omega^+) = \left(\sin\frac{\theta^+}{2}\right)^{-m^+ + m'^+}\left(\cos\frac{\theta^+}{2}\right)^{-m^+ - m'^+}\frac{d^{l^+ - m^+}}{d\left(\sin^2\frac{\theta^+}{2}\right)^{l^+ - m^+}}$$

$$\times\left[\left(\sin^2\frac{\theta^+}{2}\right)^{l^+ - m'^+}\left(\cos^2\frac{\theta^+}{2}\right)^{l^+ + m'^+}\right]e^{i(m^+\varphi^+ + m'^+\psi^+)}$$

and two of us have shown (P. H. and J.-P. V.)[5] that they constitute the only possible set. Of course, we have the same theorem for J_3^-, $J_3'^-$, $J_k^- J_k^-$.

The second remark is that if we fix l^+, l^- and m'^+, m'^- the set $Y_{l^+}^{m^+,m'^+}(\omega^+)$ $Y_{l^-}^{m^-,m'^-}(\omega^-)$ builds a vector space which transforms under an irreducible representation $\mathfrak{D}(l^+, l^-)$ of the three-dimensional complex rotation group. This result is a straightforward generalization of the classical three-dimensional theory of the rotation group. To show this it is enough to recall that if we describe a classical rigid rotator by three real axis (with components b_k^r) and determine their orientation with respect to a fixed frame a_k^r by three real Euler angles, the components of the angular momentum are just given by operators J_k and J_k' deduced from $J_k^\pm$ and $J_k'^\pm$ by putting $i\varphi_2 = i\theta_2 = i\psi_2 = 0$. The square of the angular momentum given by $J^2 = J_k J_k$ is proportional to the classical quantum Hamiltonian so that the "energy" levels are proportional to $l(l+1)$. The corresponding angular part of eigenfunctions can written $Y_l^{m,m'}(\omega)$ (ω representing the real Euler angles θ, φ, ψ) and each "level" when we fix l and m', splits into "sublevels" $Y_l^{m,m'}(\omega)$ with m taking all values $-l$, $-l+1$, $\cdots$, $l-1$, l which form a vector space transforming under an irreducible representation of the three-dimensional rotation group.

The third remark is the most important from the physical point of view. The theorem states that l^+ (or l^-) can take all possible integer and half-integer positive values so that $l^\pm = 0,\ 1/2,\ 1,\ 3/2,\ \cdots$. To make this point clear, let us go back to three-dimensional theory. In this case if we describe as before the orientation of a rigid rotator by three moving axis b_k^r, we usually eliminate all half integer levels $l = 1/2,\ 3/2,\ \cdots$ with the argument that with a rigid body we necessarily come back to the original orientation after a 2π angular rotation. As Pauli first remarked there is no mathematical reason to eliminate these levels.

Indeed, as Lochak[7] has demonstrated these levels are even necessary if the state function is to be continuous on the sphere of Cayley-Klein parameters. Physically their existence is evidently connected with non-rigid rotators. In that case, materialized for example by a liquid droplet, there is no reason why the body should not recover its original distribution after a 4π rotation only or any higher multiple of 2π. Naturally one would expect such highly excited states to be rather unstable, but they are evidently permissible. Indeed, interesting consequences can be deduced from their possible existence in the domain of atomic spectra.[7] This reasoning can evidently be extended to the relativistic domain. First, we know that in that case it is impossible to generalize in a covariant manner the classical notion of rigid body. The rotation of our b_μ^ξ frame is then evidently physically connected with a relativistic fluid droplet so that there is no reason to eliminate the half integral values of l^+ and l^- if we connect the self-dual tensors $B_k^{r\pm}$ with internal physical structures. Moreover, any disparity between l^+ and l^- must be interpreted as connected with an internal disparity or distortion of the internal matter. Any attempt to transfer without precaution into the relativistic domain results connected with the classical notion of rigidity might lead to contradictory results.

The second theorem we want to discuss here is related to the physical assumption that the Lagrangian is invariant under transformations of the full complex rotation group; that is, including reflections through the origin of the moving tetrad b_μ^ξ.

We know already that products of functions $Y_{l^+}^{m^+,m'^+}(\omega^+)\,Y_{l^-}^{m^-,m'^-}(\omega^-)$ transform under the $\mathfrak{D}(l^+, l^-)$ representation of the proper complex rotation group.

Simple considerations[6] then show the validity of the following theorem:

Theorem II: The set of eigenfunctions $Z_{l^+,l^-,s'}^{m^+,m^-,m'}(\omega^+, \omega^-)$ of the set of commuting operators $J_k^+J_k^+$, J_3^+, $J_k^-J_k^-$, J_3^-, S_kS_k' and S_3' (with $S_k'=J_k'^+ + J_k'^-$), that is, the functions satisfying

$$
\begin{aligned}
&J_k^{\pm}J_k^{\pm}Z_{l^+,l^-,s'}^{m^+,m^-,m'}(\omega^+,\ \omega^-)=l^{\pm}(l^{\pm}+1)Z_{l^+,l^-,s'}^{m^+,m^-,m'}(\omega^+,\omega^-),\\
&J_3^{\pm}Z_{l^+,l^-,s'}^{m^+,m^-,m'}(\omega^+,\ \omega^-)=m^{\pm}Z_{l^+,l^-,s'}^{m^+,m^-,m'}(\omega^+,\ \omega^-),\\
&S_k'S_k'Z_{l^+,l^-,s'}^{m^+,m^-,m'}(\omega^+,\ \omega^-)=s'(s'+1)Z_{l^+,l^-,s'}^{m^+,m^-,m'}(\omega^+,\ \omega^-),\\
&S_3'Z_{l^+,l^-,s'}^{m^+,m^-,m'}(\omega^+,\ \omega^-)=m'Z_{l^+,l^-,s'}^{m^+,m^-,m'}(\omega^+,\ \omega^-),\\
&s'=l^++l^-,\ l^++l^--1,\ \cdots,\ |l^+-l^-|,\\
&m'=-s',\ -s'+1,\ \cdots,\ s'-1,\ s',
\end{aligned}
\tag{6}
$$

transform when we fix the values of l^+, l^-, s', m' according to the representation $\mathfrak{D}(l^+, l^-)\oplus\mathfrak{D}(l^-, l^+)$ of the full complex rotation group. In other terms, the the set of functions $Z_{l^+,l^-,s'}^{m^+,m^-,m'}(\omega^+, \omega^-)$ with l^+, l^-, s', m' fixed constitute a set of independent orthogonal vectors of a finite dimensional vector space transforming into itself under the complex rotation group according to the corresponding

representation. Evidently such vector spaces are subspaces of the general numerably infinite dimensional Hilbert space containing all finite dimensional representations of the full complex rotation group.

Before we discuss this last statement, we recall[6] that these $Z_{l^+,l^-,s'}^{m^+,m^-,m'}(\omega^+,\omega^-)$ functions are given in terms of the preceding $Y_{l^+}^{m^+,m'^+}(\omega^+)$ and $Y_{l^-}^{m^-,m'^-}(\omega^-)$ functions by the expressions

$$Z_{l^+,l^-,s'}^{m^+,m^-,m'}(\omega^+,\omega^-)=\sum_{-m'^+,-m'^-}(l^+,l^-,-m'^+,-m'^-|l^+,l^-,s',-m')\times Y_{l^+}^{m^+,m'^+}(\omega^+)\,Y_{l^-}^{m^-,m'^-}(\omega^-)$$

where

$$s'=l^++l^-,\ l^++l^--1,\ \cdots,\ |l^+-l^-|$$

$$m'=-s',\ -s'+1,\ \cdots,\ s'-1,\ s'.$$

and $(l^+,l^-,-m'^+,-m'^-|l^+,l^-,s',-m')$ are the usual Clebsch-Gordan parameters.

Now the problem of defining an orthogonal basis within each subspace associated to an irreducible representation of the full complex rotation group is closely connected with the question of the metric associated with this group. In order to clarify this point we shall briefly recall the classical results of the three-dimensional real rotation group expressed in terms of real Euler angles.

Introducing as before the operators J_1, J_2, J_3 given by relations (1a), we know that the eigenfunctions of the three commuting operators J_3, J_3', and $J^2=(J_kJ_k)=(J_j'J_j')$, namely $Y_l^{m,m'}(\omega)$ (with $J_3Y_l^{m,m'}(\omega)=mY_l^{m,m'}(\omega)$, $J_3'Y_l^{m,m'}(\omega)=m'Y_l^{m,m'}(\omega)$ and $J^2Y_l^{m,m'}(\omega)=l(l+1)\,Y_l^{m,m'}(\omega)$) build, when we fix l and m', a finite dimensional subspace of a general Hilbert space, which transforms according to the irreducible representation $\mathfrak{D}(l)$ of the three-dimensional rotation group.

The metric of that Hilbert space is well known. If we take two functions $f(\omega)$ and $g(\omega)$, and call $f^*(\omega)$ the complex conjugate function of $f(\omega)$ (corresponding to $j\to -j$) we define the scalar product of f and g by the relation

$$\langle f|g\rangle=\frac{1}{2}\int_0^{4\pi}d\psi\int_0^{2\pi}d\varphi\int_0^{\pi}f^*(\omega)g(\omega)\sin\theta\,d\theta.$$

This definition implies that $\langle f|f\rangle$ is always positive (which means that the space has a positive definite metric) and that the preceding operators J_3, J_3', J^2 are Hermitian operators in the sense that they satisfy the general relation $\langle f|Ag\rangle=\langle Af|g\rangle$. We also know that the $Y_l^{m,m'}(\omega)$ vectors satisfy the general orthogonality conditions:

$$\langle Y_j^{m,m'}(\omega)\,|\,Y_l^{n,n'}(\omega)\rangle=\delta_{jl}\delta_{mn}\delta_{m'n'}.$$

Now our problem is to generalize these results to the space built on the $Z_{l^+,l^-,s'}^{m^+,m^-,m'}(\omega^+,\omega^-)$ eigenfunctions of the operators $J_3^+,(J^+)^2,J_3^-,(J^-)^2,S_3',(S')^2$. This is evidently a rather complex mathematical problem since the new space

cannot be compact because $i\varphi_2$, $i\theta_2$, $i\psi'_2$ vary from $-\infty$ to $+\infty$. It can however be solved, as two of us (P. H. and J.-P.V.)[8] have shown, if one introduces an indefinite metric in the sense recently utilized by Heisenberg[9] and other authors.[10] We shall just summarize here results (for detailed demonstration, see ref. 4)).

First let us recall two definitions. We have seen, following Møller, that we must distinguish between two types of complex quantities, the first with i resulting from Minkowski's space-time, the second with j introduced by the quantization procedure. This implies two operations corresponding to complex conjugation $i\to -i$ and $j\to -j$. The second type we shall denote by * so that

$$f^*(j, \omega^+, \omega^-) = f(-j, \omega^+, \omega^-).$$

We can now demonstrate the following theorem.

Theorem III: If we associate to the Hilbert space of functions $f(\omega^+, \omega^-)$ the indefinite metric defined by the scalar product:

$$\langle f(\omega^+, \omega^-) | g(\omega^+, \omega^-) \rangle = \lim_{\substack{B_1, B_2, B_3 \\ B_1', B_2', B_3'}\to\infty} A \int_{-B_1'}^{B_1} d\varphi_2 \int_{-B_2'}^{B_2} d\psi_2 \int_{-B_3'}^{B_3} d\theta_2$$

$$\times \left[\int_0^{4\pi} d\varphi_1 \int_0^{2\pi} d\psi_1 \int_0^{\pi} f^*(\omega^+, \omega^-) g(\omega^+, \omega^-) |1-\cosh^2\theta_2 \cos^2\theta_1| d\theta_1 \right],$$

where the [ ] under the $\int$ indicates that it is first performed with respect to θ_1, φ_1, ψ_1, A is a weighing factor defined by

$$A = e^{-(L+1)(B_3+B_3')} (B_1+B_3')(B_2+B_2'),$$

L being the maximum of the numbers l^+ and l^- of the representation on which we can project $f(\omega^+, \omega^-)$ and $g(\omega^+, \omega^-)$ and we can show that the operators J_3^+, J_3^-, S_3', $(J^+)^2$, $(J^-)^2$ and $(S')^2$ are pseudo-Hermitian operators, meaning that they satisfy $\langle f|Ag\rangle = \langle Af|g\rangle$. We can also show that if we define the average value of an operator A by

$$\overline{A} = \frac{\langle f|Af\rangle}{\langle f|f\rangle},$$

we have

$$\overline{J_3^{\pm}} = m^{\pm},$$

$$\overline{S_3'} = m',$$

$$\overline{(J^{\pm})^2} = l^{\pm}(l^{\pm}+1),$$

$$\overline{(S')^2} = s'(s'+1),$$

the numbers m^+, m^-, m', l^+, l^-, s' being all real.

One can also show that we have

$$\langle Z^{m^+,m^-,m'}_{l^+,l^-,s'}(\omega^+,\ \omega^-)\,|\,Z^{n^+,n^-,n'}_{j^+,j^-,t'}(\omega^+,\ \omega^-)\rangle = \varepsilon\,\delta_{l^+j^+}\delta_{l^-j^-}\delta_{m^+n^+}\delta_{m^-n^-}\delta_{m'n'},$$

(with $\varepsilon = \pm 1$), a relation which shows that it is possible in the vector space constructed on the functions $Z^{m^+,m^-,m'}_{l^+,l^-,s'}(\omega^+,\ \omega^-)$ to define a scalar product (and therefore a metric) in such a way that these vectors $Z^{m^+,m^-,m'}_{l^+,l^-,s'}(\omega^+,\ \omega^-)$ constitute an orthogonal and unitary basis. Naturally in this generalization of the usual Hilbert space we must distinguish two types of vectors, for they split in space-like vectors ($\langle Z|Z\rangle > 0$) and time-like ($\langle Z|Z\rangle < 0$) since the metric is indefinite.

We must remark also that the $Z^{m^+,m^-,m'}_{l^+,l^-,s'}(\omega^+,\ \omega^-)$ functions are not bounded. This raises many new physically interesting points which we shall discuss later. Evidently the introduction of this new metric is a new qualitative factor introduced in the theory. This is not very surprising. One would hardly expect that the new "internal" variables should be governed by exactly the same mechanics and interpretation as the usual "external" variables, for, according to our views, they belong to a new "subquantum level" of matter defined by distances smaller than 10^{-13} cm. Moreover, the corresponding operators are pseudo-Hermitian, that is, Hermitian with respect to the above defined indefinite metric. One notices also that this new metric is connected with the fact that the "state vectors" $Z^{m^+,m^-,m'}_{l^+,l^-,s'}(\omega^+,\ \omega^-)$ must be invariant under transformations of the full complex rotation group. It might be necessary, however, as suggested by Heisenberg, that one must restrict oneself for physical reasons to certain regions of this new Hilbert space such as the space-like part.

We shall conclude this mathematical section by a few results established by two of us (P. H. and J.P. V.) on the symmetry properties of the $Z^{m^+,m^-,m'}_{l^+,l^-,s'}(\omega^+,\ \omega^-)$ functions under certain transformations such as $P(x_k \to -x_k)$ since they will turn out later to be useful for further steps in the theory of relativistic rotators. We know, moreover, that such symmetry properties play an essential part in the theory of interactions.

First let us study the parity operation P. We have by definition

$$\begin{aligned} &Pa_\mu{}^r = -a_\mu{}^r,\ Pa^{(4)}_\mu = a^{(4)}_\mu, \\ &Pb_\mu{}^r = -b_\mu{}^r,\ Pb^{(4)}_\mu = b^{(4)}_\mu, \end{aligned} \qquad \text{so that} \qquad \begin{aligned} &PA^{r\pm}_k = A^{r\mp}_k, \\ &PB^{r\pm}_k = B^{r\mp}_k. \end{aligned}$$

$(r \sim 1, 2, 3)$

We deduce therefrom that P transforms the angles ω^+ into ω^- (and vice-versa), so that taking into account the symmetry properties of the Clebsch-Gordan coefficients we finally get

$$PZ^{m^+,m^-,m'}_{l^+,l^-,s'}(\omega^+,\ \omega^-) = (-1)^{l^+ + l^- - s'} Z^{m^-,m^+,m'}_{l^-,l^+,s'}(\omega^+,\ \omega^-).$$

By definition also the time reversal operation T implies the relations

$$\begin{aligned} Ta^{(4)}_\mu &= -a^{(4)}_\mu, \\ Tb^{(4)}_\mu &= -b^{(4)}_\mu. \end{aligned}$$

But these correspond to non-linear transformations which imply the following operation on the relativistic Euler angles:

$$\omega^+ \to \omega^-, \quad \omega^- \to \omega^+,$$

$$\theta_1 \to \theta_1 + \pi, \quad \theta_2 \to \theta_2 + i\pi,$$

so that we finally get after a short calculation

$$T Z^{m^+, m^-, m'}_{l^+, l^-, s'}(\omega^+, \omega^-) = (-1)^d Z^{-m^-, -m^+, -m'}_{l^-, l^+, s'}(\omega^+, \omega^-),$$

with

$$d = |m' - m^+ - m^-|,$$

The preceding operations evidently imply the existence of a third operation C (defined by $CPT=1$) which we shall call the transition from "state" to "anti-state".

We find after a short calculation

$$C Z^{m^+, m^-, m'}_{l^+, l^-, s'}(\omega^+, \omega^-) = (-1)^\alpha Z^{-m^+, -m^-, -m'}_{l^+, l^-, s'}(\omega^+, \omega^-),$$

with $\alpha = l^- + l^- + d - s'$. Evidently the so-called "anti-states" are just as real as the "states" themselves. They are just associated with a different irreducible representation of the complex rotation group since we pass from state $Z^{m^+, m^-, m'}_{l^+, l^-, s'}(\omega^+, \omega^-)$ to anti-state by the transformation:

$$m^+ \to -m^+, \quad m^- \to -m^-, \quad m' \to -m'.$$

We have only adopted this denomination because, as we shall see later, the spinors corresponding to particles and anti-particles are precisely obtained through the preceding transformation.

§ 3

We are now in a position to classify the "levels" associated to our new "internal" Hamiltonian H.

This classification is evidently a straightforward generalization of the classification of the angular part of the energy level of the three-dimensional rotator expressed in terms of real Euler angles θ, φ, ψ. In order to clarify our procedure, we shall once more recall very briefly certain points of that well-known case, closely connected with the four-dimensional relativistic rotator.

As we have seen in the first section the angular parts of the state functions are just the eigenfunctions $Y_l^{m,m'}(\theta, \varphi, \psi)$ corresponding to the well-known energy levels $l(l+1)$.

When we fix m', the energy levels split into sub-levels corresponding to different values of m; the corresponding vector space transforming under the irreducible representation $\mathfrak{D}(l)$ of the three-dimensional rotation group.

The corresponding classification of levels is evident. For example:

I. for $l=1/2$ $m'=-1/2$ (states 1/2), we have two sub-levels $Y_{1/2}^{-1/2,-1/2}(\theta, \varphi, \psi)$ and $Y_{1/2}^{1/2,-1/2}(\theta, \varphi, \psi)$ represented by the table:

	m	m'
$Y_{1/2}^{-1/2,-1/2}(\theta, \varphi, \psi)$	$-1/2$	$-1/2$
$Y_{1/2}^{1/2,-1/2}(\theta, \varphi, \psi)$	$+1/2$	$-1/2$

The columns characterizing the values of m and m'.

II. for $l=1/2$, $m'=1/2$ (anti-states 1/2), we obtain two sub-levels described by the table:

	m	m'
$Y_{1/2}^{1/2,1/2}(\theta, \varphi, \psi)$	1/2	1/2
$Y_{1/2}^{-1/2,1/2}(\theta, \varphi, \psi)$	$-1/2$	1/2

III. for $l=1$, $m'=1$ (states 1), we get the three sub-levels corresponding to $m=1$, 0, -1 corresponding to the table:

	m	m'
$Y_1^{1,1}(\theta, \varphi, \psi)$	1	1
$Y_1^{0,1}(\theta, \varphi, \psi)$	0	1
$Y_1^{-1,1}(\theta, \varphi, \psi)$	-1	1

and so on *ad infinitum* for all possible values of l and m'.

We can now pass to four dimensions and Minkowski space.

The "levels" associated to H are evidently also determined by a set of state (or anti-state) functions $Z_{l^+,l^-,s'}^{m^+,m^-,m'}(\omega^+, \omega^-)$, the eigenvalue associated to H being

$$l^+(l^+ +1)+l^-(l^- +1).$$

Each "level" evidently splits into "sub-levels" for if we fix l^+, l^-, s', m', we have various possibilities for m^+ and m^-. According to the results of the preceding section we shall classify the levels into families, each family being associated with a specific irreducible representation of the full complex rotation group.

Every sub-level is associated to a function $Z_{l^+,l^-,s'}^{m^+,m^-,m'}(\omega^+, \omega^-)$ and characterized in the classification by fixed values of l^+, l^-, s', m', that is to say, by table elements of the form:

	m^+	m^-	m'
$Z_{l^+,l^-,s'}^{m^+,m^-,m'}(\omega^+, \omega^-)$	m^+	m^-	m'

To every sub-level corresponds an anti-level with the same values of l^+, l^-, s', but opposite signs of m^+, m^-, m', that is, table elements:

$Z^{-m^+,-m^-,-m'}_{l^+,l^-,s'}(\omega^+,\omega^-)$	$-m^+$	$-m^-$	$-m'$

Naturally there is an infinite number of possible levels corresponding to increasing values of l^+, l^- and s'. One sees also that higher levels contain more and more sub-level families (each associated with an irreducible representation of the full complex rotation group) since m^+ and m^- vary between l^+ and $-l^+$, l^- and $-l^-$ while s' and also m' can take any value between $l^+ + l^-$ and $|l^+ - l^-|$.

As an example we shall only give here the tables corresponding to the lowest levels with minimum values of s', that is, $s' = |l^+ - l^-|$.

A. *Representation* $\mathfrak{D}(1/2, 0) \oplus \mathfrak{D}(0, 1/2)$;

For $l^+, l^- = \{1/2, 0\}$, $s' = 1/2$, $m' = -1/2$ we have a family associated with the $\mathfrak{D}(1/2, 0) \oplus \mathfrak{D}(0, 1/2)$ representation of the full complex group and the table:

	m^+	m^-	m'	$m^+ + m^- + m'$
$Z^{-1/2,0,-1/2}_{1/2,0,1/2}(\omega^+,\omega^-)$	$-1/2$	0	$-1/2$	-1
$Z^{1/2,0,-1/2}_{1/2,0,1/2}(\omega^+,\omega^-)$	1/2	0	$-1/2$	0
$Z^{0,-1/2,-1/2}_{0,1/2,1/2}(\omega^+,\omega^-)$	0	$-1/2$	$-1/2$	-1
$Z^{0,1/2,-1/2}_{0,1/2,1/2}(\omega^+,\omega^-)$	0	1/2	$-1/2$	0

the corresponding sub-space being four-dimensional. The anti-states corresponding to the same representation with $l^+, l^- = \{1/2, 0\}$ $s' = 1/2$, $m' = +1/2$. The introduction of the fourth column in which are indicated the values of $m^+ + m^- + m'$ of each sub-level $Z^{m^+,m^-,m'}_{l^+,l^-,s'}(\omega^+,\omega^-)$ will be explained later.

B. *Representation* $\mathfrak{D}(1, 0) \oplus \mathfrak{D}(0, 1)$:

For $l^+, l^- = \{1, 0\}$ $s' = 1$, $m' = 0$ we have associated the $\mathfrak{D}(1,0)$ and $\mathfrak{D}(0, 1)$ representations, that is,

m^+	m^-	m'	$m^+ + m^- + m'$
-1	0	0	-1
0	0	0	0
1	0	0	1

corresponding to $\mathfrak{D}(1, 0)$, and

m^+	m^-	m'	$m^+ + m^- + m'$
0	-1	0	-1
0	0	0	0
0	1	0	1

corresponding to $\mathcal{D}(0, 1)$, the sub-space of the general Hilbert space defined in the preceding section being six-dimensional.

C. *Representation* $\mathcal{D}(1, 1)$:
For $\mathcal{D}(1, 1)$ $s'=0$, $m'=0$ we shall not explicitly give the corresponding table.

D. *Representation* $\mathcal{D}(1/2, 1/2)$:
For $\mathcal{D}(1/2, 1/2)$ $s'=0$, $m'=0$ we have

m^+	m^-	m'	$m^+ + m^- + m'$
−1/2	−1/2	0	−1
1/2	−1/2	0	0
−1/2	1/2	0	0
1/2	1/2	0	1

the last two sub-levels being the anti-levels of the two first.

E. *Representatin* $\mathcal{D}(1, 1/2) \oplus \mathcal{D}(1/2, 1)$:
For $\{l^+, l^-\} = \{1, 1/2\}$ with $s'=1/2$, $m'=1/2$ we get, corresponding to $\mathcal{D}(1/2, 1)$,

m^+	m^-	m'	$m^+ + m^- + m'$
1/2	−1	1/2	0
1/2	0	1/2	1
1/2	1	1/2	2
−1/2	−1	1/2	−1
−1/2	0	1/2	0
−1/2	1	1/2	1

and corresponding to $\mathcal{D}(1, 1/2)$,

m^+	m^-	m'	$m^+ + m^- + m'$
−1	1/2	1/2	0
0	1/2	1/2	1
1	1/2	1/2	2
−1	−1/2	1/2	−1
0	−1/2	1/2	0
1	−1/2	1/2	1

The anti-states are obtained with $m'=-1/2$. This classification has a series of remarkable characteristics.

We know[11] that a Lorentz transform from an initial tetrad a_μ^ξ to a final tetrad b_μ^ξ does not define the latter completely since it is always possible to make afterwards an arbitrary space-like rotation in a space-like hyperplane. As a consequence the Lagrangian of the hyperspherical rotator is necessarily invariant under the transformation

$$\varphi_1 \to \varphi_1 + \alpha,$$
$$\psi_1 \to \psi_1 + \beta,$$

since φ_1 and ψ_1 precisely correspond to such rotations. Let us tentatively call such transformations gauge transformations. Under these transforms the $Z^{m^+, m^-, m'}_{l^+, l^-, s'}(\omega^+, \omega^-)$ functions become

$$e^{jS} Z^{m^+, m^-, m'}_{l^+, l^-, s'}(\omega^+, \omega^-) = e^{j[(m^+ + m^-)\alpha + m'\beta]} Z^{m^+, m^-, m'}_{l^+, l^-, s'}(\omega^+, \omega^-),$$

if we take into account a well-known property of the Clebsch-Gordan coefficients (they vanish if $m^+ + m^- \neq m'$), so that under the infinitesimal transform

$$\varphi_1 \to \varphi_1 + \varepsilon,$$
$$\psi_1 \to \psi_1 + \varepsilon,$$

we get

$$e^{jS} Z^{m^+, m^-, m'}_{l^+, l^-, s'}(\omega^+, \omega^-) = Z^{m^+, m^-, m'}_{l^+, l^-, s'}(\omega^+, \omega^-) + j(m^+ + m^- + m')\,\varepsilon Z^{m^+, m^-, m'}_{l^+, l^-, s'}(\omega^+, \omega^-)$$
$$= Z^{m^+, m^-, m'}_{l^+, l^-, s'}(\omega^+, \omega^-) + j\left[-j\frac{\partial}{\partial \varphi_1} - j\frac{\partial}{\partial \psi_1}\right]\varepsilon Z^{m^+, m^-, m'}_{l^+, l^-, s'}(\omega^+, \omega^-).$$

We deduce therefrom that the tetrad's transform

$$\varphi_1 \to \varphi_1 + \varepsilon,$$
$$\psi_1 \to \psi_1 + \varepsilon,$$

generates in the vector space of the $Z^{m^+, m^-, m'}_{l^+, l^-, s'}(\omega^+, \omega^-)$ functions the operators

$$N = -j\,\partial/\partial\psi_1,$$
$$M = -j\,\partial/\partial\varphi_1,$$
$$Q = N + M = -j(\partial/\partial\varphi_1 + \partial/\partial\psi_1).$$

When applied to any state $Z^{m^+, m^-, m'}_{l^+, l^-, s'}(\omega^+, \omega^-)$, they give

$$M Z^{m^+, m^-, m'}_{l^+, l^-, s'}(\omega^+, \omega^-) = (m^+ + m^-)\, Z^{m^+, m^-, m'}_{l^+, l^-, s'}(\omega^+, \omega^-),$$
$$N Z^{m^+, m^-, m'}_{l^+, l^-, s'}(\omega^+, \omega^-) = m' Z^{m^+, m^-, m'}_{l^+, l^-, s'}(\omega^+, \omega^-),$$
$$Q Z^{m^+, m^-, m'}_{l^+, l^-, s'}(\omega^+, \omega^-) = (m^+ + m^- + m')\, Z^{m^+, m^-, m'}_{l^+, l^-, s'}(\omega^+, \omega^-),$$

with

$$\bar{M} = m^+ + m^-,$$
$$\bar{N} = n^+ + n^-,$$
$$\bar{Q} = m^+ + m^- + m'.$$

As $m^+ + m^-$ and m' must be separately and simultaneously integer or half

integer numbers, $\overline{Q}$ is always integer. Operator Q can thus in a sense be compared to the usual charge operator with respect to our states, N and M being comparable to Fermionic and isotopic spin charge.

If we calculate matrix elements between states introducing the indefinite metric, we obtain certain rules comparable to selection laws.

For example, if we introduce the above definition of the metric (which we can call strong metric) we have shown elsewhere[8] that the matrix elements

$$\langle Z^{n^+, n^-, n'}_{l^+, l^-, s'}(\omega^+, \omega^-) \mid Z^{m^+, m^-, m'}_{l^+, l^-, s'}(\omega^+, \omega^-) \rangle$$

are zero unless

$$\Delta m^+ = 0,$$

$$\Delta m^- = 0,$$

$$\Delta m' = 0.$$

This corresponds to strong coupling.

Another definition of the metric associated to representations $\mathfrak{D}(l^+, 0) \oplus \mathfrak{D}(0, l^+)$ called weak metric[8] leads to the rules

$$\Delta(m^+ + m^-) = 0,$$

$$\Delta m' = 0.$$

All these results do not mean that one must assimilate elementary particles directly with these new rotator sub-levels; this would immediately lead to incorrect results.

As we shall see in the conclusion it seems more promising to compare them with particular sub-levels combinations. They do seem to indicate, however, that the basic physical idea from which we started (that the mutual transformations and interactions of elementary particles indicate they are excited internal states of basic physical structures in space-time) should be investigated seriously.

Conclusion

The preceding results suggest a certain number of line of research. The first is that an effort should be made to substitute systematically the relativistic rotator (equivalent as we have shown[1] to a bilocal structure), to the point particle model as classical substratum of Quantum Theory. This proposal which we developed with de Broglie[11] and Takabayasi[12] implies new researches on the classical properties of relativistic rotators. Some results have already been obtained on the Hamiltonian formalism[13] and the group theoretical properties of such models.

The second line is evidently connected with the classical and quantum interpretation of these results. We already know[14] that classically the relativistic rotators can be interpreted as describing the average external and internal behaviour of relativistic fluid masses enclosed within time-like tubes. But this is not the

only possibility. Preliminary results obtained in Bristol indicate that this model or more exactly the hyperplanes $A_k^{r\pm}$ and $B_k^{r\pm}$ can be directly connected with collective material behaviour within small regions (or cells) of a possible sub-quantum level of matter. This would directly connect hyperplanes and bilocality with basic properties of a subquantum field.

The third line of research is more directly related with the interpretation of the results of § 3. At first sight, as two of us (P. H. and J.P. V.) had proposed,[15] one is tempted to associate every elementary particle to a certain state $Z_{l^+,l^-,s'}^{m^+,m^-,m'}(\omega^+, \omega^-)$ of the relativistic rotator.

We have now reasons to believe that this view is too simple and it is better to associate elementary particles with certain state vectors belonging to specific irreducible representations of the complex rotation group. This is strongly suggested by the non-relativistic limit of the theory. Indeed, if we return to the beginning of Section 3 and examine the solution associated with $l=1/2$ we find that for $m'=1/2$ the set of two wave functions is

$$\Phi=\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix} \quad \text{with} \quad \begin{aligned}\Phi_1 &= Y_{1/2}^{1/2,1/2}(\theta,\varphi,\psi)=\cos\frac{\theta}{2}e^{(i/2)(\varphi+\psi)},\\ \Phi_2 &= Y_{1/2}^{-1/2,1/2}(\theta,\varphi,\psi)=\sin\frac{\theta}{2}e^{(-i/2)(\varphi-\psi)},\end{aligned}$$

while the set corresponding to $m'=-1/2$ can be written as

$$\widetilde{\Phi}=\begin{pmatrix}\widetilde{\Phi}_1\\ \widetilde{\Phi}_2\end{pmatrix} \quad \text{with} \quad \begin{aligned}\widetilde{\Phi}_1 &= Y_{1/2}^{-1/2,-1/2}(\theta,\varphi,\psi)=-\sin\frac{\theta}{2}e^{(i/2)(\varphi-\psi)},\\ \widetilde{\Phi}_2 &= Y_{1/2}^{1/2,-1/2}(\theta,\varphi,\psi)=\cos\frac{\theta}{2}e^{(-i/2)(\varphi+\psi)},\end{aligned}$$

This is very interesting since we recognize easily that for $l=1/2$, $m'=1/2$ the set of two eigenfunctions Φ_1, Φ_2 is identical with the classical expression of the two-component Pauli spinor expressed in terms of Euler angles by one of us (D. B.). One notices also that for $m'=-1/2$ the two eigenfunctions are the components of the so-called charge-conjugate spinor $\widetilde{\Phi}$ derived from Φ by the operation C(see reference 8)). This strongly suggests that we ought to associate particles (and antiparticles) not with specific sub-levels but with vectors (that is sub-level combinations) belonging to particular sub-spaces of our general Hilbert space, which transforms under irreducible representations of the complex rotation group.

Indeed, two of us (P. H. and J.P. V.) have already established elsewhere[16] that one can discover in the sub-space transforming under $\mathfrak{D}(1/2, 0)\oplus\mathfrak{D}(0,1/2)$ four different types of vectors or Dirac spinors. These vectors can be associated to four different types of leptons (Fermions). This will be investigated in a further paper.

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References

1) D. Bohm, P. Hillion, T. Takabayasi and J.P. Vigier, Prog. Theor. Phys. **23** (1960), 496.
2) D. Bohm and J.P. Vigier, Proceedings of the Colston Research Society (1957).
3) J. L. Synge, *Relativity—The special theory* (North Holland 1956).
C. Møller, Annales de l'Institut H. Poincaré **11–12** (1949), 251.
4) P. Hillion and J.P. Vigier, Annales de l'Institut H. Poincaré **16** (1959), 161.
5) F. Halbwachs, P. Hillion and J.P. Vigier, Annales de l'Institut H. Poincaré **16** (1959), 115.
6) C. Van Winter, Thesis (Groningen, 1957).
7) G. Lochak, Cahiers de Physique No. 102 (1959), 41.
8) P. Hillion and J.P. Vigier, to be published in Annales de l'Institut H. Poincaré.
9) W. Heisenberg, Nucl. Phys. **4** (1957), 532.
10) Complete bibliography in L. K. Pandit, Nuovo Cimento, Suppl. No. 2 (1959).
11) L. de Broglie, P. Hillion and J.P. Vigier, C. R. Acad. Sc. **249** (1959), 2251.
12) T. Takabayasi, See ref. 1).
13) F. Halbwachs, P. Hillion and J.P. Vigier, C. R. Acad. Sc. **250** (1960), 471.
14) F. Halbwachs, P. Hillion and J.P. Vigier, Nuovo Cimento **15** (1960), 209.
15) P. Hillion and J.P. Vigier, Quantification du mouvement interne des masses fluides relativistes seminaires L. de Broglie, Ed. Revue d'optique (1959).
16) P. Hillion and J.P. Vigier, Nucl. Phys. **16** (1960), 360.

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On the Nucleon-Nucleon Scattering Phase Shifts and the High Energy Nucleons Elastically Scattered from Light Nuclei

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The polarizations and triple scattering parameters of the nucleons elastically scattered from light nuclei are explained with the optical model potential by the use of the nucleon-nucleon scattering phase shifts. The set of the two-body scattering phase shifts meson-theoretically derived gives a small magnitude of the parameter of optical model potential V_{SR}. The small value of V_{SR} is mainly caused by the large positive ${}^3P_0$ state phase shift of the two-body scattering. The small value of V_{SR} gives a small polarization and a small absolute value of the triple scattering parameter for the nucleon scattered from the potential. Furthermore the value of $\frac{|V_S|}{|V_C|}(\hbar/\mu c)^2$ calculated by the use of the meson-theoretical phase shifts is smaller than the ones based on the Gammel-Thaler and Signell-Marshak phase shifts. This difference is mainly due to the difference of ${}^3P_0$ state phase shifts in the two-body scattering.

§ 1. Introduction

Recently, many authors[1),2),3)] have investigated the differential cross sections and polarizations of the high energy nucleons elastically scattered from light nuclei, using the impulse approximation with the nucleon-nucleon scattering amplitude, for the impulse approximation is equivalent to the use of the optical model potential in the first Born approximation which may be justified for the small angle scattering from light nuclei.[4),5)] The ordinary procedure is that, given several alternative sets of the nucleon-nucleon scattering phase shifts, the optical model potentials corresponding to the sets of the phase shifts are derived, and then the differential cross sections and polarizations of the nucleons scattered in small angles are calculated. The comparison of the results with the observed differential cross sections and polarizations may distinguish the various sets of the nuclear phase shifts rejecting some of them.

The best way of studying the nucleon-nucleon interactions by the use of these optical model potential is to investigate the polarizations and triple scattering parameters of the nucleons elastically scattered from light nuclei, because when one calculates the polarizations and triple scattering parameters of these nucleons through the optical model potentials, the final results are independent of the detailed structure of the target nucleus and to a certain extent free from the limitation of the Born approximation. That is, the expressions for the polarization and triple scattering parameter do not contain the nuclear form factors, i.e. the Fourier trans-

form of square of the wave function of the target nucleus which depends upon the detailed structure of the nucleus, so that the expressions for the polarization and triple scattering parameter depend essentially on the nucleon-nucleon scattering phase shifts.

The main purposes of the present paper are as follows,

(1) to construct the optical model potentials by the use of the nucleon-neucleon scattering phase shifts ; [6),7),8)]

(2) to calculate the polarizations of the nucleons elastically scattered from light nuclei through the optical model potentials, taking into account the angular dependence of the nucleon-nucleon scattering for the amplitude of the scattered nucleon ;

(3) to calculate preliminarily the triple scattering parameters of the scattered nucleons by using the optical model potential, taking into account the angular dependence of the nucleon-nucleon scattering ;

(4) to study how the large positive $^3P_0$ state phase shift meson-theoretically derived gives rise to the effects for the polarization and triple scattering parameter.

§ 2 is devoted to the expressions for the polarization and triple scattering parameter. The discussions for the results are given in § 3. § 4 contains concluding remarks.

§ 2. Expressions for the polarization and triple scattering parameter

Since the optical model potential relates to the amplitude of nucleon-nucleon scattering, we use the scattering spin matrix M to describe the amplitude of two-body scattering,[9)]

$$M=\alpha+\gamma(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\overline{\boldsymbol{n}}+\beta(\boldsymbol{\sigma}_1\overline{\boldsymbol{n}})(\boldsymbol{\sigma}_2\overline{\boldsymbol{n}})+\delta(\boldsymbol{\sigma}_1\overline{\boldsymbol{q}})(\boldsymbol{\sigma}_2\overline{\boldsymbol{q}})+\varepsilon(\boldsymbol{\sigma}_1\overline{\boldsymbol{p}})(\boldsymbol{\sigma}_2\overline{\boldsymbol{p}}), \qquad (1)$$

where $\overline{\boldsymbol{n}}$, $\overline{\boldsymbol{p}}$, $\overline{\boldsymbol{q}}$ are three mutually perpendicular unit vectors, chosen so that $\overline{\boldsymbol{n}}$ is normal to the scattering plane. The coefficients α, β, γ, δ and ε depend on the energy and the scattering angle in the center of mass of two-body system, and 1 and 2 indicate the incident and the struck nucleon, respectively.

We restrict our consideration to the nuclei with total spin zero at the ground state. When one constructs the amplitude of nucleon-nucleus scattering by the two-body scattering spin matrix M, some terms of M are cancelled out, because one must average the scattering amplitude over the spin states of the nucleons in the target nucleus. Then, the terms of M containing $\boldsymbol{\sigma}_2$ drop out. This means neglects of correlations between nucleons in the nucleus during the scattering process, since the z components of the spin of the individual nucleons in the nucleus do not change. These cancelled terms do not contribute much to the polarized cross section but contribute to the total cross section in the nucleon-nucleon scattering.[10)] Then one obtains the amplitude of the nucleon scattered from the optical model potential which gives rise to the large polarization.

Now, the optical model potential is given, under some approximations,[4),5)] by

$$\langle \boldsymbol{k}'|v|\boldsymbol{k}\rangle = \left\{-V_C+\left(\frac{\hbar}{\mu c}\right)^2 i\boldsymbol{\sigma}_1\boldsymbol{k}'\times\boldsymbol{k}\cdot V_S\right\}\left(\frac{1}{2\pi\hbar}\right)^3\int\rho_N(r)\exp[-i(\boldsymbol{k}'-\boldsymbol{k})\boldsymbol{r}]\,d\boldsymbol{r}, \tag{2}$$

where

$$\begin{aligned} V_C &= 3\frac{A}{R^3}\left(\frac{\hbar^2}{m\gamma_0}\right)\left(\frac{\gamma_0+1}{2}\right)^{1/2}\bar{\alpha}, \\ V_S &= -3\frac{A}{R^3}\left(\frac{\hbar^2}{m\gamma_0}\right)(\gamma_0+1)\frac{1}{k^2}\left(\frac{\mu c}{\hbar}\right)^2\bar{\gamma}, \end{aligned} \tag{3}$$

while $\boldsymbol{k}$ and $\boldsymbol{k}'$ are the incident and scattered nucleon momenta in the laboratory system, respectively; here $k=|\boldsymbol{k}|=|\boldsymbol{k}'|$. γ_0 is the total energy of the incident nucleon in the laboratory system divided by mc^2; E_{lab}/mc^2, and m is the nucleon mass. R is the nuclear radius and A is the mass number of the nucleus. In Eq. (2), the factor $(\hbar/\mu c)^2$ gives the parameters of optical model potential for the central part V_C and for the spin-orbit part V_S the dimensions of energy, and $\hbar/\mu c$ is the pion Compton wavelength, and μ is the pion mass. $\rho_N(r)$ is the nucleon density of the target nucleus normalized to the nuclear volume. Here it is assumed that $\rho_N(r)$ for the central part of the potential is equal to the one for the spin-orbit part of it, and that $\rho_N(r)$'s for the real terms of the potential are equal to the ones for the imaginary terms of it.

Since the differential cross section divided by the square of the Fourier transform of nuclear density is not constant even for the small angular region at interested incident energies,[11)] we shall take account of the nucleon-nucleon angular dependence for the optical model potential.* Then, in Eq. (3), we use the coefficients of scattering spin matrix which depend on the momentum transfer; $q=2k\sin(\theta/2)$,

$$\begin{aligned} \bar{\alpha} &= \frac{1}{2A}\{(A-Z)\alpha_0(q)+(A+Z)\alpha_1(q)\}, \\ \bar{\gamma} &= -\frac{1}{2iA}\{(A-Z)\gamma_0(q)+(A+Z)\gamma_1(q)\}\frac{1}{\sin\theta}, \end{aligned} \tag{4}$$

where Z is the atomic number of the nucleus, and the subscripts 1 and 0 indicate $T=1$ and 0, respectively. Here $\theta=(1/2)\theta_{2\text{-}body}$. This approximation is reasonable in the point of view of the impulse approximation at high energies.[13)]

Since $R=(\hbar/\mu c)\kappa A^{1/3}$ where κ is a dimensionless scale parameter, $\kappa\simeq 0.8$, the parameters of optical model potential V_C and V_S do not depend on A and R of

* The polarizations calculated by using the optical model potential which does not take account of the angular dependence of nucleon-nucleon scattering have been reported in the previous note.[12)] The difference between the results of the note[12)] and of the present paper is due to the fact that the nucleon-nucleon angular dependence is taken into account in the latter.

the target nucleus, when the nucleon density in the nucleus, $1/\{(\hbar/\mu c)^3\kappa^3\}$, is assumed A-independent.

In the first Born approximation, the amplitude of the nucleon scattered from α-particle type nuclei, obtained through the optical model potential, is given by

$$f_N+g_N\boldsymbol{\sigma}_1\boldsymbol{n}=\left(\frac{2k}{\hbar v}\right)\frac{1}{4\pi}F_N(q)\left\{V_C(q)+iV_S\left(\frac{\hbar}{\mu c}\right)^2k^2\sin\theta\,\boldsymbol{\sigma}_1\boldsymbol{n}\right\}, \qquad (5)$$

where $\boldsymbol{n}=\boldsymbol{k}\times\boldsymbol{k}'/|\boldsymbol{k}\times\boldsymbol{k}'|$, and $F_N(q)$ is the nuclear form factor, i.e. the Fourier transform of the nuclear density $\rho_N(r)$.

When the incident nucleon is the proton, the Coulomb scattering amplitude may be calculated in the first Born approximation as

$$f_C+g_C\boldsymbol{\sigma}_1\boldsymbol{n}=-\left(\frac{2k}{\hbar v}\right)\frac{e}{q^2}F_C(q)\left\{1-i(\gamma_0-1)\left(\mu_p-\frac{1}{2}\right)\sin\theta\,\boldsymbol{\sigma}_1\boldsymbol{n}\right\}, \qquad (6)$$

where μ_p is the magnetic moment of the proton in the units of the nuclear magneton. In Eq. (6) the phase factor $\exp[-2i\eta\ln\sin^2(\theta/2)]$ is not present which should be included in the exact Coulomb amplitude. This approximation, however, may be compensated by omitting the Coulomb phase shifts from the nuclear amplitude.[2] The charge form factor $F_C(q)$ is the Fourier transform of the charge density of the nucleus normalized to the total charge of the nucleus, Ze.

If $\rho_C(r)$ is assumed to be equal to $\rho_N(r)$, except for its normalization, from Eqs. (5) and (6), the polarization is given by[14]

$$P(\theta)=\frac{2\,\mathrm{Re}\,i\left\{\frac{R^3}{3}V_C(q)-\frac{Ze^2}{q^2}\right\}^*\left\{\frac{R^3}{3}V_S(q)\left(\frac{\hbar}{\mu c}\right)^2k^2+\frac{Ze^2}{q^2}(\gamma_0-1)\left(\mu_p-\frac{1}{2}\right)\right\}\sin\theta}{\left|\frac{R^3}{3}V_C(q)-\frac{Ze^2}{q^2}\right|^2+\left|\left\{\frac{R^3}{3}V_S(q)\left(\frac{\hbar}{\mu c}\right)^2k^2+\frac{Ze^2}{q^2}(\gamma_0-1)\left(\mu_p-\frac{1}{2}\right)\right\}\sin\theta\right|^2} \qquad (7)$$

and the triple scattering parameter is given by[14]

$$\sin\beta=\frac{-2\,\mathrm{Im}\,i\left\{\frac{R^3}{3}V_C(q)-\frac{Ze^2}{q^2}\right\}^*\left\{\frac{R^3}{3}V_S(q)\left(\frac{\hbar}{\mu c}\right)^2k^2+\frac{Ze^2}{q^2}(\gamma_0-1)\left(\mu_p-\frac{1}{2}\right)\right\}\sin\theta}{\sqrt{1-P^2(\theta)}\left[\left|\frac{R^3}{3}V_C(q)-\frac{Ze^2}{q^2}\right|^2+\left|\left\{\frac{R^3}{3}V_S(q)\left(\frac{\hbar}{\mu c}\right)^2k^2+\frac{Ze^2}{q^2}(\gamma_0-1)\left(\mu_p-\frac{1}{2}\right)\right\}\sin\theta\right|^2\right]} \qquad (8)$$

In order to point out evidently which parameters of the optical model potential play essential roles for the polarization and triple scattering parameter, it is convenient to write as follows,

$$V_C(q)=V_{SR}(q)+iV_{CI}(q),\quad V_S(q)=V_{SR}(q)+iV_{SI}(q), \qquad (9)$$

where four quantities specifying the optical model potential $V_{CR}(q)$, $V_{CI}(q)$, $V_{SR}(q)$, and $V_{SI}(q)$ connected with the nucleon-nucleon scattering amplitude are given as the functions of the nucleon-nucleon scattering phase shifts at the definite incident energy. $V_{CR}(0)$ $V_{CI}(0)$, $V_{SR}(0)$ and $V_{SI}(0)$ are the parameters of optical model potential defined by Riesenfeld-Watson.[5] It is to be noted that the parameters $V_{CR}(q)$, $V_{CI}(q)$, $V_{SR}(q)$ and $V_{SI}(q)$ depend on the factor $(\hbar/\mu c)\kappa$; however, the expressions for the polarization and triple scattering parameter do not contain $(\hbar/\mu c)\kappa$. The expressions depend only on the nucleon-nucleon scattering phase shifts.

§ 3. Results and discussions

Figs. 1 and 2 show the results for the polarizations of the protons calculated through the optical model potential by taking account of the effects of the nucleon-nucleon angular dependence at incident energies near 90 and 150 Mev, respectively. The differences between the results for the polarizations calculated with the meson-theoretically derived phase shifts and the results based on the Gammel-Thaler and Signell-Marshak phase shifts are observed at very small scattering angular regions. We shall investigate the reasons why the results calculated with the meson-theoretical phase shifts are different from those based on the Gammel-Thaler and Signell-Marshak phase shifts. The convenient way of studying the difference is to use the parameters of optical model potential, since the contributions of the particular state phase shifts to the polarization and triple scattering parameter are easily investigated in the parameters. The polarization is proportional, from Eqs. (7) and (9), to

$$P(\theta) \propto \frac{V_{CI}(0)\,V_{SR}(0) + V_{CR}(0)\,|V_{SI}(0)|}{V_{CR}^2(0) + V_{CI}^2(0) + \left(\frac{\hbar}{\mu c}\right)^2 \{V_{SR}^2(0) + V_{SI}^2(0)\}\,k^2}, \qquad (10)$$

except for the Coulomb terms. Therefore, the small value of $V_{SR}(0)$ obtained by the meson-theoretical phase shifts gives rise to the small polarization at small scattering angles even if the magnitude of $V_{CR}(0)$ is larger than that of $V_{CI}(0)$. Since $V_{CI}(0)$ corresponds to the total cross section of the two-body scattering, the magnitudes of $V_{CI}(0)$ are roughly equal to each other for the sets of the nuclear phase shifts. Then, $V_{SR}(0)$ decides the magnitude of the polarization since $V_{CR}(0)$ is nearly equal to or smaller than $V_{CI}(0)$ at high energies.

At the interested energies of the incident nucleons, the magnitudes of the parameters of optical model potential are almost determined by ${}^3P_J(J=0, 1, 2)$, ${}^3D_J(J=1, 2)$ state phase shifts and associated mixing parameters $\epsilon_J(J=1, 2)$ in each set of the two-body scattering phase shifts. Especially the difference between the magnitudes of $V_{SR}(0)$ calculated by using the meson-theoretical phase shifts and that based on the Gammel-Thaler phase shifts is mainly caused by the difference of ${}^3P_0$ state phase shifts.

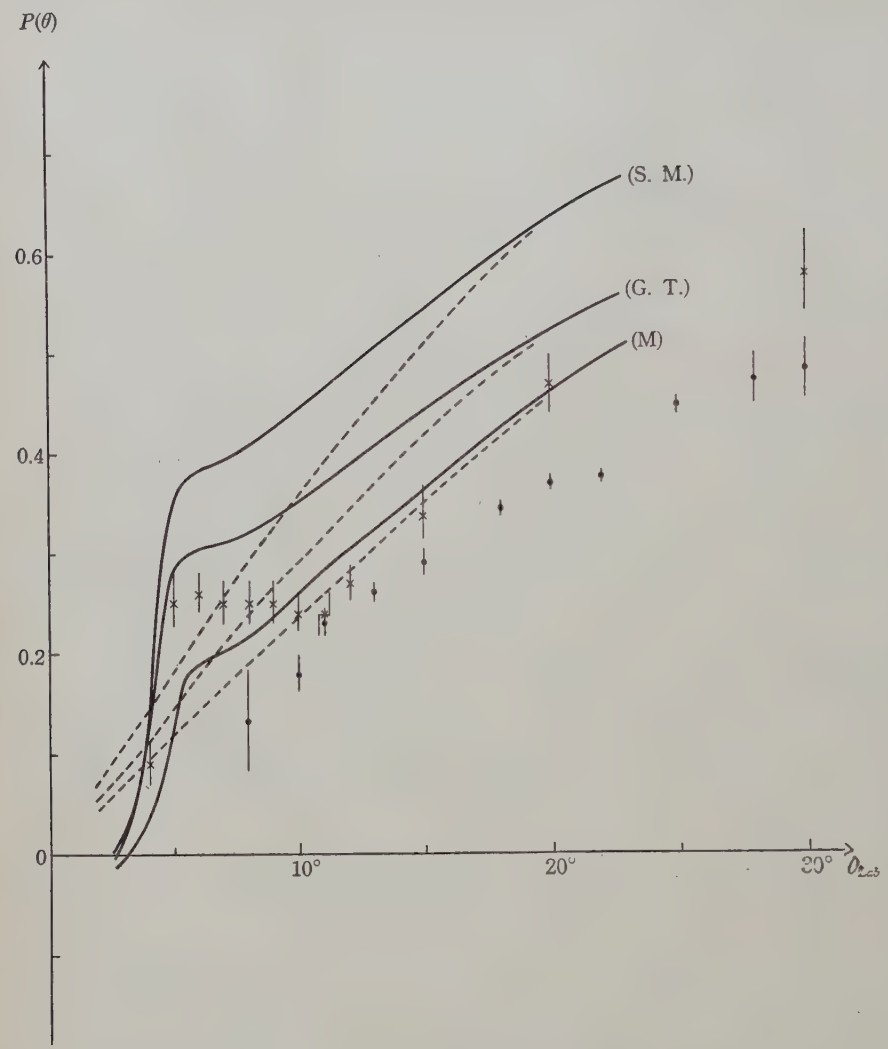

Fig. 1. Polarizations of the nucleons elastically scattered from He^4 and C^{12}. Curves are calculated by the use of the nucleon-nucleon scattering phase shifts derived from the meson-theoretical potential (M) at 90 Mev, the Gammel-Thaler (G. T.) at 90 Mev and the Signell-Marshak (S. M.) at 100 Mev. Solid curves refer to the polarizations of proton-α-particle type nuclei scattering and dashed curves to the polarizations of neutron-α-particle type nuclei scattering, the latters are obtained by setting $e \to 0$ in the expression of the polarization, Eq. (7). Experimental points ⧫ indicate the values measured on p-He^4 (96 Mev) by Gotow and ✳ the values on p-C^{12} (95 Mev) by Dickson-Salter.

The characters of the meson-theoretical, the Gammel-Thaler and Signell-Marshak potentials for the two-body system have been discussed by several authors.[6] The most visible difference is that $^3P_0$ state phase shifts differ from each other in those

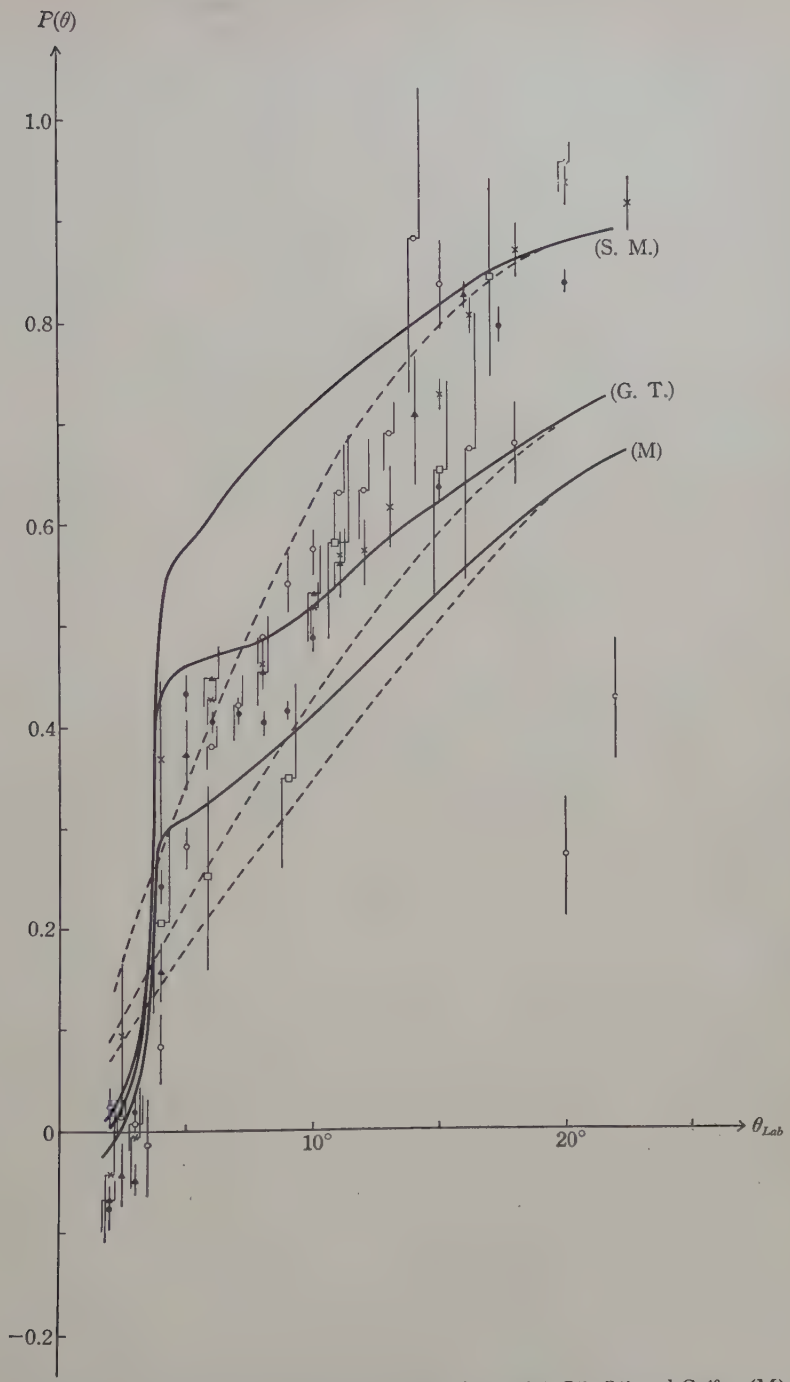

Fig. 2. Polarizations of the nucleons elastically scattered from He^4, C^{12}, O^{16} and Ca^{40}. (M) at 150 Mev, (G. T.) at 156 Mev and (S. M.) at 150 Mev. Solid and dashed curves correspond to the polarizations of proton- and neutron-α-particle type nuclei, respectively. Experimental points ⊕ indicate the values measured on p-He^4 (147 Mev) by Wilson, et al., ⊟ the values on n-C^{12} (155 Mev) by Harding and ✳, ⊥, ○ the values of p-C^{12}, p-O^{16}, p-Ca^{40} (155 Mev) by Alphonce, et al., respectively.

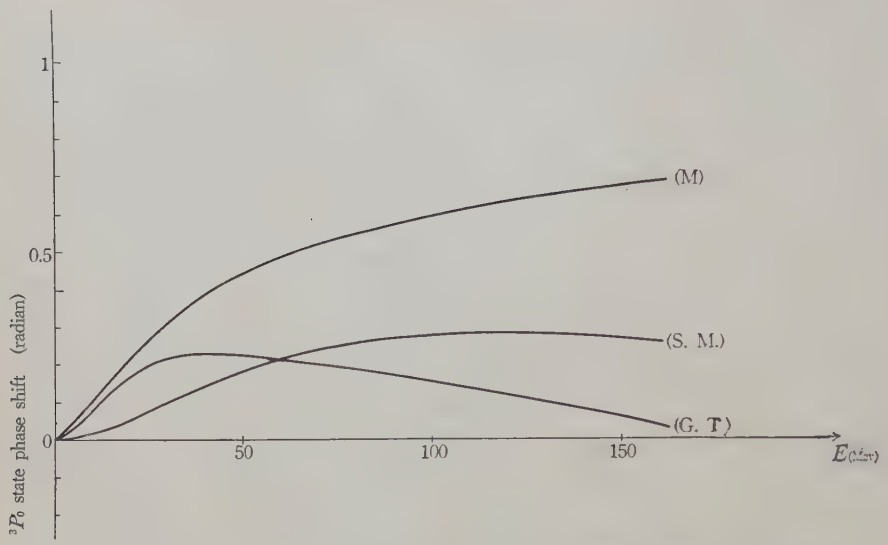

Fig. 3. $^3P_0$ state phase shifts calculated by using various two-body potentials.

sets of nuclear phase shifts. Whether the two-body potential contains the spin-orbit force or not largely affects the magnitude of $^3P_0$ state phase shift since $^3P_0$ state phase shift is determined by the two-body potential $V_C - 4V_T - 2V_{LS}$. Fig. 3 shows the differences among $^3P_0$ state phase shifts. It seems that the differences among $^3P_0$ state phase shifts may be checked by the high energy nucleon scattered from α-particle type nuclei.

Since $V_{SR}(0)$ contains the term $\{-\sin 2\delta(^3P_0)\}$ except for some factors,[5] the large positive $^3P_0$ state phase shift causes the small magnitude of $V_{SR}(0)$. Also the large positive $^3P_0$ state phase shift gives $|V_{SI}(0)|$ the large magnitude because $V_{SI}(0)$ contains the term $\{-\sin^2\delta(^3P_0)\}$ save for some factors.[5] Table I shows the contributions of all state phase shifts except for $^3P_0$ state phase shift to $V_{SR}(0)$, denoted by $V_{SR}'(0)$, and the contribution of $^3P_0$ state phase shift to $V_{SR}(0)$, denoted by $V_{SR}(^3P_0)$. The sum of $V_{SR}'(0)$ and $V_{SR}(^3P_0)$ gives $V_{SR}(0)$. The larger the positive $^3P_0$ state phase shift is, the larger $|V_{SR}(^3P_0)|$ is, and the smaller $V_{SR}(0)$ is. Also Table II shows the contributions of all state phase shifts save for $^3P_0$ state phase shift to $V_{SI}(0)$, denoted by $V_{SI}'(0)$, and the contribution of $^3P_0$ state phase shift to $V_{SI}(0)$, denoted by $V_{SI}(^3P_0)$. The sum of $V_{SI}'(0)$ and $V_{SI}(^3P_0)$ gives $V_{SI}(0)$. Then, the larger the positive $^3P_0$ state phase shift is, the larger $|V_{SI}(^3P_0)|$ is, and the larger $|V_{SI}(0)|$ is. Tables III and IV show the contributions of all state phase shifts save for $^3P_0$ state phase shift to $V_{CR}(0)$ and $V_{CI}(0)$, denoted by $V_{CR}'(0)$ and $V_{CI}'(0)$, respectively, and the contributions of $^3P_0$ state phase shift to $V_{CR}(0)$ and $V_{CI}(0)$, denoted by $V_{CR}(^3P_0)$ and $V_{CI}(^3P_0)$, respectively. The sums of $V_{CR}'(0)$ and $V_{CR}(^3P_0)$ and of $V_{CI}'(0)$ and $V_{CI}(^3P_0)$ give $V_{CR}(0)$ and $V_{CI}(0)$, respectively. It is to be noted that the large positive $^3P_0$ state phase shift

Table I. Contribution of $^3P_0$ state phase shift to $V_{SR}(0)$

	$V_{SR}'(0)$	$V_{SR}(^3P_0)$	$V_{SR}(0)$ (Mev)
(M) (90 Mev)	3.22	−2.12	1.10
(G.T.) (90 Mev)	4.60	−0.79	3.81
(S.M.) (100 Mev)	7.16	−1.24	5.92
(M) (150 Mev)	2.67	−0.95	1.72
(G.T.) (156 Mev)	3.05	−0.09	2.96
(S.M.) (150 Mev)	5.00	−0.50	4.50

Table II. Contribution of $^3P_0$ state phase shift to $V_{SI}(0)$

	$V_{SI}'(0)$	$V_{SI}(^3P_0)$	$V_{SI}(0)$ (Mev)
(M) (90 Mev)	−1.30	−1.33	−2.63
(G.T.) (90 Mev)	−1.71	−0.61	−2.32
(S.M.) (100 Mev)	−1.28	−0.35	−1.63
(M) (150 Mev)	−0.37	−0.77	−1.14
(G.T.) (156 Mev)	−0.77	−0.00	−0.77
(S.M.) (150 Mev)	−0.94	−0.14	−1.08

Table III. Contribution of $^3P_0$ state phase shift to $V_{CR}(0)$

	$V_{CR}'(0)$	$V_{CR}(^3P_0)$	$V_{CR}(0)$ (Mev)
(M) (90 Mev)	12.3	8.9	21.2
(G.T.) (90 Mev)	30.2	3.3	33.5
(S.M.) (100 Mev)	24.9	5.2	30.1
(M) (150 Mev)	13.8	6.1	19.9
(G.T.) (156 Mev)	20.8	0.6	21.4
(S.M.) (150 Mev)	15.3	3.2	18.5

Table IV. Contribution of $^3P_0$ state phase shift to $V_{CI}(0)$

	$V_{CI}'(0)$	$V_{CI}(^3P_0)$	$V_{CI}(0)$ (Mev)
(M) (90 Mev)	17.6	5.6	23.2
(G.T.) (90 Mev)	23.7	0.6	24.3
(S.M.) (100 Mev)	25.4	1.5	26.9
(M) (150 Mev)	13.6	4.9	18.5
(G.T.) (156 Mev)	18.3	0.0	18.3
(S.M.) (150 Mev)	18.6	0.9	19.5

acts to increase the magnitude of $V_{CR}(0)$ and $V_{CI}(0)$, while the large positive $^3P_0$ state phase shift acts to decrease the magnitudes of $V_{SR}(0)$ and $V_{SI}(0)$. As

shown in these Tables, the effects of the large positive ${}^3P_0$ state phase shift on $V_{CR}(0)$ and $V_{CI}(0)$ are not so visible as $V_{SR}(0)$ and $V_{SI}(0)$.

Figs. 4 and 5 show the triple scattering parameters calculated with Eq. (8) at incident energies near 90 and 150 Mev, respectively. The triple scattering parameter is, from Eqs. (8) and (9), proportional to

$$\sin\beta \propto \frac{1}{\sqrt{1-P^2(\theta)}} \frac{2\{-V_{CR}(0)\,V_{SR}(0)+V_{CI}(0)\,|V_{SI}(0)|\}}{V_{CR}^2(0)+V_{CI}^2(0)+(\hbar/\mu c)^2\{V_{SR}^2(0)+V_{SI}^2(0)\}k^2} \tag{11}$$

without the Coulomb terms.

The differences among the triple scattering parameters calculated by several sets of nucleon-nucleon scattering phase shifts are mainly caused by the magnitudes of $V_{SR}(0)$ like the polarizations. Then, the small value of $V_{SR}(0)$ obtained by using the meson-theoretical phase shifts gives the small value of $|\beta|$ at small scattering angles. At incident energy of 90 Mev, the small $V_{SR}(0)$ based on the meson-theoretical phase shifts gives β the positive value at interested angular region since $V_{CR}(0) < V_{CI}(0)$ and $V_{SR}(0) < |V_{SI}(0)|$.

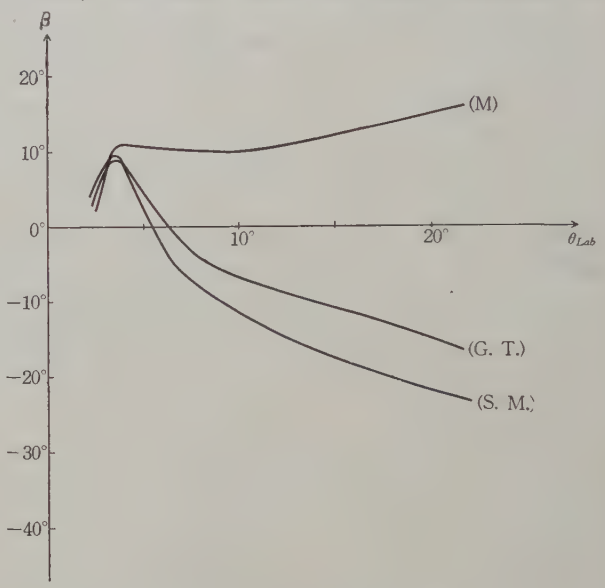

Fig. 4. Triple scattering parameters calculated by the use of the nucleon-nucleon scattering phase shifts of the meson-theoretical potential (M) at 90 Mev, the Gammel-Thaler (G. T.) at 90 Mev and the Signell-Marshak (S. M.) at 100 Mev.

Table V shows the ratios of $V_{CR}(0)$ to $V_{CI}(0)$, of $V_{SI}(0)$ to $V_{SR}(0)$ and of $|V_S(0)| \times (\hbar/\mu c)^2$ to $|V_C(0)|$. The ratios are independent of the α-particle type nuclei. These values depend only on the two-body scattering phase shifts. The small value of $(|V_S(0)|/|V_C(0)|)(\hbar/\mu c)^2$ predicts a large θ_0, the angle of the first maximum in the polarization, since[15]

$$\frac{|V_S(0)|}{|V_C(0)|}\left(\frac{\hbar}{\mu c}\right)^2 = \frac{1}{k^2\theta_0}. \tag{12}$$

The value of $(|V_S(0)|/|V_C(0)|)(\hbar/\mu c)^2$ calculated by the use of the meson-theoretical phase shifts is small. Then, the two-body scattering phase shifts derived meson-theoretically gives the large θ_0. The large θ_0 is due to the large positive ${}^3P_0$ state phase shifts.

As above mentioned, the expressions for the polarization and triple scattering

parameter do not contain any factors which depend on the characters of α-particle type nuclei. They depend only on the amplitude of the nucleon-nucleus scattering calculated by using the two-body scattering phase shifts. Then Eq. (7) gives the prediction that the nucleons elastically scattered from α-particle type nuclei must have the same polarization. The prediction is checked by the available experimental data on the polarizations of the nucleons scattered from α-particle nuclei.[11),16)] (See Figs. 1 and 2.) The similarities of these data are observed up to about 30° at 90 Mev and up to about 20° at 150 Mev. The angular region in which the similarity is observed is in less than θ_0. Eq. (7) is not connected with θ_0 since it does not contain any θ_0-dependent factors. The value of θ_0 does not depend on A but depends only on the incident energy, that is, θ_0 is proportional to $1/k$, for lighter nuclei. It seem that the above method of calculation does not apply to heavier nuclei, because in spite of the fact that the positions of the first diffraction maxima of the

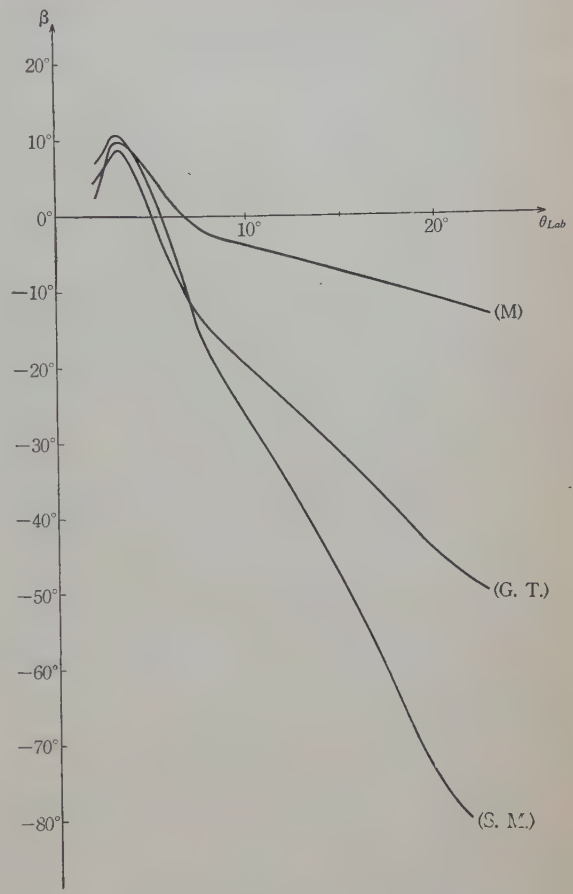

Fig. 5. Triple scattering parameters calculated by the use of the nucleon-nucleon scattering phase shifts of (M) at 150 Mev, (G. T.) at 156 Mev and (S. M.) at 150 Mev.

Table V. Comparison of the parameters of optical model potentials

	$V_{CR}(0)/V_{CI}(0)$	$V_{SI}(0)/V_{SR}(0)$	$\lvert V_S(0)\rvert/\lvert V_C(0)\rvert\times(\hbar/\mu c)^2$
(M) (90 Mev)	0.91	−2.39	0.181×10^{-26} cm^2
(G.T.) (90 Mev)	1.38	−0.62	0.217×10^{-26} cm^2
(S.M.) (100 Mev)	1.12	−0.28	0.305×10^{-26} cm^2
(M) (150 Mev)	1.08	−0.66	0.152×10^{-26} cm^2
(G.T.) (156 Mev)	1.17	−0.26	0.223×10^{-26} cm^2
(S.M.) (150 Mev)	1.00	−0.24	0.344×10^{-26} cm^2

experimental polarization depend slightly on A for heavier nuclei, the expression for the polarization cannot reproduce the A-dependence of the patterns for the polarization even at small angular regions. Since the diffraction effects due to the nuclear size and shape come into play and the multiple scattering effects play essential roles, it seems that the method of the above calculation might have no validity for such heavier nuclei. Also, since the effects of the off-energy shell of two-body scattering play important roles at large scattering angles, it seems that the above method has no justification in such angular regions.

When one carefully compares the experimental data of the polarization for p-He^4 with the ones of p-C^{12} scattering, one finds the large difference between those data at very small angular region in the incident energy near 90 Mev, though the polarization predicted by Eq. (7) for α-particle type nuclei must be equal. It is observed that the higher the incident energies are, the smaller the difference between the polarizations of p-He^4 and p-C^{12} scattering at forward small angles. Then, for the polarizations of nucleon-He^4 as well as nucleon-C^{12} scattering, the use of Eq. (2) or (7) would not be justified in the incident energies less than about 100 Mev, because, although the expression for the polarization predicts the same polarization for α-particle type nuclei, the observed polarizations show a large difference between the data of p-He^4 and p-C^{12} scattering at small angular region in the incident energy about 90 Mev. It is expected that Eq. (2) or (7) has its validity for the calculation of the polarization at such energies that there are no differences among the experimental polarizations of α-particle type nuclei.

In Eq. (7), $\rho_N(r)$'s for the real and imaginary terms of the central and of the spin-orbit potential have their radii different from each other, but $\rho_N(r)$ is assumed to be a function of the same form for these parts of the potential. The difference between the polarizations of p-He^4 and p-C^{12} at near 90 Mev might indicate that the forms of $\rho_N(r)$ for the real and imaginary terms of the central and of the spin-orbit parts of the potential differ from each other for He^4 and for C^{12}, respectively. Also it might be expected that the difference between the kinks of the experimental polarization curves of p-He^4 and p-C^{12} at near 90 Mev would be due to the influence of the Coulomb potential. These problems are left for further investigation.

§ 4. Concluding remarks

At the incident energy of 150 Mev, the experimental polarizations of the nucleons scattered from light nuclei show similar patterns at small scattering angles; this fact indicates that the origins of the polarizations of the nucleons scattered from such nuclei are the same, and this is consistent with the prediction by the results of the optical model potential explained by the two-body scattering phase shifts. Then it is expected that the optical model potential connected with the two-body scattering phase shifts would have a sufficient validity at near 150 Mev.

The agreement between the calculated results and the experimental data at 150 Mev is not so good as at the higher energies.[2),3)] This could either be due to the approximations in the derivation of the optical model potential or be due to the inadequacies in the two-body scattering phase shifts which do not give the accurate fits even to the free scattering data. However, it seems that there are more possibilities to correct the two-body scattering phase shifts since the experimental polarizations of the nucleons scattered from α-particle type nuclei show the very similar patterns.

The large positive ${}^3P_0$ state phase shift derived meson-theoretically causes a small value of $V_{SR}(0)$. The small value of $V_{SR}(0)$ gives rise to the small polarization at small scattering angles. And there are large differences among the triple scattering parameters calculated by using several sets of the two-body scattering phase shifts. The differences among the triple scattering parameters calculated are mainly caused by the magnitudes of $V_{SR}(0)$. By the comparison of the calculated results with the accurate experiments made at very small angular regions, it would become evident whether the small $V_{SR}(0)$ is favorable or not and, moreover, large positive ${}^3P_0$ state phase shifts are favorable or not.

It is also to be noted that the polarization and triple scattering parameter depend little on the spin independent distortion and absorption for the wave functions of incident and scattered nucleons, though the magnitudes of the differential cross section as well as the polarized cross section of the scattered nucleons are largely affected by the absorption. This is essentially because there are considerable cancellations of the matrix elements in the polarization and triple scattering parameter which do not occur in the differential cross section and polarized cross section. The polarization and triple scattering parameter would be slightly altered by the spin dependent distortion and absorption; however, the spin dependent distortion and absorption would be very small since the values of $V_{SR}(0)$ and $V_{SI}(0)$ are smaller than the values of $V_{CR}(0)$ and $V_{CI}(0)$. Then the above results would not be seriously altered by the effects of the distortion and absorption. Therefore the measurement of the polarization and triple scattering parameter of the nucleon scattered from nuclei would help in clarifying the nucleon-nucleon interaction, and would give the best way to discriminate the sets of the two-body scattering phase shifts.

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References

1) H. A. Bethe, Annals of Physics **3** (1958), 190.
2) S. Ohnuma, Phys. Rev. **111** (1958), 1172.
3) A. H. Cromer, Phys. Rev. **113** (1959), 1607.

4) N. Francis and K. M. Watson, Phys. Rev. **92** (1953), 291.
G. Takeda and K. M. Watson, Phys. Rev. **97** (1955), 1336.
5) W. B. Riesenfeld and K. M. Watson, Phys. Rev. **102** (1956), 1157.
6) S. Otsuki, Prog. Theor. Phys. **20** (1958), 171.
W. Watari, Prog. Theor. Phys. **20** (1958), 181.
R. Tamagaki, Prog. Theor. Phys. **20** (1958), 505.
T. Hamada, J. Iwadare, S. Otsuki, R. Tamagaki and W. Watari, Prog. Theor. Phys. **22** (1959), 301, 566; **23** (1960), 366.
7) J. L. Gammel anc R. M. Thaler. Phys. Rev. **107** (1957), 291, 1337.
8) P. S. Signell and R. E. Marshak, Phys. Rev. **109** (1958), 1229.
9) L. Wolfenstein and J. Ashkin, Phys. Rev. **85** (1952), 947.
R. H. Dalitz, Proc. Phys. Soc. **A65** (1952), 172.
10) S. Tamor, Phys. Rev. **97** (1955), 1077.
11) A. M. Cormack, J. N. Palmieri, N. F. Ramsey and R. Wilson, Phys. Rev. **115** (1959), 599.
12) Y. Sakamoto, Prog. Theor. Phys. **23** (1960), 383.
13) Y. Sakamoto and T. Sasakawa, Prog. Theor. Phys. **21** (1959), 879
14) L. Wolfenstein, *Annual Review of Nuclear Science* (Annual Review, Inc., Stanford.) Vol. 6 (1956), P. 43.
15) I. I. Levintov, Soviet Physics JETP, **1** (1956), 175.
H. S. Köhler, Nuovo Cimento **2** (1955), 911.
16) R. Alphonce, A. Johansson and G. Tibell, Nuclear Physics **3** (1957), 185; **4** (1957), 672.
J. M. Dickson and D. C. Salter, Nuovo Cimento **6** (1957), 235.
K. Gotow, unpublished.
R. S. Harding, Phys. Rev. **111** (1958), 1164.

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Photodisintegration of the Deuteron in Medium Energy Range and the Pion Theoretical Nuclear Forces*

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The static pion potential which consists of the central and tensor parts is applied to the calculations of the photodisintegration of the deuteron in the medium energy range (20~80 Mev). The angular distributions are in good agreement with the observed results. The polarizations of final proton and neutron are also calculated. There is no observation yet to be compared with them. The general formulas are given for the angular distribution and polarization with the numerical coefficients for the lower multipole transitions.

§ 1. Introduction and summary

The pion theory of nuclear forces has been extensively investigated and verified in the analyses of the deuteron problem and the nucleon-nucleon scattering. In these analyses, the following points have so far been established :[1),2)] (1) One-pion-exchange potential in the outer region ($r \gtrsim 1.5\hbar/\mu c$) can be evaluated from the present-day pion theory. This potential can reproduce all the low energy two-nucleon data, including the deuteron data, if one adopts a suitable phenomenological potential in the intermediate and inner regions ($r \leq 1.5\hbar/\mu c$).[1)] (2) The magnitude of the coupling constant for the pion-nucleon interaction is determined uniquely as $0.06 \lesssim g^2/4\pi hc \lesssim 0.09$ for the one-pion exchange potential in the deuteron state. This determination does not depend on the assumed shapes of the intermediate and inner potentials.[1)] The value of the coupling constant is consistent with other low energy data and is in good agreement with the one determined from the pion reaction data.[1)] (3) The two-pion exchange potential can be evaluated qualitatively from the pion theory and has characteristic features which are required from the properties of the two-nucleon system at low energies.[1)] (4) The nucleon-nucleon single and double scattering data up to 100 Mev can be qualitatively reproduced by a static potential which has the one-pion-exchange tail.[2)] In the intermediate region, the potential has the qualitative features which are characteristic of the two-pion exchange potential and does not contain the direct $L \cdot S$ coupling part.

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In the light of these achievement, it seems very important to test the validity of the potential in other processes. The photodisintegration of the deuteron can add a crucial test on the potential, since it has a close connection with the properties of the two-nucleon system through its initial and final two-nucleon states.

A few years ago, the angular distribution of the emitted proton in the photodisintegration of deuteron at 22.4 Mev was examined by one of the present authors (J. I.) and his collaborators with the aim of testing the validity of the pion potential.[3] The purpose of the present paper is to extend the results to the medium energy region, i.e. 20～80 Mev.

The angular distribution of the proton emitted in the reaction at *low* energies is expressed in the following form:

$$\frac{d\sigma}{d\Omega}=a+b\sin^2\theta(1+2\beta\cos\theta).$$

The $\sin^2\theta$ term is mainly due to $E1$ transition and is given by a classical argument with the neglect of the retardation effects. The β term is due to the interference effect between $E1$ and $E2$ transitions, or in other words, due to the retardation of the γ-ray.[4] If there were no tensor forces at all, a would be entirely due to $M1$ transition, and would be very small ($a/b\lesssim 0.03$).[5] Whereas experimental data revealed $a/b\sim 0.1$ and $\beta\sim 0.1$ for $E_\gamma\sim 20$ Mev.[6] As the energy becomes higher, the a/b ratio rises to $a/b\sim 1.0$ at 80 Mev, and also the total cross section becomes much higher than the value expected from the simple theory which assumes only the central forces.[7] ($\sigma_T\sim 0.1$ mb in sharp contrast with the theoretical prediction of $\sigma_T\sim 0.05$ mb at 80 Mev).[5] For a long time, these discrepancies remained as a challenge to the theory. Several authors have attempted to solve these discrepancies, assuming various final-state interactions,[8] assuming a large interaction current in the $M1$ transition,[9] or even denouncing the Siegert theorem for the electric transitions.[10]

In the previous attempt, I,[3] along the line of the pion theory of nuclear forces, the authors tried to explain the large value of a/b, taking proper account of the large d-state probability ($\sim 7\%$) of the deuteron,[1a] and the strong positive tensor potential in the triplet odd (^{3}O) final states,[11] both of which are characteristic consequences of the pion theory of nuclear forces. For 22 Mev γ-ray energy, they found that about half of the experimental value of a comes from the $E1$ transition to ^3P_J final states, where the large d-state mixing in the deuteron and the splitting of the ^3P_J final waves due to the tensor potential play decisive roles. Since the $^3d\rightarrow{}^3F_2$ $E1$ transition was ignored in their calculations, it was thought necessary to invoke a very large $M1$ transition to make up the other half of the observed value of a. This was achieved by assuming a large hard-core radius in the S-state to push the wave function out, at the price of the fit to p-p scattering at 40 Mev.

Later on, several authors pointed out that the $^3d\rightarrow{}^3F_2$ $E1$ transition is no longer negligible even at this energy and it results in an appreciable contribution

to a, if one assumes a large d-state mixing in the deuteron.[12] With this finding the main difficulty in the medium energy region were solved in principle. De Swart and Marshak and their collaborators have performed calculations along this line and obtained nice fits to the data up to 80 Mev using Signell-Marshak's potential with a large direct $L \cdot S$ coupling.[13]

In the present paper, we show that the nice fits to the data up to 80 Mev are reproduced in the presence of the pion-theoretical potential without assuming any direct $L \cdot S$ coupling in two-body force. The fits are again the natural consequences of the large d-state mixture in the deuteron and the strong positive tensor potential in the triplet odd (^{3}O) final states. The angular distribution and polarization of the emitted proton are calculated for the incidence of the unpolarized photons at 22, 48, and 79 Mev. The angular distribution and total cross section show the excellent agreement with the observations. There are no available experimental data for the polarization. These results seem to be very important. We have already shown that the two-nucleon scattering data up to 100 Mev can be well reproduced by the static pion theoretical potential.[2] Now we know that the photodisintegration of the deuteron in the corresponding energy range can be reproduced by the same potential. Further, there are several calculations[14] which are indicating the possibility of explaining a considerable part of the effective spin-orbit coupling in heavier nuclei can be explained in terms of the strong tensor potential between nucleons. If it turns out to be true then there would be no strong evidence which suggests the necessity of introducing the strong direct spin-orbit coupling in the outer and intermediate regions of the two-body forces, which is not justified from the point of view of the pion theory.[15] Although the nucleon-nucleon scattering experiments at 300 Mev region suggest the existence of some interactions other than the central plus tensor potentials,[16] this is well expected also from the pion theory of nuclear forces, because in the energy region the nucleons can approach very close to one another and thus be affected by the inner part of the interactions for which the existence of some spin-orbit coupling is highly plausible. We also improve the approximation in the previous work, **I**,[3] by including $M1$, $E1$ and $E2$ transitions and giving up the conventional $\kappa_\gamma r$ expansion in the evaluation of the transition amplitudes. It is found that the use of the conventional expression for the electric dipole interaction ($\int \psi_f r \psi_i dr$ form) results in an appreciable underestimate of the transition amplitudes.

General formula for the angular distribution and polarization are also given in the Appendices.

§ 2. Details of the calculations

2a) *Wave function of the deuteron*

Many kinds of wave functions have so far been proposed. For the deuteron state some authors have preferred to use some phenomenological wave functions.[5),8),9)]

Nevertheless, now that the validity of the pion theoretical potential has been well established, one should adopt the deuteron wave function which is consistent with the pion potential. In one of the previous works[1a] the deuteron problem was examined by one of the present authors (J.I.) and his collaborators and it was shown that the wave function of the deuteron is determined uniquely in the outer region $r \gtrsim \hbar/\mu c$ by the presence of the pion theoretical potential. Since the deuteron has a large radius compared with the range of the nuclear forces, the wave function of the deuteron is determined only by the shape of the outer potential and some boundary conditions in the asymptotic region. The boundary conditions themselves are determined by the observed values of the parameters, that is, binding energy, quadrupole moment and effective range. The detailed calculations determine the deuteron wave function in the following shape:

$$\psi(\boldsymbol{R}) = \frac{1}{\sqrt{4\pi}}\left(u + \frac{1}{\sqrt{8}} S_{12} w\right) \chi_m^1 (\mu c/\hbar)^{3/2}, \tag{1}$$

$$u = 1.039 \exp(-0.328x) - 1.392 \exp(-2.360x), \tag{2}$$

$$w = 0.0262\{1 + 3/(0.328x) + 3/(0.328x)^2\} \exp(-0.328x) - 1.298/x^2 \exp(-0.962x).$$

The wave function is normalized to $\int |\psi|^2 d\boldsymbol{R} = 1$, and $\boldsymbol{x} = (\mu c/\hbar)\boldsymbol{R}$, $\boldsymbol{R} = \boldsymbol{r}_p - \boldsymbol{r}_n$. This gives the following values of the parameters.

$$\begin{aligned} &Q = 2.74 \times 10^{-27}\,\mathrm{cm}^2, \quad \lim_{x\to\infty} (w/u) = 0.025, \\ &{}^3r_e = 1.70 \times 10^{-13}\,\mathrm{cm}, \quad g^2/4\pi\hbar c = 0.08, \\ &P_d = 0.067. \end{aligned} \tag{3}$$

Although the shape of the inner wave function is not very reliable, reflecting the lack of our knowledge about the inner potential, it does not affect the amplitude of the photodisintegration except the $M1$ transition to the ${}^1S$ final state. It is to be noted that the wave function (2) gives a large value for the d-state mixture of 7% in contrast to the conventional value of 4% which was determined on the basis of the magnetic moment of the deuteron. This large value, however, does not contradict with the observation, if one takes account of the various corrections to the predicted value of the deuteron magnetic moment,[17] and is rather supported by the observation of the hyperfine structure of the deuterium.* Various wave functions so far proposed in connection with the theory of nuclear forces seem to give the large value of the d-state mixture[8] rather unequivocally. Our wave function is very similar to that obtained by Gartenhaus[18] in the presence of his potential. However, our determination has been made on much more general

* See S. Machida, Prog. Theor. Phys. **9** (1953), 683. Recently papers with similar contents are published by A. Sessler and H. Foldy, Phys. Rev. **98** (1955), 6; **110** (1958), 995.

basis than his. The agreement with his result should be regarded as an evidence for the validity of our general argument.

2b) *Final state interaction*

The wave functions in the final state are distorted by the nuclear potential in the same way as in the nucleon-nucleon scattering at the corresponding energies. It has been shown[2] in the analyses of nucleon-nucleon scattering that the phase shifts which are given by the static pion potential without direct $L\cdot S$ coupling can reproduce the data on the single and double scattering up to 150 Mev fairly well. On the other hand, some authors have proposed to introduce strong phenomenological $L\cdot S$ couplings as the results of their analysis of the nucleon-nucleon scattering.[20] Although it is quite plausible that we would have some strong spin-orbit coupling effects at the energies as high as 300 Mev, it is not so certain that we have the strong $L\cdot S$ coupling in the intermediate region of nuclear forces as suggested by these authors. The pion theory predicts only a very weak $L\cdot S$ coupling potential in the outer and intermediate regions.[15] The strong spin-orbit coupling effects at high energies seem to arise from some interactions with very short range. Therefore it seems legitimate not to assume the existence of any strong $L\cdot S$ coupling in the analysis of medium energy phenomena. In this paper we adopt the wave functions which are calculated in reference 2) by the use of the static pion potential. The coupling of the final ${}^3S_1-{}^3D_1$ waves and ${}^3P_2-{}^3F_2$ waves are properly taken into account. Table I shows the values of the phase shifts used in the calculation. The most important final states in the photodisintegration process are the ${}^3P_J$ and ${}^3F_2$ states. The phase shifts and distortion of waves in these states will be mainly determined by the strong positive tensor potential in the triplet odd (^{3}O) state which is given by the one-pion-exchange potential.

Table I. Phase shifts by the pion-theoretical potential (radian unit).

E_{lab}(Mev)		40	90	150
for $M1$	${}^1\delta_0$	0.70	0.15	0.05
spin flip	${}^1\delta_2$		0.04	0.08
	δ_0	0.296	0.55	0.68
	δ_1	−0.122	−0.19	−0.17
for $E1$	$\delta_{2\alpha}$	0.033	0.11	0.20
	$\delta_{2\gamma}$	0	−0.02	−0.01
	ϵ_2	0	−0.52	−0.40
	$\delta_{1\alpha}$	1.00	0.73	0.43
	$\delta_{1\gamma}$	0	−0.24	−0.36
for $E2$	ϵ_1	0	0.055	0.076
	δ_2	0	0.29	0.45
	δ_3	0	0	0.10

2c) *Multipole transitions*

The angular distribution and polarization are calculated straightforwardly from the value of multipole moments by the use of general formula given in Appendix I.

The multipole moments are defined by the following formula:

$$M_l=-\frac{(2l+1)!!}{c}\frac{1}{(l+1)}\kappa^{-l-1}\times\int Y^*_{lm}\left[\omega\left\{-(l+1)j_l(\kappa r)+\kappa r j_{l+1}(\kappa r)\right\}\rho(\boldsymbol{r})+c\kappa^2 j_l(\kappa r)\left\{-\boldsymbol{r}\cdot\boldsymbol{j}(\boldsymbol{r})+c\nabla\cdot\boldsymbol{r}\times\boldsymbol{M}(\boldsymbol{r})\right\}\right]dV, \tag{4}$$

$$M_0=\frac{i(2l+1)!!}{c}\frac{1}{(l+1)}\kappa^{-l}\times\int Y^*_{lm}\left[ic\left\{(l+1)j_l(\kappa r)-\kappa r j_{l+1}(\kappa r)\right\}\nabla\cdot\boldsymbol{M}+\kappa^2 j_l(\kappa r)\left\{-ic(\boldsymbol{r}\cdot\boldsymbol{M})+i\nabla\cdot\boldsymbol{r}\times\boldsymbol{j}(\boldsymbol{r})/\kappa^2\right\}\right]dV. \tag{5}$$

For the proton emission in the photodisintegration of the deuteron we put

$$\rho(\boldsymbol{r})=8e\psi_f^*(2\boldsymbol{r})\psi_i(2\boldsymbol{r}) \tag{6}*$$

$$\boldsymbol{M}(\boldsymbol{r})=8e\hbar/(2Mc)\left\{\psi_f^*(2\boldsymbol{r})\sigma_p\mu_p\psi_i(2\boldsymbol{r})+\psi_f^*(-2\boldsymbol{r})\sigma_n\mu_n\psi_i(-2\boldsymbol{r})\right\}, \tag{7}$$

and for the neutron emission we have only to change $2\boldsymbol{r}\to-2\boldsymbol{r}$ and $\chi^s_m(p,n)\to\chi^s_m(n,p)$ (singlet spin function).

If we take only the first term in κr expansion of $j_l(\kappa r)$, then we get the conventional expressions for the multipole moments. We shall now estimate the relative magnitude of the various terms.

$$M_l\propto\frac{er^l}{(l+1)}\left\{(l+1)[1+\mathcal{O}(\kappa r)^2]+\mathcal{O}\left(\kappa r\sqrt{\frac{E_\gamma}{Mc^2}}\right)+\mathcal{O}\left(\frac{E_\gamma}{2Mc^2}\right)\right\} \tag{8}$$

$$M_0\propto\frac{er^{l-1}}{(l+1)}\left\{(l+1)[1+\mathcal{O}(\kappa r)^2]+\mathcal{O}(\kappa r)^2+\mathcal{O}(1)\right\}\frac{\hbar}{2Mc}. \tag{9}$$

Therefore it is not legitimate to invoke κr expansion when we are treating the energy as high as 80 Mev. We used the following estimation for the order of magnitude of various terms.

$$|\nabla|\sim r^{-1},\qquad \boldsymbol{r}\times\boldsymbol{j}\sim(e\hbar/M)\boldsymbol{L},\qquad \rho(\boldsymbol{r})\sim e,$$

* Our final states are normalized to $|\psi_f(R)|^2\,dR\sim(1/2)dR$.

$$\boldsymbol{j}(\boldsymbol{r}) \sim e\boldsymbol{v} \sim ec(E_\gamma/Mc^2)^{1/2}, \qquad |\boldsymbol{M}| \sim e\hbar/2Mc,$$

$$\kappa r \sim \frac{1}{2}\frac{E_\gamma}{\hbar c}\frac{1}{\gamma} = \frac{1}{2}\frac{E_\gamma}{\hbar c}\frac{\hbar}{\sqrt{BM}} = \frac{1}{2}\sqrt{\frac{E_\gamma^2}{BMc^2}},$$

$$Mv^2 \sim E_\gamma, \qquad B = (\gamma\hbar)^2/M.$$

In our calculations we consider only $E1$, $E2$ and $M1$-spin-flip-transitions since it is known that the higher multipole transitions give negligible effects on the process in the energy range of our concern.[12] We neglect the second and third terms in the electric moments because it is small compared with the first term. For the first term we do not use any expansion in terms of κr, but we treat it in the closed form. For magnetic transitions, they are separated into the spin-flip and non-flip terms. For $M1$ transition the contributions from the spin non-flip term may be neglected. For the spin-flip term we reduce the interaction into the form

$$M_0 = \int Y_{00} j_0(\kappa r)\, \boldsymbol{M}_m \rho(\boldsymbol{r})\, d\boldsymbol{r} \qquad (10)$$

and treat the interaction again in the closed form. The datailed calculations show that the conventional expression for $E1$ interaction gives rise to an underestimate of the transition amplitude.[22]* This can also be seen by comparing the particular overlap integrals with those calculated for the first term of κr expansion. Here we show the overlap integrals between ${}^3P_1$ wave and ${}^3S_1$ wave at 48 and 79 Mev.

Table II. Effect of retardation in overlap integrals.

$h\nu$	48 Mev	79 Mev
Exact interaction	0.827	0.497
κr term	0.772	0.415

In the approximation, the reaction amplitude for proton and neutron emissions are related in the following way.

$$R_1(l, m)\,\text{proton} = (-)^l R_1(l, m)\,\text{neutron}$$

$$R_0(l, m)\,\text{proton} = (-)^l R_0(l, m)\,\text{neutron.} \quad \text{(for spin-flip)} \qquad (11)$$

Thus as far as we consider $E1$ and $M1$-spin-flip transitions the angular distributions and polarizations of the proton and neutron are completely the same in reference to the direction of the emission of the respective particle.[24] If we take into account the $E2$ transition or spin non-flip $M1$ transition, then a difference appears between the polarizations of the proton and neutron. If we consider only triplet

* The conclusion obtained by Brown and Nicholson[23] that the retardation effects are still negligible at the energy of 130 Mev seems inaccurate due to their too rough estimation.

transitions and neglect all the spin flip transitions (including $M1$ transition), then the polarization of the proton emitted in the direction θ, φ and that of the neutron emitted in the direction $\pi-\theta$. $\pi+\varphi$ are again the same.

§ 3. Results and discussions

3a) *Angular distribution at 22 Mev*

In order to show some qualitative features of the process, we shall at first take up the photodisintegration at 22 Mev which has been considered in detail in our previous work **I**.[3] For the sake of simplicity, we consider only $E1$ and $M1$ transitions and neglect the $M1$ transition to ^1D state and the coupling between $^3P_2$ and $^3F_2$ final waves. Then the angular distribution of the emitted proton given in Appendix I is rewritten in the following form:

$$\frac{d\sigma}{d\Omega}=a+b\sin^2\theta, \tag{12}$$

$$a=a_{E1}+a_{M1}, \tag{13}$$

$$a_{M1}=B(k)\,(\mu/M)^2(\mu_p-\mu_n)^2 I_s^2, \tag{14}$$

$$a_{E1}=B(k)\left\{\left|\frac{1}{9}\,e^{i\delta_0}I_0-e^{i\delta_2}I_2-\sqrt{\frac{3}{2}}\,e^{i\delta_F}I_F\right|^2 + \frac{1}{4}\left|e^{i\delta_1}I_1-e^{i\delta_2}I_2+\sqrt{\frac{2}{3}}e^{i\delta_F}I_F\right|^2\right\}, \tag{15}$$

$$b=B(k)\,\frac{1}{2}\left\{\frac{1}{4}\left|e^{i\delta_1}I_1+3e^{i\delta_2}I_2-\sqrt{6}\,e^{i\delta_F}I_F\right|^2-\frac{25}{3}\,I_F^2 - \frac{1}{15}\left|e^{i\delta_0}I_0-5e^{i\delta_2}I_2+\frac{15}{\sqrt{6}}e^{i\delta_F}I_F\right|^2+\frac{1}{15}\left|e^{i\delta_0}I_0+5\sqrt{6}e^{i\delta_F}I_F\right|^2\right\}, \tag{16}$$

$$B(k)=(1/12)\,(e^2/\hbar c)\,(ME_\gamma/\hbar^2)\,(1/k)\,(\hbar/\mu c)^3, \tag{17}$$

$$I_0=I_0^s-\sqrt{2}\,I_0^d, \qquad I_2=I_2^s-(1/5\sqrt{2})\,I_2^d,$$

$$I_1=I_1^s+(1/\sqrt{2})\,I_1^d, \qquad I_F=(3\sqrt{3}/5)\,I_F^d,$$

$$I_s=\int v(^1S_0)\,j_0(\kappa R/2)\,u\,dR, \qquad I_J^{(s\text{ or }d)}=\int v(^3P_J)\,6\,j_1(\kappa R/2)\,(u\text{ or }w)\,dR.$$

δ_J and δ_F represent the phase shifts of ^3P_J and $^3F_2$ waves, $I_J^{(s\text{ or }d)}$ and $I_F^{(s\text{ or }d)}$ being the overlapping integrals between s (or d) wave of the deuteron and the ^3P_J and $^3F_2$ final waves. The various contributions to the quantities in the absolute square signs in a_{E1} are shown schematically in a complex plane in Fig. 1 for the case of a positive tensor potential in the triplet odd (^{3}O) state. From the formula we can see that a_{E1} will vanish if we neglect tensor forces completely in both the initial and final states ($\delta_0=\delta_1=\delta_2$ and $I_0=I_1=I_2$, $I_F=0$). If we take into account the tensor force in the final states, the phase shifts split one from another and

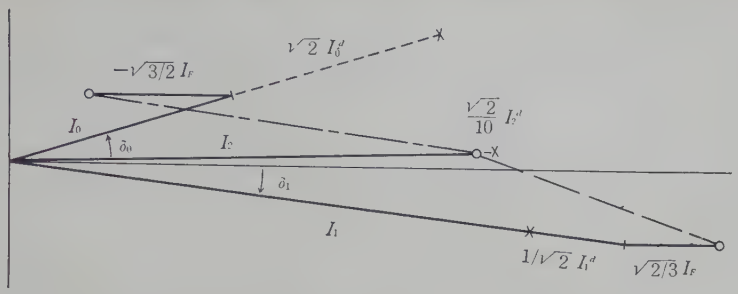

Fig. 1. Contributions of each I_J in a.
$d\sigma/d\Omega = a + b \sin^2\theta\ (1 + 2\beta \cos\theta)$
$a \propto \{(1/9)|I_0 e^{i\delta_0} - I_2 d^{i\delta_2} - \sqrt{3/2} I_F e^{i\delta_F}|^2$
$+ (1/4)|I_1 e^{i\delta_1} - I_2 e^{i\delta_2} + \sqrt{2/3} I_F e^{i\delta_F}|^2\}$.

the overlapping integrals split too. An attractive force in a particular state decreases the overlapping, while a repulsive force increases it. Then, if we include the transition from the d-wave of the deuteron, the component contributes also to a large splitting among the overlapping integrals for the ${}^3P_J$ states.

Finally we have to consider the transition to the ${}^3F_2$ state from the d-wave part of the deuteron, again having a large contribution to a_{E1}. Fig. 1 illustrates the situation in terms of the vector addition in the complex plane. From Fig. 1 and Table III we see that all of these three effects interfere constructively in reproducing a large value of a_{E1}, so long as the tensor potential in ${}^3O$ state is positive. This point together with the important role played by the large d-state probability of the deuteron predicted by the pion theory was already emphasized in our previous work I without taking account of ${}^3F_2$ transition.[3] The ${}^3F_2$ transition is also very important particularly at high energies in reproducing the large value of a_{E1}, and again the magnitude of the effect is very sensitive to the d-state probability of the deuteron. Magnitude of each transition amplitude is given in Table IV. Fig. 2 shows the results of the detailed calculations. The full line gives the angular distribution of the proton predicted by the theory. The effect

Table III. Magnitude of a_{E1} in the various approximations (in μb/sterad).

	Free final waves	With final state interaction
$S \longrightarrow P$	0	0.5
$S+D \longrightarrow P$	0.9	2.0
$S+D \longrightarrow P+F$	3.0	4.6

of $E2$ transition is to give an asymmetry around 90 degrees in the angular distribution. The fit with the observed results are very good. Inclusion of the ${}^3P_2$—${}^3F_2$ coupling does not appreciablly change the result at this energy, so far as one adopts the bar phase shifts for the uncoupled system.[16]

Table IV. Transition amplitudes and total cross sections.

E_γ(Mev)	22	48	79
$I^S{}_0$	1.472	0.410	0.121
$I^S{}_1$	1.711	0.827	0.497
$I^S{}_{p\alpha}$	1.595	0.630	0.338
$I^S{}_{p\gamma}$		0.525	0.209
$I^d{}_0$	0.505	0.133	0.057
$I^d{}_1$	0.462	0.317	0.202
$I^d{}_{p\alpha}$	0.473	0.249	0.145
$I^d{}_{p\gamma}$		0.166	0.087
$I^d{}_{F\alpha}$		−0.187	−0.146
$I^d{}_{F\gamma}$	0.367	0.281	0.281
I_0	0.758	0.222	0.041
I_1	2.038	1.051	0.640
$I_{2\alpha}$	1.528	0.401	0.166
$I_{2\gamma}$	0.382	0.732	0.489
$\sigma_T(E1)$	469 μb	178 μb	83.9 μb
$I_{1\alpha}$	0.03	0.002	−0.143
$I_{1\gamma}$	4.51	0.271	0.413
I_2	4.51	1.740	1.470
I_3	4.51	1.145	0.459
$\sigma_T(E2)$	1.1 μb	0.5 μb	0.9 μb
I_s	0.375	0.244	0.121
I_d		0.126	0.120
$\sigma_T(M1)$	11.6 μb	8.6 μb	4.2 μb
σ(Total)	481 μb	187 μb	89.0 μb

3b) *Polarization at 22 Mev*

Next we shall see the polarization of the emitted protons. The polarization of the protons (or neutrons) can be expressed in the following form in the same approximation as (12).

$$\sigma \boldsymbol{P}_{p(n)}(\theta) = B(k)\,\{c \sin 2\theta + d \sin\theta\}\,\boldsymbol{n}_{p(n)}, \tag{19}$$

$$c = -(1/6) I_0 I_2 \sin(\delta_0 - \delta_2) - (1/4) I_1 I_2 \sin(\delta_1 - \delta_2) + (1/6) I_0 I_F \sin(\delta_0 - \delta_F) - (3/8) I_1 I_F \sin(\delta_1 - \delta_F) + (5/24) I_2 I_F \sin(\delta_2 - \delta_F). \tag{20}$$

$$d = \alpha\, I_1\, I_s \sin(\delta_1 - \delta_s), \tag{21}$$

$$\alpha \equiv (\hbar\kappa/Mc)\,(\mu_p - \mu_n). \tag{22}$$

I_s is the overlapping integral between s-wave of the deuteron and the ^1S final wave. $\boldsymbol{n}_{p(n)}$ is the unit vector perpendicular to the reaction plane defined by

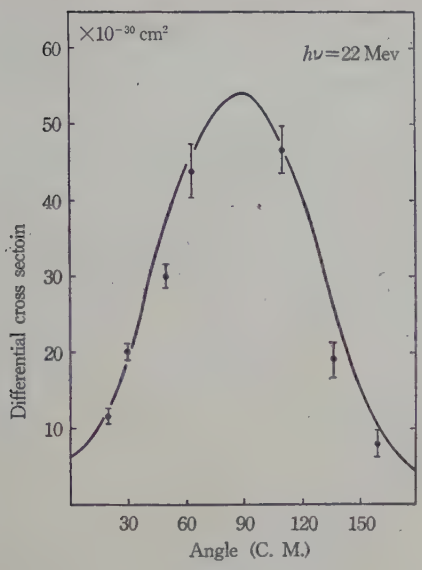

Fig. 2. The differential cross section at $h\nu=22$ Mev.
The soild curve is given by
$d\sigma/d\Omega=(5.46+48.8\ \sin^2\theta)(1+0.225\ \cos\theta)$.
$\sigma_T=469\ \mu b$, $\quad a/b=0.112$.

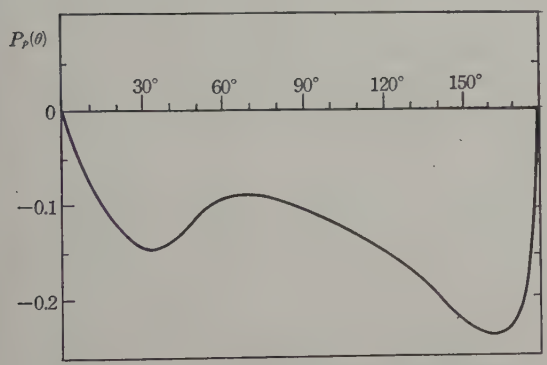

Fig. 3. The proton polarization at the photon energy of 22 Mev.

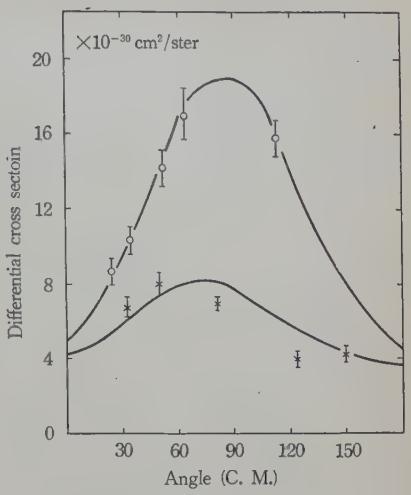

Fig. 4. Angular distributions at the photon energy 48 and 79 Mev.
The full lines are the calculated result by the present theory.
upper (48 Mev) :
$d\sigma/d\Omega=4.80+14.31\ \sin^2\theta+(0.19+3.30\ \sin^2\theta)\cos\theta$
$-0.27\ \sin^4\theta$.
$\sigma_T=\sigma_{E1}+\sigma_{E2}+\sigma_{M1}=178.0+0.5+8.6=187.1\ \mu b$.
lower (79 Mev) :
$d\sigma/d\Omega=3.91+4.11\ \sin^2\theta+(0.31+2.18\ \sin^2\theta)\cos\theta$
$-0.41\ \sin^4\theta$.
$\sigma_T=\sigma_{E1}+\sigma_{E2}+\sigma_{M1}=83.9+0.9+4.2=89.0\ \mu b$.

$\boldsymbol{n}_{p(n)}=(\boldsymbol{\kappa}\times\boldsymbol{P}_{p(n)})/|(\boldsymbol{\kappa}\times\boldsymbol{P}_{p(n)})|$. The pure $E1$ transition gives the polarization cross section of the shape of $\sin 2\theta$, while the interference term contributes a term of the shape $\sin\theta$. The polarization at 90 degrees is given only by this interference term and will reveal an information about the magnitude of $M1$ transition to the ^1S state.[2] In the presence of a strong positive tensor potential in the triplet odd (^{3}O) state, all terms except the first term give positive contributions to c. Since I_0 in the first term is known to be the smallest of I_J's, it is clear that we have $c>0$ for the potential. Concerning the second term the sign is determined by the sign of relative phase shifts and the sign of I_s overlapping integral. For the class of potentials which can reproduce the p-p scattering data at the corresponding energy, the sign of d will be negative. As the result we have the polarization as shown in Fig. 3. The curve in Fig. 3 is calculated for our potential. The sign of the polarization is determined by the interference

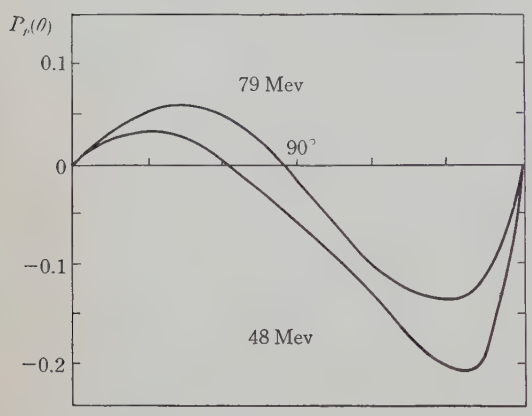

Fig. 5(a). The proton polarization at $h\nu=48$, 79 Mev.
48 Mev: $P(\theta)=[1.105\ \sin 2\theta-1.295\ \sin\theta -0.080\ \sin\ 2\theta\cdot\cos\theta]/\sigma(\theta)$
79 Mev: $P(\theta)=[0.518\ \sin 2\theta-0.035\sin\theta -0.164\ \sin 2\theta\ \cos\theta]/\sigma(\theta)$

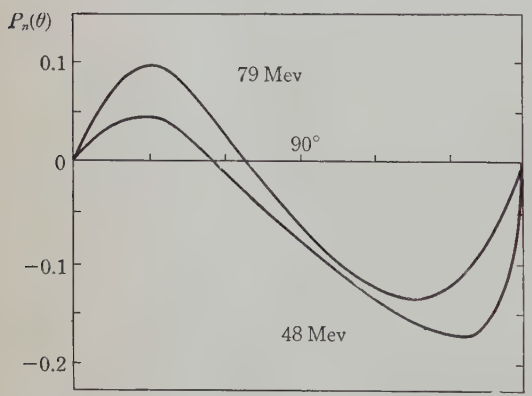

Fig. 5(b). The neutron polarization at $h\nu=48$, 79 Mev.
48 Mev: $P(\theta)=[1.104\ \sin 2\theta-1.369\ \sin\theta +0.080\sin 2\theta\cdot\cos\theta]/\sigma(\theta)$
79 Mev: $P(\theta)=[0.650\ \sin 2\theta-0.581\ \sin\theta +0.164\sin 2\theta\cdot\cos\theta]/\sigma(\theta)$

term $d\sin\theta$, while the asymmetry is due to the $c\sin 2\theta$ term. The pronounced dip around 90 degrees is due to the maximum at 90 degrees of the unpolarized cross section $a+b\ \sin^2\theta$. When the energy becomes higher, the overlapping integral I_s decreases and the polarization will change the sign in small angle region. (See the next paragraph.)

3c) *Photodisintegration at 48 and 79 Mev*

The angular distributions and polarizations at 48 and 79 Mev can be calculated straightforwardly by the use of the formula given in the Appendices. We consider the $E1$, $E2$ and spin-flip-$M1$ transitions. In the presence of $E2$ transition, we cannot make any simple argument about the cross section and polarization as in § 3. The similar mechanism is, however, working to determine the main features of the processes, since the contribution from $E2$ is very small. We have the large isotropic cross sections and the characteristic shapes for the polarizations, the latter of which have been modified in the way already remarked in § 3. Calculated cross sections and polarizations are shown in Figs. 4 and 5. The cross sections are in good agreement with the observations.

3d) *Discussions*

The results in the previous paragraphs show that the static pion potential can reproduce the angular distribution of proton in the photodisintegration of the deuteron at medium energies. It was shown that the positive tensor potential in the ^3P state is important in reproducing the large observed values of a/b below 20 Mev. At higher energies, the presence of $d\rightarrow{}^3F_2$ transition is essential for the large total cross section and the large a/b ratio. These features in the angular distribution are common for all the classes of potentials which are essentially some

modifications of the pion potential. The addition of a small $L \cdot S$ potential will not change the situation very much. The assumptions such as the Serber force or a negative tensor potential are, of course, excluded on this basis.

The general features in the polarization of the emitted proton or neutron and its energy dependence show the similar trend for the modified pion potentials. However, the details of the energy dependence particularly that of the $\boldsymbol{P}(\theta)$ at small angles, are rather sensitive to the phase shifts of ^1S- and ^3P_J-waves. Although there is no data available for the moment concerning the point, the future measurement will provide very useful information as to the shape of the inner wave functions in the ^1S state and the magnitude of the exchange effects.

Recently there have been a few works published on the same subject as treated in the present work. The main difference between the present work and those lies in that we used the closed form for each multipole interaction, while other authors resorted to the conventional power series expansion in terms of κr. It seems that there is some discrepancy between our results and those in reference 28) concerning the magnitude of the $M1$-spin-flip transition to the $^1D_2$ state. The cause of this discrepancy was not yet clarified.

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Appendix 1

We consider here the photodisintegration process of the deuteron caused by an unpolarized photon with wave number κ. The transitions are classified by the following quantities:

the electric or magnetic character of the transition
  $p=1$ for the electric radiation,
  $p=0$ for the magnetic radiation.
the order of multipole radiation: l
the total spin of the target deuteron: $I=1$
the orbital angular momentum of the target deuteron: $l_i=0, 2$
the parity of the final state: π
the total angular momentum of the final state: J

the spin angular momentum of the final state: $S=0, 1$

the orbital angular momentum of the final state: L.

From the general theory of angular distribution in photoreaction the angular distribution is readily given by the following formula,

$$\frac{d\sigma}{d\Omega}=\frac{\hbar^2}{24k^2}\sum\Big\{R(1)R^*(2)\,i^{\nu}\sqrt{(2J_1+1)(2J_2+1)(2l_1+1)(2l_2+1)}$$
$$\times C_{l_1l_2}(L0\,;\,1-1)\,W(l_1J_1l_2J_2:1L)\,Z(L_1J_1L_2J_2SL)\Big\}P_L(\cos\theta). \quad (A\cdot1)$$
$$\nu=L-l_1+l_2-p_1+p_2+2S-2I-2.$$

Here we adopted conventional definitions for the W and Z. The total cross section is obtained by putting $L=0$ in (A·1) as

$$\sigma_T=\frac{\pi}{2(2I+1)k^2}\sum(2J+1)\,|R|^2. \quad (A\cdot2)$$

The polarization cross section of the outgoing particle is given by

$$\frac{d\sigma}{d\Omega}\boldsymbol{P}=\frac{\hbar^2\sqrt{3}}{12k^2}\boldsymbol{n}\Sigma\Big\{iR(1)R^*(2)\,i^{\nu'}\sqrt{(2J_1+1)(2J_2+1)}$$
$$\times\sqrt{(2l_1+1)(2l_2+1)(2L_1+1)(2L_2+1)(2S_1+1)(2S_2+1)}$$
$$\times C_{L_1L_2}(L0\,;\,1-1)\,W(l_1J_1l_2J_2:1L)\,C_{L_1L_2}(L0:00)\,W\left(S_1\frac{1}{2}S_2\frac{1}{2};\frac{1}{2}1\right)$$
$$\times\begin{pmatrix}J_1 & L_1 & S_1\\ L & L & 1\\ J_2 & L_2 & S_2\end{pmatrix}\cdot\sqrt{\frac{(2L+1)}{2L(L+1)}}P_L^1(\cos\theta)\Big\}. \quad (A\cdot3)^*$$
$$\nu'=-l_2+l_1-p_1+p_2+L_1+L_2+2J_2+2S_2.$$

Table A-I. Coefficients in the differential cross section for the photodisintegration of the deuteron. $R(^3P_0)$ is the R-matrix for the $E1$ transition with final state $^3P_0$, R^* means the complex conjugate of R. The table gives the coefficients $C_{ll'}$ of $R(1)R^*(2)$ in the expression for the $24k^2(d\sigma/d\Omega)$. Differential cross section is given as $24k^2(d\sigma/d\Omega)=\sum_{ll'}C_{ll'}\,R(l)R^*(l')$.

($E1$—$E1$)

$R(1)$ / $R^*(2)$	$^3P_0$	$^3P_1$	$^3P_2$	$^3F_2$
$^3P_0$	$P_0(\theta)$	0	$-P_2(\theta)$	$-\frac{3}{\sqrt{6}}P_2(\theta)$
$^3P_1$	0	$3P_0(\theta)-\frac{3}{4}P_2(\theta)$	$-\frac{9}{4}P_2(\theta)$	$\frac{9}{2\sqrt{6}}P_2(\theta)$
$^3P_2$	$-P_2(\theta)$	$-\frac{9}{4}P_2(\theta)$	$5P_0(\theta)-\frac{7}{4}P_2(\theta)$	$-\frac{3}{2\sqrt{6}}P_2(\theta)$
$^3F_2$	$-\frac{3}{\sqrt{6}}P_2(\theta)$	$\frac{9}{2\sqrt{6}}P_2(\theta)$	$-\frac{3}{2\sqrt{6}}P_2(\theta)$	$5P_0(\theta)-2P_2(\theta)$

* In Eq. (2.4) of reference 26), the sign factor $(-)^{\sigma}=(-)^{J_1+L_1+S_1+2L+1+J_2+L_2+S_2}$ is missing.

(*E*2—*E*1)

$R^*(2)$ \ $R(1)$	$^3P_0$	$^3P_1$	$^3P_2$	$^3F_2$
$^3S_1$	$\sqrt{\frac{3}{2}}P_1(\theta)$	$-\frac{9}{2\sqrt{6}}P_1(\theta)$	$\frac{1}{2}\sqrt{\frac{3}{2}}P_1(\theta)$	$-3P_3(\theta)$
$^3D_1$	$\sqrt{3}P_1(\theta)$	$\frac{3}{4}\sqrt{3}P_1(\theta)$	$\frac{\sqrt{3}}{20}P_1(\theta)-\frac{9\sqrt{3}}{5}P_3(\theta)$	$\frac{9}{10\sqrt{2}}P_1(\theta)-\frac{6\sqrt{2}}{5}P_3(\theta)$
$^3D_2$	0	$\frac{9\sqrt{3}}{4}P_1(\theta)-\sqrt{3}P_3(\theta)$	$\frac{3\sqrt{3}}{4}P_1(\theta)-2\sqrt{3}P_3(\theta)$	$-\frac{3\sqrt{2}}{4}P_1(\theta)+2\sqrt{2}P_3(\theta)$
$^3D_3$	$-\sqrt{3}P_3(\theta)$	$-2\sqrt{3}P_3(\theta)$	$\frac{21}{5}\sqrt{3}P_1(\theta)-\frac{6}{5}\sqrt{3}P_3(\theta)$	$\frac{3\sqrt{2}}{10}P_1(\theta)-\frac{4\sqrt{2}}{5}P_3(\theta)$

(*E*2—*E*2)

$R^*(2)$ \ $R(1)$	$^3S_1$	$^3D_1$	$^3D_2$	$^3D_3$
$^3S_1$	$3P_0(\theta)$	$\frac{3}{4}\sqrt{2}P_2(\theta)$	$-\frac{5}{4}\sqrt{2}P_2(\theta)$	$\frac{1}{\sqrt{2}}P_2(\theta)$
$^3D_1$	$\frac{3\sqrt{2}}{4}P_2(\theta)$	$3P_0(\theta)+\frac{3}{4}P_2(\theta)$	$\frac{5}{4}P_2(\theta)$	$\frac{1}{7}P_2(\theta)-\frac{36}{7}P_4(\theta)$
$^3D_2$	$-\frac{5\sqrt{2}}{4}P_2(\theta)$	$\frac{5}{4}P_2(\theta)$	$5P_0(\theta)+\frac{25}{28}P_2(\theta)-\frac{80}{21}P_4(\theta)$	$\frac{10}{7}P_2(\theta)-\frac{100}{21}P_4(\theta)$
$^3D_3$	$\frac{1}{\sqrt{2}}P_2(\theta)$	$\frac{1}{7}P_2(\theta)-\frac{36}{7}P_4(\theta)$	$\frac{10}{7}P_2(\theta)-\frac{100}{21}P_4(\theta)$	$7P_0(\theta)+\frac{24}{7}P_2(\theta)-\frac{44}{21}P_4(\theta)$

Magnetic dipole contribution (triplet final states)

(*M*1—*M*1)

$R^*(2)$ \ $R(1)$	$^3S_1$	$^3D_1$	$^3D_2$
$^3S_1$	$-3P_0(\theta)$	$-\frac{3\sqrt{2}}{4}P_2(\theta)$	$-\frac{3\sqrt{10}}{4}P_2(\theta)$
$^3D_1$	$-\frac{3\sqrt{2}}{4}P_2(\theta)$	$-3P_0(\theta)-\frac{3}{4}P_2(\theta)$	$\frac{3\sqrt{5}}{4}P_2(\theta)$
$^3D_2$	$-\frac{3\sqrt{10}}{4}P_2(\theta)$	$\frac{3\sqrt{5}}{4}P_2(\theta)$	$-5P_0(\theta)+\frac{5}{4}P_2(\theta)$

(*E*1—*M*1)

$R^*(2)$ \ $R(1)$	$^3S_1$	$^3D_1$	$^3D_2$
$^3P_0$	$\sqrt{\frac{3}{2}}P_1(\theta)$	$\sqrt{3}P_1(\theta)$	0
$^3P_1$	$\frac{3\sqrt{6}}{4}P_1(\theta)$	$-\frac{3\sqrt{3}}{4}P_1(\theta)$	$\frac{3\sqrt{15}}{4}P_1(\theta)$
$^3P_2$	$-\frac{5\sqrt{6}}{4}P_1(\theta)$	$-\frac{\sqrt{3}}{4}P_1(\theta)$	$-\frac{3\sqrt{15}}{4}P_1(\theta)$
$^3F_2$	0	$-\frac{9\sqrt{2}}{4}P_1(\theta)$	$\frac{3\sqrt{10}}{4}P_1(\theta)$

($E2-M1$)

$R^*(2)$ \ $R(1)$	$^3S_1$	$^3D_1$	$^3D_2$
$^3S_1$	0	$-\frac{9\sqrt{2}}{4}P_2(\theta)$	$\frac{3\sqrt{10}}{2}P_2(\theta)$
$^3D_1$	$-\frac{9\sqrt{2}}{4}P_2(\theta)$	$-\frac{9}{4}P_2(\theta)$	$-\frac{3\sqrt{5}}{4}P_2(\theta)$
$^3D_1$	$-\frac{5\sqrt{2}}{4}P_2(\theta)$	$\frac{5}{4}P_2(\theta)$	$-\frac{5\sqrt{5}}{4}P_2(\theta)$
$^3D_3$	$\frac{7\sqrt{2}}{2}P_2(\theta)$	$1\ P_2(\theta)$	$\frac{20\sqrt{7}}{7}P_2(\theta)$

($M1-M1$) Singlet final states

$R^*(2)$ \ $R(1)$	$^1S_0$	$^1D_2$
$^1S_0$	$-P_0(\theta)$	$\sqrt{10}/2\cdot P_2(\theta)$
$^1D_2$	$\sqrt{10}/2\cdot P_2(\theta)$	$-5P_0(\theta)+\frac{5}{2}P_2(\theta)$

In Tables A-I and A-II we tabulated the coefficients of each contribution.*,**

Appendix 2

Evaluation of the reaction amplitude R

The reaction amplitude R is expressed with the real amplitude $\bar{R}$ between standing wave solutions and phase shifts of nucleon-nucleon scattering in the following way:[25]

i) For the uncoupled final states

$$R_{JJ}=\bar{R}_{JJ}\exp(i\delta_{JJ}).$$

ii) For the coupled final states

$$R_{JJ-1}=\cos\epsilon_J\bar{R}_{J,\alpha}\exp(i\delta_{J\alpha})-\sin\epsilon_J\bar{R}_{J,\gamma}\exp(i\delta_{J\gamma})$$
$$R_{JJ+1}=\sin\epsilon_J\bar{R}_{J,\alpha}\exp(i\delta_{J\alpha})+\cos\epsilon_J\bar{R}_{J,\gamma}\exp(i\delta_{J\gamma}),$$

where $\bar{R}_{J,\alpha}$ and $\bar{R}_{J,\gamma}$ are the amplitudes for the transitions to the standing wave solutions α and γ of the coupled wave equation. $\delta_{J,\alpha}$, $\delta_{J,\gamma}$ and ϵ_J are the corresponding Blatt-Biedenharn phase shifts and mixing parameter.[16] $\bar{R}$ is connected with the matrix element of the operator of the multipole transition as

$$\Sigma\langle J(LS)M|M_p(lm)|IM_1\rangle C_{lI}(JM;mM_1)(-2/c\hbar^2)\sqrt{M/\kappa k}$$
$$\times\left[(-)^p\kappa^{-l-1}\frac{(2l+1)!!}{c}\sqrt{\frac{l}{l+1}}\right]^{-1}\sqrt{4\pi}=R(J(LS)plI).$$

* There are some discrepancies between our results and those in reference 26).

** We did not make any detailed comparison of our results with those in reference 24).

Table A-II. Coefficients in the polarization cross section for the photodisintegration of the deuteron. Polarization cross section is given as

$$\frac{24k^3}{\sin\theta}\boldsymbol{P}\frac{d\sigma}{d\Omega}=\boldsymbol{n}\sum C_{ll'}R(l)R^*(l').$$

		E_1				$M1$ spin flip	
		$^3P_0$	$^3P_1$	$^3P_2$	$^3F_2$	$^1S_0$	$^1D_2$
$E1$	$^3P_0{}^*$	0	0	$-i\frac{3}{2}\cos\theta$	$i\frac{\sqrt{6}}{2}\cos\theta$	0	0
	$^3P_1{}^*$	0	0	$-i\frac{9}{4}\cos\theta$	$-i\frac{9\sqrt{6}}{8}\cos\theta$	$i\frac{3}{2}$	$i\frac{3\sqrt{10}}{8}$
	$^3P_2{}^*$	$i\frac{3}{2}\cos\theta$	$i\frac{9}{4}\cos\theta$	0	$i\frac{5\sqrt{6}}{8}\cos\theta$	0	$-i\frac{9}{8}\sqrt{10}$
	$^3F_2{}^*$	$-i\frac{\sqrt{6}}{2}\cos\theta$	$i\frac{9\sqrt{6}}{8}\cos\theta$	$-i\frac{5\sqrt{6}}{8}\cos\theta$	0	0	$i\frac{3\sqrt{15}}{4}$
$M1$ spin flip	$^1S_0{}^*$	0	$-i\frac{3}{2}$	0	0	0	0
	$^1D_2{}^*$	0	$-i\frac{3}{8}\sqrt{10}$	$i\frac{9\sqrt{10}}{8}$	$-i\frac{3\sqrt{15}}{4}$	0	0
$M1$	$^3S_1{}^*$	$-i\frac{\sqrt{6}}{2}$	$-i\frac{3\sqrt{6}}{8}$	$-i\frac{5\sqrt{6}}{8}$	0	0	$i\frac{3\sqrt{15}}{4}\cos\theta$
	$^3D_1{}^*$	$i\frac{\sqrt{3}}{2}$	$-i\frac{3\sqrt{3}}{4}$	$-i\frac{\sqrt{3}}{2}$	$i\frac{9\sqrt{2}}{8}$	0	$-i\frac{3\sqrt{30}}{8}\cos\theta$
	$^3D_2{}^*$	0	0	$-i\frac{3\sqrt{15}}{4}$	$-i\frac{9\sqrt{10}}{8}$	$i\frac{\sqrt{15}}{2}\cos\theta$	$i\frac{5\sqrt{6}}{8}\cos\theta$
$E2$	$^3S_1{}^*$	$-i\frac{\sqrt{6}}{2}$	$i\frac{3\sqrt{6}}{8}$	$i\frac{\sqrt{6}}{8}$	$i\frac{3}{2}(5\cos^2\theta-1)$	0	$-i\frac{3\sqrt{15}}{4}\cos\theta$
	$^3D_1{}^*$	$i\frac{\sqrt{3}}{2}$	$i\frac{3\sqrt{3}}{4}$	$-i\frac{\sqrt{3}}{2}(9\cos^2\theta-2)$	$-i\frac{3\sqrt{2}}{8}(1-2\cos^2\theta)$	0	$i\frac{3\sqrt{30}}{8}\cos\theta$
	$^3D_2{}^*$	0	0	$i\frac{5\sqrt{3}}{4}(1-2\cos^2\theta)$	$i\frac{15\sqrt{2}}{8}(1-2\cos^2\theta)$	$i\frac{5\sqrt{3}}{2}\cos\theta$	$-i\frac{5\sqrt{30}}{8}\cos\theta$
	$^3D_3{}^*$	$i\frac{\sqrt{3}}{2}(5\cos^2\theta-1)$	$i\frac{3\sqrt{3}}{4}(5\cos^2\theta-1)$	$-i\frac{3\sqrt{3}}{4}(3-\cos^2\theta)$	$-i\frac{3\sqrt{2}}{2}(1-2\cos^2\theta)$	0	$-i\sqrt{30}\cos\theta$

Table A-II.

	$M1$			$E2$			
	$^3S_1$	$^3D_1$	$^3D_2$	$^3S_1$	$^3D_1$	$^3D_2$	$^3D_3$
$^3P_0{}^*$	$i\frac{\sqrt{6}}{2}$	$-i\frac{\sqrt{3}}{2}$	0	$i\frac{\sqrt{6}}{2}$	$-i\frac{\sqrt{3}}{2}$	0	$-i\frac{\sqrt{3}}{2}(5\cos^2\theta-1)$
$^3P_1{}^*$	$i\frac{3\sqrt{6}}{8}$	$i\frac{3\sqrt{3}}{4}$	0	$-i\frac{3\sqrt{6}}{8}$	$-i\frac{3\sqrt{3}}{4}$	0	$-i\frac{3\sqrt{3}}{4}(5\cos^2\theta-1)$
$^3P_2{}^*$	$i\frac{5\sqrt{6}}{8}$	$i\frac{\sqrt{3}}{2}$	$i\frac{3\sqrt{15}}{4}$	$-i\frac{\sqrt{6}}{8}$	$i\frac{\sqrt{3}}{2}(9\cos^2\theta-2)$	$-i\frac{5\sqrt{3}}{4}(1-2\cos^2\theta)$	$i\frac{3\sqrt{3}}{4}(3-\cos^2\theta)$
$^3F_2{}^*$	0	$-i\frac{9\sqrt{2}}{8}$	$i\frac{9\sqrt{10}}{8}$	$-i\frac{3}{2}(5\cos^2\theta-1)$	$i\frac{3\sqrt{2}}{8}(1-2\cos^2\theta)$	$-i\frac{15\sqrt{2}}{8}(1-2\cos^2\theta)$	$i\frac{3\sqrt{2}}{2}(1-2\cos^2\theta)$
$^1S_0{}^*$	0	0	$-i\frac{\sqrt{15}}{2}\cos\theta$	0	0	$-i\frac{5\sqrt{3}}{2}\cos\theta$	0
$^1D_2{}^*$	$-i\frac{3\sqrt{15}}{4}\cos\theta$	$i\frac{3\sqrt{30}}{8}\cos\theta$	$-i\frac{5\sqrt{6}}{8}\cos\theta$	$i\frac{3\sqrt{15}}{4}\cos\theta$	$-i\frac{3\sqrt{30}}{8}\cos\theta$	$i\frac{5\sqrt{30}}{8}\cos\theta$	$i\sqrt{30}\cos\theta$
$^3S_1{}^*$	0	$-i\frac{9\sqrt{2}}{8}\cos\theta$	$-i\frac{3\sqrt{10}}{8}\cos\theta$	0	$-i\frac{27\sqrt{2}}{8}\cos\theta$	$-i\frac{5\sqrt{2}}{8}\cos\theta$	$-i\frac{7\sqrt{2}}{2}\cos\theta$
$^3D_1{}^*$	$i\frac{9\sqrt{2}}{8}\cos\theta$	0	$-i\frac{3\sqrt{5}}{4}\cos\theta$	$i\frac{27\sqrt{2}}{8}\cos\theta$	0	$-i\frac{5}{4}\cos\theta$	$-i\frac{5}{2}\cos\theta$
$^3D_2{}^*$	$i\frac{3\sqrt{10}}{8}\cos\theta$	$i\frac{3\sqrt{5}}{4}\cos\theta$	0	$-i\frac{3\sqrt{10}}{8}\cos\theta$	$-i\frac{3\sqrt{3}}{4}\cos\theta$	0	$-i3\sqrt{5}\cos\theta$
$^3S_1{}^*$	0	$-i\frac{27\sqrt{2}}{8}\cos\theta$	$i\frac{3\sqrt{10}}{8}\cos\theta$	0	$-i\frac{9\sqrt{2}}{8}\cos\theta$	$i\frac{25\sqrt{2}}{4}\cos\theta$	$i\frac{\sqrt{2}}{2}\cos\theta$
$^3D_1{}^*$	$i\frac{27\sqrt{2}}{8}\cos\theta$	0	$i\frac{3\sqrt{3}}{4}\cos\theta$	$i\frac{9\sqrt{2}}{8}\cos\theta$	0	$i\frac{25}{2}\cos\theta$	$-i\frac{25}{7}\cos\theta$
$^3D_2{}^*$	$i\frac{5\sqrt{2}}{8}\cos\theta$	$i\frac{5}{4}\cos\theta$	0	$-i\frac{25\sqrt{2}}{4}\cos\theta$	$-i\frac{25}{2}\cos\theta$	0	$-i\frac{30}{7}\cos\theta$
$^3D_3{}^*$	$i\frac{7\sqrt{2}}{2}\cos\theta$	$i\frac{5}{2}\cos\theta$	$i3\sqrt{5}\cos\theta$	$-i\frac{\sqrt{2}}{2}\cos\theta$	$i\frac{25}{7}\cos\theta$	$i\frac{30}{7}\cos\theta$	0

Here $M_p(lm)$ are the multipole moments defined in the conventional way.

$$M_1(l, m) = -\frac{(2l+1)!!}{c}\sqrt{\frac{l}{l+1}}\,\kappa^{-l-1}\int j_l(\kappa r)\,\boldsymbol{X}^*_{lm}(\boldsymbol{\nabla}\times\boldsymbol{j}+c\kappa^2\boldsymbol{M})\,dV$$

(for electric transitions)

$$M_0(l, m) = \frac{i(2l+1)!!}{c}\sqrt{\frac{l}{l+1}}\,\kappa^{-l}\int j_l(\kappa r)\,\boldsymbol{X}^*_{lm}(\boldsymbol{j}+c\boldsymbol{\nabla}\times\boldsymbol{M})\,dV$$

(for magnetic transitions)

$\boldsymbol{j}(r)$ and $\boldsymbol{M}(r)$ are the distributions of electric current and magnetization, and $\boldsymbol{X}_{lm}(\theta, \varphi)$ is the vector spherical harmonics

$$\boldsymbol{X}_{lm}(\theta, \varphi) \equiv \boldsymbol{Y}^m_{ll1}(\theta, \varphi) = \frac{\boldsymbol{L}Y_{lm}(\theta, \varphi)}{\sqrt{l(l+1)}}.$$

$j_l(r)$ is the spherical bessel function of order l. These multipole moment operators are reduced by the partial integration

$$\int j_l(\kappa r)\,\boldsymbol{X}^*_{lm}(\boldsymbol{\nabla}\times\boldsymbol{j}+c\kappa^2\boldsymbol{M})\,dV$$

$$\to\int\frac{\boldsymbol{Y}^*_{lm}}{\sqrt{l(l+1)}}\left[\omega\{-(l+1)j_l(\kappa r)+\kappa r j_{l+1}(\kappa r)\}\rho(\boldsymbol{r})\right.$$

$$\left.+i\kappa^2 j_l(\kappa r)\{-(\boldsymbol{r}\cdot\boldsymbol{j})+c\boldsymbol{\nabla}\cdot\boldsymbol{r}\times\boldsymbol{M}(\boldsymbol{r})\}\right]dV.$$

$$\int j_l(\kappa r)\,\boldsymbol{X}^*_{lm}(\boldsymbol{j}+c\boldsymbol{\nabla}\times\boldsymbol{M})\,dV$$

$$\to\int\frac{\boldsymbol{Y}^*_{lm}}{\sqrt{l(l+1)}}\left[ic\{(l+1)j_l(\kappa r)-\kappa r j_{l+1}(\kappa r)\}\boldsymbol{\nabla}\cdot\boldsymbol{M}\right.$$

$$\left.+\kappa^2 j_l(\kappa r)\{-ic(\boldsymbol{r}\cdot\boldsymbol{M})+i\boldsymbol{\nabla}\cdot(\boldsymbol{r}\times\boldsymbol{j})/\kappa^2\}\right]dV,$$

to give the results in (4) and (5).

After this work was completed, various evidences have been accumulated for the presence of the L-S type splitting of the ${}^3P_J$ phase shifts in nucleon-nucleon scattering. (For example, M. H. MacGregor and M. J. Moravcsik, Phys. Rev. Letters **4** (1960), 524; and K. Gotow and E. Heer, to be published.) This splitting seems to be due to an L-S type interaction in the innermost region of internucleon forces.

References

1) J. Iwadare, S. Otsuki, R. Tamagaki and W. Watari, 1a) Prog. Theor. Phys. **16** (1956), 455, 1b) 472, 1c) 604, 1d) Suppl. No. **3** (1956).
2) T. Hamada, J. Iwadare, S. Otsuki, R. Tamagaki and W. Watari, Prog. Theor. Phys. **22** (1959), 301, 566.
3) J. Iwadare, S. Otsuki, M. Sano, S. Takagi and W. Watari, Prog. Theor. Phys. **16** (1956), 658.
4) W. Heitler, "*The Quantum Theory of Radiation*" (Oxford Clarendon Press), p. 180.
5) L. I. Schiff, Phys. Rev. **78** (1950), 733.
J. F. Marshall and E. Guth, Phys. Rev. **78** (1950), 738.
6) J. Halpern and E. V. Weinstock, Phys. Rev. **91** (1953), 934.
L. Allen, Phys. Rev. **98** (1955), 705.
A. Wheststone and J. Halpern, Phys. Rev. **109** (1958), 2072.
7) E. A. Whalin, B. D. Schriever and A. O. Hanson, Phys. Rev. **101** (1956), 377.
J. C. Keck and A. V. Tollestrup, Phys. Rev. **101** (1956), 360.
8) N. Austern, Phys. Rev. **85** (1952), 283.
T. Komoda and M. Sasaki, Prog. Theor. Phys. **6** (1953), 468.
9) J. Fujimura, Prog. Theor. Phys. **9** (1953), 132.
A. Sugie and S. Yoshida, Prog. Theor. Phys. **10** (1953), 236.
10) N. Austern, Phys. Rev. **108** (1957), 937.
11) M. Taketani, S. Machida and S. Ohnuma, Prog. Theor. Phys. **7** (1952), 45.
12) N. Austern, Phys. Rev. **108** (1957), 973.
M. L. Rustgi, thesis, Louisiana State University (1957), unpublished, referred by N. Austern.
13) J. J. de Swart and R. E. Marshak, Phys. Rev. **111** (1958), 272.
J. J. de Swart, J. Sawicki and W. Czyz, Phys. Rev. Letters **2** (1959), 51.
14) T. Terasawa, Prog. Theor, Phys. **23** (1960), 87.
S. Takagi, W. Watari and M. Yasuno, Prog. Theor. Phys. **22** (1959), 549.
S. Nagata, T. Sasakawa, T. Sawada and R. Tamagaki, Prog. Theor. Phys. **22** (1959), 274.
B. Jancovici, Prog. Theor. Phys. **22** (1959), 585.
15) I. Sato and K. Itabashi, Prog. Theor. Phys. **13** (1955), 341.
S. Okubo and S. Sato, Prog. Theor. Phys. **21** (1959), 384.
16) H. P. Stapp, T. J. Ypsilantis and N. Metropolis, Phys. Rev. **105** (1957), 302.
17) H. Miyazawa, Prog. Theor. Phys. **7** (1952), 207.
S. Machida, Prog. Theor. Phys. **9** (1953), 683.
18) S. Gartenhaus, Phys. Rev. **100** (1955), 900.
19) J. Gammel and R. Thaler, Phys. Rev. **107** (1957), 291.
20) P. S. Signell and R. E. Marshak, Phys. Rev. **106** (1957), 832; **109** (1958), 1229.
21) M. Matsumoto, Prog. Theor. Phys. **23** (1960), 597.
22) This point was also remarked by S. H. Hsieh and M. Matsumoto.
S. H. Hsieh, Prog. Theor. Phys. **21** (1959), 585.
M. Matsumoto, Prog. Theor. Phys. **23** (1960), 188.
23) A. F. Nicholson and G. E. Brown, Proc. Phys. Soc. **73** (1959), 221.
24) W. Czyz and J. Sawicki, Phys. Rev. **110** (1958), 900.
35) L. D. Perlstein and A. Klein, Phys. Rev. **107** (1957), 836.
26) M. Kawaguchi, Phys. Rev. **111** (1958), 1314.
27) W. Zernik, M. L. Rustgi and G. Breit, Phys. Rev. **114** (1959), 1358.
28) J. J. de Swart and R. E. Marshak, Physica **25** (1959), 1001.
29) G. Kramer, Nuclear Physics **15** (1960), 60.

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A Quantitative Investigation on the "Compositeness" of a Particle

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A quantitative discussion is given concerning to what degree it is plausible to regard a particle as a composite one in the sense of the non-relativistic quantum mechanics of particles (the "Compositeness"). For this purpose we calculate the probability that the system is in a state that cannot be regarded as a system of the constituent particles distributed with some probability densities. The main conclusion is that such a probability is proportional to the square root of the binding energy.

§ 1. Introduction

The notion of the composite particle is very distinct between the cases of the relativistic quantum theory of fields and of the non-relativistic quantum theory of particles.

In the quantized theory of particles a composite particle is only a set of constituent particles which are distributed in a finite region of space with some probability density, and the constituent particles bear their own properties as in a free motion. Therefore there is an essential difference between an elementary and a composite particle.

In the quantum field theory, on the contrary, the difference itself between an elementary and a composite particle is somewhat ambiguous. For instance, both kinds of particles can be equally well described by means of a single local field operator. Moreover, one cannot construct such a particle as having the properties of the composite particle in the strict sense of the particle dynamics. (In the following we shall use the term "true composite particle" in such a limited sense, as in the particle dynamics). All what one can do in the quantum field theory is to construct a system which can be regarded only approximately as a true composite particle (in the narrow sense as stated above).

In order to facilitate the following discussions, we introduce the term "compositeness", as is used in the title of this paper. The term "compositeness" here means the degree to which the system under consideration can plausibly be regarded as a true composite particle.

In § 2 we shall take a view of the non-relativistic particle dynamics to find a clue to the "compositeness". § 3 will be devoted to explicit calculations in perturbation theory for several reasonable models. In § 4 we shall draw a picture of the composite particle judging from the result in § 3, and conjecture that the

conclusion in § 3, though obtained on the basis of perturbation theory, is a rather general feature of the composite particle.

§ 2. Preliminary discussions

In order to throw light on the circumstances in particle dynamics, let us first consider a non-relativistic quantum mechanical system of two particles with masses m and m' respectively. In this case we have the "probability densities" for finding the individual particles, and these quantities have well-defined physical meanings. Let $P_m(\boldsymbol{x})$ and $P'_m(\boldsymbol{x})$ denote the probability densities of the m- and m'-particles at the position $\boldsymbol{x}$ of the particle measured from the center of mass of the system. Then there holds the following relation between P_m and P'_m:

$$P_m\left(\frac{m'}{m}\boldsymbol{x}\right)\Big/P_{m'}(-\boldsymbol{x})=(m/m')^3. \tag{1}$$

The derivation of this equality is given in the Appendix. This equality means that the probability distribution of the individual particles are mutually similar to a proportion $1/m : 1/m'$ giving the value $(m/m')^3$ to the ratio of the probability densities. The relation (1) holds strictly for any two-particle system. Therefore, we would like to emphasize that when we deal with a system which can be regarded approximately as composed of two particles, the relation (1) will be usuful as a clue for defining the "compositeness" of the system.

Next we shall consider, in the relativistic quantum field theory, a model in which a particle with mass M dissociates virtually into two particles with masses m and m'. In the field theory quantities such as the probability densities of the individual particles have no physical meanings. Instead, as quantities with well-defined physical meanings, we have the "particle densities". (We assume here that the fields describing the m- and m'-particles are neither real nor Majorana ones.) The particle density and the probability density coincide with each other if and only if the effect of particle-antiparticle pairs is negligible, and it is also only in this case that the probability density has a physical meaning.

In the inner region we cannot, of course, neglect the pair effect, and in this region the probability densities are always meaningless. In the peripheral region we distinguish the following two cases. Case A: The effect of pairs is considerable even in this region, and the probability densities are still meaningless. Case B: The effect of pairs is negligible. In this case, therefore, the probability densities have physical meanings in the peripheral region.

One of the present authors (K. Y.) asserted in the previous paper[1] that this difference in the structure in the peripheral region is useful for definition of the elementary particles. In addition this difference is associated with whether there appears the so-called "anomalous threshold" in the dispersion relation for $\widetilde{F}_i(\boldsymbol{q}^2)=\int d^3\boldsymbol{x}\ \exp(-i\boldsymbol{q}\cdot\boldsymbol{x})F_i(\boldsymbol{x})$ $(i=m,\ m')$, where $F_i(\boldsymbol{x})$ is the particle density for the i-particle. Namely, if $M^2<m^2+m'^2$ the M-particle belongs to Case A,

and if $M^2>m^2+m'^2$ it belongs to Case B. Thus it may be conjectured that if $M^2>m^2+m'^2$ the M-particle state is approximately a two-particle state in the peripheral region, and that $F_i(\boldsymbol{x})$ has the meaning of probability density.

While the above discussion is only a qualitative one, a more quantitative investigation is possible if we make use of the relation (1). Namely it is possible to define the "compositeness" of the M-particle (in Case B), making use of (1) as a clue.

The explicit procedure goes as in the following. The matrix element $\langle p', M|j_\mu^i(0)|p, M\rangle$ of the i-particle density four-vector $j_\mu^i(x)$ between the M-particle states with momenta p and p', respectively, can be written in the general form as follows:

$$\langle p', M|j_\mu^i(0)|p, M\rangle=(p+p')_\mu\widetilde{F}_i(q^2)$$
for Bose M-particle, (2)

$$\langle p', M|j_\mu^i(0)|p, M\rangle=\bar{u}(p')[i\gamma_\mu\widetilde{F}_i(q^2)+i\sigma_{\mu\nu}q^\nu\widetilde{F}_i'(q^2)]u(p)$$
for Fermi M-particle, (3)

where $q^2=(p-p')^2$. We can then estimate the "compositeness" by investigating in what region and to what degree the Fourier transform $F_i(\boldsymbol{x})$ of $\widetilde{F}_i(\boldsymbol{q}^2)$ in (2) or (3) satisfies the relation (1).

$F_i(\boldsymbol{x})$ is a quantity which can, in principle, be calculated if we adopt some definite model, though the calculation may in general be cumbersome. The notion of the "compositeness", however, is rather dependent on subjective opinions. Therefore it would be unnecessary to calculate $F_i(\boldsymbol{x})$ exactly for a given M-particle and investigate it by (1). It seems sufficient to calculate $F_i(\boldsymbol{x})$ in simple examples and conjecture for more general cases.

In the following section we shall try to investigate under what condition and with what accuracy F_i's satisfy the relation (1), by explicit calculations to the lowest order of the perturbation theory in several reasonable cases, and with these concrete examples we shall define and estimate the quantity "compositeness".

§ 3. Explicit examples

The simplest vertex parts of four types as shown in Fig. 1 are calculated. Let $\widetilde{F}_i(q^2: m, m')$ be the form factor corresponding to each diagram (j=BB, FF, FB and BF) and related to the m-particle density in the state of M-particle composed of

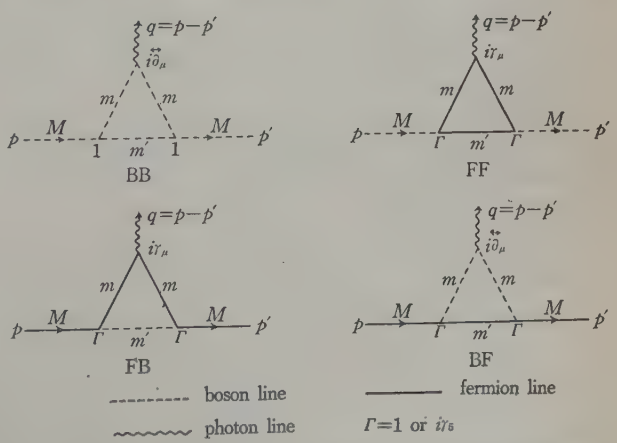

Fig. 1. The four types of vertex diagrams calculated.

m- and m'-particles. It is well known that $\widetilde{F}_j(q^2: m, m')$ has the following spectral representation:

$$\widetilde{F}_j(q^2: m, m') = \int_{\kappa_0}^{\infty} d\kappa \frac{m\sigma_j(\kappa^2: m, m')}{\kappa^2+q^2}, \tag{4}$$

where κ_0 is given by

$$\kappa_0 = 2m\nu_0, \tag{5}$$

$$\nu_0 = 1 \quad \text{for } M^2 < m^2+m'^2, \tag{6}$$

$$\nu_0 = [1-(M^2-m^2-m'^2)^2/4m^2m'^2]^{1/2} \quad \text{for } M^2 > m^2+m'^2. \tag{7}$$

Then we have

$$F(\boldsymbol{x}: m, m') = \frac{1}{(2\pi)^3}\int d^3\boldsymbol{q}\,\widetilde{F}(\boldsymbol{q}^2: m, m')\exp(i\boldsymbol{q}\cdot\boldsymbol{x})$$

$$= \frac{m^3}{2\pi m|\boldsymbol{x}|}\int_{\nu_0}^{\infty} d\nu \exp(-2m\nu|\boldsymbol{x}|)\sigma(4m^2\nu^2: m, m'). \tag{8}$$

Here it should be noted that the threshold value ν_0 is symmetric under the exchange $m \leftrightarrow m'$.

We see from Eq. (8) that the necessary and sufficient condition for $F(\boldsymbol{x}: m, m')$ to satisfy the relation (1) is

$$\sigma(4m^2\nu^2: m, m') = \sigma(4m'^2\nu^2: m', m). \tag{9}$$

For values of ν greater than 1 (i.e. $\kappa^2 > 4m^2$) the contribution of m-(or m'-) particle pairs is not negligible and we cannot expect that Eq. (9) is satisfied in this region of ν. Thus the region in which we expect the σ's to satisfy (9) is the interval $0 < \nu < 1$. This interval correspond to the region $|\boldsymbol{x}| > 1/2m$ in the ordinary space. Therefore it is necessary, for the M-particle to be regarded as composite, that the binding energy $B = M-m-m'$ is sufficiently small, giving $\nu_0^2 \ll 1$. (Note, for example, that $\nu_0^2 \approx 10^{-2}$ for deuteron.) In the following we shall confine ourselves to such cases.

The explicit calculation of the diagrams shown in Fig. 1 gives the following results for the spectral function $\sigma(4m^2\nu^2: m, m')$ in the interval $\nu_0 < \nu < 1$ apart from constant factors:

$$\begin{aligned}
m\sigma_{\mathrm{BB}} &= Z^{-3/2}(Z-2\Delta), \\
m\sigma_{\mathrm{FF}} &= Z^{-5/2}[\epsilon mm' Z(Z-2\Delta) - m'^2Z^2 + (M^2-3m^2+4m'^2)\Delta Z - 3\Delta^3], \\
m\sigma_{\mathrm{FB}} &= Z^{-5/2}[2\epsilon MmZ(Z-2\Delta) + (M^2+m^2+m'^2)Z^2 \\
&\quad -2\{3M^4+2M^2(3m'^2-2m^2)+(m^2-m'^2)^2\}Z + 12M^2\Delta^2], \\
m\sigma_{\mathrm{BF}} &= Z^{-5/2}[\epsilon Mm' Z(Z-2\Delta) + 2M^2(M^2+2m'^2-m^2)Z - 6M^2\Delta^2],
\end{aligned} \tag{10}$$

where

$$Z=4(M^2-m^2\nu^2),$$

$$\Delta=M^2+m'^2-m^2,$$

$$\epsilon=\pm 1 \quad \text{according to } \Gamma=1 \text{ or } i\gamma_5.$$

In the limits $B\to 0$, $\nu\to 0$, σ_{FF} vanishes if $\epsilon=+1$, and σ_{FB} as well as σ_{BF} vanishes if $\epsilon=-1$. This corresponds to the fact that in the non-relativistic approximation pair creation does not occur in the scalar interaction and boson emission does not in the γ_5-interaction. As the above choice of ϵ is inappropriate as a model of the composite particle, we shall confine ourselves in the following to the choice $\epsilon=-1$ for FF and $\epsilon=+1$ for FB and BF.

In the peripheral region only the values of ν in the vicinity of the threshold contribute to $F(\boldsymbol{x})$. On the other hand, it is likely that the best accuracy of (1) will be realized for the limit of zero binding energy ($M=m+m'$) in which case the constituent particles behave as free ones. These considerations let us expect that if $B=0$ the σ's satisfy (9) at the threshold $\nu=0$. It can be immediately seen that this is indeed the case:

$$\begin{aligned}
&\sigma_{\mathrm{BB}}(0:m,m')\approx\sigma_{\mathrm{BB}}(0:m',m),\\
&\sigma_{\mathrm{FF}}(0:m,m')\approx\sigma_{\mathrm{FF}}(0:m',m),\\
&\sigma_{\mathrm{FB}}(0:m,m')\approx\sigma_{\mathrm{BF}}(0:m',m), \quad \text{for } B\approx 0.
\end{aligned} \tag{11}$$

The deviation from (9) arises only in the first order in ν and B.

Next we shall investigate with what accuracy (9) is satisfied in the region $\nu_0<\nu<1$ in case $B\ll M$. For this purpose we decompose each σ into a symmetric and an antisymmetric part with respect to the exchange $m\leftrightarrow m'$. (In the boson-fermion case we need also the exchange of the diagrams BF and FB.)

$$\begin{aligned}
&\chi(\nu:m,m')=1/2[\sigma(4m^2\nu^2:m,m')+\sigma(4m'^2\nu^2:m',m)],\\
&\chi'(\nu:m,m')=1/2[\sigma(4m^2\nu^2:m,m')-\sigma(4m'^2\nu^2:m',m)].
\end{aligned} \tag{12}$$

If χ' is identically zero, the M-particle can be regarded as a composite one so that the magnitude of χ' relative to that of χ is expected to have something to do with the "compositeness".

For simplicity we shall consider in the following only the special case of $m=m'$. In this special case χ'_{BB} and χ'_{FF} vanishes. But this cannot mean that the M-particle in this case may be regarded as a two-particle system, because it is required that the relation (9) holds for $\sigma(4m^2\nu^2:m,m')$ and $\sigma(4m'^2\nu^2:m',m)$ as functions of m and m', and the coincidence only at some special values of m and m' is insufficient that if $m=m'$ the relation (1) is satisfied even in the inner region, where the effect of pairs is not negligible. As χ'_{BB} and χ'_{FF} are proportional to the mass difference $m-m'$, we may consider the coefficient of $m-m'$

as a quantity which is responsible to the deviation from (9). We define, therefore, new quantities $\hat{\chi}$ instead of χ' in case $m=m'$:

$$\hat{\chi}_j(\nu : m)=\lim_{m'\to m}\frac{m}{m-m'}[\sigma_j(4m^2\nu^2 : m, m')-\sigma_j(4m'^2\nu^2 : m', m)]/2 \quad \text{for } j=\text{BB, FF,}$$

$$\hat{\chi}_j(\nu : m)=\chi_j'(\nu : m, m) \quad \text{for } j=\text{FB.} \tag{13}$$

Assuming $B\ll M$ we expand χ and $\hat{\chi}$ in power series in $B/2m$:

$$\chi(\nu : m)=\chi^{(0)}(\nu : m)+(B/2m)\chi^{(1)}(\nu : m)+\cdots,$$
$$\hat{\chi}(\nu : m)=\hat{\chi}^{(0)}(\nu : m)+(B/2m)\hat{\chi}^{(1)}(\nu : m)+\cdots. \tag{14}$$

Adopting the normalization $\chi^{(0)}(0 : m)=1$, we have the following expressions for the χ's defined above:

$$\chi^{(0)}_{\text{BB}}=2x-x^3, \qquad \chi^{(1)}_{\text{BB}}=4x^3-3x^5,$$
$$\hat{\chi}^{(0)}_{\text{BB}}=-2x+\frac{7}{2}x^3-\frac{3}{2}x^5, \qquad \hat{\chi}^{(1)}_{\text{BB}}=-8x^3+\frac{33}{2}x^5-\frac{15}{2}x^7,$$
$$\chi^{(0)}_{\text{FF}}=2x-\frac{7}{4}x^3+\frac{3}{4}x^5, \qquad \hat{\chi}^{(1)}_{\text{FF}}=\frac{15}{2}x^3-\frac{39}{4}x^5+\frac{15}{4}x^7,$$
$$\hat{\chi}^{(0)}_{\text{FF}}=-3x+\frac{57}{8}x^3-6x^5+\frac{15}{8}x^7, \quad \hat{\chi}^{(1)}_{\text{FF}}=-\frac{41}{2}x^3+\frac{423}{8}x^5-45x^7+\frac{105}{8}x^9, \tag{15}$$
$$\chi^{(0)}_{\text{FB}}=\frac{7}{4}x-\frac{3}{4}x^3, \qquad \chi^{(1)}_{\text{FB}}=-2x+\frac{17}{4}x^3-\frac{9}{4}x^5,$$
$$\hat{\chi}^{(0)}_{\text{FB}}=\frac{3}{8}x-\frac{3}{4}x^3+\frac{3}{8}x^5, \qquad \hat{\chi}^{(1)}_{\text{FB}}=-\frac{1}{2}x+\frac{25}{8}x^3-\frac{9}{2}x^5+\frac{15}{8}x^7,$$

where $$x=[1-(\nu/2)^2]^{-1/2}.$$

For every case $\hat{\chi}^{(0)}(0 : m)=0$ which corresponds to (11). As $\chi^{(1)}$ and $\hat{\chi}^{(1)}$ are of the comparable order in magnitude with $\chi^{(0)}$ and $\hat{\chi}^{(0)}$, the latter quantities give the main information if $B/2m\ll 1$.

Let us define the following quantities by means of χ and $\hat{\chi}$:

$$\rho=\frac{1}{(2\pi)^3}\int d^3\boldsymbol{x}\int d^3\boldsymbol{q}\int_{\kappa_0}^{\infty}d\kappa\frac{\exp(i\boldsymbol{q}\cdot\boldsymbol{x})}{\kappa^2+\boldsymbol{q}^2}m\chi\left(\frac{\kappa}{2m} : m\right)=\int_{\nu_0}^{\infty}\frac{d\nu}{2\nu^2}\chi(\nu : m), \tag{16}$$

$$\hat{\rho}=\int_{\nu_0}^{\infty}\frac{d\nu}{2\nu^2}\,|\hat{\chi}(\nu : m)|. \tag{17}$$

The physical meaning of ρ or $\hat{\rho}$ is, roughly speaking, the relative probability that the M-particle is or is not considered to be a two-particle system. Therefore it seems to be plausible to define the "compositeness" through the ratio $\hat{\rho}/\rho$.

We assume that if $B/2m \ll 1$ the contribution from the region $\nu > 1$ to the integration (16) or (17) is negligible compared with that from the region $\nu_0 < \nu < 1$ and estimate the values of (16) and (17) using (15). The results are in the lowest order in B:

$$(\hat{\rho}/\rho)_{BB} \approx \frac{4}{3\sqrt{3}}\sqrt{B/m},$$

$$(\hat{\rho}/\rho)_{FF} \approx \sqrt{3}\,\sqrt{B/m},$$

$$(\hat{\rho}/\rho)_{FB} \approx \frac{1}{6\sqrt{3}}\sqrt{B/m}. \tag{18}$$

These results are the main conclusion of this section.

§ 4. Discussion

In the previous section we have seen that the quantity $\hat{\rho}/\rho$, which we call tentatively "acompositeness", is almost proportional to $B^{1/2}$. This fact originates in the following four points:

1. χ and $\hat{\chi}$ do not vary rapidly with B when $B \approx 0$.
2. χ approaches approximately to 1 with a gentle slope according as $\nu \to 0$.
3. $\hat{\chi}$ approaches approximately to 0 with a gentle slope according as $\nu \to 0$.
4. The threshold value ν_0 is approximately proportional to $B^{1/2}$.

From these facts it can be seen that ρ is almost proportional to $B^{-1/2}$ while $\hat{\rho}$ is not so sensitive to B, and thus $\hat{\rho}/\rho \propto B^{1/2}$.

These circumstances allow us to draw the following picture for a composite particle by means of the quantities ρ^{in}, $\hat{\rho}^{in}$, ρ^{per} and $\hat{\rho}^{per}$ defined below.

1. The inner structure of a composite particle is not very sensitive to variation of B about $B \approx 0$. Namely $\hat{\rho}^{in}/\rho^{in}$ is nearly constant with respect to B.
2. For small values of B, $\hat{\rho}^{per}/\rho^{per}$ almost vanishes.
3. The division into the inner and peripheral regions is almost independent of B when B is small.

Here ρ^{in}, ρ^{per}, $\hat{\rho}^{in}$ and $\hat{\rho}^{per}$ are the "inner" and "peripheral" parts of ρ and $\hat{\rho}$ and are defined respectively as the contributions to the integrals in (16) and (17) from the inner and the peripheral regions. The peripheral region here is roughly defined as the one the contribution from which to $\hat{\rho}$ is negligible. Thus we write

$$\rho = \rho^{in} + \rho^{per}, \tag{19}$$

$$\hat{\rho} = \hat{\rho}^{in} + \hat{\rho}^{per}. \tag{20}$$

As the distribution of the constituent particles extends outward according as B becomes small, ρ^{in} diminishes and also $\hat{\rho}^{in}$, since $\hat{\rho}^{in}/\rho^{in} \approx$ const. Thus the probability to be in the states other than two-particle ones is made smaller. Therefore, in such a model, that $\hat{\rho}/\rho$ is proportional to $B^{1/2}$ can be understood if it is

shown that the probability to be in two-particle states is proportional to $B^{1/2}$. This situation seems to be quite natural, for the solution of the non-relativistic Schrödinger equation is $\exp(-br)/r$ ($b \propto B^{1/2}$) in the domain where the potential vanishes, giving a much greater value when $B \approx 0$ to the probability to be in that domain than the one to be in the domain where the potential is finite.

From the discussions given above we may conjecture that it is a considerably general feature of the composite particle that the "acompositeness" is proportional to $B^{1/2}$ when B is small. The proportionality constant thereof is determined by the detail of the dynamics, but this constant is likely to be of the order of unity. Thus, we may conjecture, for instance, that the "acompositeness" of deuteron is of the order of several percent.

Appendix

Here we give a proof for the relation (1). Let $\psi(\boldsymbol{r}_1, \boldsymbol{r}_2)$ be the probability density for the two-particle system, where $\boldsymbol{r}_i (i=1, 2)$ is the position of each particle. The position of the i-particle measured from the center of mass of the system is given by

$$\begin{aligned} \bar{\boldsymbol{r}}_1 &= \boldsymbol{r}_1 - \frac{m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2}{m_1+m_2} = \frac{m_2}{m_1+m_2}(\boldsymbol{r}_1 - \boldsymbol{r}_2), \\ \bar{\boldsymbol{r}}_2 &= \boldsymbol{r}_2 - \frac{m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2}{m_1+m_2} = \frac{m_1}{m_1+m_2}(\boldsymbol{r}_2 - \boldsymbol{r}_1). \end{aligned} \tag{A·1}$$

The probability that the 1-particle is located at $\bar{\boldsymbol{r}}_1 = m_2/m_1 \cdot \boldsymbol{x}$ is then calculated as follows:

$$\begin{aligned} P_{m_1}\left(\frac{m_2}{m_1}\boldsymbol{x}\right) &= \iint d^3\boldsymbol{r}_1 d^3\boldsymbol{r}_2 |\psi(\boldsymbol{r}_1, \boldsymbol{r}_2)|^2 \delta\left(\bar{\boldsymbol{r}}_1 - \frac{m_2}{m_1}\boldsymbol{x}\right) \\ &= \int d^3\boldsymbol{r}_1 \left|\psi\left(\boldsymbol{r}_1, \boldsymbol{r}_1 - \frac{m_1+m_2}{m_1}\boldsymbol{x}\right)\right|^2 \left(\frac{m_1+m_2}{m_2}\right)^3. \end{aligned} \tag{A·2}$$

On the other hand, the probability that the 2-particle is located at $\bar{\boldsymbol{r}}_2 = -\boldsymbol{x}$ is

$$\begin{aligned} P_{m_2}(-\boldsymbol{x}) &= \iint d^3\boldsymbol{r}_1 d^3\boldsymbol{r}_2 |\psi(\boldsymbol{r}_1, \boldsymbol{r}_2)|^2 \delta(\bar{\boldsymbol{r}}_2 + \boldsymbol{x}) \\ &= \int d^3\boldsymbol{r}_1 \left|\psi\left(\boldsymbol{r}_1, \boldsymbol{r}_1 - \frac{m_1+m_2}{m_1}\boldsymbol{x}\right)\right|^2 \left(\frac{m_1+m_2}{m_1}\right)^3. \end{aligned} \tag{A·3}$$

The ratio of (A·2) and (A·3) immediately yields the relation (1).

References

1) K. Yamamoto, Prog. Theor. Phys. **23** (1960), 1204.
2) Y. Nambu, Nuovo Cimento **9** (1958), 610.
R. Karplus, C. Sommerfield and E. Wichmann, Phys. Rev. **111** (1958), 1187.
R. Oehme, Nuovo Cimento **13** (1959), 778.

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Propagation of Shock Waves in Inhomogeneous Gases. III

—*Spherical Shock Waves*—

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The propagation of spherical shock waves in the stellar interior is considered, extending the method developed in previous papers. A differential equation of the shock strength is derived, which can be applied to any stellar models. It is found that the spherical damping effect is pronounced near the center of stars, and this guarantees the stability of the stellar core in general. In the envelope, this effect becomes negligible, and it is justified near the stellar surface to treat the shock waves as plane.

In the previous papers "**I**" and "**II**",[1] we treated the propagation of *plane* shock waves in an inhomogeneous medium. The result of these papers may be applied to the phenomena in the stellar surface, i.e. stellar atmosphere and chromosphere where the effect of the curvature of the surface is relatively small. On the other hand, they may not be applicable to the phenomena which are supposed to originate in stellar interior, e.g. the supernova and the nova explosions. In fact, if the result of "**I**" is formally applied to a finite disturbance generated in the stellar interior, it follows that the disturbance always grows up to a strong shock wave due to the enormous pressure decrease from the center to the surface, although an infinitesimal disturbance does not grow as stated in "**I**".* This means that the stars must be unstable against any finite disturbances generated in the interior, but it is well known that this is not necessarily the case. This discrepancy seems to be overcome by taking into account the effect of the *spherical damping* of the spherical shock waves propagating from the interior to the surface.

Now, the propagation of spherical shock waves in a *homogeneous* medium is treated by several authors in relation to blast waves and shock tubes.[2] Especially, Chisnell[3] considered the problem of the change in the shock strength Δz for the increase in the area A of the cross section of a shock tube from A to $A+\Delta A$, and derived a relation between Δz and ΔA. Integrating the differential equation thus obtained, the connection between the strength z and the area A is derived in the first approximation. The propagation of spherical shock waves can be treated in this way, putting $A \propto r^2$, where r is the distance from the center.

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* See (4·6) and (4·7) in "**I**".

Combining this procedure with that of "**I**", we finally get the following equation for the shock propagation:

$$\frac{dz}{d\ln r}=\frac{\left(\frac{1}{n+1}+2\sqrt{\frac{1+\lambda^2 z}{z(1+\lambda^2)}}\right)V-\frac{4}{1+\frac{(z-1)(1-\lambda^2)}{\sqrt{(1+\lambda^2)z(1+\lambda^2 z)}}}}{\frac{2}{z-1}-\frac{1}{z+\lambda^2}+\frac{2}{z-1}\sqrt{\frac{1+\lambda^2 z}{z(1+\lambda^2)}}}. \quad (1)$$

The first and the second term of the numerator comes respectively from the "*pressure growing*", that is, the growth of the shock strength as the pressure decreases, and from the "*spherical damping*". Both $n+1$ and V are the functions of r specified by the initial distribution, i.e. they are determined when the initial static model is given. n is the polytropic index usually adopted in stellar models:**

$$n+1=\frac{d\log p}{d\log T}. \quad (2)$$

$1/n+1$ is usually finite (or zero) in the stellar interior. V is defined by

$$V=-\frac{d\log p}{d\log r}, \quad (3)$$

which has the meaning

$$V\propto\frac{\text{gravitational energy}^{4)}}{\text{internal energy}},$$

and so increases from the value zero at the center to infinity at the surface.† Lastly, both functions of z in the numerator, i.e.

$$2\sqrt{\frac{1+\lambda^2 z}{z(1+\lambda^2)}}^{\dagger\dagger} \quad \text{and} \quad \frac{4}{1+\frac{(z-1)(1-\lambda^2)}{\sqrt{(1+\lambda^2)z(1+\lambda^2 z)}}},$$

---

* The definition of the shock strength z and the parameter λ^2 is the same as in "**I**".

** This is related to k in "**I**" and "**II**" by $n=\frac{1}{k-1}$.

† As is known, the growth of the value of V is not always monotone, for example, when a shell source exists.

†† According to Whitham,[5] who derived the differential equation for the "*pressure growing*" by using the characteristic equation, this function is multiplied by

$$\varphi(z)=\frac{z(1-\lambda^2)\sqrt{1+\lambda^2 z}+\lambda^2(z-1)\sqrt{z(1+\lambda^2)}}{(z-1)(1-\lambda^2)\sqrt{1+\lambda^2 z}+(1+\lambda^2 z)\sqrt{z(1+\lambda^2)}},$$

for which

$$\varphi(1)=\varphi(\infty)=1.$$

In my opinion, the method of Whitham does not seem to be well grounded. Anyhow, the function $\varphi(z)$ takes the values near to unity for z between unity and infinity (the minimum value is ~ 0.91 at $z\sim 2.8$ for $\lambda^2=1/4$), and therefore we may here neglect this factor.

are finite for the values of z from unity to infinity.

Summarizing the behaviours of these terms, we can deduce the development of the shock waves generated near the center as follows: Near the center or in the core of the stars, the strength of shock waves is largely reduced due to the spherical damping effect. Then, after they pass through a critical position r_{crit}, where $dz/dr=0$, the strength rises rapidly due to the increase of V, i.e. to the pressure growing effect. In the envelope of the star, the spherical damping effect becomes negligible, and near the surface the one-dimensional treatment of shocks employed in "I" may provide with a good approximation. In Fig. 1, the above behaviour of the shocks is shown graphically for the case of Eddington model $n=3$, using the isoclinic method.

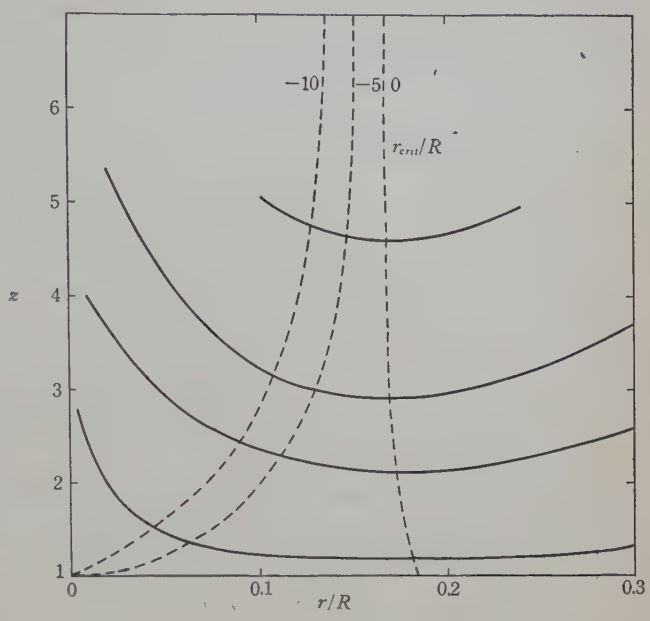

Fig. 1. Shock strength versus distance from the center (relative to the stellar radius R) calculated by the isoclinic method for the Eddington model. Dotted curves represent the isoclinic lines, where the inclination of solutions takes the value $dz/dr=0$ (for r_{crit}), -5 and -10, respectively.

The critical value of V at r_{crit}, i.e. V_{crit}, varies with both z and the model used, but the dependency on z is found to be rather slight. For polytropes, they are given by

$$\begin{aligned} V_{crit}&=1.9\sim2.0 \quad \text{for} \quad n=\infty \quad (\textit{isothermal}),\\ &=1.5\sim1.8 \qquad\quad n=3 \quad\ (\textit{Eddington}),\\ &=1.3\sim1.6 \qquad\quad n=1.5 \quad (\textit{convection}). \end{aligned}$$

It is seen that the isothermal distribution arrests the growth of disturbances most effectively.

The suppression of the shock growth may have a close connection with the stability of stars. Although all disturbances have been found to grow up after the critical position r_{crit} except for the case $z=1$, it is necessary to note that here we have dealt only with the case of stationarily generated shock waves (*quasistationary approximation*).* As a matter of fact, we must treat the shock pulses

*See the Introduction in "I".

which are damped by dissipation. In that case it might occur that the most disturbances generated at the core are reflected and dissipated in the region $r < r_{crit}$.

In order to see whether the stars are stable or unstable for the shock development, it is necessary to take an appropriate stellar (static) model and to settle the criterion of its stability. Then, the possible initial strength z_0 of a disturbance should be determined, using nuclear reaction rates and deriving the power generated. These will be treated in a separate paper.

The author wishes to express his thanks to Dr. S. Sakashita and Mr. H. Yamazaki for helpful discussions.

References

1) Y. Ôno, S. Sakashita and H. Yamazaki, **I**, Prog. Theor. Phys. **23** (1960), 294; **II**, *ibid.* **24** (1960), 155.
2) For example, R. B. Payne, J. Fluid Mech. **2** (1957), 185.
3) R. F. Chisnell, J. Fluid Mech. **2** (1957), 286.
4) See M. Schwarzschild, "*Structure and Evolution of the Stars*", 1958, Princeton.
5) G. B. Whitham, J. Fluid Mech. **4** (1958), 337.

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Symmetrical Properties of Two-Dimensional Ising Lattices

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Some symmetrical properties on the statistics of two-dimensional Ising lattices are studied. It is shown that such considerations give us very useful suggestions even for cases in which we know no exact solution. We investigate especially the cases of the square, triangular, honeycomb and generalized square lattices, with respect to the expressions of spontaneous magnetizations and susceptibilities.

§ 1. Introduction

As is well known, the exact treatments of the statistics for the two-dimensional Ising lattices have been obtained under zero magnetic field. But under finite magnetic field, it is very difficult to solve the problem. We know the internal energy, short range order, specific heat, entropy, etc., for the several types of lattices, but on their magnetic properties we know only Yang's[1] work on the spontaneous magnetization. It seems to us very difficult to extend the exact theory to know the spontaneous magnetization or the susceptibilities for other types of lattices.

The thermodynamic properties of one Ising lattice can be transformed to those of another type of lattice. These transformations enable us to know the unknown solution from the known one. In fact, when there are no magnetic fields, exact solutions for several types of lattices are derived from the known one. Under the external magnetic field, such transformations have not been successfully used.

One of us[2] and Fisher[3] derived the relations on the spontaneous magnetizations and the susceptibilities between the honeycomb lattice and the triangular lattice. But it is not sufficient for knowing the solution. Fisher[4] have obtained the exact solution for a special model by a transformation method, but his model seems to us too specialized to know a general feature of the Ising lattice. Basing on the symmetry consideration, Potts[5] derived the expression for the spontaneous magnetization for a triangular lattice from that of a square lattice.

Considerations on the symmetric properties will be very useful for the case in which we know no exact solution.

§ 2. Y-Δ transformation, quasi-spherical triangle

Y-Δ transformation for the anisotropic case seems very complicated at first sight.[6] But there are interesting symmetry properties between parameters. The equation for the Y-Δ transformation is

$$\sum_{\mu_0=\pm 1} \exp[\mu_0(H_1\mu_1+H_2\mu_2+H_3\mu_3)]=A\exp(T_1\mu_2\mu_3+T_2\mu_3\mu_1+T_3\mu_1\mu_2), \tag{1}$$

where H_1, H_2, H_3 and T_1, T_2, T_3, are the interaction parameters for the honeycomb lattice and the triangular lattice respectively, and $\mu_i(i=0, 1, 2, 3)$ are the Ising spin variables. For the possible values of $\mu_i=\pm 1$ we get the relations

$$\left.\begin{aligned} 2\,\mathrm{ch}(H_1+H_2+H_3)&=A\exp(T_1+T_2+T_3)\\ 2\,\mathrm{ch}\,(H_1-H_2+H_3)&=A\exp(-T_1+T_2-T_3)\\ 2\,\mathrm{ch}(H_1+H_2-H_3)&=A\exp(-T_1-T_2+T_3)\\ 2\,\mathrm{ch}(-H_1+H_2+H_3)&=A\exp(T_1-T_2-T_3) \end{aligned}\right\}, \tag{2}$$

which are easily deformed to

$$\left.\begin{aligned} A^4&=4(s_1^2+s_2^2+s_3^2+2c_1c_2c_3+2)\\ \coth 2T_1&=(c_1+c_2c_3)/s_2s_3\\ \coth 2T_2&=(c_2+c_3c_1)/s_3s_1\\ \coth 2T_3&=(c_3+c_1c_2)/s_1s_2 \end{aligned}\right\}, \tag{3}$$

or

$$\left.\begin{aligned} 1/\mathrm{sh}^2 2T_1&=A^4/4(s_2s_3)^2\\ 1/\mathrm{sh}^2 2T_2&=A^4/4(s_3s_1)^2\\ 1/\mathrm{sh}^2 2T_3&=A^4/4(s_1s_2)^2 \end{aligned}\right\}, \tag{4}$$

where $s_i=\mathrm{sh}\,2H_i$, $c_i=\mathrm{ch}\,2H_i$. We can write these in symmetric form

$$\frac{1}{\mathrm{sh}^2 2T_1\cdot\mathrm{sh}^2 2H_1}=\frac{1}{\mathrm{sh}^2 2T_2\cdot\mathrm{sh}^2 2H_2}=\frac{1}{\mathrm{sh}^2 2T_3\cdot\mathrm{sh}^2 2H_3}$$
$$=\frac{s_1^2+s_2^2+s_3^2+2c_1c_2c_3+2}{(s_1s_2s_3)^2}. \tag{5}$$

By the dual transformation from the triangular lattice (T_1, T_2, T_3), we get again the honeycomb lattice with interaction parameters (H_1', H_2', H_3').

$$\mathrm{sh}\,2H_i'\cdot\mathrm{sh}\,2T_i=\mathrm{ch}\,2H_i'\cdot\mathrm{th}\,2T_i=\mathrm{th}\,2H_i'\cdot\mathrm{ch}\,2T_i=1. \tag{6}$$

With (3) and (6), we have

$$\left.\begin{aligned} \mathrm{ch}\,2H_1'&=(c_1+c_2c_3)/(s_2s_3)\\ \mathrm{ch}\,2H_2'&=(c_2+c_3c_1)/(s_3s_1)\\ \mathrm{ch}\,2H_3'&=(c_3+c_1c_2)/(s_1s_2) \end{aligned}\right\}. \tag{7}$$

Rearranging the terms, we get

$$\left.\begin{aligned} c_1 &= -c_2c_3+s_2s_3c_1' \\ c_2 &= -c_3c_1+s_3s_1c_2' \\ c_3 &= -c_1c_2+s_1s_2c_3' \end{aligned}\right\}, \tag{8}$$

where $c_i'=\mathrm{ch}\,2H_i'$.

Putting $s_i'=\mathrm{sh}\,2H_i'$, with (5) and (6) we have

$$\frac{s_1'^2}{s_1^2}=\frac{s_2'^2}{s_2^2}=\frac{s_3'^2}{s_3^2}=\frac{s_1^2+s_2^2+s_3^2+2c_1c_2c_3+2}{(s_1s_2s_3)^2}\equiv k_H^2. \tag{9}$$

Eqs. (8) and (9) have similarity to the "cosine law" and the "sine law" of a spherical triangle. But the relations between H_1, H_2, H_3 and H_1', H_2', H_3' are symmetric:

$$\left.\begin{aligned} c_1' &= -c_2'c_3'+s_2's_3'c_1 \\ c_2' &= -c_3'c_1'+s_3's_1'c_2 \\ c_3' &= -c_1'c_2'+s_1's_2'c_3 \end{aligned}\right\}, \tag{10}$$

$$\frac{s_1'^2}{s_1^2}=\frac{s_2'^2}{s_2^2}=\frac{s_3'^2}{s_3^2}=\frac{(s_1's_2's_3')^2}{s_1'^2+s_2'^2+s_3'^2+2c_1'c_2'c_3'+2}$$

$$=\frac{s_1^2+s_2^2+s_3^2+2c_1c_2c_3+2}{(s_1s_2s_3)^2}=k_H^2. \tag{11}$$

From (8), we can get

$$c_1'c_2'c_3'+1=k_H^2(c_1c_2c_3+1). \tag{12}$$

(11) and (12) give the symmetric relations

$$\frac{s_1'^2}{s_1^2}=\frac{s_2'^2}{s_2^2}=\frac{s_3'^2}{s_3^2}=\frac{c_1'c_2'c_3'+1}{c_1c_2c_3+1}=\frac{c_1'c_2'c_3'+1-(s_1's_2'+s_2's_3'+s_3's_1')}{c_1c_2c_3+1-(s_1s_2+s_2s_3+s_3s_1)}=k_H^2. \tag{13}$$

According to Potts' conjecture, the spontaneous magnetization I_h for the honeycomb lattice will be

$$\mathrm{I}_h^8=(1-k_H^2). \tag{14}$$

The similar relations for the triangular lattice will be easily obtained. Let the dual lattice of the honeycomb lattice (H_1, H_2, H_3) be the triangular lattice (T_1', T_2', T_3'). Then we have

$$\mathrm{sh}\,2H_i\,\mathrm{sh}\,2T_i'=\mathrm{ch}\,2H_i\,\mathrm{th}\,2T_i'=\mathrm{th}\,2H_i\,\mathrm{ch}\,2T_i'=1. \tag{15}$$

With (11), (13) and (15), putting $c_i'=\mathrm{ch}\,2T_i'$, $c_i=\mathrm{ch}\,2T_i$, we have

$$\frac{s_1'^2}{s_1^2}=\frac{s_2'^2}{s_2^2}=\frac{s_3'^2}{s_3^2}=\frac{1}{(s_1s_2)^2+(s_2s_3)^2+(s_3s_1)^2+2(s_1s_2s_3)^2+2c_1c_2c_3s_1s_2s_3}$$

$$= (s_1's_2')^2 + (s_2's_3')^2 + (s_3's_1')^2 + 2(s_1's_2's_3')^2 + 2c_1'c_2'c_3's_1's_2's_3'$$

$$= \left(\frac{c_1'c_2'c_3' + s_1's_2's_3'}{c_1c_2c_3 + s_1s_2s_3} \right)^2 = \left(\frac{c_1'c_2'c_3' + s_1's_2's_3' - (s_1' + s_2' + s_3')}{c_1c_2c_3 + s_1s_2s_3 - (s_1 + s_2 + s_3)} \right)^2 = k_T^2. \quad (16)$$

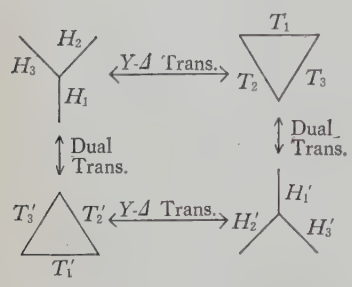


Fig. 1. Transformation of lattices.

Potts proposed that the spontaneous magnetization for the triangular lattice should be

$$I_t^8 = (1 - k_T^2). \quad (17)$$

(17) and (14) satisfy the condition derived by one of us: $I_h(H_1, H_2, H_3) = I_t(T_1, T_2, T_3)$. When $H_3 \to \infty$ or $T_3 \to 0$, (14) or (17) is reduced to the spontaneous magnetization for a square lattice:

$$I_s^8 = \left(1 - \frac{1}{s_1^2 s_2^2}\right). \quad (18)$$

§ 3. Spontaneous magnetization of the generalized square lattice

As a special case of the so-called chequer-type lattice, Utiyama[7] treated the partition function of a generalized square lattice. This lattice is reduced to a triangular, honeycomb or square lattice according as one of the parameters, for example, L_4, tends to infinity, zero or $L_4 = L_2$, and $L_1 = L_3$. As a generalization of (14) and (17) we arrive at the expression for the spontaneous magnetization for a generalized square lattice:

$$I_{g\cdot s} = \left(1 - \frac{\Sigma s_i^2 + 2c_1c_2c_3c_4 + 2s_1s_2s_3s_4 + 2}{\Sigma (s_i s_j s_k)^2 + 2(s_1s_2s_3s_4)^2 + 2c_1c_2c_3c_4s_1s_2s_3s_4 + 2s_1s_2s_3s_4}\right)^{1/8}, \quad (19)$$

where the summations extend for i, j, k from 1 to 4, excluding the congruence of them. Hereafter we will omit this notice on the summations which will appear frequently. Or with parameters for a dual lattice $(L_1^*, L_2^*, L_3^*, L_4^*)$, we can rewrite (19) as a symmetric form

$$I_{g\cdot s} = \left(1 - \frac{s_1^* s_2^* s_3^* s_4^*}{s_1 s_2 s_3 s_4} \cdot \frac{(\Sigma s_i^2 + 2c_1c_2c_3c_4 + 2s_1s_2s_3s_4 + 2)}{(\Sigma s_i^{*2} + 2c_1^*c_2^*c_3^*c_4^* + 2s_1^*s_2^*s_3^*s_4^* + 2)}\right)^{1/8}. \quad (20)$$

This symmetry on duality will suggest correctness of (19). There are several reasons to believe the correctness of (19).

i) At the Curie point of a generalized square lattice

$$c_1c_2c_3c_4 + s_1s_2s_3s_4 + 1 = \Sigma s_i s_j,$$

or in terms of Gudermannian functions $\mathrm{gd}(2L) = \arctan(\mathrm{sh}\, 2L)$,

$$\mathrm{gd}(2L_1) + \mathrm{gd}(2L_2) + \mathrm{gd}(2L_3) + \mathrm{gd}(2L_4) = \pi,$$

the expression (19) tends to zero, and at absolute zero of temperature (19) tends

to unity.

ii) When $L_4 \to 0$, (19) tends to (14), the spontaneous magnetization of a honeycomb lattice.

iii) When $L_4 \to \infty$, (19) tends to (17), the spontaneous magnetization of a triangular lattice.

iv) When $L_2 = L_4$ and $L_1 = L_3$, (19) becomes (18), the spontaneous magnetization of anisotropic square lattice.

v) Expanding (19) in power series of $x_i = \exp(-2L_i)$, we have

$$\mathrm{I}_{g.s} = 1 - 2(x_1x_2x_3x_4) - 2\Sigma x_i^2 x_j^2 x_k^2 + 2(x_1x_2x_3x_4)^2 - 6(x_1x_2x_3x_4)\Sigma x_i^2 x_j^2 + 4(x_1x_2x_3x_4)\Sigma x_i^2 x_j^2 x_k^2 - 4\Sigma x_i^2 x_j^4 x_k^4 - 30(x_1x_2x_3x_4)^2 \Sigma x_i^2 + \mathrm{O}(x^{12}), \quad (21)$$

which coincides with the exact low temperature expansion.

§ 4. Susceptibility of one-dimensional lattice

On the susceptibility of a two-dimensional Ising lattice, there is no exact derivation. As stated in the Introduction, some of the transformations will be useful for the derivation of susceptibility, but, unless, at least, one exact solution is found for a standard Ising lattice, such transformation will hardly be used. Domb and Sykes,[8] basing on the exact high temperature expansion of the susceptibility, considered the critical behavior of the ferromagnetic susceptibility by extrapolation. And also Fisher,[9] basing upon the Onsager-Kaufman's work, concluded that ferromagnetic susceptibility has the singularity of the following type, in a critical region,

$$\chi \simeq \frac{N\mu^2}{kT} \frac{C}{(1 - T_c/T)^{7/4}}, \quad (22)$$

where C is a constant which depends on lattice type and k, T, N, μ have usual meanings. Domb and Sykes[8] obtained the numerical values of C, as 0.9684 for square lattice and 0.9295 for triangular lattice.

As an introduction, we want to summarize the results for one-dimensional Ising lattice which can be solved exactly. As is well known, the susceptibility of a linear chain with interaction parameter L_1 is, putting $v_1 \equiv \mathrm{th}\, L_1$,

$$\chi_l = \frac{N\mu^2}{kT} \frac{1 + v_1}{1 - v_1}. \quad (23)$$

Or substitution of $c_1 = \mathrm{ch}\, 2L_1$, $s_1 = \mathrm{sh}\, 2L_1$, gives

$$\chi_l = \frac{N\mu^2}{kT} \frac{1}{c_1 - s_1}. \quad (24)$$

The denominator of (24) is closely related to the expression for the partition function of a linear chain:

$$\log \lambda_l \sim \int_0^{\pi} \log (c_1 - s_1 \cos \omega) \, d\omega. \tag{25}$$

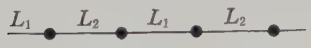


Fig. 2.

To assure the correspondence between (24) and (25), we calculate the susceptibility χ_l' for a linear chain with two interaction parameters L_1, L_2 alternatively.

$$\chi_l' = \frac{N\mu^2}{kT} \frac{c_1 + c_2 + s_1 + s_2}{c_1 c_2 + 1 - s_1 s_2}, \tag{26}$$

and the partition function is

$$\log \lambda_l' \sim \int_0^{\pi} \log (c_1 c_2 + 1 - s_1 s_2 \cos \omega) \, d\omega. \tag{27}$$

In this case also, there occurs a close relationship between the expressions (26) and (27). In the so-called "semi-ferromagnetic" case, in which the magnetic atom and non-magnetic atom alternate, we have

$$\chi_{l,semi}' = \frac{N\mu^2}{kT} \frac{c_1 + c_2}{c_1 c_2 + 1 - s_1 s_2}. \tag{28}$$

§ 5. A closed form related to the susceptibility of square and triangular lattices

Taking into consideration the relations (24), (25) and Onsager's[10] partition function,

$$\log \lambda_s \sim \int_0^{\pi} \int_0^{\pi} \log (c_1 c_2 - s_1 \cos \omega_1 - s_2 \cos \omega_2) \, d\omega_1 d\omega_2,$$

we can get a rough estimate of susceptibility of square lattice,

$$\frac{N\mu^2}{kT} \frac{1}{c_1 c_2 - s_1 - s_2}. \tag{29}$$

(The expression similar to this can be suggested from the combinatory consideration, for example, by Temperley.[11]) The expression (29) has the singularity at the critical temperature, $s_1 s_2 = 1$, but its character differs from (22). We modify (29) as

$$\chi_s = \frac{N\mu^2}{kT} \frac{(1 - s_1^2 s_2^2)^{1/4}}{c_1 c_2 - s_1 - s_2}. \tag{30}$$

Notice that the numerator of (30) is closely related to the long-range order of the dual lattice. We can rewrite (30) as

$$\chi_s = \frac{N\mu^2}{kT} \frac{(1 - v_1^2)(1 - v_2^2)}{(1 - v_1 - v_2 - v_1 v_2)^2} \left(1 - \frac{16 v_1^2 v_2^2}{(1 - v_1^2)^2 (1 - v_2^2)^2}\right)^{1/4}, \tag{31}$$

where

$$v_i=\tanh L_i,\ i=1,\ 2.$$

Naturally, if one of the parameters v_2 tends to zero, (31) tends to Eq. (23) for the one-dimensional lattice. Expanding (31) in power series of v_i, we have

$$\begin{aligned}\chi_s=&(N\mu^2/kT)\,\{1+2(v_1+v_2)+2(v_1^2+v_2^2)+8v_1v_2+2(v_1^3+v_2^3)+16(v_1^2v_2+v_1v_2^2)\\&+2(v_1^4+v_2^4)+24(v_1^3v_2+v_1v_2^3)+48v_1^2v_2^2+2(v_1^5+v_2^5)+32(v_1^4v_2+v_1v_2^4)\\&+104(v_1^3v_2^2+v_1^2v_2^3)+2(v_1^6+v_2^6)+40(v_1^5v_2+v_1v_2^5)+180(v_1^4v_2^2+v_1^2v_2^4)\\&+296v_1^3v_2^3+2(v_1^7+v_2^7)+48(v_1^6v_2+v_1v_2^6)+280(v_1^5v_2^2+v_1^2v_2^5)\\&+656(v_1^4v_2^3+v_1^3v_2^4)+O_s(v^8)\}.\end{aligned}\tag{32}$$

This coincides with the exact high temperature expansion up to the order of v^7. In the isotropic case where $v_1=v_2\equiv v$, we have

$$\begin{aligned}\chi_s&=\frac{N\mu^2}{kT}\,\frac{(1-v^2)^2}{(1-2v-v^2)^2}\left(1-\frac{16v^4}{(1-v^2)^4}\right)^{1/4}\\&=(N\mu^2/kT)\,(1+4v+12v^2+36v^3+100v^4+276v^5+740v^6+1972v^7+O_s(v^8)).\end{aligned}\tag{33}$$

In the critical region, near the Curie temperature, we have

$$\chi_s\simeq\frac{N\mu^2}{kT}\,\frac{C_s}{(1-T_c/T)^{7/4}},\qquad C_s=\frac{1}{8}\left(\frac{\sqrt{2}}{L_c}\right)^{7/4}\fallingdotseq 0.9617,\ \ (L_c=0.4407),\tag{34}$$

which has the singularity of the type (22), and the numerical value of C_s has good agreement with that given by Domb and Sykes[8] 0.9684.

For the triangular lattice, after the same reasoning, we have

$$\begin{aligned}\chi_t&=\frac{N\mu^2}{kT}\,\frac{[1-(\Sigma s_i^2s_j^2+2c_1c_2c_3s_1s_2s_3+2s_1^2s_2^2s_3^2)]^{1/4}}{c_1c_2c_3+s_1s_2s_3-s_1-s_2-s_3}\\&=\frac{N\mu^2}{kT}\,\frac{(1-v_1^2)^2(1-v_2^2)^2(1-v_3^2)^2}{(1-v_1-v_2-v_3-v_1v_2-v_2v_3-v_3v_1+v_1v_2v_3)^2}\\&\quad\times\left[1-16\,\frac{(1+v_1v_2v_3)(v_1+v_2v_3)(v_2+v_3v_1)(v_3+v_1v_2)}{(1-v_1^2)^2(1-v_2^2)^2(1-v_3^2)^2}\right]^{1/4}.\end{aligned}\tag{35}$$

The expression (35) is reduced to the susceptibility of square lattice (30) or (31), when one of the parameters tends to zero. Expanding (35) in power series of v_i, we have

$$\begin{aligned}\chi_t=&(N\mu^2/kT)\,\{1+2\Sigma v_i+2\Sigma v_i^2+8\Sigma v_iv_j+2\Sigma v_i^3+16\Sigma v_i^2v_j+36v_1v_2v_3\\&+2\Sigma v_i^4+24\Sigma v_i^3v_j+48\Sigma v_i^2v_j^2+104\Sigma v_i^2v_jv_k+2\Sigma v_i^5+32\Sigma v_i^4v_j\\&+104\Sigma v_i^3v_j^2+212\Sigma v_i^3v_jv_k+376\Sigma v_i^2v_j^2v_k+O_t(v^6)\},\end{aligned}\tag{36}$$

which coincides with the exact high temperature expansion up to the order of v^5. In the isotropic case, we have

$$\chi_t=\frac{N\mu^2}{kT}\frac{(1-v^2)^3}{(1-3v-3v^2+v^3)^2}\left(1-\frac{16v^3(1+v^3)}{(1-v)^2(1-v^2)^3}\right)^{1/4}$$
$$=(N\mu^2/kT)\,(1+6v+30v^2+138v^3+606v^4+2586v^5+O_t(v^6)). \quad (37)$$

At the neighborhood of the critical temperature, we have

$$\chi_t\simeq\frac{N\mu^2}{kT}\frac{C_t}{(1-T_c/T)^{7/4}},\quad C_t=\frac{1}{6\sqrt{3}\,L_c^{7/4}}\fallingdotseq 0.9235,\quad (L_c=0.2747). \quad (38)$$

The numerical value of C_t has good agreement with that given by Domb and Sykes[8] (0.9295).

§ 6. Extension to the honeycomb lattice

Fisher[3] derived a useful relation which connects the susceptibility of honeycomb lattice with that of triangular lattice,

$$\chi_t(T_1, T_2, T_3)=\chi_{h,semi}(H_1, H_2, H_3)=\frac{1}{2}\{\chi_h(H_1, H_2, H_3)+\chi_h(-H_1, -H_2, -H_3)\}, \quad (39)$$

where T_1, T_2, T_3 and H_1, H_2, H_3 are connected by Y-Δ transformation (2). The middle part of (39) is the susceptibility of semi-ferromagnetic honeycomb-lattice. From (35) and (39), we can get

$$\chi_{h,semi}=\frac{N\mu^2}{kT}\frac{(\Sigma s_\alpha^2+2c_1c_2c_3+2)^{1/2}}{c_1c_2c_3+1-\Sigma s_i s_j}\left(1-\frac{s_1^2s_2^2s_3^2}{\Sigma s_i^2+2c_1c_2c_3+2}\right)^{1/4}. \quad (40)$$

Extension to honeycomb lattice will be suggested by the case of linear chain. We have, comparing (40) with (26) and (28), the following expression:

$$\chi_h=\frac{N\mu^2}{kT}\frac{(\Sigma s_i^2+2c_1c_2c_3+2)^{1/2}+\Sigma s_i}{c_1c_2c_3+1-\Sigma s_i s_j}\left(1-\frac{s_1^2s_2^2s_3^2}{\Sigma s_i^2+2c_1c_2c_3+2}\right)^{1/4}$$
$$=\frac{N\mu^2}{kT}\Big[\Big\{(1+v_1v_2+v_2v_3+v_3v_1)\,(1+v_1v_2-v_2v_3-v_3v_1)\,(1-v_1v_2+v_2v_3-v_3v_1)$$
$$\times(1-v_1v_2-v_2v_3+v_3v_1)\Big\}^{1/2}+\sum v_i(1-v_j^2)(1-v_k^2)\Big]\Big/(1-v_1v_2-v_2v_3-v_3v_1)^2$$
$$\times\Big[1-16\cdot v_1^2\cdot v_2^2\cdot v_3^2\Big/\Big\{(1+v_1v_2+v_2v_3+v_3v_1)\,(1+v_1v_2-v_2v_3-v_3v_1)$$
$$\times(1-v_1v_2+v_2v_3-v_3v_1)\,(1-v_1v_2-v_2v_3+v_3v_1)\Big\}\Big]^{1/4}. \quad (41)$$

This satisfies the relation (39) and, when one of the parameters H_3 tends to zero, it is reduced to the susceptibility of a linear chain:

$$\chi_l' = \frac{N\mu^2}{kT} \frac{(s_1^2+s_2^2+2c_1c_2+2)^{1/2}+s_1+s_2}{c_1c_2+1-s_1s_2} = \frac{N\mu^2}{kT} \frac{c_1+c_2+s_1+s_2}{c_1c_2+1-s_1s_2}. \quad (42)$$

Moreover, when one of the interaction parameters H_3 tends to infinity, (41) is reduced to that of a square lattice (30) or (31). In fact, the expression (41) has a singularity of the type of (22) and has high temperature expansion:

$$\begin{aligned}\chi_h = (N\mu^2/kT)\,\{&1+\Sigma v_i+2\Sigma v_iv_j+\Sigma v_i^2v_j+6v_1v_2v_3+2\Sigma v_i^2v_j^2+6\Sigma v_i^2v_jv_k\\&+\Sigma v_i^3v_j^2+2\Sigma v_i^3v_jv_k+12\Sigma v_i^2v_j^2v_k+2\Sigma v_i^3v_j^3+10\Sigma v_i^3v_j^2v_k+24v_1^2v_2^2v_3^2\\&+\Sigma v_i^4v_j^3+3\Sigma v_i^4v_j^2v_k+18\Sigma v_i^3v_j^3v_k+30\Sigma v_i^3v_j^2v_k^2+2\Sigma v_i^4v_j^4+14\Sigma v_i^4v_j^3v_k\\&+22\Sigma v_i^4v_j^2v_k^2+54\Sigma v_i^3v_j^3v_k^2+\Sigma v_i^5v_j^4+4\Sigma v_i^5v_j^3v_k+6\Sigma v_i^5v_j^2v_k^2\\&+24\Sigma v_i^4v_j^4v_k+63\Sigma v_i^4v_j^3v_k^2+102v_1^3v_2^3v_3^3+2\Sigma v_i^5v_j^5+18\Sigma v_i^5v_j^4v_k\\&+40\Sigma v_i^5v_j^3v_k^2+96\Sigma v_i^4v_j^4v_k^2+152\Sigma v_i^4v_j^3v_k^3+O_h(v^{11})\}, \end{aligned} \quad (43)$$

which coincides with the exact one up to the order of v^{10}. Especially, in the isotropic case, we have

$$\begin{aligned}\chi_h &= \frac{N\mu^2}{kT} \frac{((1-v^2)^3(1+3v^2))^{1/2}+3v(1-v^2)^2}{(1-3v^2)^2}\left(1-\frac{16v^6}{(1-v^2)^3(1+3v^2)}\right)^{1/4}\\ &= (N\mu^2/kT)\,(1+3v+6v^2+12v^3+24v^4+48v^5+90v^6+168v^7\\ &\qquad +318v^8+600v^9+1098v^{10}+O_h(v^{11})). \end{aligned} \quad (44)$$

§ 7. Expression for generalized square lattice and consistency examination

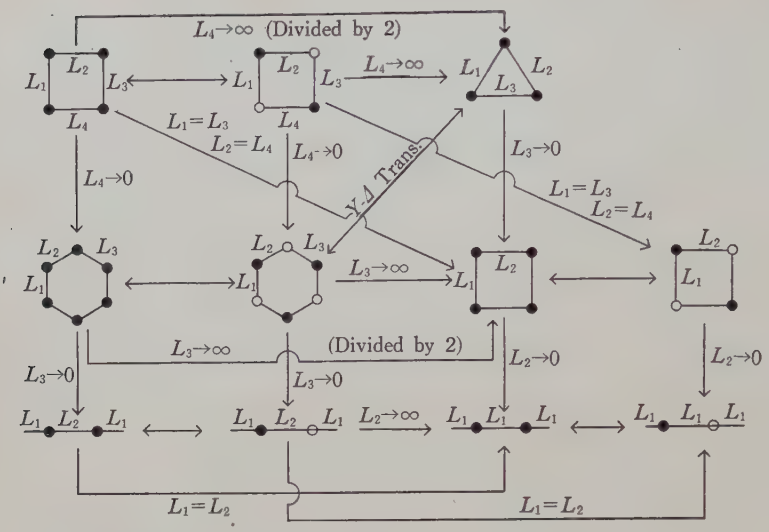
Extension to the generalized square lattice, which describe the properties for the square, honeycomb and triangular lattices, is now easy. We have

$$\begin{aligned}\chi_{g.s} = \frac{N\mu^2}{kT} &\frac{(\Sigma s_i^2+2c_1c_2c_3c_4+2s_1s_2s_3s_4+2)^{1/2}+\Sigma s_i}{c_1c_2c_3c_4+s_1s_2s_3s_4+1-\Sigma s_is_j}\\ &\times\left(1-\frac{\Sigma s_i^2s_j^2s_k^2+2s_1^2s_2^2s_3^2s_4^2+2s_1s_2s_3s_4(c_1c_2c_3c_4+1)}{\Sigma s_i^2+2c_1c_2c_3c_4+2s_1s_2s_3s_4+2}\right)^{1/4}. \end{aligned} \quad (45)$$

Of course, we have, for the semi-ferromagnetic generalized square lattice,

$$\begin{aligned}\chi_{g.s,semi} = \frac{N\mu^2}{kT} &\frac{(\Sigma s_i^2+2c_1c_2c_3c_4+2s_1s_2s_3s_4+2)^{1/2}}{c_1c_2c_3c_4+s_1s_2s_3s_4+1-\Sigma s_is_j}\\ &\times\left(1-\frac{\Sigma s_i^2s_j^2s_k^2+2s_1^2s_2^2s_3^2s_4^2+2s_1s_2s_3s_4(c_1c_2c_3c_4+1)}{\Sigma s_i^2+2c_1c_2c_3c_4+2s_1s_2s_3s_4+2}\right)^{1/4}. \end{aligned} \quad (46)$$

To show that these expressions are satisfactory in the scope of our consideration, we give a graphical representation of their consistency.



● indicates magnetic atom
○ indicates non-magnetic atom

Fig. 3. Graphical representation for the consistency examination of our expressions.

§ 8. Discussion on antiferromagnetic case

The formula mentioned above cannot be applied to the antiferromagnetic cases. As a trial to fit the expressions to the antiferromagnetic cases, we replace

the factor $(1-k^2)^{1/4}$ by $\frac{2}{\pi}\mathrm{E}(k)=\frac{2}{\pi}\int_0^{\pi/2}(1-k^2\sin^2\omega)^{1/2}\mathrm{d}\omega$, (47)

the complete elliptic integral of the second kind. For example, we have for the case of antiferromagnetic square lattice

$$\chi_s=\frac{N\mu^2}{kT}\frac{1}{c_1c_2+s_1+s_2}\cdot\frac{2}{\pi}\mathrm{E}(k),\quad k=s_1s_2. \tag{48}$$

These replacements of our formulae do not affect the range of coincidence for the high temperature expansions (32), (36) and (43) and the consistency examinations. At the Curie point, the replaced expressions remain finite, unlike (30), (41) and (45) which tend to zero for the antiferromagnetic cases. Furthermore, the expressions behave, in the critical region, as

$$\chi\simeq\frac{N\mu^2}{kT}\{\xi_c-D(T-T_c)\log|T-T_c|\}, \tag{49}$$

which can be shown by the well-known properties of elliptic functions,

$$\frac{\partial\mathrm{E}}{\partial k}=\frac{\mathrm{E}-\mathrm{K}}{k},\ \text{and for}\ k\to 1,$$

$$K(k) \simeq \log 4/|k'|, \quad k'^2 = 1 - k^2, \tag{50}$$

where K is the complete elliptic integral of the first kind. This type of critical behavior for the antiferromagnetic case has been conjectured by Fisher,[9] and Sykes and Fisher[12] and the critical value $\xi_c = 1/2\pi = 0.159$ obtained from (48) for the square lattice has also good agreement with that given by them ($kT\chi/N\mu^2 = 0.156 \sim 0.158$). For the honeycomb lattice, we have $\xi_c = 2/3\sqrt{3}\pi = 0.123$. On the other hand, for the triangular lattice which has no transition point, the expression decrease monotonically and has value $\chi_t = 0.339N\mu^2/kT$ at the temperature corresponding to its ferromagnetic Curie point.

However, these expressions diverge to infinity as $(1 - T_c/T)^{-2}$, not as $-7/4$ power, if applied to ferromagnetic cases. Below the critical temperature, we must change the expressions, about which we have no satisfactory results.

§ 9. Conclusion

There are no reasons to doubt the correctness of the expressions for the spontaneous magnetizations. However, it is obvious that the expressions for susceptibilities cannot be exact. In fact, the high temperature expansions diverge slightly as in Table I. Moreover, the discrepancy between ferromagnetic and antiferromagnetic expressions is troublesome. Although these expressions are not exact, these will be a milestone for the derivation of exact expressions for the susceptibilities of two-dimensional Ising lattices.

Table I

	Square Lattice	Triangular Lattice	Honeycomb Lattice
Exact Value	$5172v^8 + 13492v^9 + \cdots$	$10818v^6 + 44574v^7 + \cdots$	$2007v^{11} + 3696v^{12} + \cdots$
Ferro. Mag. Expr.	$5168v^8 + 13492v^9 + \cdots$	$10814v^6 + 44550v^7 + \cdots$	$2004v^{11} + 3692v^{12} + \cdots$
Antiferro. Mag. Expr.	$5180v^8 + 13540v^9 + \cdots$	$10826v^6 + 44694v^7 + \cdots$	$2004v^{11} + 3704v^{12} + \cdots$

References

1) C. N. Yang, Phys. Rev. **85** (1952), 808.
2) S. Naya, Prog. Theor. Phys. **11** (1954), 53.
3) M. E. Fisher, Phys. Rev. **113** (1959), 969.
4) M. E. Fisher, Proc. Roy. Soc. A **254** (1960), 66.
5) R. B. Potts, Phys. Rev. **88** (1952), 392.
6) R. M. F. Houtappel, Physica **16** (1950), 425.
7) T. Utiyama, Prog. Theor. Phys. **6** (1951), 907.
8) C. Domb and M. F. Sykes, Proc. Roy. Soc. A **240** (1957), 214.
9) M. E. Fisher, Physica **25** (1959), 521.
10) L. Onsager, Phys. Rev. **65** (1944), 117.
11) H. N. V. Temperley, Phys. Rev. **103** 1956, 1.
12) M. F. Sykes and M. E. Fisher, Phys. Rev. Letters **1** (1958), 321.

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P-Wave Pion-Pion Interaction

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P-Wave pion-pion resonance formula is derived by both the Chew-Mandelstam theory and the chain approximation. Adopting scattering length $a=1/5\cdot(\hbar/\mu c)^3$, we predict the resonance of 600 Mev and total width of 130 Mev, which is consistent with the results of Takeda et al. and Fulco and Frazer. The mean square radius of isovector part of magnetic moment of a nucleon is calculated by using the chain approximation.

§ 1. Introduction

The existence of pion-pion interaction has long been anticipated from the Feynman diagram occurring through the virtual nucleon-antinucleon closed loop. The experimental evidence for the pion-pion force has for the first time been suggested by Landau,[1] Takeda,[2] and Dyson[3] to explain high energy phenomena. Landau analysed jet shower by the meson fluid model which is based on pion-pion force and succeeded in the explanation of the angular spread and multiplicity of jet shower. Takeda and Dyson interpreted the second maximum in the pion-nucleon scattering cross section as the resonance scattering between incident pion and cloud pion around a nucleon. Ito[4] made also an extensive analysis of Kovacs type diagram and noticed the important role played by pion-pion force in the phenomena of Bev region.

The clearest evidence for this force seems to be the recent experiment[5] on the double pion production in the pion-nucleon collision at 250~450 Mev (lab). In this energy region the cross section for the reaction $\pi+N\rightarrow N+2\pi$ increases from 0.5 to 4.5 mb, which is considerably larger than that expected from the collision between an incident meson and a nucleon core. We should interpret this reaction as the collision between an incident pion and a cloud pion, since the pion cloud is extended up to the distance of meson Compton wavelength around a nucleon core. From the analysis of this experiment, Rodberg[6] obtained the scattering length for the S, P state. $a_0=5\lambda(\hbar/\mu c)$, $a_2=2\lambda(\hbar/\mu c)$, $\lambda=1/10$ and $a_1=1/5(\hbar/\mu c)^3$, where a_I=scattering length in isospin I state. Sato[7] has also obtained attractive scattering length $a_0=0.7\sim1.7$, $a_1=0.20\sim0.27$, using a new type of dispersion relation for π-N scattering. His value is consistent with Rodberg's. Thus the pion-pion force seems to be experimentally established.

The aim of this paper is to investigate whether the present field theory predicts the resonance in pion-pion scattering which seems to be very important in the interpretation of the second maximum of π-N scattering cross section. Takeda assumed the resonance in the $I=l=1$ state, while Dyson assumed the resonance in the S state. But the resonance in the S state seems to be very unlikely in the theory of Chew and Mandelstam[8] on the S-wave pion-pion scattering. On the other hand, the resonance in the $I=l=1$ state is very likely, because Igi-Kawarabayashi's[9] perturbation theoretical calculation and Sato's dispersion theory indicates the strong attractive force in this state. In § 2 we apply the Chew-Mandelstam* theory to P-wave pion-pion scattering. At the end of this section it is shown that the Chew-Mandelstam theory, which starts from a Hamiltonian $4\pi\lambda(\varphi_\alpha\varphi_\alpha)^2$, predicts too small P-wave scattering length compared to Rodberg's one. Instead, we start from $-2\pi a(\varphi_\alpha\partial_\mu\varphi_\beta-\varphi_\beta\partial_\mu\varphi_\alpha)^2$ and set up the C. M. equation for P-wave. This equation can be solved in the same way as the C. M. equation for S-wave. In this case we must introduce a cutoff A, twice as large as a nucleon mass, and get the resonance at about 600 Mev (c. m.) and resonance width 130 Mev, which is consistent with the one adopted in Fulco-Frazer's[10] theory on the isovector part of the magnetic moment of a nucleon, and also in Takeda, Itabashi, Kato, and Nakagawa's[11] new theory on the second and third maxima in the π-N scattering cross section. We use dispersion relation to estimate the contribution from negative energy integral which does not so much affect the location of resonance energy.

In § 3 it is shown that we can readily obtain the same result as in § 2 in the chain approximation of the Feynman diagram which has been used by Okubo[14] in the S-wave case. Our approximation in § 2 is found to be the solution of the Bethe-Salpeter equation (Fig. 1) in which the nucleon closed loops are replaced by a point interaction. If the nucleon closed loops were properly taken into account, we should get the finite result without cutoff in the P-wave problem. By the uncertainty principle, the virtual nucleon and antinucleon pair is spatially extended up to $\hbar c/2M$. Thus the integrand of virtual integral should begin to decrease at $2M$. This is the physical meaning of a cutoff A.

The advantage of the C. M. theory is to preserve unitarity and crossing symmetry, while in our chain approximation unitarity is lost by the exchange of incident and outgoing meson (the crossed diagram). The most reliable treatment of P-wave pion-pion scattering is to solve the coupled C. M. equations for $\pi+\pi\to\pi+\pi$ and $\pi+\pi\to N+\bar{N}$ which has been left untouched in this paper.

In § 4 we use our chain approximation to evaluate the mean square radius of the isovector part of anomalous magnetic moment of a nucleon. We can get in a very simple way the same result as that of Fulco and Frazer's dispersion theoretical method.[10]

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* We shall cite this paper as C. M.

§ 2. Chew-Mandelstam equation for pion-pion scattering

i) *Scattering length in the $I=l=1$ state*

We shall consider the pion-pion scattering $(q_1\alpha, q_2\beta)\to(q_3\gamma, q_4\delta)$, where q_i are the four-momenta of the pions, and α, β and γ, δ are charge states of the incoming and outgoing pions. We define the scattering amplitude in the center of mass system as

$$\langle q_3\gamma, q_4\delta|T|q_1\alpha, q_2\beta\rangle=-32\pi\sum(2l+1)P_l(\cos\theta)T^{I=1}_{l=1}\langle\gamma\delta|P_{I=1}|\alpha\beta\rangle, \quad (2\cdot1)^*$$

where

$$T^{I=1}_{l=1}=\sqrt{\frac{\nu+1}{\nu}}\sin\delta e^{i\delta} \qquad (2\cdot2)$$

and $\nu=q^2$ (q=magnitude of three-momentum). $P_{I=1}=(1/2)(\delta_{\alpha\gamma}\delta_{\beta\delta}-\delta_{\alpha\delta}\delta_{\beta\gamma})$ is the projection operator for $I=1$.

Scattering length in the $l=1$ state is defined by

$$\lim_{q\to0}\delta/q^3=a(\mu^{-3}), \quad \mu\text{: pion mass.}^{**}$$

Igi and Kawarabayashi obtained a by evaluating the Feynman diagram of nucleon closed loop.

$$a=\left(\frac{g^2}{4\pi}\right)^2\frac{2}{9\pi M^2}=0.35\mu^{-3}.$$

By the analysis of the data on double pion production, Rodberg[6] gives $a=0.2\mu^{-3}$.

Using the dispersion relation of a new type, Sato[7] gives $a=0.2\sim0.27\mu^{-3}$.

Table I. Sacttering length a.

Igi et al.	Sato	Rodberg	π-N $I=3/2$ $J=3/2$
0.35	0.2–0.27	0.2	0.1

ii) *Resonance Formula in the $I=l=1$ state*

It can be shown by Mandelstam[12] representation that $T^{I=1}_{l=1}$ has two branch cuts $\infty>\nu>0$ and $-\infty<\nu<-1$. Following Chew and Mandelstam, we put

$$T^{I=1}_{l=1}=\frac{N/\nu}{D}. \qquad (2\cdot3)$$

N is analytic except the branch cut $-\infty<\nu<-1$. D is analytic except the branch cut $\infty>\nu>0$. By assumption we have

---

* $S=1-i(2\pi)^4\delta(p_i-p_f)\frac{1}{\sqrt{(2\pi)^{12}2q_{10}2q_{20}2q_{30}2q_{40}}}\langle q_3\gamma_1, q_4\delta|T|q_1\alpha_1, q_2\beta\rangle$.

** We use the unit ($\mu=1$) hereafter.

$$\left.\begin{aligned}&\operatorname{Im} N/\nu = D \operatorname{Im} T/\nu\\&\operatorname{Im} D = 0\end{aligned}\right\} \quad -\infty < \nu < -1,$$

$$\operatorname{Im} N = \operatorname{Im} D = 0 \qquad -1 < \nu < 0,$$

$$\left.\begin{aligned}&\operatorname{Im} D = (N/\nu)\operatorname{Im}\nu/T = -N\sqrt{\frac{\nu}{1+\nu}}\\&\operatorname{Im} N = 0\end{aligned}\right\} \quad \infty > \nu > 0. \tag{2·4}$$

We normalize $\lim_{\nu\to 0} N(\nu)/\nu = a$, and $D(0)=1$. We write dispersion relations for N/ν and D using (2·4)

$$N/\nu = a + \frac{\nu}{\pi}\int_{-\infty}^{-1} d\nu' \frac{\operatorname{Im} T^{I=1}_{l=1}/\nu' D(\nu')}{\nu'(\nu'-\nu)},$$

$$D = 1 + \frac{\nu}{\pi}\int_{0}^{\infty}\left(-\sqrt{\frac{\nu'}{\nu'+1}}\right)\frac{N d\nu'}{\nu'(\nu'-\nu-i\epsilon)}. \tag{2·5}$$

Neglecting the contribution from the negative cut, we have

$$N/\nu \doteqdot a,$$

$$D = 1 - (\nu/\pi)\int_{0}^{\infty}\frac{d\nu'\sqrt{\frac{\nu'^3}{1+\nu'}}}{\nu'(\nu'-\nu-i\epsilon)}a. \tag{2·5′}$$

With the help of $\cot\delta = -\operatorname{Im} D/\operatorname{Re} D$, our resonance formula can be obtained:

$$\sqrt{\frac{\nu^3}{\nu+1}}\cot\delta = a^{-1}\left[1 - \frac{2a\nu}{\pi}\left\{\log 2A - \sqrt{\frac{\nu}{1+\nu}}\ln(\sqrt{\nu}+\sqrt{1+\nu})\right\}\right]. \tag{2·6}$$

From zero of the right hand side of (2·6), we know the resonance level ν_r, which is related with the scattering length a as follows:

$$a = (\pi/2\nu_r)\left\{\log 2A - \sqrt{\frac{\nu_r}{1+\nu_r}}\log(\sqrt{\nu_r}+\sqrt{1+\nu_r})\right\}^{-1}. \tag{2·7}$$

Comparing (2·6) with one level formula, $\cot\delta = \frac{\sqrt{\nu_r+1}-\sqrt{\nu+1}}{\Gamma/2} \doteqdot \frac{\nu_r-\nu}{\Gamma\sqrt{\nu_r+1}}$, we obtain the total width of resonance level,

$$\Gamma = \frac{\nu_r^{5/2}a}{1+\nu_r}. \tag{2·8}$$

We introduce a cutoff A to make the integral in (2·5′) convergent. If we take into account $\pi+\pi\to N+\bar{N}$, we may get a convergent result without any cutoff factor. The physical meaning of A will be clarified in the next section. In Table II, level width and scattering length are listed with resonance level as a variable.

Table II. Resonance level ν_r, level width Γ, and scattering length a. cut off $A=M$ $(2M)$. M: nucleon mass.

ν_r	$\sqrt{\nu_r+1}$ (c.m.)	a	Γ
2	480 Mev	0.47 (0.33)	120 (86) Mev
3	560 Mev	0.35 (0.24)	168 (130) Mev

For cutoff $A=2M$ and scattering length $a=0.24$ (consistent with Rodberg and Sato), there is a resonance at 560 Mev, which is consistent with the value adopted by Takeda et al.[11] in his recent theory for the explanation of the second and third maxima of π-N elastic scattering.

iii) *Estimate of the contribution from negative cut*

We write the dispersion relation to estimate the contribution from negative energy cut:

$$\mathrm{Re}\,T/\nu=a+\frac{\nu}{\pi}\int_0^{\infty}d\nu'\frac{\mathrm{Im}\,T/\nu'}{\nu'(\nu'-\nu)}+\frac{\nu}{\pi}\int_{-\infty}^{-1}d\nu'\frac{\mathrm{Im}\,T/\nu'}{\nu'(\nu'-\nu)}. \tag{2·9}$$

By crossing symmetry, Im T for $\nu<-1$ can be represented by Im T for $\nu>0$:

$$\underset{\nu<-1}{\mathrm{Im}\,T^{I=1}_{l=1}}(\nu)=\frac{1}{\nu}\int_0^{-\nu-1}d\nu'\left(1+2\frac{\nu'+1}{\nu}\right)\alpha_{11}3\left(1+\frac{\nu+1}{\nu'}\right)\mathrm{Im}\,T^{I=1}_{l=1}(\nu'). \tag{2·10}*$$

Inserting (2·10) into (2·9), the third term of (2·9) can be simplified as

$$\frac{1}{\pi}\int_0^{\infty}h(\nu,\nu')\,\mathrm{Im}\,T(\nu')\,d\nu', \tag{2·11}$$

where

$$h(\nu,\nu')=-3\left\{\frac{\nu'+2}{6(1+\nu')^2\nu'}+\frac{2}{\nu^2\nu'}(2\nu+\nu'+2)\right.$$
$$\left.+\ln\frac{1+\nu'}{1+\nu'+\nu}\left[(5+6/\nu')1/\nu^2+2/\nu'\nu+2\frac{(\nu'+1)(\nu'+2)}{\nu'\nu^3}\right]\right\}.$$

(2·11) is roughly $h(\nu,\nu_r)\Gamma(1+\nu_r)/\nu_r^{1/2}$, which is almost equal to $\sqrt{(1+\nu_r)/\nu_r}^3\times\sin\delta\cos\delta$ at resonance, since the sum of first and second terms vanishes in our approximation. δ is shifted by only 6° near resonance ($\nu=3$) due to the contribution from negative cut.

iv) *S-wave pion-pion interaction*

Two S-wave scattering lengths for $I=0$ and 2 are defined by a Hamiltonian $4\pi\lambda(\varphi_\alpha\varphi_\alpha)^2$ as follows:

* Refer to the formula (V.8) of C. M.

$$a_0=-5\lambda, \text{ and } a_2=-2\lambda. \qquad (2\cdot12)$$

Several authors adopted various values of λ, which are listed in Table III.

Table III. coupling constant λ.

Rodberg	Sato	Miyazawa et al.[13]	Sakuma[19]
0.1	$-0.14\sim-0.34$	-0.34	1.1

S-wave phase shift is obtained by Chew and Mandelstam in the same way as in (ii):

$$\sqrt{\frac{\nu}{1+\nu}}\cot\delta_I=1/a_I+\frac{2}{\pi}\sqrt{\frac{\nu}{1+\nu}}\ln(\sqrt{\nu}+\sqrt{1+\nu}). \qquad (2\cdot13)$$

No resonance can be expected in this formula. Finally we must point out that a Hamiltonian of $4\pi\lambda(\varphi_\alpha\varphi_\alpha)^2$ predicts too small *P*-wave scattering length. We write a dispersion relation to estimate *P*-wave scattering length a_1 in terms of a_0 and a_2:

$$T^{I=1}_{l=1}=\frac{\nu}{\pi}\int_0^\infty\frac{d\nu'\,\mathrm{Im}\,T^{I=1}_{l=1}}{(\nu'-\nu)\nu'}+\frac{\nu}{\pi}\int_{-\infty}^{-1}\frac{d\nu'\,\mathrm{Im}\,T^{I=1}_{l=1}}{\nu'(\nu'-\nu)}. \qquad (2\cdot14)$$

In the limit $\nu\to0$, we get the scattering length a_1 as follows:

$$a_1=\frac{1}{\pi}\int_{-\infty}^{-1}d\nu'\frac{\mathrm{Im}\,T^{I=1}_{l=1}}{\nu'^2} \qquad (2\cdot15)$$

Im $T^{I=1}_{l=1}$ on negative cut is represented by Im $T^{I=0,2}_{l=0}$ on positive cut

$$\underset{\nu<-1}{\mathrm{Im}\,T^{I=1}_{l=1}}=\frac{1}{\nu}\int_0^{-\nu-1}d\nu'\left\{1+2\frac{\nu'+1}{\nu}\right\}\{\alpha_{10}\,\mathrm{Im}\,T^{I=0}_{l=0}+\alpha_{12}\,\mathrm{Im}\,T^{I=0}_{l=0}\}, \qquad (2\cdot16)^*$$

where Im $T^{I=0}_{l=0}$ is approximated by effective range theory. Then we have

$$a_1=\frac{1}{6\pi}\int_0^\infty\frac{d\nu'}{(1+\nu')^2}\{\alpha_{10}\,\mathrm{Im}\,T^{I=0}_{l=0}+\alpha_{12}\,\mathrm{Im}\,T^{I=0}_{l=0}\}$$

$$=\frac{1}{3\pi}\left\{\frac{2}{3}\left(1-\frac{1}{a_0}\tan^{-1}a_0\right)-\frac{5}{3}\left(1-\frac{1}{a_2}\tan^{-1}a_2\right)\right\}=3\times10^{-3}\quad\left(\lambda=\frac{1}{10}\right). \qquad (2\cdot17)$$

For $\lambda=1/10$, a_1 is quite small compared to Rodberg's value. This leads us to the conclusion that we should treat *S*- and *P*-wave separately and solve *P*-wave pion-pion interaction simultaneously with $\pi+\pi\to N+\bar{N}$, since the latter process

* Refer to formula (V. 8) of C. M.

gives a fairly good scattering length even in the perturbation theory as shown by Igi and Kawarabayashi.

§ 3. Chain approximation

The physical meaning of the approximation in the foregoing section will be made clear by the method of chain approximation which has been adopted by Okubo[14] in S-wave pion-pion scattering.

In Fig. 2 we replace a nucleon closed loop in the Bethe-Salpeter equation in Fig. 1 by a point interaction, and take the Hamiltonian $-2\pi a(\varphi_\alpha \partial_\mu \varphi_\beta - \varphi_\beta \partial_\mu \varphi_\alpha{}^2)$. Then we can define the potential, where two incoming pions with four-momenta q_1, q_2 and isospins α, β are scattered into a final state with four-momenta q_3, q_4 and isospins γ, δ, as

Fig. 1. Diagram for Bethe-Salpeter equation of pion-pion scattering. Straight line and dotted line represent nucleon and pion respectively.

Fig. 2. Diagram for chain approximation of pion-pion scattering.

$$\langle q_3\gamma,\ q_4\delta|V|q_1\alpha, q_2\beta\rangle = -24\pi a(q_1-q_2,\ q_3-q_4)\langle\gamma\delta|P_1|\alpha\beta\rangle. \tag{3·1}$$

In the chain approximation, the transition matrix is

$$\langle q_3\gamma,\ q_4\delta|T|q_1\alpha,\ q_2\beta\rangle = 12a\pi i(q_3-q_4)_i\langle 0|(\varphi_\alpha\partial_i\varphi_\beta-\varphi_\beta\partial_i\varphi_\alpha)_0|\alpha,\beta\rangle \tag{3·2}$$

which satisfies the following equation

$$\begin{aligned}\langle q_3\gamma,\ q_4\delta|T|q_1\alpha,\ q_2\beta\rangle &= \langle q_3\gamma,\ q_4\delta|V|q_1\alpha,\ q_2\beta\rangle \\ &\quad + i(q_3-q_4)_s X_{st}\langle 0|(\varphi_\alpha\partial_t\varphi_\beta-\varphi_\beta\partial_t\varphi_\alpha)_0|\alpha\beta\rangle,\end{aligned} \tag{3·3}$$

where

$$X_{st}=24\pi ai\int dx'\left(\Delta_F(-X')\partial_s'\partial_i\Delta_F(X')-\partial_s'\Delta_F(-X')\partial_i\Delta_F(X')\right)e^{i(q_1+q_2,X')}, \tag{3·4}*$$

$$X=\frac{-16\pi ai}{(2\pi)^4}\int\frac{d^4q\,d^4q'}{(q^2+\mu^2)(q'^2+\mu^2)}\boldsymbol{q}^2\delta^4(q+q'-q_1-q_2)$$

$$=\frac{2}{\pi}\int_0^1 du\int_0^\Lambda\frac{\boldsymbol{q}^4\,d\boldsymbol{q}}{(\boldsymbol{q}^2+\alpha^2)^{5/2}},\quad \alpha^2=1+(q_1+q_2)^2u(1-u). \tag{3·4}'$$

(3·3) can be easily solved as

$$\langle q_3\gamma,\ q_4\delta|T|q_1\alpha,\ q_2\beta\rangle = -24\pi a(q_1-q_2,\ q_3-q_4)\langle\gamma\delta|P_1|\alpha\beta\rangle/(1-X). \tag{3·5}$$

---

* X_{st}=comes from the contraction of $(\varphi_\alpha\partial_i\varphi_\beta-\varphi_\beta\partial_i\varphi_\alpha)H$. In the barycentric system $X_{st}=X\delta_{st}$ which facilitates the evaluation.

After the quadratic divergence in X is renormalized in a, we get the same result as in (2·6).

In the original Bethe-Salpeter equation there is no divergence in the P-wave scattering, and no need of renormalization. If the effect of nucleon closed loop is properly taken into account, there appears a natural cutoff at $q=2M$ in the integrand of X. In the nucleon closed loop the uncertainty in energy is $\Delta E=2M$, the uncertainty in time is $\Delta t=1/2M$, and the uncertainty in spatial extention is $c/2M$. Then the range of pion-pion potential is roughly $1/2M$. Therefore in the original B. S. equation, the integrand begins to decrease at about $1/2M$. In our chain approximation where the nucleon closed loop is replaced by a point interaction, we should introduce a cutoff at $2M$ to get a rough agreement with the solution of B. S. equation.

In our approximation at low energy we might include the effect of inelastic pion-pion scattering in the scattering length "a" by taking into account a correction due to annihilation of virtual nucleon-antinucleon pairs.

Finally, we mention the crossed diagram in our chain approximation, which corresponds to the contribution from negative cut in the Chew-Mandelstam theory. In the crossed diagram, the unitarity is lost, while in the C. M. theory, the unitarity analyticity and crossing symmetry is preserved, though the latter symmetry is taken into account only by iteration.

§ 4. Mean square radius of the isovector part of magnetic moment of a nucleon

There is a wide discrepancy in the perturbation theoretical value $0.16\cdot10^{-26}$ cm^2[15]* and the experimental value $0.64\cdot10^{-26}$ cm^2 for the mean square radius of the isovector part of magnetic moment of a nucleon. Fulco and Frazer[10] introduce a resonance at total center-of-mass energy 600 Mev in $I=l=1$ pion-pion interaction to bring about the agreement with experiment. They employed a rather complicated dispersion theoretical method. We recalculate this problem in our chain approximation to get a more physical insight.

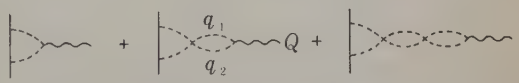

Fig. 3. Diagram for isovector magnetic moment in the chain approximation. The wavy line represents a virtual photon.

The Feynman diagrams in the chain approximation are given in Fig. 3. The first diagram corresponds to the first term of perturbation expansion:

$$g\frac{e\hbar}{2M}\tau_3. \qquad (4\cdot1)$$

* The authors of ref. 15) obtain this value by making a cutoff at $2M$ in the virtual integral, and by including π-N rescattering correction.

Our chain approximation gives

$$g\frac{e\hbar}{2M}\tau_3(1+X+X^2+\cdots),$$

which can be easily summed up:

$$g\frac{e\hbar}{2M}\tau_3\frac{1}{1-X}. \tag{4·2}$$

After the renormalization of quadratic divergence in charge e and scattering length a, $1/(1-X)$ can be replaced by

$$1/D(\nu)=\frac{1}{1-\frac{2a\nu}{\pi}\left[\log 2A-\sqrt{\frac{\nu}{1+\nu}}\log(\sqrt{\nu}+\sqrt{1+\nu})\right]} \tag{4·3}$$

where $D(\nu)$ is given in (2·6), and $\nu=-(q_1+q_2)^2/4-1$, where q_1 and q_2 are the four-momenta of virtual mesons. But in our problem, we should put

$$\nu=-Q^2/4-1,$$

where Q is the four-momenta of a virtual photon. The form factor $F(Q^2)$ is

$$F(Q^2)=D(-1)/D(-1-Q^2/4).$$

The mean square radius of isovector magnetic moment is given by

$$\langle r^2\rangle=\frac{3a}{\pi}(\log 2A-4/3)\left\{1+\frac{2a}{\pi}(\log 2A-1)\right\}^{-1}. \tag{4·4}$$

Table IV.* $\langle r^2\rangle$, m.s.r. of isovector magentic moment.

ν_r	$\langle r^2\rangle$
2	0.38 (0.43)
3	0.30 (0.33)

As shown in Table IV, the position of resonance seems to be about 500 Mev, to give agreement with experimental value.

§ 5. Summary

As shown in Table I, pion-pion force is twice as strong as pion-nucleon force in the $I=J=3/2$ state, and plays an important role in pion physics. We summarize the pheonomena which can be considered as related to pion-pion force.

i) *S-wave π-N scattering.* By including pion-pion force, Miyazawa and Kawarabayashi[13] point out the possibility of the reduction of a large isotopic independ-

* The value with (without) parenthesis corresponds to a cutoff $A=M$ $(2M)$.

ent term, to get an agreement with experiment on S-wave scattering length, but it seems very difficult to disentangle the effect of pion-pion force from that of a nucleon core. It is rather preferable to test pion-pion force in the D-wave π-N scattering, since pion-pion interaction takes place at a larger distance from a nucleon core.

ii) *Electromagnetic form factor.* As explained in § 4, Fulco and Frazer introduce a resonance in pion-pion scattering to bring about the agreement with m.s.r. of the isovector part of a magnetic moment. Takeda, Kato and Itabashi[16] also show the possibility of the enhancement of the isoscalar part by this resonance. But we cannot overlook the possibility that a large contribution to the form factor of a nucleon comes from cloud of mesons other than pion, such as π_0' meson of the Sakata model[17] and ρ meson of Nambu.[18]

iii) *Bev energy phenomena.* It is rather startling that Takeda et al.[11] show the possibility of the explanation of the 2nd and the 3rd maxima of π-N scattering by the potential due to the intermediary of $l=I=1$ resonance in pion-pion scattering. It will be an important problem to settle whether the second maximum is correctly described by the Takeda model, or by a more fundamental idea, such as the composite model* of Sakata, must be required. Pion-pion force will play a role in the angular correlation and multiplicity of mesons in the N-$\bar{N}$ annihilation. The maintenance of constant total cross section of π-N scattering indicates the diffraction scattering of pion by a cloud pion, because of strong pion-pion force. We must investigate wide angle π-N scattering to get some information on the internal structure of a nucleon core.

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* Matumoto, Sawada and Yonezawa[17] have discovered the mass formula of elementary particles which also predicts all the observed resonance levels in π-N and K-N scattering.

References

1) L. D. Landau, Izv. Akad. Nauk S. S. S. R. **17** (1953), 51.
2) G. Takeda, Phys. Rev. **100** (1955), 440.
3) F. J. Dyson, Phys. Rev. **99** (1955), 1037.
4) D. Ito, T. Kobayashi, T. Yamazaki and S. Minami, Prog. Theor. Phys. **18** (1957), 264.
5) W. A. Perkins et al., Phys. Rev. Letters **3** (1959), 56.
6) L. S. Rodberg, Phys. Rev. Letters **3** (1959), 58.
7) I. Sato, A. Takahashi and Y. Ueda, Prog. Theor. Phys. **22** (1957), 617.
8) G. F. Chew and S. Mandelstam, UCRL-8728. We cite this paper as C. M.; Phys. Rev. **119** (1960), 467.
9) K. Igi and K. Kwarabayashi, Prog. Theor. Phys. **20** (1958), 576.
10) J. R. Fulco and W. R. Frazer, Phys. Rev. Letters **2** (1959), 365; Phys. Rev. **117**(1960), 1609.
11) K. Itabashi, M. Kato, K. Nakagawa, and G. Takeda, Prog. Theor. Phys. **24** (1960), 529.
12) S. Mandelstam, Phys. Rev. **112** (1958), 1344.
13) H. Miyazawa and K. Kawarabayashi, Prog. Theor. Phys. **23** (1960), 490.
14) S. Okubo, Phys. Rev. **118** (1960), 357.
15) P. Federbush, M. L. Goldberger and S. B. Treiman, Phys. Rev. **112** (1958), 642.
16) G. Takeda, M. Kato, K. Itabashi, Lecture at Annual Meeting of Physical Society (1960).
17) S. Sakata, Prog. Theor. Phys. **16** (1956), 686.
K. Matumoto, Prog. Theor. Phys. **16** (1956), 583.
M. Ikeda, S. Ogawa, and Y. Ohnuki Prog. Theor. Phys. **22** (1959), 715.
S. Sawada and M. Yonezawa, Prog. Theor. Phys. **23** (1960), 662.
18) Y. Nambu, Phys. Rev. **106** (1957), 1366.
19) T. Sakuma, preprint. He determines λ by Chew-Low method(Phys. Rev. **113** (1959), 1640).

Note added in proof : As for the quadratic divergence in § 4, there is no quadratic divergence in the last loop which is connected with a photon line, since this divergence can be shown to vanish by reason of gauge invariance when the contribution from $16\pi ae(\varphi\partial_\mu\varphi^*-\varphi^*\partial_\mu\varphi)A_\mu\varphi^*\varphi$ is added. The remaining quadratic divergence is renormalized in a, as stated in §4. In our chain approximation the magnitude of magnetic moment is not changed by pion-pion scattering correction.

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The Meissner Effect in the Quasi-Chemical Equilibrium Theory of Superconductivity

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Based on the quasi-chemical equilibrium theory of superconductivity, we give an explicitly gauge-invariant proof of the Meissner-Ochsenfeld effect (magnetic field expulsion from a superconductor). The essential condition for a Meissner effect is the Bose-Einstein condensation of electron pairs. Other aspects of the problem, such as the energy gap and plasma waves, are not necessary for the Meissner effect and do not appear in our calculation. We do not make use of the random phase approximation.

§ 1. Introduction

In spite of the considerable progress which has been made in recent years in the theory of superconductivity, the theory of the electromagnetic effects in a superconductor is still not in a completely satisfactory state. There are two main electromagnetic effects:

(1) The expulsion of a magnetic field from all but a thin surface layer of the material, known as the Meissner-Ochsenfeld effect, and

(2) The existence of extremely long-lived "persistent currents".

This paper is devoted to the first of these, and aims to give a proof of the field expulsion free from the difficulties of alternative proofs which have appeared in the literature. Furthermore, our method of calculation allows us to determine just what feature of the theory of superconductivity is essential for the occurrence of a Meissner effect.

Long before there was any detailed microscopic theory of superconductivity, there were suggestions regarding the origin of the Meissner effect. The first London equation[1]

$$\operatorname{curl}(\lambda \boldsymbol{j}_s) = -\boldsymbol{B}/c \tag{1·1}$$

is often written in an alternative form by introducing the vector potential $\boldsymbol{A}$ through

$$\boldsymbol{B} = \operatorname{curl} \boldsymbol{A} \tag{1·2}$$

and "cancelling the curls" in (1·1) to get

$$\boldsymbol{j}_s = -\frac{1}{\lambda c}\boldsymbol{A}. \tag{1·3}$$

This equation depends on the specification of a particular gauge for $\boldsymbol{A}$, namely the London gauge

$$\operatorname{div} \boldsymbol{A}=0. \tag{1·4}$$

In any other gauge, (1·3) fails to hold.

One suggestion concerning the origin of the Meissner effect was made by London[1] and has been elaborated since by Bardeen and others[2]: the idea of "stiff" wave functions. The expression for the expectation value of the current operator in the presence of a vector potential $\boldsymbol{A}$ is, for a single particle

$$\boldsymbol{j}(\boldsymbol{r})=-\frac{e^2}{mc}\boldsymbol{A}(\boldsymbol{r})|\psi(\boldsymbol{r})|^2-\frac{i\hbar}{2m}(\psi^*\operatorname{grad}\psi-\psi\operatorname{grad}\psi^*). \tag{1·5}$$

Both terms vanish in the absence of a vector potential, the first because it is proportional to $\boldsymbol{A}$, the second because the unperturbed wave function ψ_0 cannot lead to a current. When the field is switched on, both terms make contributions. In normal materials, these contributions very largely cancel, so that $\boldsymbol{j}$ is proportional, not to $\boldsymbol{A}$, but to curl curl $\boldsymbol{A}$ (i.e. $\boldsymbol{j}=c\operatorname{curl}\boldsymbol{M}$, and the magnetization density $\boldsymbol{M}$ is proportional to $\boldsymbol{B}=\operatorname{curl}\boldsymbol{A}$). London suggested that, in a superconductor, the wave function may be "stiff", in the sense that ψ in (1·5) is much more similar to the unperturbed ψ_0 than in a normal material. Since ψ_0 gives zero for the second term in (1·5), such a stiff wave function would lead to a small result for the second term, not enough to cancel the first term; thus we would arrive at Eq. (1·3). The extension of this argument to a many-particle system is straightforward. One then arrives at a numerical estimate for the material constant λ in the London equation, namely

$$\frac{1}{\lambda c}=\frac{Ne^2}{Vmc}, \tag{1·6}$$

where N is the number of "superconducting electrons", e and m their charge and mass, and V the volume of the specimen (N/V is the number density). This estimate is in reasonable agreement with measurements on superconductors, if we use the ordinary electronic mass and charge for m and e, and a number N of the same order of magnitude as the number of conduction electrons in the metal.

However, Eq. (1·3) holds only in *one* specific gauge for A, namely (1·4). If the wave function ψ is "stiff" in any one gauge, it must *ipso facto* be "flexible" in all other gauges. The argument as given here is at least seriously incomplete, since there appears to be no special reason for preferring the London gauge to any other gauge. The vector potential, and with its gauge, is not a physical quantity at all, and all physical quantities must be independent of the gauge used (i.e. must depend only on the actual field $\boldsymbol{B}$, not on the vector potential $\boldsymbol{A}$).

When evidence started to accumulate for an "energy gap" in superconductors, it was suggested that this energy gap was responsible for producing stiffness in

the wave function.[2] This suggestion was disputed by Buckingham,[3] who called attention to the existence of exact identities inconsistent with this point of view. Bardeen[4] countered Buckingham's criticism by calling attention to the role of the "collective oscillations" in metals; according to Bardeen, the collective oscillations play an essential role in preserving the gauge invariance of the theory.

Thus, one strand of argument leading to a Meissner effect can be summarized as: energy gap plus collective effects.

An entirely different approach to an understanding of the Meissner effect is based on the fact[5),6),7] that the ideal Bose-Einstein gas below its condensation point exhibits a Meissner effect. This Meissner effect is obtained from a straightforward and completely gauge-invariant calculation, and is a consequence of the fact that a macroscopic number of particles occupy the same single-particle quantum state. This fact makes the many-particle wave function Ψ both "stiff" and "flexible" at the same time: It is flexible because the single-particle wave function has the full gauge-flexibility; it is "stiff" because of the (gauge-invariant) restriction that a macroscopic number of particles must occupy this one state. It was then suggested[8),9] that superconductivity in metals can be understood as a Bose-Einstein condensation of correlated electron pairs.

Whereas a theory of superconductivity can (and present theories do) lead to *both* an energy gap and the existence of correlated pairs of electrons, it must be realized that these two lines of argument are quite distinct. The energy-gap argument is based on a stiffness of the single-particle (or perhaps pair) wave function; the energy gap is considered essential, and collective excitations must be brought in to preserve gauge invariance. In the other line of argument, both the energy gap and the collective excitations are considered irrelevant, and the essential thing is the Bose-Einstein condensation. No attempt is made to keep the pair wave function "stiff"; the "stiffness" arises as a many-particle effect entirely, and is of statistical origin (Bose-Einstein condensation), *not* of dynamical origin (energy gap). It has been shown[10] that the Bose-Einstein condensation of correlated pairs does not depend on the existence of an energy gap, but takes place even if the first excited state has an excitation energy of the order of $\hbar^2/mL^2$, L being the linear dimension of the container.

The energy gap line of argument for the Meissner effect is inherently weak. We know exceptions in both directions: Systems with energy gap and no Meissner effect (all normal insulators), and systems with a Meissner effect and no energy gap (the ideal Bose-Einstein gas). Nevertheless, in the absence of any detailed calculation based entirely on the Bose-Einstein condensation point of view, the energy gap argument has found many adherents. The present paper supplies this missing detailed calculation, and shows that the Bose-Einstein condensation by itself (without any further assumptions) suffices to yield a gauge-invariant Meissner effect. Our calculation makes no use of the energy gap, nor of the collective excitations. In fact, at no time do we use an expansion in terms of

excited states of the system. Gauge invariance is complete to all orders (not merely for linear terms).

We emphasize that we do not deny the existence of an energy gap in superconductors, nor of plasma waves. Rather, we deny that they have any importance for an understanding of the Meissner effect.

Since we are interested primarily in the fundamental question of what causes the Meissner effect, we have simplified the calculation by assuming a rather schematic interaction between electrons in a metal, namely just an ordinary and non-singular potential $V(r-r')$. We do not, however, restrict ourselves to any "reduced" interaction, i.e. we do not throw away the vast majority of matrix elements before we start.[11] Furthermore, our calculation does not make any assumptions concerning the details of the potential function $V(r-r')$; all that we require is that it be sufficiently attractive to lead to a solution of Bose-Einstein condensation type.

There is a serious question whether such a solution, even if established formally, is really consistent with an ordinary potential.[12] The true interaction producing superconductivity is of course the Fröhlich interaction, which is quite different from an ordinary potential.[13] We prefer to leave that question open, since in any case it appears to be possible to extend the methods used in the present paper so as to allow explicit inclusion of the Fröhlich interaction. The calculation is in progress and will be reported later.

There are a large number of calculations in the literature concerning the Meissner effect, and we cannot discuss them in detail here. The early calculations[11),14),15)] were not gauge invariant. Wentzel's[16)] calculation was gauge-invariant, but he used an expansion in powers of the electron-phonon coupling constant. His result disagrees with the result of all the other calculations. A variational approach suggested by Blatt, Matsubara, and May[17)] was used in an explicit calculation by May and Schafroth.[18)] This calculation has been criticized by Wentzel[19)] on the ground that the calculation exhibits gauge invariance only in the limit of infinite order, and this limit is highly singular. Calculations using collective excitations explicitly were made by Anderson,[20)] Rickayzen,[20a)] and by Blatt and Matsubara.[21)] The latter authors believed that the collective excitations in question are characteristic of the superconducting state, and represent "moving bosons". This has not been borne out by later calculations.[22)] While Anderson's calculation is fully gauge-invariant, it makes use of the random phase approximation; the use of this approximation in fundamental questions in the theory of superconductivity has been criticized severely by Wentzel,[19)] on the ground that it represents an additional assumption of the theory, an assumption whose physical meaning is not obvious. Bogoliubov[23)] has extended his original theory so as to permit a fully gauge-invariant calculation. We believe that Bogoliubov's new method is closely related to ours, although we have not established any equivalence formally. However, Bogoliubov's calculation also was criticized by Wentzel[19)] for the use of the random phase approximation in later stages of the calculation. Since no random phase approximation is used in the present work, it is likely that Bogoliubov's calculation could be modified, so as to avoid use of this approximation; but this has not yet been done. Nambu[24)] has applied methods taken from quantum field theory to the discussion of the Meissner effect. As he remarks, "although the proof (of gauge invariance) is thus rigorous, it is still somewhat disturbing since $K^{(1a)}$, $K^{(1b)}$, and $K^{(2)}$ are all infinite." The cancellation of infinite terms against each other may be a necessary evil in quantum field theory, in its present form. In the theory of the solid state, however, it should be possible to make calculations in which everything stays finite at all stages. The most recent work of which we are aware is by Kadanoff and Mar-

tin.[24a] Their basic approach is to break off a series of equations for Green's functions at the Green's function for motion of a pair of electrons. This is of course identical with the old quasi-chemical equilibrium picture,[8),9)] expressed in a different formalism. We feel that their paper is not altogether conclusive, for two reasons: (i) Terms are discarded at various stages of the calculation, so that it is not altogether apparent what the overall approximation is; (ii) more important, the paper contains no proof of the Meissner effect, but rather a discussion of the electromagnetic response of a superconductor subjected to external fields. Now it is the essence of the Meissner effect that the field expulsion occurs even when the material is placed between the poles of a magnet at a temperature higher than the transition temperature (so that the magnetic field penetrates), and is *then* cooled down. That is, the Meissner effect is an *equilibrium* phenomenon, not merely a phenomenon "independent of relaxation times". The distinction is of major importance, precisely because superconductors also exhibit *non*-equilibrium phenomena with infinite (or at least extremely long) relaxation times, i. e. persistent currents. The point has been stressed particularly by London.[1)]

This discussion of previous work is not meant to be complete, but merely to indicate that there is some doubt, at least, about all the calculations published so far; and that one calculation[16)] leads to a result at variance with all the others.

§ 2. The model

a) The Hamiltonian:

The Hamiltonian in the absence of a magnetic field is taken to be

$$H_0=\sum_{i=1}^{N}\frac{\boldsymbol{p}_i^2}{2m}+\sum_{i<j=1}^{N}V(\boldsymbol{r}_i-\boldsymbol{r}_j) \tag{2·1}$$

where V is some potential function which we shall not need to specify in detail. The choice of an ordinary potential in (2·1) makes the Hamiltonian gauge invariant, but is somewhat unrealistic, as mentioned in the Introduction. Extension of this work to include the electron-phonon interaction explicitly is now in progress. The only condition on V which we shall have to require is that its volume integral converge. This condition excludes an unscreened Coulomb interaction, but allows a screened Coulomb repulsion between electrons.

The magnetic field is introduced through the usual device of replacing the operator $\boldsymbol{p}_i$ by $\boldsymbol{p}_i-(e/c)\boldsymbol{A}(\boldsymbol{x}_i)$. This gives*

$$H=H_0+H_1+H_2 \tag{2·2a}$$

$$H_1=-\frac{e}{2mc}\sum_{i=1}^{N}[\boldsymbol{p}_i\cdot\boldsymbol{A}(\boldsymbol{x}_i)+\boldsymbol{A}(\boldsymbol{x}_i)\cdot\boldsymbol{p}_i] \tag{2·2b}$$

$$H_2=\frac{e^2}{2mc^2}\sum_{i=1}^{N}|\boldsymbol{A}(\boldsymbol{x}_i)|^2. \tag{2·2c}$$

We shall use second-quantization notation, in k-space, through most of the calculation. The operator H_0 then becomes:

$$H_0=\sum_k\varepsilon_k a_k^+ a_k+\sum_{k,l,k',l'}K_{kl,k'l'}a_k^+ a_l^+ a_{l'}a_{k'} \tag{2·3}$$

---

* We ignore spin currents, since the Meissner-Ochsenfeld field expulsion is produced by the orbital currents.

with

$$\varepsilon_k = \hbar^2 k^2/2m \tag{2·4}$$

and

$$K_{kl,k'l'} = \langle \boldsymbol{k}s, \boldsymbol{l}\sigma | V | \boldsymbol{k}'s', \boldsymbol{l}'\sigma' \rangle = \delta_{ss'}\delta_{\sigma\sigma'}\delta(\boldsymbol{k}+\boldsymbol{l}-\boldsymbol{k}'-\boldsymbol{l}')G(\boldsymbol{k}-\boldsymbol{k}') \tag{2·5a}$$

$$G(k) = \frac{1}{V}\int \exp(-i\boldsymbol{k}\cdot\boldsymbol{r})\, V(\boldsymbol{r})\, d^3r. \tag{2·5b}$$

In (2·5b) the V in front of the integral is the normalization volume.

We shall prove gauge invariance exactly, in section 4, for arbitrary $A(x)$. For actual calculations, however, we shall restrict ourselves to fields involving one Fourier component only, i.e.

$$\boldsymbol{A}(\boldsymbol{x}) = \boldsymbol{A}(\boldsymbol{q})\,(2/V)^{1/2}\cos(\boldsymbol{q}\cdot\boldsymbol{x}). \tag{2·6}$$

The operators H_1 and H_2 then become*

$$H_1 = -c^{-1}\boldsymbol{j}_1(\boldsymbol{q})\cdot\boldsymbol{A}(\boldsymbol{q}) \tag{2·7a}$$

$$\boldsymbol{j}_1(\boldsymbol{q}) = \frac{e\hbar}{2m}\left(\frac{2}{V}\right)^{1/2}\sum_k \boldsymbol{k}\,(a^+_{k+(1/2)q}a_{k-(1/2)q} + a^+_{k-(1/2)q}a_{k+(1/2)q}) \tag{2·7b}$$

and

$$H_2 = -c^{-1}\boldsymbol{j}_2(\boldsymbol{q})\cdot\boldsymbol{A}(\boldsymbol{q}) \tag{2·8a}$$

$$\boldsymbol{j}_2(\boldsymbol{q}) = -\frac{e^2}{2mc}\frac{\boldsymbol{A}(\boldsymbol{q})}{V}\sum_k (2a^+_k a_k + a^+_k a_{k+2q} + a^+_k a_{k-2q}). \tag{2·8b}$$

b) The Wave Function:

We allow an arbitrary, adjustable wave function for a correlated electron pair, which we shall call $\varphi(k, k')$. This wave function is normalized through

$$\sum_{k,k'} |\varphi(k, k')|^2 = 1. \tag{2·9}$$

We define the operator b by

$$b = 2^{-1/2}\sum_{k,k'}\varphi^*(k, k')\, a_k a_{k'}. \tag{2·10}$$

This operator destroys one electron pair in the quantum state $\varphi(k, k')$. The wave function Ψ of the many-particle system is then taken to be of the form

$$\Psi = C(b^+)^{N_2}|0\rangle \tag{2·11}$$

where C is a normalization constant, $N_2 = 1/2\cdot N$ is the number of electron pairs, and $|0\rangle$ is the true vacuum state (*not* a filled Fermi sea).

The assumption (2·11) means that all electrons are paired up, and all pairs occupy the same pair quantum state $\varphi(k, k')$, i.e. an extreme Bose-Einstein con-

* The index $k+q$ is understood to mean a momentum $\boldsymbol{k}+\boldsymbol{q}$ with the same spin s as in the index $k=(\boldsymbol{k}, s)$.

densation. It is quite simple to generalize the calculation so as to allow for a filled Fermi sea below the pairs. The general expressions for matrix elements in the quasi-chemical equilibrium theory[25] apply also to that case. However, we prefer to keep things simple here, and to avoid all unnecessary complications, i.e. complications not essential for understanding the *qualitative* origin of the Meissner effect. The assumption (2·11) includes,[26] as special case, the wave function for the original Bogoliubov theory[15),27] and its equivalents.[28),11] It is likely (but not proved) that (2·11) in its full generality is equivalent to Bogoliubov's second Ansatz.[23] The specialization in Bogoliubov's first theory is equivalent to the assumption of "simple" pairs, namely

$$\varphi(\boldsymbol{k}s, \boldsymbol{k}'s') = 0 \text{ unless } \boldsymbol{k}' = -\boldsymbol{k},\ s' = -s. \qquad (2\cdot12)$$

No such restriction is implied in our present Ansatz (2·11), and in fact we shall work with pair wave functions $\varphi(k, k')$ which violate (2·12).

The Hamiltonian and the wave function described in this section represent our assumptions in this paper. From here on, all calculations are exact.

§ 3. The current operator and expectation values

The operator for the current is obtained by differentiating the Hamiltonian H with respect to the vector potential. The result is, for the special case of (2·6),

$$\boldsymbol{j}(\boldsymbol{q}) = \boldsymbol{j}_1(\boldsymbol{q}) + \boldsymbol{j}_2(\boldsymbol{q}) \qquad (3\cdot1)$$

where $\boldsymbol{j}_1(\boldsymbol{q})$ is given by (2·7b) and $\boldsymbol{j}_2(\boldsymbol{q})$ by (2·8b). The London equation (1·3) then assumes the form

$$\langle \boldsymbol{j}(\boldsymbol{q}) \rangle = \frac{(\Psi, \boldsymbol{j}(\boldsymbol{q})\Psi)}{(\Psi, \Psi)} = -\frac{1}{\lambda c}\boldsymbol{A}_\perp(\boldsymbol{q}) \qquad (3\cdot2)$$

where $\boldsymbol{A}_\perp(\boldsymbol{q})$ denotes the component of $\boldsymbol{A}(\boldsymbol{q})$ perpendicular to the vector $\boldsymbol{q}$. Equation (3·2) is supposed to hold for small $A_\perp(q)$, small q, and arbitrary $A_{//}(q)$: small $A_\perp(q)$, because the London equation determines the *linear* terms only; currents proportional to the square or higher powers of the magnetic field are excluded *ab initio*; small q, because we are interested in the law of field penetration at large distances x within the superconducting specimen; it is well known that the London equation in its original form does not give the correct field penetration law in the immediate neighbourhood of the surface, i.e. (3·2) is incorrect for q comparable to, or larger than, d^{-1}, where d is Pippard's coherence length.[29] The parallel component $A_{//}(q)$ can, of course, take arbitrarily large values since this component can be modified at will by a gauge transformation.

To derive (3·2) from the model of Section 2, we need expressions for the expectation values of typical single-particle and two-particle operators over the

wave function (2·11). This calculation is being published separately[25] and we only quote the results here.

We introduce an *un*normalized wave function for the pair by*

$$\chi(k, k') = C\varphi(k, k') \tag{3·3}$$

where C is an adjustable parameter. Thus $\chi(k, k')$ is not subject to the normalization condition (2·9). Next we introduce the one-particle operator $\rho_{kk'}$ by**

$$\rho_{kk'} = \sum_{k''} \chi(k, k'')\chi^*(k', k''). \tag{3·4}$$

This is, except for normalization, the reduced one-particle density matrix for one of the two particles of the pair. ρ is Hermitean and non-negative. We introduce the one-particle operator $h_{kk'}$ by

$$h_{kk'} = \left(\frac{\rho}{1+\rho}\right)_{kk'} \tag{3·5}$$

and the two-particle operator $p_{kl,k'l'}$ by

$$p_{kl,k'l'} = h_{kk'}h_{ll'} - h_{kl'}h_{k'l}. \tag{3·6}$$

We use the symbols " tr " and " tr_2 " for traces over the one-particle and two-particle spaces, respectively, that is,

$$\mathrm{tr}(h) = \sum_k h_{kk} \qquad \mathrm{tr}_2(p) = \sum_{kl} p_{kl,kl}. \tag{3·7}$$

We also use the ordinary notation for expectation values over a two-particle wave function, e.g.

$$(\varphi, p\varphi) = \sum_{k,l,k',l'} \varphi^*(k, l)\, p_{kl,k'l'}\, \varphi(k', l'). \tag{3·8}$$

We define a series of unnormalized wave functions $\chi_n(k, k')$ by

$$\chi_0(k, k') = \chi(k, k') \tag{3·9a}$$

$$\chi_n(k, k') = -\sum_{k''} \rho_{kk''}\chi_{n-1}(k'', k'). \tag{3·9b}$$

Since $\chi(k, k')$ is antisymmetric (Pauli principle), we can write the functions (3·9) in an expanded form, as follows:

$$\chi_1(k, k') = \sum_{k''k'''} \chi(k, k'')\chi^*(k'', k''')\chi(k''', k') \tag{3·10a}$$

$$\chi_2(k, k') = \sum_{k_1 k_2 k_3 k_4} \chi(k, k_1)\chi^*(k_1, k_2)\chi(k_2, k_3)\chi^*(k_3, k_4)\chi(k_4, k') \tag{3·10b}$$

and so on.***

---

* The constant which we call C here is called $(2vt)^{1/2}$ in reference 25).

** In terms of the quantities of reference 25), $\rho = -2vtq$.

*** In terms of reference 25), $\chi_n(k, k') = (2vt)^{n+1/2}\varphi_n(k, k')$.

Finally, we introduce the unnormalized function $\psi(k, k')$ by

$$\psi(k,k') = \sum_{n=0}^{\infty} \chi_n(k, k') = \sum_{k''} \left(\frac{1}{1+\rho}\right)_{kk''} \chi(k'', k'). \tag{3·11}$$

We write the typical one-particle operator J in the form

$$J = \sum_{k,k'} J_{kk'} a_k^+ a_{k'} \tag{3·12}$$

and the typical two-particle operator K as

$$K = \sum_{k,l,k',l'} K_{kl,k'l'} a_k^+ a_l^+ a_{l'} a_{k'}. \tag{3·13}$$

It is then shown in reference 25) that the following expressions for the expectation values hold up to errors of relative order $1/N$:

$$\langle J \rangle = \frac{(\Psi, J\Psi)}{(\Psi, \Psi)} = \mathrm{tr}(Jh) \tag{3·14}$$

and

$$\langle K \rangle = \frac{(\Psi, K\Psi)}{(\Psi, \Psi)} = \mathrm{tr}_2(K\rho) + (\psi, K\psi). \tag{3·15}$$

If we make the substitution $J_{kk'} = \delta_{kk'}$, J becomes the operator for the number of particles. Thus (3·14) gives the condition

$$\mathrm{tr}(h) = N = 2N_2. \tag{3·16}$$

This condition determines the adjustable constant C in (3·3).

As far as the expectation value of the Hamiltonian is concerned, the first sum in (3·3), and the quantities H_1 and H_2, (2·7) and (2·8), are single-particle operators, of type J. The second term on the right of (2·3) is a two-particle operator. Thus Eqs. (3·14) and (3·15) suffice to determine the expectation value $\langle H \rangle$, and this is what we intend to do, treating the pair wave function $\chi(k, k')$ as a variational trial function.

§ 4. Gauge invariance

We now demonstrate the gauge invariance of the formalism. The gauge transformation is given by

$$\boldsymbol{A}'(\boldsymbol{x}) = \boldsymbol{A}(\boldsymbol{x}) + \mathrm{grad}\Lambda(\boldsymbol{x}) \tag{4·1}$$

where $\Lambda(\boldsymbol{x})$ is an arbitrary function. The pair wave function $\varphi(\boldsymbol{x}_1, \boldsymbol{x}_2)$ transforms according to

$$\varphi'(\boldsymbol{x}_1, \boldsymbol{x}_2) = \exp[(ie/\hbar c)(\Lambda(\boldsymbol{x}_1) + \Lambda(\boldsymbol{x}_2))]\varphi(\boldsymbol{x}_1, \boldsymbol{x}_2). \tag{4·2}$$

We are now required to show that the expectation value of the Hamiltonian, $\langle H \rangle$, is invariant under the simultaneous replacement of A by A' and φ by φ'.

The proof is easiest in x-space. The operator ρ is given in x-space by*

$$\langle x|\rho|x'\rangle=\int d^3x''\chi(x, x'')\chi^*(x', x'') \tag{4·3}$$

and therefore transforms into

$$\langle x|\rho'|x'\rangle=\exp[(ie/\hbar c)(\Lambda(x)-\Lambda(x'))]\langle x|\rho|x'\rangle. \tag{4·4}$$

Similarly,

$$\langle x|h'|x'\rangle=\exp[(ie/\hbar c)(\Lambda(x)-\Lambda(x'))]\langle x|h|x'\rangle. \tag{4·5}$$

The two-particle operator p transforms according to

$$\langle x_1x_2|p'|x_1'x_2'\rangle=\exp[(ie/\hbar c)(\Lambda(x_1)+\Lambda(x_2)-\Lambda(x_1')-\Lambda(x_2'))]\langle x_1x_2|p|x_1'x_2'\rangle. \tag{4·6}$$

We have

$$\mathrm{tr}_2(Kp)=\int d^3x_1d^3x_2\,V(x_1-x_2)\langle x_1x_2|p|x_1x_2\rangle \tag{4·7}$$

and this is clearly invariant under the transformation (4·6). Furthermore, the function $\psi(x_1, x_2)$ transforms just like $\varphi(x_1, x_2)$, (4·2). Hence

$$(\psi, K\psi)=\int d^3x_1d^3x_2\,V(x_1-x_2)|\psi(x_1, x_2)|^2 \tag{4·8}$$

is also invariant. If $H=J+K$ is the break-up of the Hamiltonian into single-particle and two-particle operators, then clearly

$$J=-(\hbar^2/2m)\left(\nabla-\frac{ie}{\hbar c}\boldsymbol{A}\right)^2. \tag{4·9}$$

Using the identity

$$\langle x|h|x'\rangle=\int d^3x''\chi(x, x'')\psi^*(x', x''), \tag{4·10}$$

we can write

$$\mathrm{tr}(Jh)=\frac{\hbar^2}{2m}\int d^3x\,d^3x''\left[\left(\nabla_x-\frac{ie}{\hbar c}\boldsymbol{A}(\boldsymbol{x})\right)\psi(\boldsymbol{x}, \boldsymbol{x}'')\right]^* \times\left[\left(\nabla_x-\frac{ie}{\hbar c}\boldsymbol{A}(\boldsymbol{x})\right)\chi(\boldsymbol{x}, \boldsymbol{x}'')\right] \tag{4·11}$$

where ∇_x means the gradient operator with respect to the variable x. We now make the gauge transformation, and get from (4·1) and (4·2)

$$\left[\nabla_x-\frac{ie}{\hbar c}\boldsymbol{A}'(\boldsymbol{x})\right]\psi'(\boldsymbol{x}, \boldsymbol{x}'')=\exp\left(\frac{ie}{\hbar c}\Lambda(\boldsymbol{x}'')\right)\left[\nabla_x-\frac{ie}{\hbar c}\boldsymbol{A}(\boldsymbol{x})\right]\psi(\boldsymbol{x}, \boldsymbol{x}'') \tag{4·12}$$

---

* A sum over spin indices s'' is understood in $\int d^3x''\cdots$. We ignore spin indices in this section, since they do not affect the argument.

as well as an analogous equation for the other factor in the integrand of (4·11). Substitution of (4·12) into (4·11) then gives the identity

$$\mathrm{tr}(J'h') = \mathrm{tr}(Jh). \tag{4·13}$$

Combination of (4·13) with the invariance of (4·7) and (4·8) establishes the gauge identity

$$(\Psi', H'\Psi') = (\Psi, H\Psi). \tag{4·14}$$

Since all our subsequent equations are determined from the condition that $\langle H \rangle$ be a minimum, this concludes the demonstration that our formalism is fully gauge invariant.

The general gauge transformation is not altogether trivial in k-space. We quote one example. Let $\Lambda(x)$ be chosen so as to yield the $A_{//}(q)$ just as in (2·6), i. e.

$$\Lambda(x) = \frac{A_{//}(q)}{q} (2/V)^{1/2} \sin(\boldsymbol{q}\cdot\boldsymbol{x}) \tag{4·15}$$

and define γ by

$$\gamma = (2/V)^{1/2} \frac{eA_{//}(q)}{2\hbar c q}. \tag{4·16}$$

Then the gauge transformation changes the pair wave function φ into its gauge transform

$$\varphi'(k, k') = \sum_{n,n'=-\infty}^{+\infty} J_n(2\gamma) J_{n'}(2\gamma)\, \varphi(k - nq, k' - n'q) \tag{4·17}$$

where J_n is the Bessel function of order n. What we have shown above is that the wave function Ψ, (2·11), constructed from *this* φ' instead of the original φ gives the same expectation value for the transformed Hamiltonian as the original wave function (2·11) gives for the original Hamiltonian. It is apparent that a direct proof of this statement within the k-space formalism would be lengthy.

(4·17) also shows that the "simple" pair condition (2·12) is inconsistent with gauge invariance. If the wave function φ obeys (2·12) in one gauge, its gauge transforn φ' fails to obey (2·12). Hence the condition of "simple" pairs must of necessity lead to a gauge-dependent formalism. This purely formal restriction, rather than any real physics, requires the bringing-in of "collective excitations" and the use of the random phase approximation. Once this formal restriction is removed, gauge invariance is immediate and explicit, requiring no additional approximations or assumptions.

For later use, we also quote the special case of the infinitesimal gauge transformation corresponding to (4·15). If $A_{//}(q)$ is considered to be small, and thus only terms linear in γ are retained in (4·17), we get the transformation

$$\varphi'(k, k') = \varphi(k, k') + \delta\varphi(k, k') + \cdots \tag{4·18a}$$

with

$$\delta\varphi(k, k') = \gamma[\varphi(k-q, k') + \varphi(k, k'-q) - \varphi(k+q, k') - \varphi(k, k'+q)]. \tag{4·18b}$$

Substitution of these expressions into our later equations results in some useful checks, as shown in the Appendix.

§ 5. Variational equation for the pair wave function

In order to get an equation for the pair wave function $\varphi(k, k')$ or its unnormalized counterpart $\chi(k, k')$, we minimize the expectation value $\langle H \rangle$ subject to the condition that the number of particles N is given, i.e. subject to (3·16). By using Lagrange multipliers, the variation principle becomes

$$\langle H\rangle-\mu\,\mathrm{tr}(h)=\text{minimum} \tag{5·1}$$

where μ, the chemical potential, appears as a Lagrange multiplier. It should be noted that our basic wave function (2·11) describes a state with a definite, given number of particles. The appearance of μ in (5·1) is only a mathematical device; there is no failure of the law of conservation of particles in our formalism.

Since $\chi(k, k')$ is complex in general, its real and imaginary parts can be varied independently. It is well known that one gets equivalent equations by varying $\chi(k, k')$ and $\chi^*(k, k')$ independently; furthermore, the two equations so obtained are complex conjugates of one another. Hence we shall vary $\chi^*(k, k')$ only, leaving $\chi(k, k')$ unchanged, and determine the first-order variation of the left side of (5·1). We write

$$H-\mu N_{op}=J+K \tag{5·2}$$

where J is a one-particle operator (differing from the J of the previous two sections by the replacement $\varepsilon_k\to\varepsilon_k-\mu$) and K is a two-particle operator.

The variation of $\langle J\rangle$ is

$$\Delta\langle J\rangle=\mathrm{tr}(J\Delta h). \tag{5·3}$$

Straightforward power series expansion gives

$$\Delta h=(1+\rho)^{-1}\Delta\rho(1+\rho)^{-1} \tag{5·4}$$

and

$$(\Delta\rho)_{kk'}=\sum_{k''}\chi(k, k'')\Delta\chi^*(k', k''). \tag{5·5}$$

Substitution into (5·3) gives, using (3·11),

$$\begin{aligned}\Delta\langle J\rangle&=\mathrm{tr}[(1+\rho)^{-1}J(1+\rho)^{-1}\Delta\rho]\\&=\sum_{k,k',m,m'}\left(\frac{1}{1+\rho}\right)_{mk}J_{kk'}\psi(k', m')\Delta\chi^*(m, m').\end{aligned} \tag{5·6}$$

Equations (5·4) and (5·5) can be used also to find the variation of $\mathrm{tr}_2(Kp)$ in (3·15); the result can be written as follows:

$$\Delta\,\mathrm{tr}_2(Kp)=2\sum_{k,k',m,m'}\left(\frac{1}{1+\rho}\right)_{mk}L_{kk'}\psi(k', m')\Delta\chi^*(m, m') \tag{5·7}$$

where

$$L_{kk'}=\sum_{ll'}h_{l'l}(K_{k'l',kl}-K_{k'l',lk}). \tag{5·8}$$

Finally we have

$$\Delta(\psi, K\psi)=\sum_{k,l,k',l'}[\Delta\psi^*(k, l)K_{kl,k'l'}\psi(k', l')+\psi^*(k, l)K_{kl,k'l'}\Delta\psi(k', l')]. \tag{5·9}$$

Since we are varying χ^* but not χ, the variation $\Delta\psi^*$ is not the complex conjugate of $\Delta\psi$. Rather, direct substitution into the expanded formulas of type (3·10) yields

$$\Delta\psi(k, l) = \sum_{m, m'} \psi(k, m)\Delta\chi^*(m, m')\psi(m', l) \tag{5·10}$$

$$\Delta\psi^*(k, l) = \sum_{m, m'} \left(\frac{1}{1+\rho}\right)_{mk} \left(\frac{1}{1+\rho}\right)_{m'l} \Delta\chi^*(m, m'). \tag{5·11}$$

Using the fact that $\psi(k, l)$ is antisymmetric, we get

$$\Delta(\psi, K\psi) = \sum_{m, m', k, k', l, l'} \left\{ \left(\frac{1}{1+\rho}\right)_{mk} \left(\frac{1}{1+\rho}\right)_{m'l} K_{kl,k'l'}\psi(k', l') \right.$$
$$\left. -\psi^*(k, l)K_{kl,k'l'}\psi(k', m)\psi(l', m') \right\} \Delta\chi^*(m, m'). \tag{5·12}$$

Equation (5·1) then becomes

$$\Delta\langle J\rangle + \Delta \operatorname{tr}_2(Kp) + \Delta(\psi, K\psi) = 0. \tag{5·13}$$

We can select out the coefficient of $\Delta\chi^*(m, m')$ in this, since the $\Delta\chi^*(m, m')$ are independent of one another. We do this, premultiply by $(1+\rho)_{\bar{m}m}(1+\rho)_{\bar{m}'m'}$ and sum over m and m' to get the final equation

$$\sum_k (J_{mk} + 2L_{mk})\chi(k, m') + \sum_{k'l'} K_{mm',k'l'}\psi(k', l')$$
$$- \sum_{k, l, k', l'} \psi^*(k, l)K_{kl,k'l'}\chi(k', m)\chi(l', m') = 0. \tag{5·14}$$

This is the basic equation determining the pair wave function $\chi(k, k')$. It is highly non-linear: ψ is related non-linearly to χ, and the quantity $h_{kk'}$ which appears in the definition of L, Eq. (5·8), is also a non-linear functional of χ.

It should be noted that the full interaction operator $K_{kl,k'l'}$ appears in (5·14). There is no restriction to a "reduced" interaction in our formalism. As it turns out, Eq. (5·14) is no more difficult to solve than the simplified equations obtained from a "reduced" interaction. There is no justification *a priori* for the use of a reduced interaction in the theory of superconductivity. There *is* an *a priori* justification for the use of a trial wave function of Bose-condensed type (2·11), namely the calculation of reference 10). We emphasize that this *a priori* justification does *not* depend on the existence of an energy gap; Bose-Einstein condensation of electron pairs can occur even if the first excited state has an energy of the order of $\hbar^2/mL^2$. It is a statistical phenomenon, not (primarily) a dynamical effect. If attractively correlated pairs exist at all (due to the interaction), they will Bose-condense, energy gap or not.

Since (5·14) has been derived from a gauge-invariant equation, without restrictive assumptions in the derivation, (5·14) must also be gauge invariant.

§ 6. Solution in the absence of external fields

In the absence of external fields the Hamiltonian H_0, (2·1), has a number of symmetry properties which restrict the complexity of the wave function $\varphi(k_1, k_2)$ of the pair.

1) Translation invariance implies that $\boldsymbol{K}=\boldsymbol{k}_1+\boldsymbol{k}_2$ is a constant of motion, i.e. $\varphi(\boldsymbol{k}_1 s_1, \boldsymbol{k}_2 s_2)$ contains a factor $\delta(\boldsymbol{K}-\boldsymbol{k}_1-\boldsymbol{k}_2)$.

2) Galilean invariance of (2·1) implies that the lowest energy must be associated with a wave function belonging to the set $\boldsymbol{K}=0$. In a solid, unlike our model (2·1), full Galilean invariance does not exist, since the solid lattice selects out a preferred frame of reference. However, it is highly reasonable to assume that $\boldsymbol{K}=0$ still leads to the lowest state of the system. Solutions with $\boldsymbol{K}\neq 0$ condensed pairs exist, of course, and are presumably required for the discussion of persistent currents. But the Meissner effect is an equilibrium phenomenon, and thus we are concerned only with the lowest state of the system.

3) There is no spin-orbit coupling in (2·1). Hence the pair states are pure singlet and triplet spin states. Other things being equal, the Pauli principle leads to the conclusion that attractive couplings are easier in singlet states, since the space wave function of the pair can then have $L=0$.

The arguments (1)–(3) lead directly to the "simple" pairs of Eq. (2·12). Finally, rotation invariance of H_0 implies that the pair wave function must correspond to an S-state, i.e.

$$\varphi(\boldsymbol{k}\uparrow, -\boldsymbol{k}\downarrow)=-\varphi(-\boldsymbol{k}\downarrow, \boldsymbol{k}\uparrow)=\varphi(|\boldsymbol{k}|). \tag{6·1}$$

This assumption is to be substituted into Eq. (5·14). With simple pairs, $h_{kk'}$ is diagonal, and equal to

$$h_{kk'}=\delta_{kk'}\frac{|\chi(k)|^2}{1+|\chi(k)|^2}. \tag{6·2}$$

$L_{kk'}$, (5·8), is also diagonal and given by

$$L_{kk'}=\delta_{kk'}\sum_{l}[2G(0)-G(\boldsymbol{k}-\boldsymbol{l})]h_{ll} \tag{6·3}$$

where $G(\boldsymbol{k})$ is defined by (2·5b). Note that the sum over $\boldsymbol{l}$ in (6·3) does *not* include a sum over spin indices. $J_{kk'}$ is of course diagonal right from the start:

$$J_{kk'}=\delta_{kk'}(\varepsilon_k-\mu)=\delta_{kk'}\left(\frac{\hbar^2k^2}{2m}-\mu\right). \tag{6·4}$$

If we put $m=(\boldsymbol{m}, \uparrow)$ in Eq. (5·14), then the first sum over k contains only a single term $k=m$, and the simple pairing condition for $\chi(k, m')$ implies that $m'=(-\boldsymbol{m}, \downarrow)$. With these values of the indices m and m', the second sum becomes

$$\sum_{k'l'}K_{mm',k'l'}\psi(k', l')=\sum_{k'}G(\boldsymbol{m}-\boldsymbol{k}')\psi(|\boldsymbol{k}'|)\equiv -C(m). \tag{6·5}$$

This equation acts as the definition of $C(m)$. The function $\psi(k)$ is related to $\chi(k)$ by

$$\psi(k)=\frac{\chi(k)}{1+|\chi(k)|^2} \tag{6·6}$$

and therefore has its maximum where $\chi(k)=1$. This is in the neighbourhood of the Fermi momentum k_F. The sum $C(k)$ is a slowly varying function of k, and can be replaced by a constant in first approximation. Except for the change in the nature of the assumed interaction, our $C(k)$ corresponds precisely to the $C(k)$ in Bogoliubov's theory.[15] Our $\psi(k)$ is called $u_k v_k$ by Bogoliubov. Following Bogoliubov's notation, we introduce the abbreviation

$$\xi(k)=\varepsilon_k-\mu+2L_{kk}. \tag{6·7}$$

This $\xi(k)$ includes the term $2L_{kk}$, which is a slowly varying function of k. This term may be considered a renormalization of the single-particle energies ε_k as a result of the interaction. This renormalization term is the *only* effect of replacing the "reduced" interaction by the full interaction. It has no qualitative influence on the nature of the solution. This explains why the completely unphysical "reduced" interaction of reference 11) nevertheless gave reasonable results.

With these definitions and relations, Eq. (5·14) becomes:

$$\xi(m)\chi(m)-C(m)+[\chi(m)]^2C(m)=0 \tag{6·8}$$

with the immediate solution

$$\chi(k)=-\frac{\xi(k)}{2C(k)}\pm\left[1+\left(\frac{\xi(k)}{2C(k)}\right)^2\right]^{1/2}. \tag{6·9}$$

Substitution of (6·9) into (6·6) gives

$$\psi(k)=\pm\frac{1}{2}\left[1+\left(\frac{\xi(k)}{2C(k)}\right)^2\right]^{-1/2} \tag{6·10}$$

with the $\pm$ signs in (6·9) and (6·10) correlated. Substitution of (6·10) into the defining equation (6·5) then yields the Bogoliubov integral equation

$$C(m)=-\sum_k \frac{G(\boldsymbol{m}-\boldsymbol{k})}{2\left[1+\left(\frac{\xi(k)}{2C(k)}\right)^2\right]^{1/2}}. \tag{6·11}$$

Here we have chosen the plus sign in (6·10), and hence also in (6·9). This is consistent if $C(k)$ turns out to be positive, for large k, for then $\chi(k)$ becomes small for large k, where $\xi(k)$ is positive and large, and thus the pair wave function is normalizable. On the other hand, a negative value of (6·11) (for large m) gives a non-normalizable pair wave function, no matter which sign is chosen in (6·9). Thus, the condition of positive $C(k)$, i.e. a sufficiently attractive interaction, emerges naturally from this formalism, without having to consider questions of stability and of excited states.

Integral equations of type (6·11) have been discussed in detail by Bogoliubov[15] and we have nothing to add to this work. Bogoliubov has shown that the solution of (6·11) is non-analytic in the coupling constant of the interaction, i.e. $C(k)$

is proportional to $\exp(-1/g)$ where g is proportional to the strength parameter in the potential.

If a solution exists, i.e. if $C(k)$ is positive, the behaviour of the un-normalized pair wave function $\chi(k)$ for small k and large k, respectively, is:

$$\chi(k) = -\xi(k)/C(k) \quad \text{for } k \ll k_F \tag{6·12a}$$

$$\chi(k) = C(k)/\xi(k) \quad \text{for } k \gg k_F. \tag{6·12b}$$

The first of these is large and positive, the second is positive and goes down roughly like $1/k^3$, leading to a normalizable wave function. Thus the pair wave function obtained from the initial assumption (2·11) is by no means unreasonable, physically. An even more reasonable result would have been obtained if we had allowed for a filled Fermi sea up to some momentum $k_0 < k_F$, with pairs made up only out of states with $k > k_0$. We have not done so purely in order to avoid trivial complications.

§ 7. The variational equation in a weak external field

We now consider Eq. (5·14) in the presence of a weak external field. This means adding the terms H_1, (1·7), and H_2, (2·8), to the Hamiltonian H_0. If we restrict ourselves to terms linear in $\boldsymbol{A}(\boldsymbol{q})$, we can omit H_2 as being quadratic.*

In terms of Eq. (5·14), the external field then leads to a change in the quantities J_{mk}. To be precise,

$$J_{kk'} = J^{(0)}_{kk'} + (\delta J)_{kk'} \tag{7·1}$$

where

$$(\delta J)_{kk'} = -\frac{e\hbar}{4mc}\left(\frac{2}{V}\right)^{1/2}(\boldsymbol{A}(\boldsymbol{q})\cdot(\boldsymbol{k}+\boldsymbol{k}'))\,\delta_{ss'}[\delta(\boldsymbol{k}-\boldsymbol{k}'+\boldsymbol{q}) + \delta(\boldsymbol{k}-\boldsymbol{k}'-\boldsymbol{q})] \tag{7·2}$$

and $J^{(0)}_{kk'}$ is given by (6·4).

We shall now make a perturbation treatment of Eq. (5·14). That is, we assume that the solution $\chi_0(k, k')$ of the unperturbed equation (5·14) is known, and we write

$$\chi(k, k') = \chi_0(k, k') + \delta\chi(k, k') \tag{7·3a}$$

$$\chi^*(k, k') = \chi_0^*(k, k') + \delta\chi^*(k, k'). \tag{7·3b}$$

We intend to substitute this into (5·14) and keep only linear terms in $\delta\chi$. Note

* This would be wrong if we wished to compute the perturbed energy; however, we merely need the perturbed *wave function* in order to find the expectation value of the current operator (3.1) over that wave function. The quadratic term in the Hamiltonian, H_2, which we neglect in this section, does contribute a term $\boldsymbol{j}_2(\boldsymbol{q})$ to the current operator, and this latter term will be retained in § 8.

that this variation has nothing to do with the formal variation $\Delta\chi^*$ used in Section 5. Since the basic Hamiltonian has altered, both χ and χ^* must alter, and certainly $\delta\chi^*=(\delta\chi)^*$.

Straightforward substitution into the definitions gives

$$\delta\psi(k,k')=\sum_{l,l'}\left[\left(\frac{1}{1+\rho_0}\right)_{kl}\left(\frac{1}{1+\rho_0}\right)_{k'l'}\delta\chi(l,l')-\psi_0(k,l)\psi_0(k',l')\delta\chi^*(l,l')\right]. \tag{7·4}$$

Furthermore, $\delta\psi^*=(\delta\psi)^*$. The variation of the operator h is:

$$\delta h=\frac{1}{1+\rho_0}\delta\rho\frac{1}{1+\rho_0} \tag{7·5}$$

with

$$(\delta\rho)_{kk'}=\sum_{k''}[\delta\chi(k,k'')\chi_0^*(k',k'')+\chi_0(k,k'')\delta\chi^*(k',k'')]. \tag{7·6}$$

Substitution of these expressions into the defining equation (5·8) gives

$$(\delta L)_{kk'}=-\sum_{l,l',m,m'}(K_{k'l',kl}-K_{k'l',lk})\times\left[\left(\frac{1}{1+\rho_0}\right)_{l'm}\delta\chi(m,m')\psi_0^*(m',l)+\psi_0(l',m)\delta\chi^*(m,m')\left(\frac{1}{1+\rho_0}\right)_{m'l}\right]. \tag{7·7}$$

Retaining all first-order variations in (5·14), we then get

$$\begin{aligned}\sum_k[&(\delta J)_{mk}\chi_0(k,m')+2(\delta L)_{mk}\chi_0(k,m')+(J^{(0)}_{mk}+2L^{(0)}_{mk})\delta\chi(k,m')]\\&+\sum_{k'l'}K_{mm',k'l'}\delta\psi(k',l')-\sum_{k,l,k',l'}[\delta\psi^*(k,l)K_{kl,k'l'}\chi_0(k',m)\chi_0(l',m')\\&+\psi_0^*(k,l)K_{kl,k'l'}\delta\chi(k',m)\chi_0(l',m')\\&+\psi_0^*(k,l)K_{kl,k'l'}\chi_0(k',m)\delta\chi(l',m')]=0.\end{aligned} \tag{7·8}$$

This is an inhomogeneous linear equation for $\delta\chi$; the first term is the inhomogeneity, proportional to the vector potential according to (7·2); all the other terms contain $\delta\chi$ or $\delta\chi^*$ linearly.

As an immediate check on this equation, we can use gauge invariance. For the special case of a longitudinal vector potential, $\boldsymbol{A}(\boldsymbol{q})=\boldsymbol{A}_{//}(\boldsymbol{q})$, the first-order variation in the pair wave function is given by (4·18b). Since χ differs from the normalized pair wave function φ merely by a constant factor, we can put

$$\delta\chi(k,k')=\gamma[\chi_0(k-q,k')+\chi_0(k,k'-q)-\chi_0(k+q,k')-\chi_0(k,k'+q)] \tag{7·9}$$

where γ is given by (4·16). This value of $\delta\chi$ must give an *exact* solution of (7·8), no matter what potential function we assume. The explicit proof of this statement is contained in the Appendix.

The inhomogeneous term $(\delta J)_{mk}$ connects the index $m=(\boldsymbol{m}, \uparrow)$ with $k=(\boldsymbol{m}\pm\boldsymbol{q}, \uparrow)$. If $\chi_0(k, m')$ corresponds to "simple" pairs as it does in our case, the first term of (7·8) connects this value of m with a value $m'=(-\boldsymbol{m}\pm\boldsymbol{q}, \downarrow)$. Thus, to first order already, components are introduced into the wave function in violation of the "simple pairing" condition. This is true *even in the London gauge*. The cause has nothing to do with the gauge as such; rather, we used translation invariance in order to get "simple" pairs, and the sinusoidal vector potential (2·6) destroys translation invariance of the Hamiltonian, in any gauge.

We now use the fact that the zero-order solution obeys the "simple pairing" condition in order to reduce (7·8) to a more manageable form. We introduce a number of notations, as follows:

$$\delta_+(k)\equiv\delta\chi\left(\boldsymbol{k}+\frac{1}{2}\boldsymbol{q}\uparrow,\ -\boldsymbol{k}+\frac{1}{2}\boldsymbol{q}\downarrow\right)+\delta\chi\left(\boldsymbol{k}-\frac{1}{2}\boldsymbol{q}\uparrow,\ -\boldsymbol{k}-\frac{1}{2}\boldsymbol{q}\downarrow\right) \tag{7·10a}$$

$$\delta_-(k)\equiv\delta\chi\left(\boldsymbol{k}+\frac{1}{2}\boldsymbol{q}\uparrow,\ -\boldsymbol{k}+\frac{1}{2}\boldsymbol{q}\downarrow\right)-\delta\chi\left(\boldsymbol{k}-\frac{1}{2}\boldsymbol{q}\uparrow,\ -\boldsymbol{k}-\frac{1}{2}\boldsymbol{q}\downarrow\right). \tag{7·10b}$$

The components of $\delta\chi$ which enter into (7·10) are the only non-vanishing components of that function. Next, we define:

$$F_+(k)=\frac{\hbar^2}{2m}\left(k^2-\frac{1}{4}q^2\right)-\mu+L^{(0)}_{k+(1/2)q}+L^{(0)}_{k-(1/2)q}+\chi_0\left(k+\frac{1}{2}q\right)C_0\left(k+\frac{1}{2}q\right)$$
$$+\chi_0\left(k-\frac{1}{2}q\right)C_0\left(k-\frac{1}{2}q\right) \tag{7·11a}$$

$$F_-(k)=\frac{\hbar^2}{2m}(\boldsymbol{k}\cdot\boldsymbol{q})+L^{(0)}_{k+(1/2)q}-L^{(0)}_{k-(1/2)q} \tag{7·11b}$$

where we have omitted the second index on $L^{(0)}_{kk'}$ since $L^{(0)}$ is diagonal, see (6·3), and where C_0 is defined by (6·5) with $\chi=\chi_0$. Further definitions are

$$T_1=\sum_{k'}[2G(\boldsymbol{q})-G(\boldsymbol{k}-\boldsymbol{k}')]\frac{\chi_0\left(\boldsymbol{k}'+\frac{1}{2}\boldsymbol{q}\right)+\chi_0\left(\boldsymbol{k}'-\frac{1}{2}\boldsymbol{q}\right)}{(1+\rho^{(0)}_{k'+(1/2)q})\,(1+\rho^{(0)}_{k'-(1/2)q})}\delta_+(\boldsymbol{k}')$$
$$+\sum_{k'}G(\boldsymbol{k}-\boldsymbol{k}')\frac{\chi_0\left(\boldsymbol{k}'+\frac{1}{2}\boldsymbol{q}\right)-\chi_0\left(\boldsymbol{k}'-\frac{1}{2}\boldsymbol{q}\right)}{(1+\rho^{(0)}_{k'+(1/2)q})\,(1+\rho^{(0)}_{k'-(1/2)q})}\delta_-(\boldsymbol{k}') \tag{7·12}$$

$$T_2^{(\pm)}=\sum_{k'}G(\boldsymbol{k}-\boldsymbol{k}')\,(1+\rho^{(0)}_{k'+(1/2)q})^{-1}\,(1+\rho^{(0)}_{k'-(1/2)q})^{-1}\delta_\pm(\boldsymbol{k}') \tag{7·13}$$

$$T_3^{(\pm)}=\sum_{k'}G(\boldsymbol{k}-\boldsymbol{k}')\psi_0\left(\boldsymbol{k}'+\frac{1}{2}\boldsymbol{q}\right)\psi_0\left(\boldsymbol{k}'-\frac{1}{2}\boldsymbol{q}\right)\delta_\pm(\boldsymbol{k}'). \tag{7·14}$$

With these definitions, the set of Eqs. (7·8) can be transformed into two simultaneous linear integral equations for $\delta_+(k)$ and $\delta_-(k)$, namely,

$$F_+(k)\delta_+(k)+F_-(k)\delta_-(k)+\left[\chi_0\left(k+\frac{1}{2}q\right)+\chi_0\left(k-\frac{1}{2}q\right)\right]T_1$$
$$+\left[1-\chi_0\left(k+\frac{1}{2}q\right)\chi_0\left(k-\frac{1}{2}q\right)\right][T_2^{(+)}-T_3^{(+)}]$$
$$=(e\hbar/2mc)\,(2/V)^{1/2}\left[\chi_0\left(k+\frac{1}{2}q\right)+\chi_0\left(k-\frac{1}{2}q\right)\right](\boldsymbol{k}\cdot\boldsymbol{A}(\boldsymbol{q})) \tag{7·15a}$$

$$F_-(k)\delta_+(k)+F_+(k)\delta_-(k)-\left[\chi_0\left(k+\frac{1}{2}q\right)-\chi_0\left(k-\frac{1}{2}q\right)\right]T_1$$
$$+\left[1+\chi_0\left(k+\frac{1}{2}q\right)\chi_0\left(k-\frac{1}{2}q\right)\right][T_2^{(-)}+T_3^{(-)}]$$
$$=-(e\hbar/2mc)\,(2/V)^{1/2}\left[\chi_0\left(k+\frac{1}{2}q\right)-\chi_0\left(k-\frac{1}{2}q\right)\right](\boldsymbol{k}\cdot\boldsymbol{A}(\boldsymbol{q})). \tag{7·15b}$$

In terms of this notation, the gauge solution (7·9) is given by:

$$\delta_+(k)=0 \tag{7·16a}$$

$$\delta_-(k)=(e/\hbar cq)\,(2/V)^{1/2}A_\parallel(q)\left[\chi_0\left(k+\frac{1}{2}q\right)+\chi_0\left(k-\frac{1}{2}q\right)\right]. \tag{7·16b}$$

It can be shown (by specialization of the general methods used in the Appendix) that (7·16) is the exact solution of (7·15) if the vector potential is purely longitudinal.

Since our primary interest is the Meissner effect, we wish to investigate the limiting forms of all the quantities in (7·15) as the wave number q of the applied field becomes small. Keeping lowest-order terms only, we get in this limit:

$$F_+(k)\to\varepsilon_k-\mu+2L_k^{(0)}+2\chi_0(k)\,C_0(k)=\frac{1+\chi_0^2(k)}{\chi_0(k)}\,C_0(k) \tag{7·17}$$

where we have used (6·8) in the last step.

$$F_-(k)\to(\boldsymbol{k}\cdot\boldsymbol{q})\left(\frac{\hbar^2}{2m}+\frac{1}{k}\frac{dL^{(0)}}{dk}\right) \tag{7·18}$$

$$T_1\to\sum_{k'}[2G(\boldsymbol{q})-G(\boldsymbol{k}-\boldsymbol{k}')]2\psi_0(k')\delta_+(k')[1+\rho_{k'}^{(0)}]^{-1}$$
$$+\sum_{k'}G(\boldsymbol{k}-\boldsymbol{k}')\frac{\boldsymbol{q}\cdot\boldsymbol{k}'}{[1+\rho_{k'}^{(0)}]^2}\frac{1}{k'}\frac{d\chi_0}{dk'}\delta_-(k') \tag{7·19}$$

$$T_2^{(\pm)}\to\sum_{k'}G(\boldsymbol{k}-\boldsymbol{k}')[1+\rho_{k'}^{(0)}]^{-2}\delta_\pm(k') \tag{7·20}$$

$$T_3^{(\pm)}\to\sum_{k'}G(\boldsymbol{k}-\boldsymbol{k}')[\psi_0(k')]^2\delta_\pm(k'). \tag{7·21}$$

In this same limit, the gauge solution (7·16) becomes

$$\delta_+(k)=0 \tag{7·22a}$$

$$\delta_-(k)=(2e/\hbar cq)\,(2/V)^{1/2}A_{\parallel}(q)\chi_0(k). \tag{7·22b}$$

As a function of the direction of the vector $\boldsymbol{k}$, this limiting form of the gauge solution is *angle-independent*. Such a solution can exist *only* for the longitudinal part of the vector potential, since only for this part do we get an angle dependence of form $\boldsymbol{k}\cdot\boldsymbol{q}$ on the right-hand side of Eq. (7·15a).*

We now turn our attention to the transverse part $\boldsymbol{A}_\perp(\boldsymbol{q})$ of the vector potential; the angle-dependence of the right-hand side of (7·15a) is $\boldsymbol{k}\cdot\boldsymbol{A}_\perp$ (in the limit of small q), of (7·15b) is $(\boldsymbol{k}\cdot\boldsymbol{q})\,(\boldsymbol{k}\cdot\boldsymbol{A}_\perp)$. We therefore attempt a solution of the following form:

$$\delta_+(k)=(\boldsymbol{k}\cdot\boldsymbol{A}_\perp)R_+(k) \tag{7·23a}$$

$$\delta_-(k)=(\boldsymbol{k}\cdot\boldsymbol{q})\,(\boldsymbol{k}\cdot\boldsymbol{A}_\perp)R_-(k) \tag{7·23b}$$

where the functions $R(k)$ depend only on the absolute value of the vector $\boldsymbol{k}$, in the limit of small q, and (in that same limit) have a magnitude proportional to $|\boldsymbol{A}_\perp(\boldsymbol{q})|$ but independent of q. We substitute this Ansatz into Eq. (7·19) to obtain the limiting form:

$$T_1=-2(\boldsymbol{k}\cdot\boldsymbol{A}_\perp)S_1+\text{Order } q^2 \tag{7·24a}$$

where

$$\boldsymbol{k}S_1(k)\equiv\sum_{k'}G(\boldsymbol{k}-\boldsymbol{k}')\,\frac{\psi_0(k')R_+(k')}{1+\rho_{k'}^{(0)}}\boldsymbol{k}'. \tag{7·24b}$$

The fact that the sum on the right of (7·24b) has the same direction as the vector $\boldsymbol{k}$ follows immediately from the assumed symmetry of $R_+(k)$. By a similar argument, we obtain

$$T_2^{(+)}-T_3^{(+)}=(\boldsymbol{k}\cdot\boldsymbol{A}_\perp)S_2+\text{Order } q^2 \tag{7·25a}$$

where

$$\boldsymbol{k}S_2(k)=\sum_{k'}G(\boldsymbol{k}-\boldsymbol{k}')\,\frac{(1-\rho_{k'}^{(0)})R_+(k')}{(1+\rho_{k'}^{(0)})^2}\boldsymbol{k}' \tag{7·25b}$$

and

$$T_2^{(-)}+T_3^{(-)}=(\boldsymbol{k}\cdot\boldsymbol{q})\,(\boldsymbol{k}\cdot\boldsymbol{A}_\perp)S_4(k)+\text{Order } q^3 \tag{7·26a}$$

where S_4 is defined by (μ and ν are Cartesian components)

$$\sum_{k'}G(\boldsymbol{k}-\boldsymbol{k}')\,\frac{R_-(k')}{1+\rho_{k'}^{(0)}}k_\mu' k_\nu'=S_3(k)\delta_{\mu\nu}+S_4(k)k_\mu k_\nu. \tag{7·26b}$$

---

* In Eq. (7.15b), there are terms independent of angle which cancel identically with this solution; terms with angle-dependence $(\boldsymbol{k}\cdot\boldsymbol{q})^2$ need more accuracy than the simple limits (7·17)–(7·22) can provide, but we know from the Appendix that the gauge solution is exact.

With these limiting forms, Eqs. (7·15) become, except for terms of relative order q^2,

$$\frac{1+\chi_0^2(k)}{\chi_0(k)} C_0(k) R_+(k) - 4\chi_0(k) S_1(k) + [1-\chi_0^2(k)] S_2(k) = (e\hbar/mc)(2/V)^{1/2}\chi_0(k) \tag{7·27a}$$

and

$$\left(\frac{\hbar^2}{2m} + \frac{1}{k}\frac{dL^{(0)}}{dk}\right) R_+(k) + \frac{1+\chi_0^2(k)}{\chi_0(k)} C_0(k) R_-(k) + \frac{2}{k}\frac{d\chi_0}{dk} S_1(k) + [1+\chi_0^2(k)] S_4(k) = -(e\hbar/2mck)\frac{d\chi_0}{dk}. \tag{7·27b}$$

The angle dependences $\boldsymbol{k}\cdot\boldsymbol{A}_\perp$ and $(\boldsymbol{k}\cdot\boldsymbol{q})(\boldsymbol{k}\cdot\boldsymbol{A}_\perp)$ have cancelled out in these equations, and there is no explicit dependence on the quantity q; hence the Ansatz (7·23) provides a consistent solution in the limit of small q. We shall not need to solve the system of Eqs. (7·27), since we shall obtain all we need for the Meissner effect from the *form* (7·23).

§ 8. The Meissner effect

We now use the weak-field solutions obtained in Section 7 to calculate the expectation value of the current operator (3·1). Keeping terms linear in $A(q)$ only, and remembering that the field-free state carries no current, we get

$$\langle j\rangle = \mathrm{tr}(\boldsymbol{j}(\boldsymbol{q})h) = \mathrm{tr}(\boldsymbol{j}_1(\boldsymbol{q})\delta h) + \mathrm{tr}(\boldsymbol{j}_2(\boldsymbol{q})h^{(0)}) + \text{Order } A^2. \tag{8·1}$$

Straightforward use of (2·8b), (6·2), and (3·16) gives for the second term on the right:

$$\mathrm{tr}(\boldsymbol{j}_2(\boldsymbol{q})h^{(0)}) = -\frac{Ne^2}{Vmc}\boldsymbol{A}(\boldsymbol{q}). \tag{8·2}$$

This is the part of the current corresponding to the first term on the right of (1·5). The result (8·2) by itself is gauge-dependent, and not in agreement with the London equation (3·2).

The first term on the right of (8·1) is associated with the "flexibility" of the wave function Ψ, (2·11). Straightforward use of (2·7b), (7·5), (7·6), and (7·10) gives

$$\mathrm{tr}(\boldsymbol{j}_1(q)\delta h) = -\frac{e\hbar}{m}\left(\frac{2}{V}\right)^{1/2}\sum_{\boldsymbol{k}} \frac{\boldsymbol{k}\left(\chi_0\left(k+\frac{1}{2}q\right)-\chi_0\left(k-\frac{1}{2}q\right)\right)}{(1+\rho^{(0)}_{k+(1/2)q})(1+\rho^{(0)}_{k-(1/2)q})}\delta_-(k). \tag{8·3}$$

Since this expression is linear in the wave function perturbation contained in $\delta_-(k)$, we can add the results obtained from the gauge perturbation, (7·16), and from the transverse vector potential perturbation, (7·23).

It is shown in the Appendix that substitution of (7·16b) into (8·3) leads exactly, for all q, to the result

$$\mathrm{tr}(\boldsymbol{j}_1(\boldsymbol{q})\delta h_{\parallel}) = +\frac{Ne^2}{Vmc}\boldsymbol{A}_{\parallel}(\boldsymbol{q}). \tag{8·4}$$

This contribution therefore cancels exactly the undesirable portion of (8·2); the wave function is "flexible" with respect to a longitudinal vector potential.

The situation is completely different for the perturbation induced by a transverse vector potential, (7·23). When we substitute (7·23b) into (8·3) and go to the limit of small q, we are led to a sum of the form

$$-\frac{e\hbar}{m}\left(\frac{2}{V}\right)^{1/2}\sum_{k} k_\mu k_\nu (\boldsymbol{k}\cdot\boldsymbol{q})^2 \frac{R_-(k)}{(1+\rho_k^{(0)})^2}\frac{1}{k}\frac{d\chi_0}{dk} = S_5 q^2\delta_{\mu\nu} + S_6 q_\mu q_\nu. \tag{8·5}$$

The quantities S_5 and S_6 defined by this equation are independent of q, since $R_-(k)$ is itself independent of q. Using this, we obtain the result:

$$\mathrm{tr}(j_1(q)\delta h_{\perp}) = S_5 q^2 \boldsymbol{A}_{\perp}(\boldsymbol{q}) + \text{Order } q^4. \tag{8·6}$$

Since $q^2\boldsymbol{A}_{\perp}(\boldsymbol{q})$ is the Fourier transform of $\nabla\times\nabla\times\boldsymbol{A} = \nabla\times\boldsymbol{B}$, this term is just ordinary magnetism, of no interest for the Meissner effect. In the limit of small q, we therefore obtain:

$$\langle \boldsymbol{j}\rangle = \mathrm{tr}(\boldsymbol{j}(\boldsymbol{q})h) = -\frac{Ne^2}{Vmc}\boldsymbol{A}_{\perp}(\boldsymbol{q}) + \text{Order } q^2. \tag{8·7}$$

This is precisely the London equation (3·2).

§ 9. Discussion

The proof of the Meissner effect in the quasi-chemical equilibrium theory of superconductivity, contained in this paper, is subject only to the initial assumptions listed in Section 2. *No approximation of any kind was made thereafter.* In particular, the restriction to small q in Sections 7 and 8 is not an approximation, but is a restriction imposed by the nature of the phenomenon: The Meissner effect is related to field expulsion at large distances x into the specimen; in Fourier transform language, large x means small q. The integral equations (7·15) are valid for arbitrary q, and their solution would yield the law of field penetration not only at large distances x, but also within the London skin depth region, i.e. x of order 10^{-5} cm. We prefer not to go into this here, since the initial assumption of an ordinary potential $V(\boldsymbol{r}-\boldsymbol{r}')$ as electron-electron interaction is too unrealistic to make such a calculation worth the trouble. Extension of this calculation to include the electron-phonon interaction is now under way, and will be reported later.

The discussion of the field-free case, in Section 6, is of interest in connection with the problem of "dirty" superconductors.[30] In Section 6, we used full trans-

lation invariance of the Hamiltonian H_0 as one of the assumptions in the proof of the "simple pairing" condition on the pair wave function. Anderson[30] has conjectured that the electron pairs in "dirty" superconductors also have "simple" pair wave functions, with the pairing involving pairs of single-particle states which are inverse under the time reversal operation. We cannot agree with this: Once translation invariance, and hence conservation of $\boldsymbol{k}_1+\boldsymbol{k}_2$, is lost, there is no reason to expect "simple" pair wave functions $\varphi(k_1, k_2)$; in fact, our own calculation already provides a counter-example, since the introduction of the vector potential, which destroys full translation invariance, also destroys the "simple pairing" property of the pair wave function. The Bose-Einstein condensation of electron pairs into one pair state $\varphi(k_1, k_2)$ is not equivalent to a simple pairing of one-particle states; thus, the methods developed here and in reference 25) are necessary, not only for a discussion of the Meissner effect in "clean" superconductors, but also for many other aspects of the theory of superconductivity.

In conclusion, it is our pleasant duty to acknowledge useful and informative discussions on these matters with Professors S. T. Butler, H. Fröhlich, R. M. May, J. E. and M. G. Mayer, and G. Wentzel.

Appendix

In this Appendix we give an explicit treatment of the infinitesimal gauge transformation (7·9). In order to get the variation of ψ', we can use the x-space result

$$\psi'(x_1, x_2)=\exp[(ie/\hbar c)(\Lambda(x_1)+\Lambda(x_2))]\psi(x_1, x_2) \qquad (A\cdot 1)$$

together with (4·15) and the assumption that $A_{ij}(q)$ is small; or alternatively we may use the general expression (7·4) together with the Fourier transform of the identity (4·10):

$$h_{kk'}=\sum_{k''}\chi(k, k'')\psi^*(k', k'') \qquad (A\cdot 2)$$

and the related identity:

$$h_{kk'}=\sum_{k''}\psi(k, k'')\chi^*(k', k''). \qquad (A\cdot 3)$$

By either method, we arrive at the result:

$$\delta\psi(k, k')=\gamma[\psi_0(k-q, k')+\psi_0(k, k'-q)-\psi_0(k+q, k')-\psi_0(k, k'+q)]. \qquad (A\cdot 4)$$

Let us now discuss the various terms in Eq. (7·8). Consider the middle term (sum over k' and l') for a start. We define $\Phi_{mm'}$ by

$$\Phi_{mm'}=\sum_{k'l'}K_{mm',k'l'}\psi_0(k', l'). \qquad (A\cdot 5)$$

There is only one property of K which we need to establish gauge invariance, namely the "translation" property:

$$K_{k,l;k'+q,l'}=K_{k-q,l;k',l'},\quad K_{k,l;k',l'+q}=K_{k,l-q;k',l'}. \tag{A·6}$$

This property holds for arbitrary q. It is a direct consequence of (2·5). Substitution of (A·4) and the use of (A·6) leads to the result:

$$\sum_{k',l'} K_{mm',k'l'}\,\delta\psi(k',l')=\gamma(\Phi_{m-q,m'}+\Phi_{m,m'-q}-\Phi_{m+q,m'}-\Phi_{m,m'+q}). \tag{A·7}$$

Similarly, let us define $Q_{mm'}$ by

$$Q_{mm'}=\sum_{k,l,k',l'}\psi_0^*(k,l)\,K_{kl,k'l'}\,\chi_0(k',m)\,\chi_0(l',m'). \tag{A·8}$$

Then a straightforward calculation gives, for the quadruple sum in (7·8),

$$\text{Last sum in (7·8)}=\gamma(Q_{m-q,m'}+Q_{m,m'-q}-Q_{m+q,m'}-Q_{m,m'+q}). \tag{A·9}$$

Next, we define $P_{mm'}$ by

$$P_{mm'}=\sum_k L^{(0)}_{mk}\chi_0(k,m'). \tag{A·10}$$

Then a somewhat more complicated calculation, using the identities (A·2) and (A·3) as well as the translation property (A·6) and the variation (7·7), leads to the result:

$$\sum_k[(\delta L)_{mk}\chi_0(k,m')+L^{(0)}_{mk}\delta\chi(k,m')]=\gamma(P_{m-q,m'}+P_{m,m'-q}-P_{m+q,m'}-P_{m,m'+q}). \tag{A·11}$$

When we consider the term in (7·8) involving $J^{(0)}_{mk}$, the translation property no longer holds. Rather, direct use of (6·4) gives the identities:

$$J^{(0)}_{k,k'-q}=J^{(0)}_{k+q,k'}-\delta_{k+q,k'}(\hbar^2/m)\,\boldsymbol{q}\cdot\left(\boldsymbol{k}-\frac{1}{2}\boldsymbol{q}\right) \tag{A·12}$$

$$J^{(0)}_{k,k'+q}=J^{(0)}_{k-q,k'}+\delta_{k-q,k'}(\hbar^2/m)\,\boldsymbol{q}\cdot\left(\boldsymbol{k}+\frac{1}{2}\boldsymbol{q}\right). \tag{A·13}$$

With the definition

$$I_{mm'}=\sum_k J^{(0)}_{mk}\chi_0(k,m') \tag{A·15}$$

we get the result

$$\sum_k J^{(0)}_{mk}\delta\chi(k,m')=\gamma(I_{m-q,m'}+I_{m,m'-q}-I_{m+q,m'}-I_{m,m'+q})$$
$$+(\gamma\hbar^2/m)\left[\boldsymbol{q}\cdot\left(\boldsymbol{m}-\frac{1}{2}\boldsymbol{q}\right)\chi_0(m-q,m')+\boldsymbol{q}\cdot\left(\boldsymbol{m}+\frac{1}{2}\boldsymbol{q}\right)\chi_0(m+q,m')\right]. \tag{A·15}$$

Finally, we get from (7·2)

$$\sum_k(\delta J)_{mk}\chi_0(k,m')=-(e\hbar/2mcq)A_{\parallel}(q)\,(2/V)^{1/2}$$
$$\times\left[\boldsymbol{q}\cdot\left(\boldsymbol{m}+\frac{1}{2}\boldsymbol{q}\right)\chi_0(m+q,m')+\boldsymbol{q}\cdot\left(\boldsymbol{m}-\frac{1}{2}\boldsymbol{q}\right)\chi_0(m-q,m')\right]. \tag{A·16}$$

We now have all the terms we need for (7·8). When everything is substituted, the result is identically zero, because the zero-order function satisfies (5·14) by assumption, i.e.

$$I_{mm'}+2P_{mm'}+\Phi_{mm'}-Q_{mm'}=0 \qquad (A\cdot 17)$$

for *all* values of m and m', including, for example, $m-q$, m', etc. The only terms not covered by this argument are the second line of (A·15) and the whole of (A·16); these, however, just cancel identically when use is made of the definition (4·16) of γ.

Our proof here uses only one property of the interaction, namely (A·6). The "translation" involved here is a translation in k-space, not in x-space. Thus, in particular, we need not assume that $V(\boldsymbol{r}, \boldsymbol{r}')$ is invariant with respect to translation in x-space, in order to establish gauge invariance. The assumption $V(\boldsymbol{r}, \boldsymbol{r}')=V(\boldsymbol{r}-\boldsymbol{r}')$ does simplify other aspects of the calculations, however, since it is necessary to get the "simple pairing" condition for the zero-order pair wave function.

In conclusion, we establish the identity (8·4) for this gauge solution. Substitution of (7·16b) into (8·3) gives, using (6·2),

$$\mathrm{tr}\,(\boldsymbol{j}_1(\boldsymbol{q})\,\delta h)=\frac{2e^2A_{\parallel}}{Vmcq}\sum_{\boldsymbol{k}}\boldsymbol{k}(h^{(0)}_{\boldsymbol{k}-(1/2)\boldsymbol{q}}-h^{(0)}_{\boldsymbol{k}+(1/2)\boldsymbol{q}})=\frac{2e^2A_{\parallel}}{Vmcq}\,\boldsymbol{q}\sum_{\boldsymbol{k}}h^{(0)}_{\boldsymbol{k}}. \qquad (A\cdot 18)$$

Since $\boldsymbol{A}_{//}(\boldsymbol{q})=(\boldsymbol{q}/q)A_{//}(q)$ and since the sum over $\boldsymbol{k}$ in the last line of (A·18) is just half the trace of the operator h which equals N by (3·16), the relation (8·4) follows.

References

1) F. London, *Superfluids*, (John Wiley and Sons, New York, 1950) Vol. 1.
2) J. Bardeen, Phys. Rev. **81** (1951), 469; article in *Encyclopedia of Physics*.
3) M. J. Buckingham, Nuovo Cimento **5** (1957), 1763.
4) J. Bardeen, Nuovo Cimento **5** (1957), 1764.
5) V. L. Ginzburg, Usp. Fiz. Nauk **48** (1952), 25; Translation in Fortschritte der Physik (Berlin) **1** (1953), 101.
6) R. P. Feynman, Proc. Kyoto Conference (1953).
7) M. R. Schafroth, Phys. Rev. **100** (1955), 463.
8) M. R. Schafroth, Phys. Rev. **96** (1954), 1442, L.
9) M. R. Schafroth, S. T. Butler and J. M. Blatt, Helv. Phys. Acta **30** (1957), 93.
10) T. Matsubara and J. M. Blatt, Prog. Theor. Phys. **23** (1960), 451.
11) J. Bardeen, L. N. Cooper and J. R. Schrieffer, Phys. Rev. **108** (1957), 1175.
12) We are grateful to Professors H. Fröhlich and J. E. Mayer for discussions concerning this point.
13) H. Fröhlich, Phys. Rev. **79** (1950), 845.
14) G. Rickayzen, Phys. Rev. **111** (1958), 817.
15) Bogoliubov, Tolmachev and Shirkov, Dubna (1958).
16) G. Wentzel, Phys. Rev. **111** (1958), 1488.
17) J. M. Blatt, T. Matsubara and R. M. May, Prog. Theor. Phys. **21** (1959), 745.
18) R. M. May and M. R. Schafroth, Phys. Rev. **115** (1959), 1446.

19) G. Wentzel, Report at the Cambridge Conference on Superconductivity (1959).
20) P. Anderson, Phys. Rev. **110** (1958), 827; **112** (1958), 1900.
20a) G. Rickayzen, Phys. Rev. **115** (1959), 795.
21) J. M. Blatt and T. Matsubara, Prog. Theor. Phys. **20** (1958), 781, L.
22) K. Yosida, Prog. Theor. Phys. **21** (1959), 731.
23) N. Bogoliubov, Soviet Physics-Uspekhi **67** (1959), 236.
24) Y. Nambu, Phys. Rev. **117** (1960), 648.
24a) L. P. Kadanoff and P. C. Martin, to be published. The author is grateful to Drs. Kadanoff and Martin for sending him a preprint.
25) J. M. Blatt, to be published in the Journal of the Australian Mathematical Society.
26) J. M. Blatt, Prog. Theor. Phys. **23** (1960), 447.
27) J. J. Valatin, Nuovo Cimento **7** (1957), 843.
28) K. Yosida, Phys. Rev. **111** (1958), 1255.
29) A. B. Pippard, Proc. Roy. Soc. (London) **A216** (1953), 547.
30) P. Anderson, Report at the Cambridge Conference on Superconductivity (1959).

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Elastic Ether Theory of Elementary Particles. I

——*Classical Theory*——

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A unified model of elementary particles based upon a renewed idea of "ether" is proposed. In this paper a general classical theory of the relativistic elastic continuum is developed. From the result of the general theory a hypothetical relativistic elastic medium "ether" is introduced. It is an unusual elastic medium which has no mass and no energy in the undeformed state but has mass and energy in the deformed states which are due entirely to the elastic self-interaction of it. It is shown that if the ether is elastically isotropic then it has a conserving quantity called "the Lagrange spin" which has very similar property as that of the iso-spin. It is not conserved if the elastic property of the ether is anisotropic. In our model the elementary particles are viewed as the excited states of the ether and the strong, the electro-magnetic and the weak interactions of them are ascribed to the isotropic, cylindrically symmetric and anisotropic elastic self-interaction of the ether, respectively.

Introduction

In the present day formulation of elementary particle theory every elementary particle is represented by a wave field. To represent an elementary particle by a wave field has its origin in the classical Faraday-Maxwell theory of electromagnetism. Since the discovery of the de Broglie wave and the establishment of quantum mechanics it has become the general trend to interpret the wave field as the probability field representing the state of the elementary particle.

However, in the old time before the appearance of special theory of relativity the concept of wave field was often considered in connection with the motion of the continuously extended substance such as the elastic medium or fluid. In order to interpret the optical and the electro-magnetic phenomena the hypothetical medium "ether" was introduced and the light was interpreted as the elastic wave in it.[1)]

Since the appearance of special theory of relativity this line of thought together with the ether has been abandoned and disregarded for long time.

However, the present day field theory of elementary particles has completely failed to explain the hierarchy of the elementary particles and their interactions and, moreover, it has the fundamental divergence difficulty inherent to it. Consequently, several attempts to revise the present day concept of matter and to construct the unified theory of elementary particles have been proposed.

We can find one of the oldest approaches to the unified theory of matter in the attempt of the vortex atom.[2)] It was proposed to interpret the atom as the

vortex motion of the ether. Although the attempt was then not successful it seems to us an attractive idea in constructing the unified theory of elementary particles.

The wave as the probability wave and the wave as the motion of the continuous medium are essentially different concepts. The former is an essentially quantum mechanical concept and the latter is completely classical. We have the probability wave even if we quantize the point particle. Hence if we start with a local field which is to be interpreted as the probability wave as the basis of the unified theory it seems to be very difficult to escape from the point particle character underlying it. On the other hand, the wave field representing the motion of the continuous matter is related to the spacially extended substance even if it is a local field and the degrees of freedom of the field is larger than the local probability field. So we may expect essentially different results in this case.

In this and subsequent papers we will try to reexamine the old idea of ether from the modern point of view and to construct a unified theory of elementary particles in terms of the ether.

In this paper we develop a general classical theory of relativistic elastic continua. The method used in this paper is in close analogy with the non-relativistic theory of elasticity with finite deformation. A modern version of it can be found in the book by Green and Zerna[3] (especially Chapter II).

In § 1 the coordinate systems which are suitable for the description of the relativistic continuous medium are introduced. They are the Euler system of coordinates and the Lagrange system of coordinates. In § 2 the definition of the strain tensor is generalized to the relativistic case. In § 3 the general form of the energy momentum tensor of the relativistic continuous medium is derived. In § 4 we derive the condition for the existence of the action from which the equations of motion, which represent the conservation of energy and momenta of the medium, is derived by the variation principle. Such class of media is called "elastic". In § 5 and § 6 more detailed discussions on the energy momentum tensor and the equation of motion of the elastic medium are given. A generalization of the theory to include the effects of gravitational field is suggested. In § 7 we discuss on the conserving quantities of the elastic medium. It is shown that if it is isotropic, then it has another conserving quantity called "the Lagrange spin" besides the four momenta and the angular momenta. The Lagrange spin is invariant under the Lorentz transformation of the Euler coordinates but behaves as an angular momentum under the rotation of the Lagrange coordinates. If the medium is not isotropic then it does not conserve. In § 8 a possible model of "ether" is given. The ether is the hypothetical relativistic medium which fills whole space and it has no mass and no energy in the undeformed state but it may have mass and energy owing to its elastic self-interaction in the deformed states. In § 9 a detailed discussion on the conserving quantities is given. The motion of the elastic medium is separated into the motion of the proper center of mass and the internal

motion. The expressions for the mass, the spin and the Lagrange spin in terms of the variable of the internal motion are derived. In § 10 the canonical formalism of the relativistic elastic medium is given. The expressions of the mass, the spin and the Lagrange spin in terms of the canonical variables are given. Finally, in § 11 a unified model of the elementary particles and their interactions in terms of the ether is proposed. The elementary particles are viewed as the excited states of the ether and the Lagrange spin is identified with the isospin. The strong, the electromagnetic and the weak interactions are attributed to isotropic, the cylindrically symmetric and the anisotropic elastic self-interactions of the ether respectively.

In this paper we will not discuss the quantum theory of the ether. It shall be given in the subsequent paper.

Throughout this paper the Greek indices of vectors and tensors run from 1 to 4 and the Latin ones from 1 to 3. The usual summation convention for the dummy indices is used. The Minkowskian covariant vector x_μ is related to the contravariant vector x^μ by $x_i=x^i$ and $x_4=-x^4$ where $x^4=t$ and we use the scale of time in which the light velocity is unity. Occasionally, a four vector (x^μ) is written by a single letter x and the spacial three vector (x^i) by a Gothic letter $\mathfrak{x}$.

§ 1. Deformation of a relativistic continuous medium

Let us consider a relativistic continuous medium with three dimensional extension. The medium is called undeformed, if there is an inertial system of coordinates relative to which every points of the medium are at rest throughout all time. Let the medium be undeformed and consider an inertial system of coordinates relative to which the medium is at rest. Then let ξ^i be the spacial cartesian coordinates of a typical point of the medium in this inertial system. The point of the medium at $\boldsymbol{\xi}=(\xi^i)$ is called the point $\boldsymbol{\xi}$. Now, let the medium be in a deformed and moving state. Then consider an arbitrary inertial system of coordinates x^μ. Let the equation of the trajectory of the point $\boldsymbol{\xi}$ in the x system of coordinates be

$$x^\mu=x^\mu(\boldsymbol{\xi}, \xi^4) \tag{1·1}$$

where ξ^4 is a time parameter measured by an arbitrary coordinate clock. Solving (1·1) with respect to ξ^μ, we have

$$\xi^i=\xi^i(x), \quad \xi^4=\xi^4(x). \tag{1·2}$$

The first equations of (1·2) give which point of the medium is at the world point x and the second equation gives the equation of the spacelike surfaces on which the coordinate clock shows the equal time.

The coordinate system of x^μ is an arbitrarily chosen inertial system of coordinates without regard to the state of the medium and we call it "Euler frame (*E*-frame)". Any inertial system of coordinates may be used as an *E*-frame and

the theory must be covariant under the inhomogeneous Lorentz transformation between E-frames. The coordinate system of $\tilde{\xi}^\mu$ is an accelerated system of coordinates fixed to the medium and moving along with the motion of it. The space part $\boldsymbol{\xi}$ of the $\tilde{\xi}$ coordinate system is the name given for the point of the medium rather than the point in the abstract space. We call the coordinate system of $\tilde{\xi}$ "Lagrange frame (L-frame)". There are also arbitrarinesses in the choice of L-frame. Namely any $\tilde{\xi}'$ which is related to $\tilde{\xi}$ by

$$\tilde{\xi}'^i = a^{ij}\tilde{\xi}^j + b^i \tag{1·3}$$

$$\tilde{\xi}'^4 = \tilde{\xi}'^4(\tilde{\xi}) \tag{1·4}$$

may equally be used as a L-frame where (a^{ij}) is an arbitrary orthogonal transformation of the spacial axes of L-frame. The transformation (1·4) represents a change of the rate of the coordinate clock and the theory must be invariant under the transformation (1·4) since the coordinate clock may be chosen arbitrarily. However, the theory is not necessarily covariant under the transformation (1·3) since (1·3) represents a change of the name of the points of the medium and it may cause non-covariant effects if the medium has anisotropic mechanical properties. It must be noted that the coordinate transformations in E-frame and L-frame are made independently.

If the medium is in the undeformed state, then by choosing suitable clock (1·1) is given in the form

$$x^\mu = \alpha_\nu{}^\mu \tilde{\xi}^\nu + \beta^\mu \tag{1·5}$$

where $(\alpha_\nu{}^\mu)$ is the Lorentz transformation which represents the uniform motion of the medium.

By the transformation (1·2) the line element of E-frame is transformed to

$$(ds)^2 = dx^\mu dx_\mu = g_{\alpha\beta} d\tilde{\xi}^\alpha d\tilde{\xi}^\beta \tag{1·6}$$

where $g_{\alpha\beta} = \partial x^\mu/\partial\tilde{\xi}^\alpha \cdot \partial x_\mu/\partial\tilde{\xi}^\beta$ is the metric tensor of L-frame. The four-velocity of the medium is given by

$$U^\mu = \frac{1}{\sqrt{-g_{44}}} \frac{\partial x^\mu}{\partial \tilde{\xi}^4}. \tag{1·7}$$

Let us consider two points $\boldsymbol{\xi}$ and $\boldsymbol{\xi}+d\boldsymbol{\xi}$ of the medium with an infinitesimal separation. In order to correctly measure the rate of deformation of the relativistic medium we have to eliminate the effect of Lorentz contraction which does not represent a real deformation. This is accomplished by measuring the rate of deformation at a point of the medium in the spacial orientations which are orthogonal to the four velocity U^μ of the medium at the point under consideration. The separation dx^μ between the point $\boldsymbol{\xi}$ at time $\tilde{\xi}^4$ and the point $\boldsymbol{\xi}+d\boldsymbol{\xi}$ at time $\tilde{\xi}^4+d\tilde{\xi}^4$ is

$$dx^\mu = \frac{\partial x^\mu}{\partial \tilde{\xi}^\alpha} d\tilde{\xi}^\alpha.$$

We choose $d\hat{\xi}^4$ so as to let dx^μ be orthogonal to U^μ and write the value of dx^μ with this orientation by $dx_\perp^\mu$. Then from the equation

$$U_\mu dx_\perp{}^\mu = U_\mu \frac{\partial x^\mu}{\partial \hat{\xi}^\alpha} d\hat{\xi}^\alpha = 0,$$

we have

$$d\hat{\xi}^4 = \frac{1}{\sqrt{-g_{44}}} U_\mu \frac{\partial x^\mu}{\partial \hat{\xi}^i} d\hat{\xi}^i,$$

so we have

$$dx_\perp^\mu = A_i{}^\mu d\hat{\xi}^i, \quad A_i{}^\mu = \frac{\partial x^\mu}{\partial \hat{\xi}^i} + U^\mu \gamma_i, \tag{1·8}$$

where

$$\gamma_i = U_\mu \frac{\partial x^\mu}{\partial \hat{\xi}^i} = \frac{g_{4i}}{\sqrt{-g_{44}}}.$$

This $A_i{}^\mu$ correctly represents the rate of deformation in the infinitesimal neighbourhood of the point under consideration.

From the definition of A_i^μ it is easy to see

$$A_i{}^\mu \frac{\partial \hat{\xi}^j}{\partial x^\mu} = \delta_i^j, \quad A_i{}^\mu \frac{\partial \hat{\xi}^i}{\partial x^\nu} = \delta_\nu{}^\mu + U^\mu U_\nu, \tag{1·9}$$

hence the inverse relation of (1·8) is given by

$$d\hat{\xi}^i = \frac{\partial \hat{\xi}^i}{\delta x^\mu} dx_\perp^\mu. \tag{1·10}$$

§ 2. Strain tensor

Although the quantity $A_i{}^\mu$ correctly represents the rate of deformation, its value depends on the choice of E-frame. In order to eliminate this defect it suffices to use the length of the vector $dx_\perp^\mu$ as the measure of deformation. From (1·8) we have

$$dx_\perp^\mu dx_{\perp\mu} = \gamma_{ij} d\hat{\xi}^i d\hat{\xi}^j \tag{2·1}$$

where

$$\gamma_{ij} = A_i{}^\mu A_{\mu j} = g_{ij} + \gamma_i \gamma_j. \tag{2·2}$$

This γ_{ij} coincides with the covariant spacial metric tensor of L-frame. We call γ_{ij} " the covariant strain tensor ". γ_{ij} is invariant for the inhomogeneous Lorentz transformation of E-frame and the transformation (1·4) of L-frame and transforms as a symmetric tensor for the transformation (1·3) of L-frame. Let us define the tensor η^{ij} by

$$\eta^{ij} = \frac{\partial \hat{\xi}^i}{\partial x^\mu} \frac{\partial \hat{\xi}^j}{\partial x_\mu}. \tag{2·3}$$

From (1·11) we see

$$\gamma_{ij}\eta^{jk}=\delta_i^{\ k}. \tag{2·4}$$

This tensor η^{ij} is called "contravariant strain tensor" and coincides with the contravariant spacial metric tensor of L-frame. If the medium is undeformed, then we have

$$\gamma_{ij}=\eta^{ij}=\delta_{ij}. \tag{2·5}$$

Since the strain tensor γ_{ij} is symmetric, we may construct from γ_{ij} three independent invariants for the transformation (1·3) of L-frame. They are

$$J_1=\gamma_{ii},\quad J_2=1/2(J_1^2-\gamma_{ij}\gamma_{ij}),\quad J_3=\gamma=\det(\gamma_{ij}). \tag{2·6}$$

Also, from η^{ij} we have

$$K_1=\eta^{ii},\quad K_2=1/2(K_1^2-\eta^{ij}\eta^{ij}),\quad K_3=\eta=\det(\eta_{ij}). \tag{2·6}'$$

J_i and K_i are not independent:

$$K_1=\frac{J_2}{J_3},\quad K_2=\frac{J_1}{J_3},\quad K_3=\frac{1}{J_3}.$$

We call these invariants "the strain invariants".

Since the matrix (γ_{ij}) is a symmetric and positive definite, it has three real positive eigenvalues ε_i. These eigenvalues, ε_i, are called "principal strains". The strain invariants J_i depend only on ε_i.

$$J_1=\varepsilon_1+\varepsilon_2+\varepsilon_3,\quad J_2=\varepsilon_1\varepsilon_2+\varepsilon_2\varepsilon_3+\varepsilon_3\varepsilon_1,\quad J_3=\varepsilon_1\varepsilon_2\varepsilon_3, \tag{2·7}$$

hence they are symmetric function of the principal strains.

Let Δ be the Jacobian of the transformation from L-frame to E-frame:

$$\Delta=\det\left(\frac{\partial x^\mu}{\partial\xi^\alpha}\right),\quad d^4x=\Delta d^4\xi. \tag{2·8}$$

Then it is easy to see

$$\Delta=\sqrt{\gamma}\sqrt{-g_{44}},\quad \gamma=\det(\gamma_{ij}). \tag{2·9}$$

The volume element in L-frame and the invariant volume element in E-frame are related by the relation

$$U^4d^3x=\sqrt{\gamma}\,d^3\xi, \tag{2·10}$$

so $\sqrt{\gamma}$ is the expansion ratio of the volume of E-frame to the volume of L-frame.

§ 3. Energy momentum tensor[4)]

Let the system of the medium be closed. As is well known, for any relativistic closed system there is a symmetric energy momentum tensor $T^{\mu\nu}$ satisfying the equation

$$\frac{\partial T^{\mu\nu}}{\partial x^{\mu}}=0. \tag{3·1}$$

Let $t^{\mu\nu}(x)$ be the energy momentum tensor in the local rest inertial frame of the point of the medium at x. Then

$$t^{4i}=t^{i4}=0. \tag{3·2}$$

Denote t^{44} by ρ. ρ is the proper mass density of the medium. Using $t^{\mu\nu}$, $T^{\mu\nu}$ is expressed in the form

$$T^{\mu\nu}=L_{\lambda}{}^{\mu}t^{\lambda\sigma}L_{\sigma}{}^{\nu} \tag{3·3}$$

where $(L_{\nu}{}^{\mu})$ is the Lorentz transformation from the local rest inertial frame to E-frame. The explicit forms of $L_{\nu}{}^{\mu}$ are

$$L_{4}{}^{\mu}=U^{\mu},\quad L_{\mu}{}^{4}=-U_{\mu},\quad L_{j}{}^{i}=\delta_{j}{}^{i}+\frac{u_{i}u_{j}}{u^{2}}\left(\frac{1}{\sqrt{1-u^{2}}}-1\right), \tag{3·4}$$

where u_i is the velocity of the medium. From (3·2), (3·3) and (3·4) we have

$$T^{\mu\nu}=\rho U^{\mu}U^{\nu}+L_{i}{}^{\mu}t^{ij}L_{j}{}^{\nu}. \tag{3·5}$$

From (3·5) we see

$$U_{\mu}T^{\mu\nu}=-\rho U^{\nu},\quad U_{\mu}T^{\mu\nu}(\delta_{\nu}{}^{\lambda}+U_{\nu}U^{\lambda})=0,$$

hence we have

$$T^{\mu\nu}=\rho U^{\mu}U^{\nu}+(\delta_{\lambda}{}^{\mu}+U^{\mu}U_{\lambda})T^{\lambda\sigma}(\delta_{\sigma}{}^{\nu}+U_{\sigma}U^{\nu}).$$

Using (1·9), we have

$$T^{\mu\nu}=\rho U^{\mu}U^{\nu}+\frac{1}{\sqrt{\gamma}}A_{i}{}^{\mu}\tau^{ij}A_{j}{}^{\nu}, \tag{3·6}$$

where

$$\tau^{ij}=\sqrt{\gamma}\,\frac{\partial\xi^{i}}{\partial x^{\mu}}T^{\mu\nu}\frac{\partial\xi^{j}}{\partial x^{\nu}} \tag{3·7}$$

is the stress tensor measured in L-frame. (3·6) is the general form of the energy momentum tensor of the relativistic continuous medium.

Define the tensor $\widetilde{T}^{\alpha\nu}$ by

$$\widetilde{T}^{\alpha\nu}=\Delta\frac{\partial\xi^{\alpha}}{\partial x^{\mu}}T^{\mu\nu}. \tag{3·8}$$

The explicit form of $\widetilde{T}^{\alpha\nu}$ are

$$\widetilde{T}^{i\nu}=\sqrt{-g_{44}}\,\tau^{ij}A_{j}{}^{\nu},\quad \widetilde{T}^{4\nu}=\kappa U^{\nu}+\gamma_{i}\tau^{ij}A_{j}{}^{\nu}, \tag{3·9}$$

where $\kappa=\rho\sqrt{\gamma}$ is the proper mass density measured in L-frame.

By definition of $\widetilde{T}^{\alpha\nu}$ holds the relation

$$T^{\mu\nu}=\frac{1}{\Delta}\frac{\partial x^{\mu}}{\partial\xi^{\alpha}}\widetilde{T}^{\alpha\nu}.$$

Then we have

$$\frac{\partial T^{\mu\nu}}{\partial x^{\mu}}=\frac{\partial\xi^{\alpha}}{\partial x^{\mu}}\frac{\partial}{\partial\xi^{\alpha}}\left(\frac{1}{\Delta}\frac{\partial x^{\mu}}{\partial\xi^{\beta}}\widetilde{T}^{\beta\nu}\right)$$
$$=\frac{1}{\Delta}\frac{\partial\widetilde{T}^{\alpha\nu}}{\partial\xi^{\alpha}}+\frac{1}{\Delta}\frac{\partial\xi^{\alpha}}{\partial x^{\mu}}\frac{\partial^{2}x^{\mu}}{\partial\xi^{\alpha}\partial\xi^{\beta}}\widetilde{T}^{\beta\nu}-\frac{1}{\Delta}\frac{\partial\Delta}{\partial\xi^{\alpha}}\widetilde{T}^{\alpha\nu}.$$

Using Eq. (3·1) and the identity

$$\frac{1}{\Delta}\frac{\partial\Delta}{\partial\xi^{\alpha}}=\frac{\partial\xi^{\beta}}{\partial x^{\mu}}\frac{\partial^{2}x^{\mu}}{\partial\xi^{\beta}\partial\xi^{\alpha}},$$

we have

$$\frac{\partial T^{\mu\nu}}{\partial x^{\mu}}=\frac{1}{\Delta}\frac{\partial\widetilde{T}^{\alpha\nu}}{\partial\xi^{\alpha}}=0. \tag{3·10}$$

The tensor $\widetilde{T}^{\alpha\nu}$ is not symmetric and the index α is referred to L-frame while the index ν to E-frame.

§ 4. Variation principle, elastic mass density

Let us derive the condition that the equation of motion (3·10) of the medium is derived by the variation principle.

Let $\delta x^{\nu}(\xi)$ be any small virtual displacement that vanishes on the boundary of the medium and at infinite past and future.

Consider the action of the virtual displacement,

$$\delta^{*}L=-\int\frac{\partial T^{\alpha\nu}}{\partial\xi^{\alpha}}\delta x_{\nu}\,d^{4}\xi, \tag{4·1}$$

where the integration is made over the interior of the medium and from the infinite past to future. By the integration by part, (4·1) becomes

$$\delta^{*}L=\int\widetilde{T}^{\alpha\nu}\,\delta\left(\frac{\partial x_{\nu}}{\partial\xi^{\alpha}}\right)d^{4}\xi, \tag{4·1}'$$

where we have used the boundary condition for the virtual displacement. Inserting (3·9), (4·1)′ becomes

$$\delta^{*}L=\int\left[\sqrt{-g_{44}}\,\tau^{ij}A_{j}{}^{\nu}\,\delta\left(\frac{\partial x_{\nu}}{\partial\xi^{i}}\right)+(\kappa U^{\nu}+\gamma_{i}\tau^{ij}A_{j}{}^{\nu})\,\delta\left(\frac{\partial x_{\nu}}{\partial\xi^{4}}\right)\right]d^{4}\xi. \tag{4·1}''$$

Using the relations

$$\frac{\partial x^{\mu}}{\partial\xi^{4}}=\sqrt{-g_{44}}\,U^{\mu},\quad A_{i}{}^{\mu}U_{\mu}=0\quad\text{and}\quad U^{\mu}U_{\mu}=-1, \tag{4·2}$$

we have the relations

$$U^\nu \delta\left(\frac{\partial x_\nu}{\partial \xi^4}\right) = -\delta(\sqrt{-g_{44}}), \quad A_j^\nu \delta\left(\frac{\partial x_\nu}{\partial \xi^4}\right) = \sqrt{-g_{44}} A_j^\nu \delta U_\nu. \tag{4·3}$$

Inserting the relations (4·2) into (4·1)″, we have

$$\delta^* L = \int \left[\sqrt{-g_{44}}\, \tau^{ij} \left(A_j^\nu \delta\left(\frac{\partial x_\nu}{\partial \xi^i} \right) + A_j^\nu \gamma_i \delta U_\nu \right) - \kappa \delta(\sqrt{-g_{44}}) \right] d^4\xi. \tag{4·4}$$

Using again the second relation of (4·2), we have

$$A_j^\nu \gamma_i \delta U_\nu = A_j^\nu \delta(\gamma_i U_\nu). \tag{4·5}$$

Noting (1·8), (2·2) and (4·5), we have

$$\delta^* L = \int \left[-\kappa \delta(\sqrt{-g_{44}}) + \frac{1}{2} \sqrt{-g_{44}}\, \tau^{ij} \delta \gamma_{ij} \right] d^4\xi, \tag{4·6}$$

where we have used that the stress tensor τ^{ij} is symmetric.

Suppose that the proper mass density κ in L-frame is a function of ξ^i and γ_{ij},

$$\kappa = \kappa(\xi^i, \gamma_{ij}).$$

Then Eq. (4·6) shows that the action $\delta^* L$ is a complete differential if the proper mass density κ and the stress tensor γ_{ij} are related by the relation

$$\tau^{ij} = -2 \frac{\partial \kappa}{\partial \gamma_{ij}}. \tag{4·7}$$

Then the action $\delta^* L$ becomes

$$\delta^* L = \delta L = -\delta \int \kappa \sqrt{-g_{44}}\, d^4\xi. \tag{4·8}$$

The medium satisfying the relation (4·7) is called "elastic". For the elastic medium the equation of motion is derived by the variation of the Lagrangian L

$$L = -\int \kappa \sqrt{-g_{44}}\, d^4\xi. \tag{4·9}$$

Write the value of the proper mass density κ when the medium is undeformed as κ_0:

$$\kappa_0 = \kappa(\xi^i, \delta_{ij}).$$

κ_0 is a function depending only on ξ^i and represents the proper mass density of the medium of its own. We can split the proper mass density κ in the deformed state as

$$\kappa = \kappa_0 + e, \tag{4·10}$$

where the function e satisfies *a fortiori*

$$e(\xi^i, \delta_{ij}) = 0. \tag{4·11}$$

The stress tensor is given in terms of e as

$$\tau^{ij}=-2\frac{\partial e}{\partial \gamma_{ij}}. \qquad (4\cdot7)'$$

If we assume that the stress in the undeformed state vanishes, then e has to satisfy

$$\left.\frac{\partial e}{\partial \gamma_{ij}}\right|_{\gamma_{ij}=\delta_{ij}}=0. \qquad (4\cdot12)$$

The function e is the density of the mass measured in L-frame which is produced by the elastic force. So we call it "the elastic mass density". It is what is called "the elastic potential" in the non-relativistic theory of elasticity.

Since the matrix (γ_{ij}) is the inverse of the matrix (η^{ij}) the proper mass density κ in L-frame may be regarded as the function of ξ^i and η^{ij}. Hence by the coordinate transformation from L-frame to E-frame the Lagrangian (4·9) may be rewritten as

$$L=-\int \rho\, d^4x, \qquad (4\cdot9)'$$

where $\rho\equiv\rho(\xi^i, \eta^{ij})=\kappa/\sqrt{\gamma}$ is the proper mass density measured in E-frame. This form of the Lagrangian is sometimes more convenient than (4·9).

§ 5. Stress strain relation in the elastic medium

From Eqs. (3·6) and (4·7) we have the energy momentum tensor of the elastic medium in terms of the strain as

$$T^{\mu\nu}=\frac{1}{\sqrt{\gamma}}\left(\kappa U^\mu U^\nu-2A_i^\mu\frac{\partial\kappa}{\partial\gamma_{ij}}A_j^\nu\right). \qquad (5\cdot1)$$

Substituting $\kappa=\sqrt{\gamma}\,\rho$, the formula (5·1) becomes

$$T^{\mu\nu}=\rho\left(U^\mu U^\nu-\frac{2}{\sqrt{\gamma}}A_i^\mu\frac{\partial\sqrt{\gamma}}{\partial\gamma_{ij}}A_j^\nu\right)-2A_i^\mu\frac{\partial\rho}{\partial\gamma_{ij}}A_j^\nu.$$

Using the relations

$$\frac{\partial\sqrt{\gamma}}{\partial\gamma_{ij}}=\frac{1}{2}\eta^{ij}\sqrt{\gamma},\quad A_i^\mu\eta^{ij}=\frac{\partial\xi^j}{\partial x^\mu} \qquad (5\cdot2)$$

and the second equation of (1·9), this becomes

$$T_\nu^\mu=-\delta_\nu^\mu\rho-2A_i^\mu\frac{\partial\rho}{\partial\gamma_{ij}}A_{\nu j}. \qquad (5\cdot3)$$

Using the relation

$$\frac{\partial\rho}{\partial\gamma_{ij}}=-\eta^{ik}\frac{\partial\rho}{\partial\eta^{kl}}\eta^{lj}$$

and the second equation of (5·2) again, we have

$$T_\nu{}^\mu = 2\frac{\partial \xi^i}{\partial x_\mu}\frac{\partial \rho}{\partial \eta^{ij}}\frac{\partial \xi^j}{\partial x^\nu} - \delta_\nu{}^\mu \rho. \tag{5·4}$$

This form of the energy momentum tensor coincides with the one derived from the Lagrangian (4·9)′ by the usual procedure.

In order to get more detailed information on the relation between the energy momentum tensor and the strain it is necessary to make a further assumption about the functional dependence of the elastic mass density e on the strain tensor.

The elastic medium is called "elastically homogeneous" if the elastic mass density e does not depend on ξ^i but depends only on the strain tensor. The elastic medium is called "elastically isotropic" if the elastic mass density depends on the strain tensor only through the strain invariants and is invariant for the rotation of the spacial axes of L-frame, i.e. it depends on ξ^i through scalar functions of ξ^i. If the medium is elastically homogeneous and isotropic, then the elastic mass density is a function of the strain invariants J_i (or K_i) only,

$$e = e(J_i).$$

In the following we assume that the medium is elastically isotropic.

From the definition (2·6) of the strain invariants we have

$$\frac{\partial J_1}{\partial \gamma_{ij}} = \delta_{ij}, \quad \frac{\partial J_2}{\partial \gamma_{ij}} = J_1\delta_{ij} - \gamma_{ij}, \quad \frac{\partial J_3}{\partial \gamma_{ij}} = \eta^{ij} J_3. \tag{5·5}$$

Accordingly, we have the stress tensor as

$$\tau^{ij} = -2\left(\frac{\partial e}{\partial J_1} + J_1\frac{\partial e}{\partial J_2}\right)\delta_{ij} + 2\frac{\partial e}{\partial J_2}\gamma_{ij} - 2J_3\frac{\partial e}{\partial J_3}\eta^{ij}. \tag{5·6}$$

Inserting Eq. (5·6) into Eq. (3·6) and using the relation

$$A_i{}^\mu \eta^{ij} A_{\nu j} = \delta_\nu{}^\mu + U^\mu U_\nu,$$

we have

$$T_\nu{}^\mu = \left(\rho - 2\sqrt{J_3}\,\frac{\partial e}{\partial J_3}\right)U^\mu U_\nu - 2\sqrt{J_3}\,\frac{\partial e}{\partial J_3}\delta_\nu{}^\mu - \frac{2}{\sqrt{J_3}}\left(\frac{\partial e}{\partial J_1} + J_1\frac{\partial e}{\partial J_2}\right)Q_\nu{}^\mu + \frac{2}{\sqrt{J_3}}\frac{\partial e}{\partial J_2}Q_\lambda{}^\mu Q_\nu{}^\lambda, \tag{5·7}$$

where $Q_\nu{}^\mu = A_i{}^\mu A_{\nu i}$. Assume further that the elastic mass density e depends on the strain tensor only through J_3

$$e = e(\xi^i, J_3)$$

and put

$$p = -2\sqrt{J_3}\frac{\partial e}{\partial J_3}.$$

Then the formula (5·7) becomes

$$T_\nu{}^\mu=(\rho+p)\,U^\mu U_\nu+\delta_\nu{}^\mu p. \tag{5·8}$$

This is just the energy momentum tensor of the relativistic perfect fluid with the pressure p.[5]

§ 6. The equation of motion

The equation of motion of the elastic medium can be derived by the variation of the Lagrangian (4·9) or (4·9)′. The equation of motion derived from the Lagrangian (4·9) coincides *a fortiori* with the equation of motion (3·10). The equation of motion derived from the Lagrangian (4·9)′ is

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial\rho}{\partial\eta^{ij}}\frac{\partial\xi^j}{\partial x_\mu}\right)-\frac{\partial\rho}{\partial\xi^i}=0. \tag{6·1}$$

This form of the equation of motion is simpler than Eq. (3·10).

Now we shall show that we can completely eliminate the coordinates of E-frame from the equation of motion of the elastic medium.

Define the symmetric tensor $\breve{T}^{\alpha\beta}$ by

$$\breve{T}^{\alpha\beta}=\frac{\partial\xi^\alpha}{\partial x^\mu}T^{\mu\nu}\frac{\partial\xi^\beta}{\partial x^\nu}. \tag{6·2}$$

$\breve{T}^{\alpha\beta}$ is the energy momentum tensor in L-frame. Then it is easy to see that $\breve{T}^{\alpha\beta}$ satisfies the equation

$$\frac{\partial\breve{T}^{\alpha\beta}}{\partial\xi^\alpha}+\Gamma_{\alpha\lambda}^{\alpha}\breve{T}^{\lambda\beta}+\Gamma_{\alpha\lambda}^{\beta}\breve{T}^{\alpha\lambda}=0. \tag{6·3}$$

where $\Gamma_{\beta\lambda}^{\alpha}$ is Christoffel's symbol in L-frame:

$$\Gamma_{\beta\lambda}^{\alpha}=\frac{\partial^2 x^\mu}{\partial\xi^\beta\partial\xi^\lambda}\frac{\partial\xi^\alpha}{\partial x^\mu}.$$

It is well known that Christoffel's symbols $\Gamma_{\beta\lambda}^{\alpha}$ can be written in terms of the metric tensor $g_{\alpha\beta}$ of L-frame as

$$\Gamma_{\beta\lambda}^{\alpha}=\frac{1}{2}g^{\alpha\nu}\left(\frac{\partial g_{\nu\lambda}}{\partial\xi^\beta}+\frac{\partial g_{\nu\beta}}{\partial\xi^\lambda}-\frac{\partial g_{\beta\lambda}}{\partial\xi^\nu}\right). \tag{6·4}$$

The explicit form of the tensor $\breve{T}^{\alpha\beta}$ for the elastic medium is given by

$$\begin{aligned}&\breve{T}^{ij}=-\frac{2}{\sqrt{\gamma}}\frac{\partial\kappa}{\partial\gamma^{ij}},\quad \breve{T}^{i4}=\breve{T}^{4i}=-\frac{2}{\sqrt{\gamma}\sqrt{-g_{44}}}\frac{\partial\kappa}{\partial\gamma_{ij}}\gamma_j,\\ &\breve{T}^{44}=-\frac{1}{g_{44}\sqrt{\gamma}}\left(\kappa-2\gamma_i\frac{\partial\kappa}{\partial\gamma_{ij}}\gamma_j\right).\end{aligned} \tag{6·5}$$

Noting that

$$\gamma_{ij}=g_{ij}-\frac{g_{4i}g_{4j}}{g_{44}},\quad \gamma_i=\frac{g_{4i}}{\sqrt{-g_{44}}}, \tag{6·6}$$

and the proper mass density κ in L-frame is a function of ξ^i and γ_{ij}, we can see that the tensor $\overset{\vee}{T}{}^{\alpha\beta}$ depends only on the metric tensor $g_{\alpha\beta}$ and ξ^i. So the equation of motion (6·3) with (6·4) can be regarded as the equation which determines the metric tensor of L-frame. However, Eq. (6·3) is insufficient to determine $g_{\alpha\beta}$ completely. To determine $g_{\alpha\beta}$ completely it suffices to add the equation which secure that the metric of L-frame is integrable. This is accomplished by adding the equations which assert that the curvature tensor $R^{\alpha\beta}$ in L-frame is zero:

$$R^{\alpha\beta}=0. \tag{6·7}$$

From Eqs. (6·7) we have that $g_{\alpha\beta}$ is of the form $g_{\alpha\beta}=\partial x^\mu/\partial\xi^\alpha\cdot\partial x_\mu/\partial\xi^\beta$ and inserting this form of $g_{\alpha\beta}$ into Eq. (6·3) we can completely determine the motion of the elastic medium. In Eqs. (6·3) and (6·7) the coordinates of E-frame are completely eliminated.

This formulation of the theory of the relativistic elastic continua suggests itself the way of the generalization of the theory to include the effect of the permanent gravitational field produced by the elastic medium. This is done by replacing Eq. (6·7) with the Einstein equation

$$R^{\alpha\beta}-\frac{1}{2}g^{\alpha\beta}R=-k\overset{\vee}{T}{}^{\alpha\beta} \tag{6·8}$$

where $R=R_\alpha{}^\alpha$ and k is the gravitational constant.

Eq. (6·8) contains Eq. (6·3) since the covariant divergence of the left-hand side of Eq. (6·8) always vanishes so that from the right-hand side of Eq. (6·8) Eq. (6·3) follows. Thus, Eq. (6·8) with (6·5) and (6·6) is a closed system of equation for the metric tensor $g_{\alpha\beta}$ and determines the motion of the elastic medium together with the gravitational field produced by it. It must be noted that Eq. (6·8) is referred to a special system of coordinates (the Lagrange frame) relative to which the medium is at rest throughout all time, so that the state of motion of the medium is represented by the time variation of the metric tensor. $g_{\alpha\beta}$ statisfying (6·8) cannot be integrable and in this case we cannot define E-frame as an inertial coordinate system.

Noting (6·6), we can write the expression (6·5) for $\overset{\vee}{T}{}^{\alpha\beta}$ compactly as

$$\overset{\vee}{T}{}^{\alpha\beta}=-\frac{2}{\sqrt{-g}}\frac{\partial}{\partial g_{\alpha\beta}}(\kappa\sqrt{-g_{44}}), \tag{6·9}$$

where $g=\det(g_{\alpha\beta})$. So Eq. (6·8) can be derived by the variation of the Lagrangian

$$L=\int(R\sqrt{-g}+2k\kappa\sqrt{-g_{44}})\,d^4\xi, \tag{6·10}$$

where we have used that[6]

$$\delta\int R\sqrt{-g}\,d^4\xi=-\int\left(R^{\alpha\beta}-\frac{1}{2}g^{\alpha\beta}R\right)\delta g_{\alpha\beta}\sqrt{-g}\,d^4\xi.$$

§ 7. Conservation laws

There are several conserving quantities for the elastic medium. Since we have had the Lagrangian for the elastic medium, we may obtain them by applying Noether's principle[7] for it.

The total four momentum P^μ and the total angular momentum $M^{\mu\nu}$ are conserving quantities for any elastic medium:

$$P^\mu = \int T^{\mu\lambda} n_\lambda d\sigma \tag{7·1}$$

$$M^{\mu\nu} = \int (x^\mu T^{\nu\lambda} - x^\nu T^{\mu\lambda}) n_\lambda d\sigma \tag{7·2}$$

where the integrations are made over an arbitrary spacelike hypersurface, $d\sigma$ is the surface element of the hypersurface and n_λ is the timelike unit vector pointing the normal direction of the hypersurface. If we take the hypersurface as the one on which the coordinate clock used in L-frame shows the equal time, then we have

$$n_\lambda d\sigma = \frac{\partial \xi^4}{\partial x^\lambda} \Delta d^3\xi . \tag{7·3}$$

Using the formula (7·3) we have another expression for P^μ and $M^{\mu\nu}$:

$$P^\mu = \int \widetilde{T}^{4\mu} d^3\xi , \tag{7·1}'$$

$$M^{\mu\nu} = \int (x^\mu \widetilde{T}^{4\nu} - x^\nu \widetilde{T}^{4\mu}) d^3\xi . \tag{7·2}'$$

If the medium is elastically isotropic and further the undeformed proper mass density κ_0 of the medium is spherically symmetric, then the Lagrangian L is invariant for the rotation of the spacial axes of L-frame. Thus, we have an important conserving quantity Ω_i:

$$\Omega_i = 2 \int \epsilon_{ijk} \xi^j \frac{\partial \rho}{\partial \eta^{kl}} \frac{\partial \xi^l}{\partial x_\mu} n_\mu d\sigma \tag{7·4}$$

where ϵ_{ijk} is the antisymmetric unit tensor. These Ω_i transform like an angular momentum vector under the rotation of the spacial axes of L-frame and are invariant for the Lorentz transformation of E-frame. We call this polar vector in L-frame "Lagrange spin (L-spin)". Using the formula (7·3) and the relations

$$\frac{\partial \rho}{\partial \eta^{ij}} = -\gamma_{ik} \frac{\partial \rho}{\partial \gamma_{kl}} \gamma_{lj}, \quad \gamma_{ij} \frac{\partial \xi^j}{\partial x_\mu} = A_i{}^\mu, \tag{7·5}$$

$$\frac{\partial \rho}{\partial \gamma_{kl}} = \frac{1}{\sqrt{\gamma}} \left(\frac{\partial \kappa}{\partial \gamma_{kl}} - \frac{1}{2} \kappa \eta^{kl} \right), \quad A_i{}^\mu \frac{\partial \xi^4}{\partial x^\mu} = \frac{\gamma_i}{\sqrt{-g_{44}}}, \tag{7·5}$$

we have another expression for Ω_i:

$$\Omega_i = \int \epsilon_{ijk}\xi^j \left(\kappa\gamma_k - 2\gamma_{kl}\frac{\partial\kappa}{\partial\gamma_{lm}}\gamma_m\right) d^3\tilde{\xi}\,. \qquad (7\cdot4)'$$

If the proper mass density κ of the medium is only cylindrically symmetric, i.e. invariant only for the rotation of L-frame around, say, $\tilde{\xi}^3$ axis, then only the third component Ω_3 of the L-spin is the conserving quantity.

If the medium is elastically anisotropic, i.e. the proper mass density κ is not invariant for the rotation of the spacial axes of L-frame, then the L-spin is not a conserving quantity.

§ 8. The ether

Let us imagine an elastic medium such that the undeformed proper mass density κ_0 is zero but the elastic mass density e has non-zero values when the medium is in strained states and also assume that the stress of the medium in the undeformed state vanishes. Such medium has no mass of its own and if it is undeformed then it has no mass, no energy, no momentum and all other physically observable quantities vanish. However, if it deforms in any way, then it may have a finite mass, energy and momentum and nonvanishing values for the other physically observable quantities. Moreover, it may extend over whole space without making its total energy infinite. This is impossible for the medium with an undeformed mass density which is finite and constant. Such a hypothetical elastic medium filling whole space is called "ether". Possibility of the ether rests on the special theory of relativity which asserts the equivalence of mass and energy. Through the motion of the relativistic elastic medium the proper mass is not conserved because the elastic force produces mass. This makes it possible that the ether which has no mass of its own becomes to have a mass in the deformed states. The mass and energy of the ether are entirely due to its elastic self-interaction.

In the undeformed state of the ether all physically observable quantities vanish, so we may regard such state as the vacuum. The ether in the undeformed state may move uniformly relative to an inertial system of coordinates. So the vacuum has degeneracy corresponding to the possible uniform motions of the ether.

In order that the ether in the deformed state has finite energy it has to be undeformed at the infinitely remote place. The inertial system of coordinates relative to which the ether is at rest at infinity may be called "the absolute inertial system of coordinates" (briefly we call it "the absolute inertial frame"). The Lagrange system of coordinates which is fixed to the ether and moves along its motion may be regarded as "the absolute coordinate system". In the deformed state the Lagrange coordinate system is an accelerated coordinate system and it is in principle possible to distinguish one mode of acceleration of the ether from the other because they produce different distributions of mass, energy and other physical observables. However, it is impossible to determine the absolute inertial

frame from physical observations because it is related only to the undeformed portion of the ether where all physical observables vanish. Moreover, the dynamics of the ether is constructed so as to be covariant with respect to the Lorentz transformation between E-frames. Therefore, though the absolute inertial frame has a conceptional meaning there is no way to determine it by physical observation. The principle of the special relativity (equivalence of any inertial frame) is maintained in our ether.

The ether has another conserving quantity besides the ones introduced in § 5 if we assume the homogeneity of the elastic property of the ether. For the ether the undeformed mass density κ_0 is zero, so if the ether is elastically homogeneous, then the Lagrangian density (the proper mass density) does not depend on ξ^i and is invariant for the translation of the spacial axes of L-frame. Hence we have a conserving quantity Θ_i.

$$\Theta_i=2\int\frac{\partial\rho}{\partial\eta^{ij}}\frac{\partial\xi^j}{\partial x_\mu}n_\mu d\sigma \tag{8·1}$$

This Θ_i transforms as a vector for the rotation of L-frame and has the property of momentum (it is related to the translation of L-frame). We call this vector Θ_i "Lagrange momentum (L-momentum)". The L-momentum conserves for any homogeneous ether irrespective of its rotational symmetry.

Using the relations (7·5), we have another expression for Θ_i:

$$\Theta_i=\int\left(\kappa\gamma_i-2\gamma_{ij}\frac{\partial\kappa}{\partial\gamma_{jk}}\gamma_k\right)d^3\xi. \tag{8·1}'$$

In the case of the homogeneous ether the proper mass density depends only on the strain tensor and the equation of motion is given by

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial\rho}{\partial\eta^{ij}}\frac{\partial\xi^j}{\partial x_\mu}\right)=\frac{\partial\rho}{\partial\eta^{ij}}\frac{\partial^2\xi^i}{\partial x^\mu\partial x_\mu}+2\frac{\partial^2\rho}{\partial\eta^{ij}\partial\eta^{kl}}\frac{\partial^2\xi^k}{\partial x_\mu\partial x_\nu}\frac{\partial\xi^j}{\partial x^\mu}\frac{\partial\xi^l}{\partial x^\nu}=0 \tag{8·2}$$

where $\rho(\eta_{ij})$ has to satisfy

$$\rho(\delta_{ij})=0,\quad \left.\frac{\partial\rho}{\partial\eta^{ij}}\right|_{\eta^{ij}=\delta_{ij}}=0.$$

§ 9. Conserving quantities-resumed

Now let us make clear the physical meanings of conserving quantities introduced in the previous sections.

We define the mass m and the spin S^μ of the elastic medium by

$$m^2=-P^\mu P_\mu, \tag{9·1}$$

$$S^\mu=\frac{1}{m}\widetilde{M}^{\mu\nu}P_\nu, \tag{9·2}$$

where $\widetilde{M}^{\mu\nu}$ is the dual tensor of $M^{\mu\nu}$.

Let X^μ be the proper center of mass[8] of the medium. Then

$$\frac{dX^\mu}{d\tau}=\frac{P^\mu}{m},$$

where τ is the proper time of the proper center of mass. Now, decompose the Euler coordinates x^μ of the medium into the coordinates of the proper center of mass and the relative coordinates r^μ:

$$x^\mu=X^\mu+r^\mu. \tag{9·3}$$

In the E-frame in which the proper center of mass is at rest we have $r^4=0$ since in such E-frame $x^4=X^4=\tau$. So, in arbitrary E-frame we have

$$P^\mu r_\mu=0. \tag{9·4}$$

Choose the standard clock of the proper center of mass, as the coordinate clock namely let ξ^4 be the proper time τ of the proper center of mass. Then we have

$$\left.\begin{aligned}&\frac{\partial x^\mu}{\partial\xi^i}=\frac{\partial r^\mu}{\partial\xi^i},\quad \frac{\partial x^\mu}{\partial\tau}=\frac{P^\mu}{m}+\frac{\partial r^\mu}{\partial\tau},\\ &U^\mu=\frac{1}{\sqrt{1-\dfrac{\partial r^\mu}{\partial\tau}\dfrac{\partial r_\mu}{\partial\tau}}}\left(\frac{P^\mu}{m}+\frac{\partial r^\mu}{\partial\tau}\right).\end{aligned}\right\} \tag{9·5}$$

Accordingly, the strain tensor γ_{ij} and g_{44} become

$$\left.\begin{aligned}&-g_{44}=1-\frac{\partial r^\mu}{\partial\tau}\frac{\partial r_\mu}{\partial\tau},\\ &\gamma_{ij}=\frac{\partial r^\mu}{\partial\xi^i}\frac{\partial r_\nu}{\partial\xi^j}+\frac{1}{1-\dfrac{\partial r^\mu}{\partial\tau}\dfrac{\partial r_\mu}{\partial\tau}}\frac{\partial r^\nu}{\partial\xi^i}\frac{\partial r_\nu}{\partial\tau}\frac{\partial r^\lambda}{\partial\xi^j}\frac{\partial r_\lambda}{\partial\tau},\end{aligned}\right\} \tag{9·6}$$

and γ_i and A_i^μ are given by

$$\left.\begin{aligned}&\gamma_i=\frac{1}{\sqrt{1-\dfrac{\partial r_\mu}{\partial\tau}\dfrac{\partial r^\mu}{\partial\tau}}}\frac{\partial r^\mu}{\partial\xi^i}\frac{\partial r_\mu}{\partial\tau}\\ &A_i^\mu=\frac{\partial r^\mu}{\partial\xi^i}+\gamma_i\frac{1}{\sqrt{1-\dfrac{\partial r^\mu}{\partial\tau}\dfrac{\partial r_\mu}{\partial\tau}}}\left(\frac{P^\mu}{m}+\frac{\partial r^\mu}{\partial\tau}\right).\end{aligned}\right\} \tag{9·7}$$

Inserting (9·5) and (9·7) into (3·9), we have

$$\widetilde{T}^{4\mu}=\frac{\kappa+\gamma_i\tau^{ij}\gamma_j}{\sqrt{1-\dfrac{\partial r^\mu}{\partial\tau}\dfrac{\partial r_\mu}{\partial\tau}}}\cdot\frac{P^\mu}{m}+q^\mu \tag{9·8}$$

where

$$q^\mu=\frac{\kappa+\gamma_i\tau^{ij}\gamma_j}{\sqrt{1-\frac{\partial r^\mu}{\partial\tau}\frac{\partial r^\mu}{\partial\tau}}}\cdot\frac{\partial r^\mu}{\partial\tau}+\gamma_i\tau^{ij}\frac{\partial r^\mu}{\partial\xi^j}, \tag{9·9}$$

and from (9·4) q^μ satisfies $q^\mu P_\mu=0$. From (9·8) and (7·1)′ we have

$$m=\int\frac{\kappa+\gamma_i\tau^{ij}\gamma_j}{\sqrt{1-\frac{\partial r^\mu}{\partial\tau}\frac{\partial r_\mu}{\partial\tau}}}d^3\xi \tag{9·10}$$

and

$$\int q^\mu d^3\xi=0. \tag{9·11}$$

(9·11) shows that q^μ may be interpreted as the internal momentum flux which does not contribute to the total momentum.

Substituting (9·8) in (7·2)′ and (9·2), we have

$$S^\mu=\frac{1}{m}\tilde{n}^{\mu\nu}P_\nu \tag{9·12}$$

where $\tilde{n}^{\mu\nu}$ is the dual of $n^{\mu\nu}$ and $n^{\mu\nu}$ is given by

$$n^{\mu\nu}=\int(r^\mu q^\nu-r^\nu q^\mu)d^3\xi. \tag{9·13}$$

In the inertial rest frame of the proper center of mass (9·12) becomes

$$S^i=\epsilon_{ijk}\int r^j q^k d^3\xi,\quad S^4=0. \tag{9·12}$$

Namely the spin S^i is the moment of the internal momentum with respect to the proper center of mass, so that it deserves the name of spin.

From (9·9), (9·6) and (9·7) we have

$$q^\mu\frac{\partial r_\mu}{\partial\xi^i}=\theta_i, \tag{9·14}$$

where θ_i is

$$\theta_i=\kappa\gamma_i+\gamma_{ij}\tau^{jk}\gamma_k.$$

As is seen from (8·1)′ θ_i coincide with the densities of the L-momenta, hence (9·14) shows that the L-mementum density θ_i is the projection of the internal momentum flux q^μ into L-frame. Thus the L-momentum is the total internal momentum measured in L-frame,

$$\Theta_i=\int q^\mu\frac{\partial r_\mu}{\partial\xi^i}d^3\xi. \tag{9·15}$$

Contrary to (9·11), the L-momentum Θ_i does not necessarily vanish. However,

for many cases θ_i also vanish. For example, if the deformation is symmetric with respect to the proper center of mass, i.e. if $r^i(\boldsymbol{\xi}, \tau)$ satisfy $r^i(-\boldsymbol{\xi}, \tau) = -r^i(\boldsymbol{\xi}, \tau)$ then θ_i vanish.

From (9·14) we have

$$\Omega_i = \epsilon_{ijk} \int \xi^j q^\mu \frac{\partial r_\mu}{\partial \xi^k} d^3\xi, \tag{9·16}$$

namely the L-spin is the internal angular momentum measured in L-frame.

§ 10. Canonical formalism

In order to proceed to quantum theory it is necessary to write down the theory in the canonical formalism. Let us choose E-frame in such a way that the proper center of mass is at rest. Then $r^4=0$ and $x^4=\tau$. Since $\xi^i(x)$ do not change their values along the trajectory of the point under consideration, their derivatives with respect to the proper time of the point are zero. Therefore the values of $\xi^i(x)$ at the proper center of mass do not depend on τ, i.e. $\xi^i(X)$ are constant. Thus we may choose the origin of L-frame in such a way that $\xi^i(X)$ are zero. Then, of course, $r^i(0, \tau)=0$.

Now we expand $r_i(\boldsymbol{\xi}, \tau)$ by spherical harmonics as follows:

$$r_i(\boldsymbol{\xi}, \tau) = \sum_{N=1}^{\infty} C^{(N)}_{i,\underbrace{j\cdots n}_{N \text{ members}}}(\tilde{\xi}, \tau) \frac{(-1)^{2N+1}}{N!} \xi^{2N+1} \frac{\partial^N}{\partial \xi^j \cdots \partial \xi^n} \left(\frac{1}{\xi}\right), \tag{10·1}$$

where $\tilde{\xi}$ denotes the length of $\boldsymbol{\xi}$; $\tilde{\xi} = \sqrt{\xi^i \xi^i}$ and the functions $C^{(N)}_{i,j\cdots k}(\tilde{\xi}, \tau)$ are symmetric and have vanishing traces with respect to the suffices after comma.

When the medium is the undeformed state, $r_i(\boldsymbol{\xi}, \tau)$ have to be in the form

$$r_i(\boldsymbol{\xi}, \tau) = a_{ij} \xi^j$$

where the matrix (a_{ij}) is a constant orthogonal matrix. So, if we assume that the deformation of the medium is restricted in finite region, then the functions $C^{(N)}_{i,j\cdots n}(\tilde{\xi}, \tau)$ have to satisfy the boundary conditions

$$C^{(1)}_{i,j}(\tilde{\xi}, \tau) \to a_{ij}\tilde{\xi},$$

$$C^{(N)}_{i,j\cdots n}(\tilde{\xi}, \tau) \to 0, \quad (N>1)$$

when $\tilde{\xi} \to \infty$, and they may be expanded as follows:

$$\begin{aligned} C^{(1)}_{i,j}(\tilde{\xi}, \tau) &= a_{ij}\tilde{\xi} + \sum_{A=1}^{\infty} C^{(1,A)}_{i,j}(\tau) f_A(\tilde{\xi}), \\ C^{(N)}_{i,j\cdots n}(\tilde{\xi}, \tau) &= \sum_{A=1}^{\infty} C^{(N,A)}_{i,j\cdots n}(\tau) f_A(\tilde{\xi}), \quad (N>1) \end{aligned} \tag{10·2}$$

where $\{f_A(\tilde{\xi})\}$ is an arbitray orthogonal complete system of functions satisfying $f_A(0)=0$ and $f_A(\infty)=0$. We use these $C^{(N,A)}_{i,j\cdots n}(\tau)$ as the canonical coordinates of the elastic medium.

The conjugate momenta $\pi^{(N,A)}_{i,j\cdots n}(\tau)$ of $C^{(N,A)}_{i,j\cdots n}(\tau)$ are given by

$$\pi^{(N,A)}_{i,j\cdots n}(\tau)=\frac{\partial \mathcal{L}}{\partial \dot{C}^{(N,A)}_{i,j\cdots n}(\tau)}=-\int \frac{\partial}{\partial \dot{r}_i}(\kappa\sqrt{-g_{44}})\frac{\partial \dot{r}_i}{\partial \dot{C}^{(N,A)}_{i,j\cdots n}(\tau)}d^3\xi \qquad (10\cdot3)$$

(not summed for i)

where the dot denotes the derivative with respect to τ, and $\mathcal{L}$ is the Lagrangian density

$$\mathcal{L}=-\int \kappa\sqrt{-g_{44}}\,d^3\xi.$$

Using (10·1), (10·2) and the relation

$$-\frac{\partial}{\partial \dot{r}_i}(\kappa\sqrt{-g_{44}})=q_i, \qquad (10\cdot4)$$

(10·3) becomes

$$\pi^{(N,A)}_{i,j\cdots n}(\tau)=\int q_i f_A(\xi)\,(-1)^{2N+1}\xi^{2N+1}\frac{\partial^N}{\partial\xi^j\cdots\partial\xi^n}\left(\frac{1}{\xi}\right)d^3\xi. \qquad (10\cdot5)$$

Substituting (10·1) and (10·2) in (9·12)′ and using (10·5), the spin is represented by the canonical variables as follows:

$$S_i=\sum_{N,A=1}^{\infty}\frac{1}{N!}\epsilon_{ijk}C^{(N,A)}_{j,l\cdots n}\pi^{(N,A)}_{k,l\cdots n}. \qquad (10\cdot6)$$

Noting the relation

$$\epsilon_{ijk}C^{(N,A)}_{a,lm\cdots n}\xi^j\frac{\partial}{\partial\xi^k}\left[\xi^{2N+1}\frac{\partial^N}{\partial\xi^l\partial\xi^m\cdots\partial\xi^n}\left(\frac{1}{\xi}\right)\right]$$
$$=-N\epsilon_{ijk}C^{(N,A)}_{a,jm\cdots n}\xi^{2N+1}\frac{\partial^N}{\partial\xi^k\partial\xi^m\cdots\partial\xi^n}\left(\frac{1}{\xi}\right),$$

we similarly have the expression for the L-spin as follows:

$$\Omega_i=-\sum_{N,A=1}^{\infty}\frac{1}{(N-1)!}\epsilon_{ijk}C^{(N,A)}_{l,jm\cdots n}\pi^{(N,A)}_{l,km\cdots n}. \qquad (10\cdot7)$$

The Hamiltonian in our theory coincides with the mass and it is given as usual by

$$m=\sum_{N,A=1}^{\infty}\frac{1}{N!}\pi^{(N,A)}_{i,j\cdots n}\dot{C}^{(N,A)}_{i,j\cdots n}-\mathcal{L}. \qquad (10\cdot8)$$

It is easy to see that the expression (10·8) for the mass coincides with the expression (9·10) if we note the relation

$$q_i\dot{r}_i=\frac{\kappa+\gamma_i\tau^{ij}\gamma_i}{\sqrt{1-\dot{r}_i\dot{r}_i}}-\kappa\sqrt{1-\dot{r}_i\dot{r}_i}.$$

The formulae (10·6), (10·7) and (10·8) provide the clue for the transition to quantum theory. However, we will not enter into quantum theory in this

paper. It shall be given in a subsequent paper.

Contrary to the expressions (10·6) and (10·7) for the spin and the L-spin which are given in terms of the canonical variables only, it is difficult to derive an explicit form of the functional dependence of the mass on the canonical variables and to eliminate the time derivatives of them from (10·8) because of the complicated non-linear character of the dynamics of the relativistic elastic medium.

§ 11. A model of elementary particles and their interactions in terms of the ether

As was shown in § 8 we may imagine a relativistic elastic medium "ether" which fills the whole space. The elastic wave of the ether may propagates through space, and since it obeys a non-linear equation it interacts with itself. Although the ether itself has no mass of its own, it may have mass in the deformed states owing to the elastic self-interaction. The most important fact is that if the elastic property of the ether is homogeneous and isotropic, then it has conserving quantities such as the L-momentum and the L-spin which are invariant for the Lorentz transformation of E-frame. Especially, the L-spin has the property of the angular momentum for the rotation of the spacial axes of L-frame. Therefore, if we quantize the homogeneous and isotropic ether, we may have an array of particles which have a symmetry property resembling that of the baryon family of elementary particles. Moreover, the conservation of the L-spin is destroyed if the elastic property of the ether deviates from isotropy. If the elastic property of the ether is only cylindrically symmetric with respect to the rotation around the third axis of L-frame, then only the third component of the L-spin is conserved, and if it is anisotropic then the L-spin is not conserved.

This strongly suggests the following speculation. Suppose that the proper mass density κ of the ether is composed of three parts: the large isotropic part κ_i, the cylindrically symmetric part κ_c of an intermediate magnitude and the small anisotropic part κ_a, so that

$$\kappa=\kappa_i+\kappa_c+\kappa_a.$$

The elementary particles are considered to be the excited states of the quantized ether, and the L-spin is identified with the isospin. The strong interaction, the electro-magnetic interaction and the weak interaction between them are considered to arise from the elastic self-interaction of the ether which originates from κ_i, κ_c and κ_a respectively.

If we assume the homogeneity of the ether, then the L-momentum conserves without regard to the rotational symmetry of the ether, so the transition between states which have different values of the L-momentum does not occur. Therefore we may assume that only the states which have zero L-momenta are realized in nature. This is not a severe restriction but in a very large class of deformations the condition is fulfilled.

We shall give the detailed discussions on this model in the subsequent paper.

Acknowledgment

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References

1) We can find an excellant historical survey on the theories of ether in the following book: E. Whittaker, Aether and Electricity, Vol. 1 (1951).
2) Cf. loc. cit. Chap. IX.
3) A. E. Green, W. Zerna, *Theoretical Elasticity* (1954);
cf. also F. D. Murnaghan, *Finite Deformation of an Elastic Solid* (1951).
4) Cf. C. Møller, *The Theory of Relativity* (Chap. VI) (1952).
5) Cf. loc. cit. (4) Chap. VI, §67.
6) Cf. loc. cit. (4) Chap. XI, §125.
7) E. Noether, Gött. Nachr. (1918), 211.
8) Cf. loc. cit. (4) Chap. VI, §64,
C. Møller, Ann. Inst. Henri Poincaré XI (1949), 251.

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Collective Excitation of Electrons in Degenerate Bands. II

——*Collective Excitations in a Metallic p-Band**——

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Various collective oscillations of electrons are examined in an idealized metallic p-band model. It is concluded that the other oscillations than the density oscillation are unstable unless rather inconceivable conditions are satisfied.

When a metallic band contains an orbital degeneracy (cf. p-band, d-band, and so forth), one may expect that various bound states of an electron and a hole might be obtained in addition to the plasmon states. In this report an idealized metallic p-band model is taken as an example in order to investigate the possibilities of the various bound states. The model considered here is as follows:

(I) A simple cubic array of p-orbitals with lattice constant a. The three p-orbitals at each lattice point are directed along the three crystal axes respectively.

(II) The average number of the valence p-electrons belonging to each lattice point is taken to be three.

(III) The core electrons are supposed to contribute only to the electro-static field

$$U(\boldsymbol{x}) = -\sum_{R} \frac{3e^2}{|\boldsymbol{x}-a\boldsymbol{R}|}$$

for the valence p-electrons. ($a\boldsymbol{R}$ denotes the position vector of a lattice point.) Thus, representing the p-electron field by $\psi(\boldsymbol{x}, \zeta)$ ($\boldsymbol{x}$ and ζ being respectively the space- and the spin-coordinate), we start from the Hamiltonian

$$H = \int \psi^*(\boldsymbol{x}, \zeta)\left[-\frac{\hbar^2}{2m}\Delta + U(\boldsymbol{x})\right]\psi(\boldsymbol{x}, \zeta)\, d\tau$$
$$+\frac{1}{2}\iint \psi^*(\boldsymbol{x}, \zeta)\psi^*(\boldsymbol{x}', \zeta')\frac{e^2}{|\boldsymbol{x}-\boldsymbol{x}'|}\psi(\boldsymbol{x}', \zeta')\psi(\boldsymbol{x}, \zeta)\, d\tau\, d\tau', \qquad (1)$$

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* An outline of this work was reported in Progress Report, Research Group for the Study of Molecular Structure, (1959).

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where $d\tau = dx\, d\zeta$, and

$$\psi(x, \zeta) = \sum_{\lambda, R}^{\sigma} a_\sigma(\lambda, R) p_R^\lambda(x) \sigma(\zeta),$$

where σ is a spin function and $p_R^\lambda(\lambda = x, y \text{ or } z)$ is the p-orbital of the R-th atom. (p_R^x, p_R^y, p_R^z) transform like $(x-X, y-Y, z-Z)$ under a three-dimensional rotation around the lattice point $R=(X, Y, Z)$. The three cartesian coordinate axes are taken along the three crystal axes respectively.

(IV) The overlaps of the form

$$p_R^\lambda(x)\, p_{R'}^{\lambda'}(x) \quad (R \neq R')$$

are neglected except in the following integral:

$$B = \int p_R^{\lambda_r}(x) \left(-\frac{\hbar^2}{2m} \Delta\right) p_{R+r}^{\lambda_r}(x)\, d^3x,$$

where r is a unit vector of the cartesian representation and λ_r denotes the coordinate axis indicated by the unit vector r.

(V) The intra-atomic repulsion of our valence electrons is neglected, since we here consider only the collective oscillations with long wavelengths which are induced obviously by the long range part of the Coulomb interaction between valence electrons. (This last restriction imposed on our model might be open to question. Such questions, however, belong to another kind of problem, which is much more complicated.)

We adopt the following transformation:

$$a_\sigma(\lambda, p) = \frac{1}{\sqrt{G}} \sum_R \exp(-ip \cdot R)\, a_\sigma(\lambda, R)$$

and

$$J_{\lambda\mu}^{\lambda'\mu'}(q) = \frac{1}{G} \sum_{R(\neq 0)} J_{\lambda\mu}^{\lambda'\mu'}(R) \exp(iq \cdot R),$$

where

$$J_{\lambda\mu}^{\lambda'\mu'}(R) = \iint \frac{e^2}{|x_1 - x_2|}\, p_{R_1}^\lambda(x_1)\, p_{R_1}^\mu(x_1)\, p_{R_1+R}^{\lambda'}(x_2)\, p_{R_1+R}^{\mu'}(x_2)\, dx_1\, dx_2$$

and G is the number of atoms in our crystal. Then Hamiltonian (1) may be written as follows:

$$H = H_0 + H_c, \tag{2}$$

$$H_0 = \sum_{\lambda, p, \sigma} E_\lambda(p)\, a_\sigma^*(\lambda, p)\, a_\sigma(\lambda, p), \tag{3}$$

$$H_c = \frac{1}{2} \sum_{\lambda\mu\lambda'\mu'} \sum_{\sigma\sigma'} \sum_{q \neq 0} \sum_{p, p'} J_{\lambda\mu}^{\lambda'\mu'}(q)\, a_\sigma^*(\lambda, p+q)\, a_{\sigma'}(\mu, p)\, a_{\sigma'}^*(\lambda', p')\, a_{\sigma'}(\mu', p'+q), \tag{4}$$

where we have omitted constant terms. In the above

$$E_\lambda(\boldsymbol{p}) = B\cos p_\lambda \quad (\lambda = x, y \text{ or } z),$$
$$-\pi \leqq p_\lambda \leqq \pi,$$

constitutes the energy-momentum relation of the p-band. If we neglect H_c, the ground state of our p-electron system is characterized by the occupation number representation as follows:

[The occupation number of the one-electron level $(\lambda, \boldsymbol{p})$]

$$= \begin{cases} 2, & \text{if } |p_\lambda| \leqq \dfrac{\pi}{2}, \\ 0, & \text{if } |p_\lambda| > \dfrac{\pi}{2}. \end{cases}$$

The Fermi levels are now given by

$$E_\lambda(\boldsymbol{p}) = 0.$$

Now, we will adopt the following conventional notation:

$$c_\sigma(\lambda, \boldsymbol{p}) \begin{cases} \equiv a_\sigma(\lambda, \boldsymbol{p}) & \text{if } E_\lambda(\boldsymbol{p}) > 0, \\ \equiv 0 & \text{if } E_\lambda(\boldsymbol{p}) \leqq 0, \end{cases}$$

$$b_\sigma(\lambda, \boldsymbol{p}) \begin{cases} \equiv 0 & \text{if } E_\lambda(\boldsymbol{p}) > 0, \\ \equiv a_\sigma^*(\lambda, \boldsymbol{p}) & \text{if } E_\lambda(\boldsymbol{p}) \leqq 0, \end{cases}$$

and $$\vartheta_\sigma(\lambda_1, \boldsymbol{p}_1|\lambda_2, \boldsymbol{p}_2) \equiv b_\sigma(\lambda_1, \boldsymbol{p}_1) c_\sigma(\lambda_2, \boldsymbol{p}_2).$$

In quantitative discussions, we adopt Sawada's approximation,[1] i.e.

$$H_c \to \mathcal{H}_c = \frac{1}{2} \sum_{q \neq 0}' \sum_{\lambda\mu\lambda'\mu'} \mathrm{J}_{\lambda\mu}^{\lambda'\mu'}(\boldsymbol{q}) \sum_{p}' \sum_{\sigma} \{\vartheta_\sigma^*(\mu, \boldsymbol{p}|\lambda, \boldsymbol{p}+\boldsymbol{q}) + \vartheta_\sigma(\lambda, -\boldsymbol{p}|\mu, -\boldsymbol{p}-\boldsymbol{q})\}$$
$$\times \sum_{p'}' \sum_{\sigma'} \{\vartheta_{\sigma'}^*(\mu', -\boldsymbol{p}'|\lambda', -\boldsymbol{p}'-\boldsymbol{q}) + \vartheta_{\sigma'}(\lambda', \boldsymbol{p}'|\mu', \boldsymbol{p}'+\boldsymbol{q})\}, \tag{5}$$

$$[\vartheta_\sigma(\lambda_1, \boldsymbol{p}_1|\lambda_2, \boldsymbol{p}_2),\ \vartheta_{\sigma'}^*(\lambda_1', \boldsymbol{p}_1'|\lambda_2', \boldsymbol{p}_2')] = \delta(\sigma|\sigma')\delta(\lambda_1|\lambda_1')\delta(\lambda_2|\lambda_2') \times \delta(\boldsymbol{p}_1|\boldsymbol{p}_1')\delta(\boldsymbol{p}_2|\boldsymbol{p}_2'),$$
$$[\vartheta_\sigma(\lambda_1, \boldsymbol{p}_1|\lambda_2, \boldsymbol{p}_2),\ \vartheta_{\sigma'}(\lambda_1', \boldsymbol{p}_1'|\lambda_2', \boldsymbol{p}_2')] = 0. \tag{6}$$

In order to simplify mathematical treatments, here we consider the collective oscillations travelling along the z-axis. Further, by means of New Tam-Dancoff's picture, we seek the eigenstates of the Hamiltonian given by (2), (3), (4) and (5) which correspond to these collective oscillations. Then we get the following collective modes.

[1] *Sound waves* (Ionicity-propagating waves)

The sound wave with wave vector $\boldsymbol{q}$ is represented by the following state vector:

$$\Psi_q = \Xi_q \Psi_0,$$

where

$$\Xi_q = \sum_{\mu,\boldsymbol{p},\sigma} \{f_q(\mu,\boldsymbol{p})\vartheta_\sigma^*(\mu,\boldsymbol{p}|\mu,\boldsymbol{p}+\boldsymbol{q}) + g_q(\mu,\boldsymbol{p})\vartheta_\sigma(\mu,-\boldsymbol{p}|\mu,-\boldsymbol{p}-\boldsymbol{q})\} \quad (7)$$

and

Ψ_0=State vector representing the ground state.

The frequency ω of this sound wave is given by the equation of motion

$$[H, \Xi_q] = \frac{\hbar}{i}\dot{\Xi}_q = \hbar\omega\,\Xi_q. \quad (8)$$

Now, using (5) and (6), we get

$$[\mathcal{H}_c, \vartheta_\sigma^*(\mu,\boldsymbol{p}|\mu,\boldsymbol{p}+\boldsymbol{q})] = \sum_{\lambda'\mu'} \mathrm{J}_{\mu\mu'}^{\lambda'\mu'}(\boldsymbol{q}) \cdot \sum_{\boldsymbol{p}'\sigma'} \{\vartheta_{\sigma'}^*(\mu',\boldsymbol{p}'|\lambda',\boldsymbol{p}'+\boldsymbol{q}) + \vartheta_{\sigma'}(\lambda',\boldsymbol{p}'|\mu',\boldsymbol{p}'-\boldsymbol{q})\}.$$

Since we are now studying the collective oscillations whose wave vector $\boldsymbol{q}$ is parallel to the z-axis, we may use the following relation:

$$\mathrm{J}_{\mu\mu}^{\lambda'\mu'}(\boldsymbol{q}) = \mathrm{J}_{\mu\mu}^{\lambda'\lambda'}(\boldsymbol{q}) \cdot \delta(\lambda'|\mu').$$

Further, for sufficiently small $|\boldsymbol{q}|$,

$$\mathrm{J}_{\mu\mu}^{\lambda\lambda}(\boldsymbol{q}) \fallingdotseq \frac{1}{\mathrm{G}}\left(\frac{4\pi e^2}{aq^2}\right).$$

Thus, we get

$$[\mathcal{H}_c, \vartheta_\sigma^*(\mu,\boldsymbol{p}|\mu,\boldsymbol{p}+\boldsymbol{q})] = \frac{4\pi e^2}{Gaq^2} \sum_{\lambda\boldsymbol{p}'\sigma'} \{\vartheta_{\sigma'}^*(\lambda,\boldsymbol{p}'|\lambda,\boldsymbol{p}'+\boldsymbol{q}) + \vartheta_{\sigma'}(\lambda,\boldsymbol{p}'|\lambda,\boldsymbol{p}'-\boldsymbol{q})\}. \quad (9)$$

Similarly, we obtain

$$[\mathcal{H}_c, \vartheta_\sigma(\mu,-\boldsymbol{p}|\mu,-\boldsymbol{p}-\boldsymbol{q})] = -\frac{4\pi e^2}{Gaq^2} \sum_{\lambda\boldsymbol{p}'\sigma'} \{\vartheta_{\sigma'}^*(\lambda,\boldsymbol{p}'|\lambda,\boldsymbol{p}'+\boldsymbol{q}) + \vartheta_{\sigma'}(\lambda,\boldsymbol{p}'|\lambda,\boldsymbol{p}'-\boldsymbol{q})\}. \quad (9)'$$

Further, it is obvious that

$$\begin{aligned}
&[\mathcal{H}_0, \vartheta_\sigma^*(\mu,\boldsymbol{p}|\mu,\boldsymbol{p}+\boldsymbol{q})] = E(\mu,\boldsymbol{p}|\mu,\boldsymbol{p}+\boldsymbol{q})\vartheta_\sigma^*(\mu,\boldsymbol{p}|\mu,\boldsymbol{p}+\boldsymbol{q}),\\
&[\mathcal{H}_0, \vartheta_\sigma(\mu,-\boldsymbol{p}|\mu,-\boldsymbol{p}-\boldsymbol{q})] = -E(\mu,-\boldsymbol{p}|\mu,-\boldsymbol{p}-\boldsymbol{q})\vartheta_\sigma(\mu,-\boldsymbol{p}|\mu,-\boldsymbol{p}-\boldsymbol{q}),
\end{aligned} \quad (10)$$

where $$E(\lambda,\boldsymbol{p}_1|\mu,\boldsymbol{p}_2) \equiv E_\mu(\boldsymbol{p}_2) - E_\lambda(\boldsymbol{p}_1).$$

Substituting (7), (9), (9)′ and (10) in (8), we get

$$\left.\begin{aligned}
&f_q(\mu,\boldsymbol{p})[\hbar\omega - E(\mu,\boldsymbol{p}|\mu,\boldsymbol{p}+\boldsymbol{q})] = \sum_{\lambda,\boldsymbol{p}'\sigma'}^{\dagger} \{f_q(\lambda,\boldsymbol{p}') - g_q(\lambda,\boldsymbol{p}')\}\frac{4\pi e^2}{Gaq^2},\\
\text{and}\quad &g_q(\mu,\boldsymbol{p})[\hbar\omega + E(\mu,\boldsymbol{p}|\mu,\boldsymbol{p}+\boldsymbol{q})] = \sum_{\lambda\boldsymbol{p}'\sigma'}^{\dagger} \{f_q(\lambda,\boldsymbol{p}') - g_q(\lambda,\boldsymbol{p}')\}\frac{4\pi e^2}{Gaq^2}.
\end{aligned}\right\}$$

From these, we obtain

$$\left.\begin{aligned} f_q(\mu, \boldsymbol{p}) &= C_q \frac{1}{\hbar\omega - E(\mu, \boldsymbol{p}|\mu, \boldsymbol{p}+\boldsymbol{q})}, \\ g_q(\mu, \boldsymbol{p}) &= C_q \frac{1}{\hbar\omega + E(\mu, \boldsymbol{p}|\mu, \boldsymbol{p}+\boldsymbol{q})} \end{aligned}\right\}$$

and the dispersion relation

$$\frac{1}{G}\left(\frac{4\pi e^2}{aq^2}\right)\sum_{\mu,\boldsymbol{p},\sigma}^{\dagger}\left\{\frac{1}{\hbar\omega - E(\mu,\boldsymbol{p}|\mu,\boldsymbol{p}+\boldsymbol{q})} - \frac{1}{\hbar\omega + E(\mu,\boldsymbol{p}|\mu,\boldsymbol{p}+\boldsymbol{q})}\right\} = 1. \quad (11)$$

In the above, C_q is a constant independent of $\boldsymbol{p}$, and the summation over $\boldsymbol{p}$ should read

$$\sum_{\mu,\boldsymbol{p}}^{\dagger} = \sum_{\mu,\boldsymbol{p}} (E_\mu(\boldsymbol{p}) \leqq 0 \cap E_\mu(\boldsymbol{p}+\boldsymbol{q}) > 0).$$

Eq. (11) has a solution for $\hbar\omega$ whose value is separated from the level continuum given by $E(\mu, \boldsymbol{p}|\mu, \boldsymbol{p}+\boldsymbol{q})$, or

$$\hbar\omega > \mathrm{Max}\{E(\mu, \boldsymbol{p}|\mu, \boldsymbol{p}+\boldsymbol{q})\}, \text{ for } \boldsymbol{p}(E_\mu(\boldsymbol{p}) \leqq 0 \cap E_\mu(\boldsymbol{p}+\boldsymbol{q}) > 0),$$

at least in the case of $\boldsymbol{q} \to 0$. This solution corresponds to a density oscillating wave. For sufficiently small $\boldsymbol{q}$, we obtain

$$\hbar\omega = \sqrt{12(e^2/a)|\mathrm{B}|} + O(q^2).$$

[2] *Quadrupole polarization wave of 1st kind*

In this case we adopt

$$\begin{aligned} \Xi_q^{(xy)} = \sum_{\boldsymbol{p},\sigma} \{ & f_q(xy, \boldsymbol{p})\vartheta_\sigma^*(x, \boldsymbol{p}|y, \boldsymbol{p}+\boldsymbol{q}) + f_q(yx, \boldsymbol{p})\vartheta_\sigma^*(y, \boldsymbol{p}|x, \boldsymbol{p}+\boldsymbol{q}) \\ & + g_q(xy, \boldsymbol{p})\vartheta_\sigma(x, -\boldsymbol{p}|y, -\boldsymbol{p}-\boldsymbol{q}) + g_q(yx, \boldsymbol{p})\vartheta_\sigma(y, -\boldsymbol{p}|x, -\boldsymbol{p}-\boldsymbol{q})\} \end{aligned} \quad (12)$$

instead of (7), and seek solutions of Eq. (8) by means of the method used in [1].

Now,

$$[\mathcal{H}_e, \vartheta_\sigma^*(x, \boldsymbol{p}|y, \boldsymbol{p}+\boldsymbol{q})] = \sum_{\lambda\mu} \mathrm{J}_{xy}^{\lambda\mu}(\boldsymbol{q}) \sum_{\boldsymbol{p}'\sigma'} \{\vartheta_{\sigma'}^*(\mu, \boldsymbol{p}'|\lambda, \boldsymbol{p}'+\boldsymbol{q}) + \vartheta_{\sigma'}(\mu, \boldsymbol{p}'|\lambda, \boldsymbol{p}'-\boldsymbol{q})\}.$$

Since $\boldsymbol{q}$ is now taken to be parallel to the z-axis, we get from a symmetry consideration the following relation:

$$\mathrm{J}_{\lambda'\mu'}^{\lambda'\mu'}(\boldsymbol{q}) = (\delta(\lambda|\lambda')\delta(\mu|\mu') + \delta(\lambda|\mu')\delta(\mu|\lambda'))\mathrm{J}_{\lambda\mu}^{\lambda\mu}(\boldsymbol{q}). \quad (\lambda \neq \mu). \quad (13)$$

Therefore, we get

$$\begin{aligned} [\mathcal{H}_e, \vartheta_\sigma^*(x, \boldsymbol{p}|y, \boldsymbol{p}+\boldsymbol{q})] &= [\mathcal{H}_e, \vartheta_\sigma^*(y, \boldsymbol{p}|x, \boldsymbol{p}+\boldsymbol{q})] \\ &= \mathrm{J}_{xy}^{xy}(\boldsymbol{q}) \sum_{\boldsymbol{p}'\sigma'} \{\vartheta_{\sigma'}^*(x, \boldsymbol{p}'|y, \boldsymbol{p}'+\boldsymbol{q}) + \vartheta_{\sigma'}^*(y, \boldsymbol{p}'|x, \boldsymbol{p}'+\boldsymbol{q}) \\ &\quad + \vartheta_{\sigma'}(x, -\boldsymbol{p}'|y, -\boldsymbol{p}'-\boldsymbol{q}) + \vartheta_{\sigma'}(y, -\boldsymbol{p}'|x, -\boldsymbol{p}'-\boldsymbol{q})\} \\ &= -[\mathcal{H}_e, \vartheta(x, -\boldsymbol{p}|y, -\boldsymbol{p}-\boldsymbol{q})] = -[\mathcal{H}_e, \vartheta(y, -\boldsymbol{p}|x, -\boldsymbol{p}-\boldsymbol{q})]. \end{aligned} \quad (14)$$

Further, it is obvious that

$$\left.\begin{aligned}&[H_0, \vartheta_\sigma^*(\lambda, \boldsymbol{p}|\mu, \boldsymbol{p}+\boldsymbol{q})]=E(\lambda, \boldsymbol{p}|\mu, \boldsymbol{p}+\boldsymbol{q})\vartheta_\sigma^*(\lambda, \boldsymbol{p}|\mu, \boldsymbol{p}+\boldsymbol{q}),\\&[H_0, \vartheta_\sigma(\lambda, -\boldsymbol{p}|\mu, -\boldsymbol{p}-\boldsymbol{q})]=-E(\lambda, \boldsymbol{p}|\mu, \boldsymbol{p}+\boldsymbol{q})\vartheta_\sigma(\lambda, -\boldsymbol{p}|\mu, -\boldsymbol{p}-\boldsymbol{q}).\end{aligned}\right\} \quad (15)$$

Substituting (13), (14) and (15) in Eq. (8), we obtain the following dispersion relation:

$$\mathrm{J}_{xy}^{xy}(\boldsymbol{q})\sum_{\boldsymbol{p},\sigma}{}^{\dagger}\left\{\frac{1}{\hbar\omega-E(x, \boldsymbol{p}|y, \boldsymbol{p}+\boldsymbol{q})}+\frac{1}{\hbar\omega-E(y, \boldsymbol{p}|x, \boldsymbol{p}+\boldsymbol{q})}\right.$$
$$\left.-\frac{1}{\hbar\omega+E(x, \boldsymbol{p}|y, \boldsymbol{p}+\boldsymbol{q})}-\frac{1}{\hbar\omega+E(y, \boldsymbol{p}|x, \boldsymbol{p}+\boldsymbol{q})}\right\}=1. \quad (16)$$

Thus, the state represented by (12) has been completely separated from the density oscillating state represented by (7).

Now

$$\mathrm{J}_{xy}^{xy}(\boldsymbol{R})=\frac{2\mathrm{Q}^2}{(aR)^5}\left\{4-\frac{5Z^2}{R^2}-35\frac{X^2Y^2}{R^4}\right\},$$

where $\boldsymbol{R}=(X, Y, Z)\neq 0$ and Q is the quadrupole moment arising from the overlap charge $ep_{\boldsymbol{R}}^{x}(\boldsymbol{x})p_{\boldsymbol{R}}^{y}(\boldsymbol{x})$. Then we adopt the following continuum approximation

$$\mathrm{J}_{xy}^{xy}(\boldsymbol{q})=\frac{2\mathrm{Q}^2}{\mathrm{G}}\int_{R\geq 1}d\boldsymbol{R}\,\frac{1}{(aR)^5}\left\{4-\frac{5Z^2}{R^2}-35\frac{X^2Y^2}{R^4}\right\}\exp(i\boldsymbol{q}\cdot\boldsymbol{R})$$
$$\xrightarrow{q\to 0}-\frac{1}{\mathrm{G}}\frac{8\pi}{105}\frac{\mathrm{Q}^2}{a^5}q^2<0, \quad \text{(cf. Appendix)}. \quad (17)$$

Hence, Eq. (15) has no solution for $\hbar\omega$ whose value is separated from the level continuum given by $E(x, \boldsymbol{p}|y, \boldsymbol{p}+\boldsymbol{q})$ or $E(y, \boldsymbol{p}|x, \boldsymbol{p}+\boldsymbol{q})$. Thus we may reasonably conclude that the polarization wave considered above will not form a stable collective oscillations.

[3] *Quadrupole polarization wave of 2nd kind*

Here we consider the one-pair states $\Psi_q^{(xz)}$ and $\Psi_q^{(yz)}$ defined by

$$\Psi_q^{(\lambda\mu)}=\Xi_q^{(\lambda\mu)}\Psi_0.$$

$$\Xi_q^{(\lambda\mu)}=\sum_{\boldsymbol{p},\sigma}\{f_q(\lambda\mu, \boldsymbol{p})\vartheta_\sigma^*(\lambda, \boldsymbol{p}|\mu, \boldsymbol{p}+\boldsymbol{q})+f_q(\mu\lambda, \boldsymbol{p})\vartheta_\sigma^*(\mu, \boldsymbol{p}|\lambda, \boldsymbol{p}+\boldsymbol{q})$$
$$+g_q(\lambda\mu, \boldsymbol{p})\vartheta_\sigma(\lambda, -\boldsymbol{p}|\mu, -\boldsymbol{p}-\boldsymbol{q})+g_q(\mu\lambda, \boldsymbol{p})\vartheta_\sigma(\mu, -\boldsymbol{p}|\lambda, -\boldsymbol{p}-\boldsymbol{q})\}, \quad (18)$$

where $(\lambda\mu)=(xz)$ or (yz). $\Psi_q^{(xz)}$ and $\Psi_q^{(yz)}$ are mutually orthogonal and constitute a degenerate state. They are orthogonal to the states considered in [1] and [2].

Substituting (18) in Eq. (8), and using (13) as in [2], we obtain

$$\left.\begin{aligned} f_q(\lambda\mu, \boldsymbol{p}) &= \mathrm{C}_q \frac{1}{\hbar\omega - E(\lambda, \boldsymbol{p}|\mu, \boldsymbol{p}+\boldsymbol{q})}, \\ g_q(\lambda\mu, \boldsymbol{p}) &= \mathrm{C}_q \frac{1}{\hbar\omega + E(\lambda, \boldsymbol{p}|\mu, \boldsymbol{p}+\boldsymbol{q})}, \end{aligned}\right\} \tag{19}$$

where C_q is a constant independent of $\boldsymbol{p}$, and

$$\mathrm{J}_{xz}^{xz}(\boldsymbol{q}) \sum_{\boldsymbol{p}}^{\mathrm{N}} \left\{ \frac{1}{\hbar\omega - E(x, \boldsymbol{p}|z, \boldsymbol{p}+\boldsymbol{q})} + \frac{1}{\hbar\omega - E(z, \boldsymbol{p}|x, \boldsymbol{p}+\boldsymbol{q})} - \frac{1}{\hbar\omega + E(x, \boldsymbol{p}|z, \boldsymbol{p}+\boldsymbol{q})} - \frac{1}{\hbar\omega + E(z, \boldsymbol{p}|x, \boldsymbol{p}+\boldsymbol{q})} \right\} = 1. \tag{20}$$

Now for $\boldsymbol{R} \neq 0$

$$\mathrm{J}_{xz}^{xz}(\boldsymbol{R}) = -\frac{2\mathrm{Q}^2}{(aR)^5}\left\{1 - \frac{5}{R^2}(X^2+Z^2) + 35\frac{X^2Z^2}{R^4}\right\}.$$

Then

$$\mathrm{J}_{xz}^{xz}(\boldsymbol{q}) = -\frac{1}{\mathrm{G}} \cdot \frac{2\mathrm{Q}^2}{a^5} \int_{R \geq 1} d\boldsymbol{R} \frac{1}{R^5}\left\{1 - \frac{5}{R^2}(X^2+Z^2) + 35\frac{X^2Z^2}{R^4}\right\} \exp(i\boldsymbol{q}\cdot\boldsymbol{R}),$$

under the continuum approximation. Thus, we obtain

$$\lim_{q\to 0} \mathrm{J}_{xz}^{xz}(\boldsymbol{q}) = \frac{1}{\mathrm{G}} \cdot \frac{16\pi}{105}\left(\frac{e^2}{a}\right)\delta^2 \times q^2, \quad \text{(cf. Appendix)} \tag{21}$$

where δ is a dimensionless constant defined by

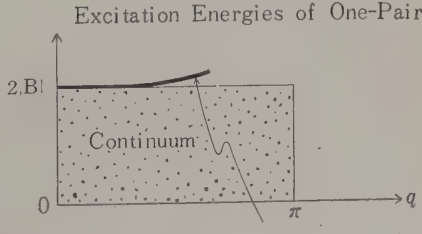
$$Q = ea^2\delta.$$

δ is expected to be much smaller than unity.

Substituting (21) in (20), we obtain the energy-momentum relation of the quanta of the quadrupole-polarization waves as shown in Figs. 1 and 2. (The dispersion relations plotted in these figures are valid only for sufficiently small q as compared with π).

In case of $|\mathrm{B}| \gg (e^2/a)\delta^2$ (Fig. 1), the excitation energies of the states under consideration practically contact with the continuum of the pair excitation energies of scattering states. Then, from (19), we should say that the collective nature of those states is almost faded away.

In case of $|\mathrm{B}| \ll (e^2/a)\delta^2$ (Fig. 2), the excitation energy of the polarization wave with $q \gg \sqrt{|\mathrm{B}|/(e^2/a)\delta^2}$ is fairly apart from the continuum.



$\hbar w = 2|\mathrm{B}|\{1 + \mathrm{const} \times \exp[-\beta/q^2]\}$,
$\beta \equiv (105/16)(|\mathrm{B}|/(e^2/a))\delta^{-2}$

Fig. 1. In case of $|\mathrm{B}| \gg (e^2/a)\delta^2$.

However, the condition $|B| \ll (e^2/a)\delta^2$ is rather inconceivable in metals. Under this condition, electrons will be localized to each lattice point one by one because of the intra-atomic Coulomb repulsion which has been neglected in this paper.

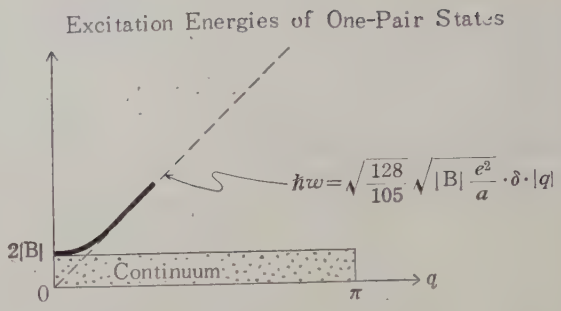


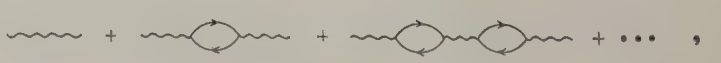
Fig. 2. In case of $|B| \ll (e^2/a)\delta^2$.

In any case $\lim_{q \to 0} \hbar\omega_q$ coincides with the maximum value of the pair excitation energies. Therefore, the oscillation of the quadrupole-polarization wave with a long wavelength will have a short life in any case. Thus, the quadrupole polarization waves, even if they could exist, would play minor roles in physical phenomena. (The waves which can be detected by optical absorptions or characteristic energy losses of electron beams are limited to those with extremely long wavelengths.)

The sound wave is the propagation of the sheets of negatively ionized atoms and the sheets of positively ionized atoms, the sheets being perpendicular to the direction of the propagation. If the wavelength is extremely long, a sufficiently ionized negative sheet should be extremely separated from a sufficiently ionized positive sheet. However, the attractive force between both sheets which are mutually parallel is irrespective of their separation. This is the reason why the excitation energy of the sound wave with a long wavelength can be separated from the continuous excitation energies of the scattering states. In the case of quadrupole polarization waves, the sheet of ionized atoms should be replaced by a sheet of neutral atoms with quadrupole-polarizations. Then the electro-static interaction between two sheets becomes smaller and smaller as the distance of the two sheets becomes larger. Therefore, this interaction is too weak to separate appreciably the excitation energies of the polarization waves from the continuum of the excitation energies of the pair scattering states.

Our result suggests that the anomalous characteristic energy losses[2] of energy lower than the plasmon energy, which have been found in some metals, could not be attributed to any bulk oscillations of electrons in the metals. Ritchie[3] and Ferrell[4] have attributed the anomalous characteristic energy losses to quantized surface waves of the degenerate electron gas. Their idea is preferable from theoretical point of view as well as experimental one.[5]

Finally, we wish to mention the following fact: $J^{xy}_{xy}(\boldsymbol{q})$, $J^{zx}_{zx}(\boldsymbol{q})$, etc., with sufficiently small $\boldsymbol{q}$ ($\boldsymbol{q}$ being parallel to z-axis) cannot be screened by means of the process



while $J^{xx}_{xx}(q)$ is strongly screened by means of the above process.

Acknowledgment

I wish to express my sincere thanks to Prof. Y. Kitano and Prof. Y. Toyozawa for useful discussions.

Appendix

$$\begin{Bmatrix} J^{xy}_{xy}(q) \\ J^{xz}_{xz}(q) \end{Bmatrix} = \frac{1}{G}\frac{2Q^2}{a^5}\begin{Bmatrix} -4\pi \\ 8\pi \end{Bmatrix}\int_1^\infty \frac{dR}{R^3} j_4(qR) = \frac{1}{G}\frac{2Q^2}{a^5}\begin{Bmatrix} -4\pi \\ 8\pi \end{Bmatrix}\int_q^\infty \frac{dx}{x^3} j_4(x)\cdot q^2$$

$$\int_q^\infty \frac{dx}{x^3} j_4(x) = -\int_q^\infty dx \cdot \frac{d}{dx}[x^{-3} j_3(x)] = q^{-3} j_3(q)$$

$$\xrightarrow{q\to 0} \frac{1}{1\cdot 3\cdot 5\cdot 7} = \frac{1}{105}$$

References

1) K. Sawada, Phys. Rev. **106** (1957), 372.
K. Sawada, K. A. Brueckner, N. Fukuda and R. Brout, Phys. Rev. **108** (1957), 507.
R. Brout, Phys. Rev. **108** (1957), 515.
2) H. Watanabe, J. Phys. Soc. Japan **11** (1956), 112.
3) R. H. Ritchie, Phys. Rev. **106** (1957), 874.
4) E. A. Stern and R. A. Ferrell, to be published in Phys. Rev.
I am indebted to Dr. Stern and Prof. Ferrell for sending me their preprint copy in advance of publication.
5) C. J. Powell and J. B. Swann, Phys. Rev. **115** (1959), 869; **116** (1959), 81.

Letters to the Editor

The opinions expressed in these columns do not necessarily reflect those of the Board of Editors. Communications should be submitted in duplicate and should be held to within 1000 lines (pica type) on standard size letter paper (approx. 21×30 cm.), so that each letter will be arranged into two pages when printed. Do not forget to count in enough space for formulas, figures and tables.

Remarks on the Viscous Dissipation in Relativistic Hydrodynamics

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May 17, 1960

For the description of dynamical system in relativistic hydrodynamics, we make use of the covariant energy-momentum tensor[1] containing the viscous stress. We can derive the equation of motion in hydrodynamic process from the 4-divergence of this tensor. Some remarks on the order estimation of viscous terms in the above derivation will be given in this note.

At the initial stage of high energy nucleon-nucleon collisions, the system has the shape of an oblate spheroid concentrated in the forward direction by the factor $\gamma=2Mc^2/E$, where M and E are the nucleon mass and its energy in c-m-system, respectively. Its volume is given by $V\approx\pi a^3\gamma$, using the transversal spread of system $a(\sim\hbar/\mu c)$. As the development of the time, the system will expand into the direction of the x-axis parallel to the forward direction of incident nucleon. Now, the spread of its system in the forward direction at a given time t being assumed as $x(x\geq a\gamma)$, the system has the volume $V\approx\pi a^2x$ at this time. Here, we may use the mean energy density of the system $\varepsilon=E/V=E/(\pi a^2x)$* instead of the energy density measured in a local system at rest of fluid element. Supposing the dependence of energy density on the temperature approximately as $\varepsilon\propto T^4$, we obtain $T\propto x^{-1/4}$. In the case in which the T-dependence[2] of viscous coefficients is assumed as $\eta\propto T^\alpha$, the x-dependence of η is automatically decided, $\eta\sim D\cdot x^{-\alpha/4}$, where D is a numerical constant.

Generally, the momentum transport by the viscous effects is given with the quantity $\eta\cdot\partial v/\partial x=-\eta\cdot\partial v/\partial\xi$, introducing the new coordinate variable $\xi=t-x$. A one-dimensional equation of motion[3] will contain the derivatives of $(\eta\cdot\partial v/\partial t)$, besides the derivatives of the above transport capacity with respect to new space-time variables (ξ, t). In the ultra-relativistic region considered, the former can be neglected by the relation $(\partial v/\partial\xi)/(\partial v/\partial t)\approx t/\xi\approx v^2\gg 1$, induced from the condition $v_0\sim v_1\equiv v\gg 1$, $(t\gg\xi\gg\Delta\equiv a\gamma)$, in which v_0, v_1 are the components of

* This approximation is not always rigorous because of the non-uniformity of energy density in the system, but we may think that its settlement is still valid for the hydrodynamic expansion, where Reynolds number is large. For in the domain considered there is only a small fluctuation of energy density.

four velocity. In the derivatives of $(\eta \cdot \partial v/\partial \tilde{\xi})$ by $(t, \tilde{\xi})$, the inequality

$$|\partial/\partial t(\eta \cdot \partial v/\partial \tilde{\xi})| \ll |\partial/\partial \tilde{\xi}(\eta \cdot \partial v/\partial \tilde{\xi})|$$

holds also by the same discussion as the above. Then, we can conclude that the most remarkable viscous effect in one-dimensional motion is brought by the term $\partial/\partial \tilde{\xi}(\eta \cdot \partial v/\partial \tilde{\xi})$.

Further, we pay attention to the following serious facts as regards the viscous term. We shall discuss the order of magnitude of two terms in the right-hand side of the expression

$$\partial/\partial \tilde{\xi}(\eta \cdot \partial v/\partial \tilde{\xi}) = \partial \eta/\partial \tilde{\xi} \cdot (\partial v/\partial \tilde{\xi}) + \eta \cdot \partial/\partial \tilde{\xi}(\partial v/\partial \tilde{\xi}).$$

Here, if we take into account $\eta \sim D \cdot x^{-\alpha/4} = D \cdot (t-\xi)^{-\alpha/4}$ mentioned above, these terms become as follows:

$$\left.\begin{aligned} &\partial \eta/\partial \tilde{\xi} \cdot (\partial \eta/\partial \tilde{\xi}) \\ &\quad = \eta \cdot (-\alpha/4)/(t-\xi) \cdot (\partial v/\partial \tilde{\xi}), \\ &\eta \cdot \partial/\partial \tilde{\xi}(\partial v/\partial \tilde{\xi}) \\ &\quad = \eta \cdot (-3/2)/\tilde{\xi} \cdot (\partial v/\partial \tilde{\xi}). \end{aligned}\right|$$

Applying the condition $t \gg \tilde{\xi} \gg \Delta$ in the above expressions, we can explicitly set up a relation

$$\eta \cdot \partial/\partial \tilde{\xi}(\partial v/\partial \tilde{\xi}) \gg \partial \eta/\partial \tilde{\xi} \cdot (\partial v/\partial \tilde{\xi}).$$

This narrates the fact that the $\tilde{\xi}$-dependence of velocity is stronger than that of the viscous coefficient "even if the viscous coefficient depends on the temperature", that is, the change of velocity with the drop of temperature is very intense. Therefore, we can expect the reasonable results in the derivation and the solution of a one-dimensional equation of motion even when the viscous coefficients are treated just as the constant parameters.

The models of elementary particles and those multiple production using such a simplified viscous fluid model as mentioned above will be given in a full paper in near future.

1) L. D. Landau and E. M. Lifshitz, *Theory of Elasticity*. G. A. Kluitenberg, *Relativistic Thermodynamics* (Printed in Leiden). M. Hamaguchi, Nuovo Cimento **4** (1956), 1242; Prog. Theor. Phys. **19** (1958), 741. Referred to as I, III, respectively.

2) In the case of $\alpha=3$, the viscous coefficients strongly depend on the temperature, $\eta \propto T^3$. See, C. Iso et al., Prog. Theor. Phys. **22** (1959), 403.

3) See, Eq. (8), in I and Eq.(1) in III.

A Note on Exciton Multiplicities*

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July 20, 1960

In a recent publication in a Supplement to this Journal, Takeuti[1] has commented on the relative positions of singlet and triplet exciton states. He maintains that it is impossible to find a case in which a singlet state will lie below a triplet (a violation of an exciton "Hund's Rule"). Since the author has found such a case in solid argon,[2] and has discussed it recently

* Based on work performed under the auspices of the U. S. Atomic Energy Commission.

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in another connection,[3] some doubt is cast upon his calculation and it appears worthwhile to examine Takeuti's comment carefully.

Let us rewrite Eq. (4·14) of reference 1), the energy of an exciton of multiplicity M, as follows:

$$ {}^{M}V_{\boldsymbol{K}}{}^{\lambda\mu}(0,0)=\delta_{\lambda\mu}({}^{M}V_{C}{}^{\lambda}+\delta_{M}2V_{EX}^{\lambda} +\delta_{M}2V_{D}{}^{\lambda}+{}^{M}V_{OL}^{\lambda}). \qquad (4\cdot14') $$

Here $\delta_1=1$, $\delta_3=0$, $\boldsymbol{K}$ is the wave vector of the exciton, and λ and μ label various states of polarization. This equation differs from (4·14) in that V_C^{λ} and V_{OL}^{λ} are allowed to depend on multiplicity. The remaining notation follows reference 1). Now

$$ {}^{1}V_{K}^{\lambda\lambda}(0,0)-{}^{3}V_{K}^{\lambda\lambda}(0,0) =2(V_{EX}^{\lambda}+V_{D}^{\lambda})+\Delta V^{\lambda}, \qquad (4\cdot20'') $$

where $\Delta V^{\lambda}={}^{1}V_{C}^{\lambda}-{}^{3}V_{C}^{\lambda}+{}^{1}V_{OL}^{\lambda}-{}^{3}V_{OL}^{\lambda}$. Takeuti has shown that $V_{EX}^{\lambda}+V_{D}^{\lambda}$ is always positive;[1] but the term ΔV^{λ} and others which appear in the complete formalism make it possible for (4·20″) to be negative. ΔV^{λ} does not appear in reference 1) because the same Wannier functions were used in the triplet and singlet states and V_{OL} was neglected. ΔV^{λ} does appear in the tight-binding model when proper Hartree-Fock atomic functions are used for states of different multiplicity, and it will arise in the effective mass model when different Wannier functions are assumed for the two branches ($j=1/2$ and $3/2$) of the valence band. In the author's previous work[2),3)] $2V_{EX}^{\lambda}$ and ΔV^{λ} were essentially considered as a single term and $2V_{D}^{\lambda}$ was discussed separately. The possibility of a Hund's rule violation does not depend, as implied in reference 1) on the use of the dipole wave sum approximation to $2V_{D}^{\lambda}$; neither is it caused solely by $2V_{D}^{\lambda}$ itself, as implied in reference 2).

It is to be emphasized that the author agrees with Takeuti's result within the framework of the standard approximations, but wishes to make clear that inequality (4·20) does not imply the general inviolability of the exciton Hund rule.

1) Y. Takeuti, Prog. Theor. Phys. Suppl. No. **12** (1959), 75.
2) R. S. Knox, J. Phys. Chem. Solids **9** (1959), 265.
3) R. S. Knox, Phys. Rev. **116** (1959), 1093.

On Mass Formula and New Image of Elementary Particles

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July 25, 1960

In order to pick up hidden characteristic elements of the internal structure of elementary particle, we carefully examined the experimental results concerning the total cross sections of high energy pion-proton scattering.[1] When we arranged the resonance levels in order of magnitudes of energies regardless of the charge of incident pion, we could discover the following remarkable rule among the magnitudes of each total

energy in the centre-of-mass system:

$$R_N\{1/4^2-1/(4+m)^2\},$$
$$m=1, 2, 3, 4, \cdots, \qquad (1)$$

where $R_N=36\sim37$ Bev. $m=2$ corresponds to the 1st, $m=3$ to the 2nd, $m=4$ to the 3rd resonance levels, and $m=5\sim7$ to the 4th one with broad width respectively. This rule is analogous to a series of line-spectrum which was established in a period of the old quantum theory. We shall call this series the "Nucleon-series". Nucleon-series is also applicable to the resonance energies (total in C. M. system) of the measured total cross sections of high energy $\gamma+p\to\pi^0+p$ process.[2] If such a series has some connection with the internal structure of elementary particle, we can expect that similar formula must come into existence, not only for the resonance levels, but also for the well-known elementary particles.

Now, let us turn our attention only to the elementary particles possessing strangeness. Marking only the rest masses of particles[3] regardless of the statistics, we found out the following series:

$$R_K\{1/8^2-1/(8+m)^2\},\ m=1, 2, 3, \cdots, \qquad (2)$$

where R_K is about 150 Bev. and K-meson, Λ particle, and Ξ particle have $m=1$, 3, and 4 respectively. The particle with $m=2$ has not yet been observed. Moreover, it is especially interesting that the observed 1st resonance level (K-meson kinetic energy is about 25 Mev) of the total cross section of K^-+p scattering[4] has $m=5$. We call this series the "Kaon-series". This Kaon-series explains the observed masses of K, Λ, and Ξ particles and K^-+p resonance energy within the error of 1%. These very accurate formulas, (1) and (2), suggest the existence of a new kind of quantum number. Especially we should like to note the following relation,

$$R_K/R_N\approx 4. \qquad (3)$$

Here we can expect that a new constant R_N will play an important role for elementary particle physics just as the role of Rydberg's constant in atomic physics. With the help of constant R_N, we find that pion mass is almost expressed by the form

$$R_N\{1/8^2-1/(8+1)^2\}. \qquad (4)$$

Formula (4) also suggests existence of "Pion-series" similar to the proceeding series (1) and (2). But others of those levels have not yet been observed.

The appearance of the above series suggests us taking and checking the corresponding relation to Bohr's theory of hydrogen atom. As is well known, in the case of hydrogen atom, Rydberg's constant is given by

$$R_e=2\pi^2 m_e e^4/h^3 c=1.097\times10^5\ \mathrm{cm}^{-1}, \qquad (5)$$

where m_e and e are electron mass and electric charge respectively. This wave number R_e is about 14 ev. in energy unit. Corresponding to this R_e, we tentatively assume our constant R_N to be as follows:

$$R_N=2\pi^2 Mg^4/h^3 c, \qquad (6)$$

If g is the strong coupling constant in

ps-ps meson theory and M is a reduced mass of two bodies in a composite system (presumably $M \approx 500\, m_e$), then $R_N/R_c \approx 2 \times 10^9$. Accordingly, R_N is about 28 Bev. This order of magnitude is consistent with that of constant R_N(36~37 Bev) in (1). This consistency is interesting enough for us to expect a characteristic length corresponding to Bohr's radius.

Here, let us consider the following for interpreting the above series. If we suppose a composite model whose constituent particles are strongly bounded probably by Yukawa force, the fact that Yukawa potential is approximately equal to Coulomb's one at small distance will permit us applying Bohr's quantum condition to the particle making the orbital motion in the internal domain of elementary particle. Then the second term of the above formulas seems to come out from Bohr's condition and therefore corresponds to the binding energy of composite particles, and then the first term will be a sum of rest masses of constituent particles. But in this case the character of the new quantum number in the first term becomes obscure. Therefore, in order to revive the new quantum number, as it was done in Bohr's theory, in the equal capacity for both terms in the above formula, *it will be necessary to introduce the drastic new concept for an image of elementary particle*, i.e. we can understand only the observed particle by means of the difference between two terms. When we advance the theory following the corresponding logic, we must notice that it may be similar to Bohr's theory in form but it may not be so in reality. We can conceive the following: For example, there exist plausible levels in ether or vacuum (In the present field theory, we cannot understand these kinds of state in vacuum. These states just correspond to those of electron in atomic physics) and the observed real particle is created or annihilated by a transition between them as light is so in atom. But now we do not know the selection rules of these transitions. Finally, we have a fancy that a new universal constant will probably appear in R_N instead of Plank's constant h.

The author would like to express his sincere thanks to Professor K. Sakuma, Drs. T. Ouchi and K. Senba for their helpful discussions. Also he would like to express his thanks to Drs. S. Machida, M. Sawamura and other members of Rikkyo University for their kind advices and encouragement.

1) H. C. Burrowes, D. O. Caldwell, D. H. Frisch, D. A. Hill, D. M. Ritson, R. A. Schluter and M. A. Wahlig. Phys. Rev. Letters **2** (1959), 119.
J. C. Brisson, J. Detoef, P. Falk-Vairaut, L. van Rossum, G. Valladas and L. C. L. Yuan, Phys. Rev. Letters **3** (1959), 561.
M. J. Longo, J. A. Helland, W. N. Hess, B. J. Moyer and V. Perez-Meudez, Phys. Rev. Letters **3** (1959), 568.
2) Report by G. Bernaldini at Kiev Conf. on High Energy Nuclear Physics (1959).
3) M. Gell-Mann and A. H. Rosenfeld, Annual Review of Nuclear Science **7** (1957), 411. Report by L. W. Alvarez at Kiev Conf. (1959).
4) See Reports at Geneva Conf. (1958) and Kiev. Conf. (1959).

On the New Constant and the Strangeness

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In the previous note[1] we described that there exist a new constant R_N and a new quantum number. In this note we refine on our mass formula by means of the introduction of correction corresponding to Rydberg's correction in atom. Here it comes into question how to determine R_N. To answer this, we proceed to the following consideration. When we adopt $R_N=37$ Mev., $R_N\{1/(8+1/2)^2-1/(9+1/2)^2\}=102$ Mev. This value is approximately equal to the μ-meson mass. So far as we know, muon which belongs to the lepton family probably has the simpler internal structure than nucleon. In our formula nucleon is $R_N\{1/(4+\delta')^2-1/(5+\delta)^2\}=2^2R_N\{1/(8+2\delta')^2-1/(10+2\delta)^2\}$, where δ' and δ are corrections and 2^2 has a correspondence to Z^2 (Z is atomic number) in atom. The magnitude of Rydberg's correction in atom generally has the following characteristic feature: When Z becomes large, correction increases. Along this line, let us fix R_N so as to take the larger correction on nucleon than that on μ-meson. Here, for convenience sake, we choose zero for correction on muon. Thus, utilizing the experimental value of muon mass 105.70 Mev., we decide R_N for the present as follows:

$$R\equiv R_N=38.29 \text{ Bev.}$$

Now, the rest masses of the well-known all elementary particles above muon mass and the total energies at resonances of total cross sections of $\pi+p$ and K^-+p scatterings are written as

$$\mathfrak{M}=R\left\{\frac{1}{(n+\frac{1}{2}s'+\delta')^2}-\frac{1}{(m+l+\frac{1}{2}s+\delta)^2}\right\},^*$$

where n and m, l, s and s', and δ and δ' are named respectively to be the mass quantum numbers, the internal azimutal quantum number**, the strangeness spin quantum numbers, and their corrections. Tables I, II and III were obtained under the following conditions (a) $\delta'\geq 0$ and $\delta\geq 0$, and (b) the same value was taken on all $\delta's$ at $n=4$ and $s'=0$. As is shwon in Table I, corrections δs which correspond to those due to the screening effect of electrons in atom, have the characteristic feature somewhat similar to Rydberg's ones: When l becomes large, δ decreases. We also see in Table II that statistics is independent of the above quantum numbers and the strangeness of elementary particle is just $s'-s$ which corresponds to representing the attribute of elementary particle by the difference between two terms as was pointed out in the previous note. (But there exists a possibility of defining the strangeness by $-s$, as s' is zero except muon at present stage.) By

* At present we know only $n=4$ or 8 and $\Delta n=m-n=1$.

** Of course, it is also possible to consider l as mass quantum number.

Table I

experimental results				our formula			
	scattering (isotopic spin, spin)	pion kinetic energy (Lab.) (Mev)	Total energy (C. of M.) (Mev)	$n=4$, $m=5$, and $s'=s=0$			
				l	δ'	δ	$\mathfrak{m}$ (Mev)
nucleon	$(\frac{1}{2}, \frac{1}{2})$		940	0	0.10	0.35	940
the 1st resonance	$\pi^+ + p$ (3/2, 3/2) and $\gamma + p \to \pi^0 + p$	190 ~ 200	1233 ~ 1241	1	0.10	0.07	1238
the 2nd resonance	$\pi^- + p$ (1/2, 3/2) and $\gamma + p \to \pi^0 + p$	570 ~ 620	1494 ~ 1526	2	0.10	0.01	1499
the 3rd resonance	$\pi^- + p$ (1/2, >5/2) and $\gamma + p \to \pi^0 + p$	880 ~ 900	1678 ~ 1689	3	0.10	0	1679
the 4th resonance (broad width)	$\pi^+ + p$ (3/2, ?)	1100 ~ 1450	1797 ~ 1971	4 5 6	0.10 0.10 0.10	0 0 0	1805 1895 1961
Total cross sections become almost constant				7	0.10	0	2012
				limit ∞	0.10	0	2278

Table II

		mass (Mev)[3]	$n=4$, $m=5$, and $s'=0$					
			l	s	δ'*)	δ*)	$\frac{1}{2}s$	$\mathfrak{m}$ (Mev)
K-meson	K^+, $\widetilde{K}^-$	494.0 ±0.20	0	−1	0.10	0.13	$-\frac{1}{2}$	492
	K^0, $\widetilde{K}^0$	497.9 ±0.6				0.14		499
Λ-particle	Λ^0	1115.45±0.12	0	+1	0.10	0.24	$+\frac{1}{2}$	1116
Σ-particle	Σ^+	1189.55±0.21	0	+1	0.10	0.43	**) $-\frac{1}{2}+\frac{1}{2}+\frac{1}{2}$	1189
	Σ^0	1191.94±0.53						
	Σ^-	1196.39±0.34				0.45		1196
Ξ-particle	Ξ^0	1311 ±8	0	+2	0.10	0.32	$+\frac{1}{2}+\frac{1}{2}$	1319
	Ξ^-	1319.1 ±0.5						
the 1st resonance of $K^- + p$ scattering		Total energy (C. M.) 1449 K-meson kinetic energy 25 Mev (Lab.)	1	+1	0.10	***) 0.30	$+\frac{1}{2}$	1450

*) If we omit condition (b), then the smaller corrections are possible, e.g. when $\delta'=\delta=0$, our formulas explain, within 1% error, the observed masses of K, Λ and Ξ particles in Table II. But as δ' is correction of $n=4$, in Table II we took $\delta'=0.10$ equal to the value of δ' in Table I.

**) As δ of Σ particle is large, we consider tentatively $\frac{1}{2}s=-\frac{1}{2}+\frac{1}{2}+\frac{1}{2}$.

***) Correction δ of (1, 1), which means (l, s), is nearly equal to a sum of correction δs of Λ (0, 1) and the first resonance of $\pi+p$ scattering belonging to (1, 0).

Table III

		mass[3] (Mev)	n	m	l	s'	s	δ'	δ	$\mathfrak{m}$ (Mev)
μ-meson	μ^-	105.70±0.06	8	9	0	1	1	0	0	105.7
π-meson	π^0	135.04±0.16	8	9	0	0	0	0*)	0.09*)	134.9
	$\pi^\pm$	139.63±0.06						0	0.14	140.1

*) These are trial errors. At present stage we cannot give definite corrections.

this fact it may be possible to express and reinterplet the strangeness in the representation different from the present one. We consider that the attribute of the antiparticle is obtained by mutual exchange between both terms in the above mass formula, though the difficulty in the sign of the rest mass remains.

1) A. Ohmae, Prog. Theor. Phys. **24** (1960), 910. As for the experimental results, see the references of this note.

Birkhoff's Theorem for Electromagnetic Fields in General Relativity

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Exact solutions for combined electromagnetic and gravitational field equations with non-static spherical symmetry are obtained. Besides electric fields, the contributions of the magnetic pole are included. It is shown that the solutions may be transformed to the static form, and thus Birkhoff's theorem[1] is extended from the purely gravitational case to the combined electromagnetic and gravitational case.

In our units, $c=8\pi G=1$. The Roman indices will stand for space-time components and commas and strokes will denote partial and covariant derivatives respectively.

The combined Maxwell-Einstein field equations in the absence of matter are

$$D^j \equiv F^{ij}{}_{/i}=0,$$

$$E^i \equiv \frac{1}{2}\varepsilon^{jkli}F_{kl,j}=F_{[kl,i]}=0 \qquad (1)$$

$$Q^i_{.j} \equiv R^i_{.j}-\frac{1}{2}\delta^i_j R-F^{ik}F_{jk}$$

$$+\frac{1}{4}\delta^i_{.j}F^{kl}F_{kl}=0.$$

Now, the most general non-static spherically symmetric metric form can be reduced to[2]

$$\Phi=-e^{\lambda(r,t)}dr^2-r^2d\theta^2-r^2\sin^2\theta\,d\phi^2$$

$$+e^{\nu(r,t)}dt^2. \qquad (2)$$

From considerations of spherical symmetry we retain only the radial components of electromagnetic fields, and put

$$F_{12}=F_{13}=F_{24}=F_{34}=0. \qquad (3)$$

Without this choice, the transverse components would define a physically distinguishable direction on the surface of a sphere, which would destroy the spherical symmetry of the field. It may be noted that we demand no other

symmetry conditions on the surviving components F_{14} and F_{23}.

We shall briefly outline the main steps of our argument, without showing explicit calculations. By virtue of (2) and (3), the field equations $Q_{12}=Q_{13}=Q_{24}=Q_{34}=Q_{23}=0$ are identically satisfied.

From $Q_4^1=0$ and $Q_4^4-Q_1^1=0$, one gets respectively

$$\lambda_{,4}=0, \tag{4}$$

$$(\lambda+\nu)_{,1}=0,\ \lambda+\nu=f(t). \tag{5}$$

By the choice (3), the electromagnetic field equations $D^1=D^4=E^2=E^3=0$ and $D^2=D^3=E^1=E^4=0$ yield

$$F_{14}=\frac{\varepsilon}{r^2}e^{1/2\cdot f(t)},\qquad F_{23}=\mu\sin\theta, \tag{6}$$

where ε, μ are constants of integration.

Utilizing (3), (4), (5) and (6), one obtains from the equations $Q_2^2=Q_3^3=0$,

$$e^{\lambda}=\left[1-\frac{2m}{r}+\frac{(\varepsilon^2+\mu^2)}{2r^2}\right]^{-1},\qquad e^{\nu}=\left[1-\frac{2m}{r}+\frac{(\varepsilon^2+\mu^2)}{2r^2}\right]e^{f(t)}. \tag{7}$$

Introducing a transformation $dt'=e^{1/2\cdot f(t)}dt$, and dropping primes afterwards, we pass over from (6) and (7) to the corresponding static field strengths and metric form

$$F_{14}=\varepsilon/r^2,\ F_{23}=\mu\sin\theta,$$

$$\Phi=-\left[1-\frac{2m}{r}+\frac{(\varepsilon^2+\mu^2)}{2r^2}\right]^{-1}dr^2-r^2d\theta^2-r^2\sin^2\theta\,d\phi^2+\left[1-\frac{2m}{r}+\frac{(\varepsilon^2+\mu^2)}{2r^2}\right]dt^2. \tag{8}$$

The constants ε and μ can be interpreted as the electric charge and magnetic pole strength of the point source. It was possible to include the magnetic contribution because we worked directly with the field strengths F_{14}, F_{23} instead of potentials and did not assume that they have any spatial symmetry. When μ vanishes the set (8) go over to the Nordström-Reissner-Jeffrey solutions.

The existence of Birkhoff's theorem, that is, the possibility of transforming away the explicit time-dependence, is due to the fact that an electromagnetic monopole cannot radiate.

In the case of a vector-meson field[3] the corresponding $Q_4^1=0$ gives $e^{-\lambda}\cdot\lambda_{,4}/r=\mu^2\phi^1\phi_4\neq 0$, so that the present method fails to prove the existence of Birkhoff's theorem. This failure can be understood by the argument that in the vector-meson case even in the absence of transverse fields the longitudinal component can carry out energy. Birkhoff's theorem does not hold also in the case of a scalar field with or without rest mass[4], because $Q_4^1=0$ gives $e^{-\lambda}\cdot\lambda_{,4}/r=\phi^1\phi_{,4}\neq 0$.

My thanks are due to Professor J. L. Synge, F. R. S., for his kind interest and discussions.

1) G. D. Birkhoff, *Relativity and Modern Physics* (Harvard University Press, 1923).
2) R. C. Tolman, *Relativity, Thermodynamics, and Cosmology* (Oxford, 1944).
3) R. Utiyama, Progr. Theor. Phys. **2** (1947), 38.
4) O. Bergmann and R. Leipnik, Phys. Rev. **107** (1958), 1157.

Moving Effect of Nucleon to Meson Field

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August 1, 1960

Following the proposal by Taketani et al.,[1] Iwadare and others[2] determined the coupling constant from the analysis of nuclear force at below 20 Mev. Later, they extended their work to 150 Mev.[3] These papers are based on the static potential, of which the inner part is given phenomenologically and its tail by the pion theory. Their recent prediction[4] on the negative depolarization seems, however, to suggest the importance of the non-static effect.

In order to begin to examine the non-static effect, here we shall discuss the moving effect of a nucleon to the meson field. To get a direct insight into the effect, it would be rather suitable to treat the classical scalar field. We neglect all such effects as recoil, relativity, heavy meson and eigen-frequency of the nucleon. Here we shall consider only the low frequency part.

Let the nucleon move with a constant velocity $\boldsymbol{v}_0$, its direction be z-axis, the initial position of the center of the nucleon be $-|z_0|$, and its density be $\sigma(r, \theta, z)$ in the cylindrical coordinate.

Solving the field equation with this source and with the initial conditions, $\phi=\partial\phi/\partial t=0$ at $t=0$, we obtain

$$\phi(\boldsymbol{r}, t)=(g/4\pi)\int d\theta''\int r''\,dr''\cdot\int dz''\,\sigma(r'', \theta'', z'')\int dt'\int d\boldsymbol{v}'\int d\omega\cdot\delta(-v_0t'+z'+|z_0|-z'')\cdot\frac{\exp[i\{\sqrt{(\omega/c)^2-\kappa^2}R-\omega\tau\}]}{2\pi R},\quad(1)$$

where $R=|\boldsymbol{r}-\boldsymbol{r}'|$ and $\tau=t-t'$. If we neglect the high frequency part, $\omega/c\gg\kappa$, Eq. (1) is reduced to

$$\phi(\boldsymbol{r}, t)\simeq(g/4\pi)\int d\theta''\int r''\,dr''\cdot\int dz''\,\sigma(r'', \theta'', z'')\cdot\frac{\exp[-\kappa|\boldsymbol{r}-(v_0t-|z_0|+z'')\boldsymbol{n}_z|]}{|\boldsymbol{r}-(v_0t-|z_0|+z'')\boldsymbol{n}_z|}.\quad(2)$$

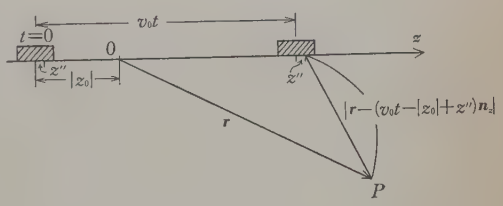

Fig. 1.

Assuming the source to be a sphere, of which the radius is a, and putting

$$\boldsymbol{r}-(v_0t-|z_0|)\boldsymbol{n}_z=\boldsymbol{\rho},$$

we obtain from (2),

$$\phi(\boldsymbol{r}, t)=(g/4\pi)\frac{2\pi}{(4\pi/3)a^3}\cdot\int_{-a}^{a}\frac{dz''}{2}(a^2-z''^2)\cdot\frac{\exp[-\kappa|\boldsymbol{\rho}-z''\boldsymbol{n}_z|]}{|\boldsymbol{\rho}-z''\boldsymbol{n}_z|}.$$

At a point removed far from the source, it becomes

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$$\phi(\boldsymbol{r}, t) = (g'/4\pi)\frac{e^{-\kappa\rho}}{\rho}, \qquad (3)$$

with use of

$$|\boldsymbol{\rho} - z''\boldsymbol{n}_z| \simeq \rho - z''\cos(\boldsymbol{\rho}\cdot\boldsymbol{n}_z).$$

Here

$$g' = g(3/(\kappa' a)^2)\cdot\left(\cosh\kappa' a - \frac{\sinh\kappa' a}{\kappa' a}\right) \qquad (4)$$

and

$$\kappa' = \kappa\cos(\boldsymbol{\rho}\cdot\boldsymbol{n}_z). \qquad (5)$$

In its form, (3) is identical with the conventional solution. However, g' depends on $\boldsymbol{\rho}$ through (4) and (5). This suggests that the nuclear force in the pseudo-scalar field contains not only the usual central and tensor potentials but also the potential derived from the gradient of g'. The coupling constant of the central and tensor potentials itself should be modified from g to g'.

Evidently from (4) and (5), only when $\boldsymbol{\rho}$ is perpendicular to the z-axis, we get the conventional field. In general, g'/g is given by Fig. 2. When $\boldsymbol{\rho}$ is parallel to the z-axis, the field increases about 10% compared with the usual one. This suggests an increase of the nuclear potential of about 20%. This effect becomes significant with increasing energy. Then we may be careful of literally taking the value[2] $g^2 = 0.080 \pm 0.010$ of the coupling constant as a true constant independent of energy.

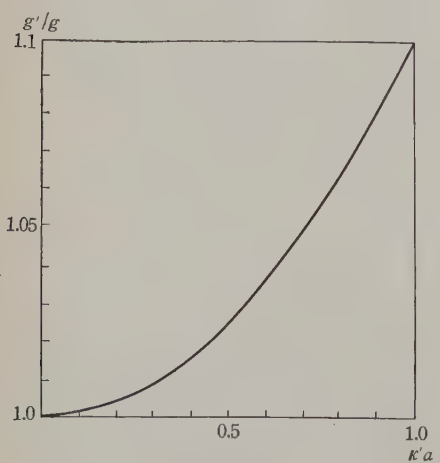

Fig. 2.

The above stated is only a crude story. However, it may be taken as meaning the significance of the non-static effect in general.

1) Taketani, Nakamura and Sasaki, Prog. Theor. Phys. **6** (1951), 581.
2) Iwadare, Otsuki, Tamagaki and Watari, Supplement of Prog. Theor. Phys. **3** (1956), 32.
3) S. Otsuki, Prog. Theor. Phys. **20** (1958), 171. W. Watari, Prog. Theor. Phys. **20** (1958), 181. R. Tamagaki, Prog. Theor. Phys. **20** (1958), 505.
4) Hamada, Iwadare, Otsuki, Tamagaki and Watari, Prog. Theor. Phys. **23** (1960), 366.

0^+ Second Excited State of ^{32}S

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The spins and parities of the ground state and the first excited state at 2.24 Mev of ^{32}S have been assigned as 0^+ and 2^+ respectively, while the second

excited state at 3.78 Mev was assigned as 1^+.

The appearance of 1^+ in low-lying excited state of ^{32}S is difficult to understand.[1)7)]

This note will briefly describe some of the results of our experiments in which we could conclude that the spin parity of the second excited state of ^{32}S is 0^+ and not 1^+.

The experiments which led to the conclusion of 1^+ for the second state of ^{32}S were as follows. El Bedewi and El Wahal[2)] have measured the angular distribution of neutrons from $^{31}P(d, n)^{32}S$ reaction. The neutron group corresponding to the second excited state of ^{32}S showed the angular distribution consistent with the assignment of $l_p=0$. As the ground state of ^{31}P is $1/2^+$, it is concluded that the spin-parity for the second state of ^{32}S is 0^+ or 1^+. On the other hand, there are several reports that the gamma ray of 3.8 Mev energy was observed in the reactions, $^{31}P(p, \gamma)^{32}S$[6)] and $^{32}Cl(\beta^+)^{32}S(\gamma)$.[3)4)] The authors of these papers concluded that this gamma ray was due to the direct transition from the second to the ground state. Therefore, 0^+ was discarded and 1^+ was assigned to the second excited state of ^{32}S.

Huby and Newns[1)] suggested that the second excited state of ^{32}S is a doublet of spacing closer than 20 Kev which is difficult to separate by the magnetic analysis.[5)] The spins and parities were suggested to be 0^+ for the one level and 2^+, $4^+\cdots$ for the other. They thought the former is responsible for the angular distribution of neutrons while the latter is responsible for the gamma ray transition.

We should, however, suspect that the gamma ray of 3.8 Mev energy is not due to the transition from the second state, but to the transition between the higher excited states of ^{32}S.

We have observed gamma ray spectra in the reaction $^{32}S(p, p'\gamma)$ at proton energy of 5.7 Mev using single and three crystal scintillation spectrometers.

An intense 2.24 Mev gamma ray corresponding to the first to ground state transition and a 1.54 Mev gamma ray corresponding to the second to first state transition were observed. The peak corresponding to the 3.78 Mev cross over gamma ray was not found in the pulse height distribution of the scintillater and its intensity relative to the 1.54 Mev and 2.24 Mev gamma rays was estimated to be less than 2×10^{-3} and 2×10^{-4} respectively.

The gamma-gamma cascade coincidence measurement and the sum coincidence measurement also indicate that the decay of the second state is of cascade through the first state.

Next, we observed the angular distribution of the 1.54 Mev gamma ray in coincidence with the 2.24 Mev gamma ray, rotating two scintillaters for the two gamma rays with fixed angle to each other about the axis through the target position and perpendicular to the plane which includes both detectors and the incident proton direction. The distribution was isotropic. Next, one detector was moved on to the rotating axis and another was rotated about the same axis. In this case also, the distri-

bution was found to be isotropic.

These results indicate that the nuclei followed by the gamma emission are not polarized and gamma-gamma angular correlation can be treated in the same way as for radioactive nuclei.

The 1.54 Mev–2.24 Mev gamma-gamma angular correlation data shows good agreement with the theoretical calculation for $0^+(E2)2^+(E2)0^+$ as shown in Fig. 1. One cannot fit the data with $1^+(M1, E2)2^+(E2)0^+$ even if any mixing ratio is assumed.

Thus it is concluded that the spin-parity of the second state of ^{32}S is 0^+.

The details of the experiment will be published in the Journal of the Physical Society of Japan.

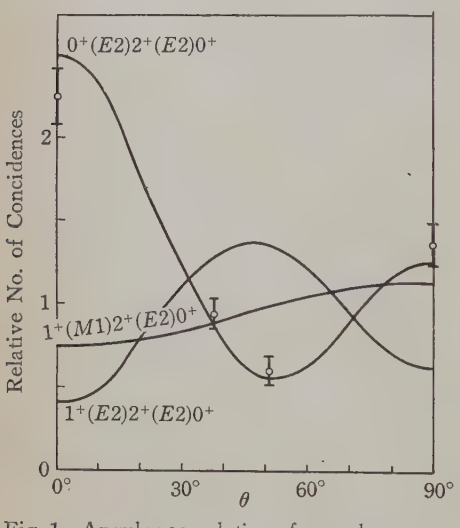

Fig. 1. Angular correlation of cascade gamma rays (1.54 Mev−2.24 Mev) from the second excited state of ^{32}S.

1) R. Huby and H. C. Newns, Proc. Phys. Soc. **68A** (1955), 758.
2) F. A. El Bedewi and M. A. El Wahal, Proc. Phys. Soc. **68A** (1955), 754.
3) Breckon, Henrikson, Martin and Foster, Can. J. Phys. **32** (1954), 223.
4) N. W. Glass and J. R. Richardson, Phys. Rev. **98** (1955), 1251.
5) P. M. Endt, C. H. Paris, A. Sperduto and W. W. Buechner, Phys. Rev. **103** (1956), 961.
6) B. D. Kern and L. W. Cockran, Phys. Rev. **104** (1956), 711.
7) A. Arima, Prog. Theor. Phys. **19** (1958), 421.

Distribution of Paticles in a Low Density Fermion System with Attractive Interactions

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Theories of nuclear matter developed by Brueckner and various other authors are established upon two assumptions.[1] One of them is the restriction to consider only the processes arising from the couplings of particle-particle and hole-hole pairs of the same total momentum. The other assumption is that the configuration of the system is not altered so drastically from that of degenerate Fermi gas. Although these two assumptions do not seem to be independent we may suppose the former one to be correct in order to investigate the validity of the latter. Among the interaction Hamiltonian we shall particularly take into account the terms for the pairs of particles with the total momentum zero.[2] The Hamiltonian thus reduced can be rewritten in terms of spin variables assigned to each one pair state $\boldsymbol{k}$.[3] The classical minimum of the reduced Hamiltonian is given

by the spin orientations determined by

$$\cos\theta_k=\varepsilon(\boldsymbol{k})/\sqrt{\varepsilon(\boldsymbol{k})^2+f(\boldsymbol{k})^2},$$

where

$$\varepsilon(\boldsymbol{k})=(\boldsymbol{k}^2-k_F^2)/2m,$$

m is the effective mass and k_F is the Fermi momentum. The one particle state which is definitely occupied is characterized by $\theta_{\boldsymbol{k}}=\pi$ and the one which is not occupied is specialized by $\theta_{\boldsymbol{k}}=0$. Function $f(\mathbf{k})$ satisfies the integral equation[4]

$$f(\boldsymbol{k})=-(1/2\Omega)\cdot\sum_{k}V(\boldsymbol{k}-\boldsymbol{k}')f(\boldsymbol{k}')/\sqrt{\varepsilon(\boldsymbol{k}')^2+f(\boldsymbol{k}')^2}.$$

Ω means the normalization volume and $V(\mathbf{k})$ is the Fourier transform of two-body potential. As $f(\boldsymbol{k})$ is expected to be a slowly varying function of $\boldsymbol{k}$, $f(\boldsymbol{k}')^2$ in the denominator may be replaced by a constant $\varepsilon_0^2=f(\boldsymbol{k}_F)^2$. Now this denominator will be further simplified. The most important feature of this denominator is that it does not vanish at $\boldsymbol{k}'=\boldsymbol{k}_F$. If $\varepsilon_0=0$, we have a divergence in the $\boldsymbol{k}'$-integral at $\boldsymbol{k}'=\boldsymbol{k}_F$. At $\boldsymbol{k}'$ far from $\boldsymbol{k}_F$, ε_0^2 may be neglected in comparison with $\varepsilon(\boldsymbol{k}')^2$. These two properties can be reproduced by making use of $|\varepsilon(\boldsymbol{k}')|+\varepsilon_0$ instead of $\sqrt{\varepsilon(\boldsymbol{k}')^2+\varepsilon_0^2}$. The weaker the coupling, that is, the smaller the value of ε_0, the better the replacement is. Physically speaking, this replacement changes the exponential decrement in the region of large distance of the correlation function $\chi(\boldsymbol{x})$ whose Fourier transform is defined by

$$\chi(\boldsymbol{k})=\frac{1}{2}\frac{f(\boldsymbol{k})}{\sqrt{\varepsilon(\boldsymbol{k})^2+f(\boldsymbol{k})^2}}$$

to the inverse power decrement. However, the correlation function can be well reproduced in the region of small distance even with this replacement. Thus the integral equation takes the form[5]

$$|\boldsymbol{k}^2-k_F^2|\chi(\boldsymbol{k})=-\frac{m}{\Omega}\sum_{k'}V(\boldsymbol{k}-\boldsymbol{k}')\chi(\boldsymbol{k}')-2m\varepsilon_0\chi(\boldsymbol{k}). \quad (1)$$

Let us first consider S-wave solution of Eq. (1). If we put

$$\chi(\boldsymbol{x})=u(r)/r,\ \boldsymbol{r}=k_F\boldsymbol{x},$$

Eq. (1) is found to be[6]

$$\frac{d^2u}{dr^2}+u=\frac{m}{k_F^2}\cdot\left[v(r)u(r)-2\int_0^\infty dsv(s)u(s)\chi(r,s)\right]+\beta^2\left[u(r)-2\int_0^\infty dsu(s)\chi(r,s)\right], \quad (2)$$

where

$$\chi(r,s)=\frac{1}{\pi}\left\{\frac{\sin(r-s)}{r-s}-\frac{\sin(r+s)}{r+s}\right\},$$

and

$$\beta^2=2m\varepsilon_0/k_F^2.$$

The two-body potential $V(\boldsymbol{x})\equiv v(r)$ is now assumed to have a hard core at $0<r<c$ and a square well with depth V at $c<r<a=b+c$. In a low density system range parameters a, b and c are small as compared with unity, but the well depth parameter $\alpha^2=mV/k_F^2$ becomes large. In this case the solution of Eq. (2) can be obtained by means of sine transform as follows:

$$u(r)=(A/\kappa_0)\sin\kappa_0(r-c), \text{ if } c<r<a,$$

$$=\int_0^1 d\kappa \frac{F\kappa}{1+\beta^2-\kappa^2}\sin\kappa r$$

$$+\frac{2}{\pi}C\int_1^\infty d\kappa \frac{\kappa}{\kappa^2-1+\beta^2}\sin\kappa r$$

$$\text{if } a<r,$$

where A and C are arbitrary constants and

$$\kappa_0^2=\alpha^2+1-\beta^2,$$

$$F=-\frac{2}{\pi}\left[\frac{2A}{\alpha}(\kappa_0 a\cos\kappa_0 b-\sin\kappa_0 b)+c\right].$$

The two boundary conditions, continuities of u and u' at $r=a$, give the condition for the existence of the non-trivial solution for Eq. (2) and it is shown to be

$$[\alpha a\sqrt{1+\beta^2}-\frac{1}{2}(\sqrt{1+\beta^2}+\sqrt{1-\beta^2})\cdot\tan\alpha b]\log\beta^2=\pi\alpha/2+\text{slowly varying function of } \beta^2 \text{ which is independent of density in the low density limit.}$$

The coefficient of $\log\beta^2$ on the left-hand side has a negative value at $\beta^2=0$ if $\alpha a-\tan\alpha b<0$. In such a case Eq. (2) always has a solution. On the other hand, the scattering lengh of the two particle system is now nothing but $(\alpha a-\tan\alpha b)/\sqrt{mV}$. Therefore we can conclude that if the scattering length of two particle system is negative the low density system must have a distribution which is drastically changed from that of degenerate Fermi gas. This conclusion may be valid irrespective of the interaction force. Since the singlet scattering length of two nucleons is actually negative, the low density nuclear matter must have such a distribution. This result is qualitatively in accordance with that of numerical calculations by Emery and Sessler[7].

Although the actual nuclear matter is of rather high density, it might be somewhat interesting to estimate the magnitudes of β^2 and the shift of the ground state energy from that of Fermi energy by means of the above results. The hard core radius is taken to be 0.40×10^{-13} cm and the other two parameters are adjusted to give the low energy data of the singlet scattering and the Fermi momentum is taken to be $k_F=1.48\times10^{13}\,\text{cm}^{-1}$. Then β^2 is found to have the magnitude $\beta^2=0.47$. The shift of the ground state energy per particle can be estimated by making use of the expression given in CMS[4] and is given by

$$\Delta E=-0.04\,k_F^2/2m.$$

1) B. M. Galitzkij, Zh. E. T. F. **34** (1958), 151. F. Iwamoto, Prog. Theor. Phys. **23** (1960), 871.
2) Bardeen, Cooper and Schrieffer, Phys. Rev. **108** (1957), 1175.
3) Y. Wada and N. Fukuda, Prog. Theor. Phys. **22** (1959), 775.
4) Cooper, Mills and Sessler, Phys. Rev. **114** (1959), 1377.
5) N. N. Bogoliubov, DAN. **119** (1958), 52.
6) H. A. Bethe and J. Goldstone, Proc. Roy. Soc. **A238** (1956), 551.
7) V. J. Emery and A. M. Sessler, the Energy Gap in Nuclear Matter, preprint.

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On the Screening of Magnetic Interaction by Degenerate Free Electrons

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A detailed investigation is made on the self-consistent determination of the vector potential due to a magnetic point dipole imbedded in a medium of degenerate free electrons. The Thomas-Fermi method shows that the screening by an induced electric current is Landau-diamagnetic, and practically negligible. A dynaminal method is then applied to the same problem so as to investigate the effect of a flip motion of the point dipole on the screening. In this case another type of the screening arises which is due to the inability of some electrons to follow the non-adiabatic change of force induced by the flip motion. Some discussion is given of the discrepancy between the result and that of Abrahams, and of Bohm and Pines. The derived potential is used for the calculation of the spin lattice relaxation time in ferromagnetic metals, and Kasuya's estimation of its order of magnitude is proved to be reasonable. As to its temperature dependence, some discussion is also given.

§ 1. Introduction

In the previous article,[1]* it was shown that the Thomas-Fermi method can be used so as to include some customarily neglected correlation between electrons, and that this method is equivalent to the familiar method of many particle problem within its present framework of usage. This implies that the conditions for the applicability of both methods are essentially the same. For example, it can be said that the conditions for discarding certain diagrams in, say, Gellmann and Bruckner's method[2] are the conditions as they are for the definition of field variables in Maxwell's equation in matter.

There is another method which is equivalent to the above. This is a transformation method in the field theory, which can be applied with a slight modification to the self-consistent determination of the electro-magnetic interaction in a deformable medium. The renormalization of the electron-phonon interaction as well as of the impurity potential was made by this method.[3] This method (will often be referred to as a dynamical method in the following) has an advantage over G-B's in that one can reach the result rather quickly, but it makes it difficult strictly to examine the legitimacy of some simplifying assumptions introduced in the derivation. Such an examination is, however, a fundamental problem to be in-

* Hereafter referred to as I.

vestigated separately, and will not be made in this article.

In this article, the aforementioned two methods, T-F's and the field theoretical one, will be applied to the detailed investigation of a self-consistent magnetic interaction in an assembly of degenerate free electrons. The same problem was investigated by Bohm and Pines.[4] They derived the screened residual interaction between orbital motions of the electrons. In the present article, the original interaction is taken to be due to a magnetic point dipole imbedded in a medium of free electrons. The same interaction was taken up independently by Kasuya[5] and Abrahams[6] in their calculations of the spin lattice relaxation time T_1 in ferromagnetic metals. Kasuya used the bare Biot-Savart interaction, whereas Abrahams took over Bohm and Pines' result with some modification. Of course, their estimates of T_1 contradicted each other.

If the interaction adopted by Abrahams is correct, an unacceptable result follows. Suppose that two magnetic point dipoles are imbedded in a medium of free electrons, then his interaction implies that the magnetic dipole-dipole interaction between these two dipoles is screened by an induced orbital motion of electrons just as the Coulomb interaction between two point charges in the medium is screened by a density modification of electrons. This leads furthermore to the fact that in a ferromagnetic metal, the corrective Lorentz field as well as the demagnetization field is reduced to a great extent from their values in a ferromagnetic insulator, a fact which seems highly improbable in a normal metal. At present, such a reduction of a magnetic interaction by conduction electrons is known only in superconducting materials. It might well be that in Abrahams' interaction, some spurious supercurrent plays an essential role in the screening.

Meanwhile, the bare Biot-Savart interaction cannot be accepted as it stands, because it is obvious that the screening due to the normal diamagnetic current exists, and because it is possible that some yet unknown effects modify the original interaction so much that the bare potential is hardly applicable. It is one of the main objects of the present article to clear up these points by treating the screening more rigorously.

In § 2, the Thomas-Fermi method is applied to the problem in hand. Instead of the scalar potential treated in I, the vector potential now comes into play. The calculation is rather straightforward. This is due to the fact that all the assumptions are introduced *a priori* in the formalism itself, and no further assumptions are introduced in the way of the calculation. It is shown that the behaviour of the screening is quite normal, and no absurdity as described above comes about. (The superconductivity will not be considered.) However, in order for this method to be precisely applicable, a motion of a perturber, a flip motion of the point dipole in the present case, must be adiabatic. This is a restriction inherent to the statistical method. Otherwise, some other method must be employed.

In § 3, the transformation method is applied to the same problem, treating the interaction from the atomistic point of view. Thereby, another difficulty

arises, which is due to the fact that the rigorous form of the interaction among electron's orbital motions is yet unknown. Therefore, a guiding principle is introduced in the calculation so that when the method is applied to the problem for which the Thomas-Fermi method is also in its realm of applicability, the results of the two methods should agree. Then the same procedure is applied to the problem to which the Thomas-Fermi method is no more applicable. This prescription seems to us not unpromising, but its legitimacy naturally depends on the reasonableness of the results obtained. It is shown that for those processes in which the flip motion of the point dipole occurs, the screening effect alters to some extent from the result of § 2. A similar alteration in the screening was already known in the case of the electron-(longitudinal) phonon interaction, but in this case the alteration was slight owing to the presence of an otherwise large screening factor as well as to the characteristic dispersion formula for the longitudinal phonons. In the present case the alteration is more remarkable so that for a nearly forward scattering there comes out a screening whose extent depends on the Zeeman energy transferred at the flip motion.

In § 4, the method of Bohm and Pines is examined, and it is found that if their treatment is supposed to be of a macroscopic nature, it is not self-consistent in that only the spurious supercurrent prevails in the screening.

In § 5, the derived effective potential is used for the calculation of the spin lattice relaxation time in ferromagnetic metals. It is found that about its order of magnitude Kasuya's estimation is reasonable, but as for its temperature dependence, further consideration is needed. In this connection some speculation is given in analogy with the anomalous skin effect. That is, using Herring's formula of spin wave's velocity, it is pointed out that the mean free path of those spin waves responsible for the ferromagnetic resonance absorption in metals may, at moderately low temperatures, be longer than the skin depth. If this is the case, a special treatment is needed as in the anomalous skin effect. This is not tried in the present article, but a qualitative argument shows that the phenomenon is not so simple as is implied by an ordinary theory.

§ 2. Statistical method

Suppose that a magnetic point dipole with moment vector $\boldsymbol{M}$ is imbedded in a medium of degenerate free electrons and that this vector is fixed both in space and time at the origin of a Cartesian coordinate system. Thus a field of the vector potential is introduced, and it is required to calculate the vector potential, which is determined by solving the Poisson equation

$$\Delta \boldsymbol{A}(\boldsymbol{x}_0) = -\frac{4\pi}{c}\boldsymbol{j}(\boldsymbol{x}_0), \qquad (2\cdot1)$$

where $\boldsymbol{A}(\boldsymbol{x}_0)$ is the vector potential at $\boldsymbol{x}_0$, and $\boldsymbol{j}(\boldsymbol{x}_0)$ is the electric current density at the same place. In the use of phenomenological equations, any quantity ap-

pearing therein should be understood as averaged in a suitable manner. $\boldsymbol{j}(\boldsymbol{x}_0)$ is formally divided into two constituents:

$$\boldsymbol{j}(\boldsymbol{x}_0)=\boldsymbol{j}^m(\boldsymbol{x}_0)+\boldsymbol{j}^e(\boldsymbol{x}_0), \tag{2·2}$$

where $\boldsymbol{j}^m(\boldsymbol{x}_0)$ is the magnetic current of the point dipole

$$\boldsymbol{j}^m(\boldsymbol{x}_0)=c\,\mathrm{rot}[\boldsymbol{M}\delta(\boldsymbol{x}_0)] \tag{2·3}$$

and $\boldsymbol{j}^e(\boldsymbol{x}_0)$ is the orbital electric current. The absence of $\boldsymbol{j}^e(\boldsymbol{x}_0)$ in (2·2) would give the solution

$$\boldsymbol{A}^0(\boldsymbol{x}_0)=\mathrm{rot}\left\{\frac{\boldsymbol{M}}{|\boldsymbol{x}_0|}\right\}=\frac{[\boldsymbol{M}\times\boldsymbol{x}_0]}{|\boldsymbol{x}_0|^3}. \tag{2·4}$$

It is expected that $\boldsymbol{j}^e(\boldsymbol{x}_0)$ screens (2·4). Qualitatively, this is the same as in the case of the scalar potential, but quantitatively quite different from it as one will see below.

Let $f(\mathcal{H})$ be the distribution function of an electron with one particle Hamiltonian $\mathcal{H}$, then $\boldsymbol{j}^e(\boldsymbol{x}_0)$ is calculated from

$$\boldsymbol{j}_\mu^e(\boldsymbol{x}_0)=-e\,\mathrm{Tr}\{v_\mu(\boldsymbol{x}_0), f(\mathcal{H})\}, \tag{2·5}$$

where $\boldsymbol{v}_\mu(\boldsymbol{x}_0)$ is μ-component of the velocity density operator

$$v(\boldsymbol{x}_0)=\{v, \delta(\boldsymbol{x}-\boldsymbol{x}_0)\}=\frac{1}{m}\left\{\left(\boldsymbol{p}+\frac{e}{c}\boldsymbol{A}\right), \delta(\boldsymbol{x}-\boldsymbol{x}_0)\right\} \tag{2·6}$$

and

$$\mathcal{H}=\frac{1}{2m}\left(\boldsymbol{p}+\frac{e}{c}\boldsymbol{A}\right)^2. \tag{2·7}$$

All the other parameters have usual meanings, and the curly brackets mean the symmetrized products. From (2·5) and (2·7) one gets

$$\boldsymbol{j}^e(\boldsymbol{x}_0)=\boldsymbol{j}_K^e(\boldsymbol{x}_0)+\boldsymbol{j}_D^e(\boldsymbol{x}_0), \tag{2·8}$$

with

$$\boldsymbol{j}_K^e(\boldsymbol{x}_0)=-\frac{ne^2}{mc}\boldsymbol{A}(\boldsymbol{x}_0) \tag{2·9}$$

and

$$\boldsymbol{j}_D^e(\boldsymbol{x}_0)=-\frac{e}{m}\,\mathrm{Tr}\{\{\boldsymbol{p}, \delta(\boldsymbol{x}-\boldsymbol{x}_0)\}, f(\mathcal{H})\}. \tag{2·10}$$

Hereafter, $\boldsymbol{j}_K^e(\boldsymbol{x}_0)$ and $\boldsymbol{j}_D^e(\boldsymbol{x}_0)$ will be called respectively the "kinematical" and the "dynamical" current. In the expression of $\boldsymbol{j}_K^e(\boldsymbol{x}_0)$, n denotes the number of the free electrons in unit volume.

Now, let us try to see what happens if we tentatively retain only $\boldsymbol{j}_K^e(\boldsymbol{x}_0)$ in (2·8). (2·1) is written in this case as

$$\Delta \boldsymbol{A}(\boldsymbol{x}_0) = \frac{4\pi n e^2}{mc^2} \boldsymbol{A}(\boldsymbol{x}_0) - 4\pi \operatorname{rot}[\boldsymbol{M}, \delta(\boldsymbol{x}_0)], \tag{2·11}$$

whence we obtain

$$\boldsymbol{A}(\boldsymbol{x}_0) = \operatorname{rot}\left\{\boldsymbol{M} \cdot \frac{e^{-q|\boldsymbol{x}_0|}}{|\boldsymbol{x}_0|}\right\}, \tag{2·12}$$

with

$$q^2 = \frac{4\pi n e^2}{mc^2}. \tag{2·13}$$

Thus in the presence of the free electrons with the spurious current (2·9), the interaction is altered from a long-range nature to a short-range one. The screening constant q is the same as in the simplest theory of the superconductivity. This is natural, because what we have retained is nothing but the simplest equation known as the "London equation" in this phenomenon.[7] In this sense, the screening may be called "superconductive".* (2·12) is the same in character as what Bohm and Pines derived in their first paper as the residual magnetic interaction among the free electrons. This means that in their case the screening is "superconductive", and this is in disagreement with the result which will be obtained below in the normal case. The detail of this problem will be discussed in § 4.

It is well known that in the normal state, the dynamical current cancels out the kinematical one, and there remains only weak, Landau-diamagnetic current.[8] In the linear approximation one obtains after some simple manipulations

$$(\boldsymbol{j}_D^e(\boldsymbol{x}_0))_\mu = -\frac{e^2}{m^2 c} \sum_{kk'} \frac{f(E_{k'}) - f(E_k)}{E_{k'} - E_k} \times \frac{1}{2} \{\langle \boldsymbol{k}| \{\boldsymbol{p}_\nu, \boldsymbol{A}_\nu\} |\boldsymbol{k}'\rangle \langle \boldsymbol{k}'|\boldsymbol{p}_\mu(\boldsymbol{x}_0)|\boldsymbol{k}\rangle + \text{c. c.}\}. \tag{2·14}$$

In the above expresson, E_k is the kinetic energy of a free electron whose state is specified as usual by a wave vector $\boldsymbol{k}$ (spin is disregarded), and $\langle \boldsymbol{k}|F|\boldsymbol{k}'\rangle$ is the matrix element of an operator F. The convention is used for the summation over the repeated Greek indices. Using the explicit expressions

$$\left.\begin{aligned} E_k &= \frac{\hbar^2 \boldsymbol{k}^2}{2m}, \\ \langle \boldsymbol{k}'|\boldsymbol{p}_\mu(\boldsymbol{x}_0)|\boldsymbol{k}\rangle &= \frac{\hbar}{2}(\boldsymbol{k}+\boldsymbol{k}')_\mu e^{i(\boldsymbol{k}-\boldsymbol{k}')\boldsymbol{x}_0}, \\ \text{and} \quad \langle \boldsymbol{k}|\boldsymbol{A}_\nu|\boldsymbol{k}'\rangle &= \int \boldsymbol{A}_\nu e^{-i(\boldsymbol{k}-\boldsymbol{k}')\boldsymbol{x}} d\boldsymbol{x}, \end{aligned}\right\} \tag{2·15}$$

* This terminology may not be properly chosen, for the treatment cannot in any way lead to the supercurrent because of the lack of the interaction responsible for the phenomenon. However, in the lack of a suitable way of expression, we dare to use the word "superconductive" here and hereafter. This word should be read with due reserve.

(2·14) is written down as

$$(\boldsymbol{j}_D^s(\boldsymbol{x}_0))_\mu = -\frac{e^2\hbar^2}{m^2c}\sum_{kk'}\frac{f(E_{k'})-f(E_k)}{E_{k'}-E_k}$$
$$\times\frac{\boldsymbol{k}_\nu+\boldsymbol{k}_\nu'}{2}\frac{\boldsymbol{k}_\mu+\boldsymbol{k}_\mu'}{2}\cdot e^{i(k-k')x_0}\int \boldsymbol{A}_\nu(\boldsymbol{x})\,e^{-i(k-k')x}\,d\boldsymbol{x} \qquad (2\cdot16)$$

(wave functions are taken to be normalized in unit volume). Let the Fourier transform of $\boldsymbol{A}(\boldsymbol{x})$ be introduced:

$$\boldsymbol{\alpha}(\boldsymbol{\kappa}) = \int \boldsymbol{A}(\boldsymbol{x})\,e^{i\kappa x}\,d\boldsymbol{x}. \qquad (2\cdot17)$$

Then, from (2·1), (2·8), (2·9), (2·16) and (2·17), we obtain an equation for $\boldsymbol{\alpha}(\boldsymbol{\kappa})$,

$$-\kappa^2\boldsymbol{\alpha}_\mu(\boldsymbol{\kappa}) = q^2\boldsymbol{\alpha}_\mu(\boldsymbol{\kappa}) + 4\pi i[\boldsymbol{\kappa}\times\boldsymbol{M}]_\mu$$
$$+\frac{4\pi e^2\hbar^2}{m^2c^2}\sum_k\frac{f(E_{k+\kappa})-f(E_k)}{E_{k+\kappa}-E_k}\left(\boldsymbol{k}+\frac{\boldsymbol{\kappa}}{2}\right)_\nu\left(\boldsymbol{k}+\frac{\boldsymbol{\kappa}}{2}\right)_\mu\boldsymbol{\alpha}_\nu(\boldsymbol{\kappa}). \qquad (2\cdot18)$$

Let the gauge of $\boldsymbol{A}$ be chosen in such a way that **div** $\boldsymbol{A}=0$, or, $\kappa_\nu\alpha_\nu(\kappa)=0$, then it is easily shown by a symmetry reasoning that the last term of the right side of (2·18) has only diagonal elements. Consequently, we obtain

$$\boldsymbol{\alpha}(\boldsymbol{\kappa}) = -4\pi i\frac{[\boldsymbol{\kappa}\times\boldsymbol{M}]}{\kappa^2+q^2(1-F(\kappa))}, \qquad (2\cdot19)$$

where

$$F(\kappa)\equiv F(\lambda) = \frac{-\hbar^2}{2mn}\sum\frac{f(E_{k+\kappa})-f(E_k)}{E_{k+\kappa}-E_k}\times|\boldsymbol{k}_\perp|^2$$
$$= \frac{3}{16}\left[\frac{10}{3}-2\lambda^2+\frac{(1-\lambda^2)^2}{\lambda}\log\left|\frac{\lambda+1}{\lambda-1}\right|\right]. \qquad (2\cdot20)$$

In this expression, $\boldsymbol{k}_\perp$ denotes the component of $\boldsymbol{k}$ perpendicular to $\boldsymbol{\kappa}$, and

$$\lambda = \frac{\kappa}{2k_0}, \qquad (2\cdot21)$$

where k_0 is the Fermi wave number. In calculating $F(\lambda)$, the perfect degeneracy of the electron was assumed.

The electric current induced by the inhomogenous field $\boldsymbol{\alpha}(\boldsymbol{\kappa})$ is, from (2·9), (2·16) and (2·17),

$$\boldsymbol{J}^e(\kappa) = -\frac{c}{4\pi}q^2[1-F(\lambda)]\boldsymbol{\alpha}(\boldsymbol{\kappa}). \qquad (2\cdot22)$$

The same formula was obtained by Ferrell[9] in another connection,* who pointed

* The function $1-F(\lambda)$ corresponds to his function $z^2L(z)$. There is a trivial misprint in his Eq. (19).

out its importance in screening the local magnetic interaction.

For small λ, $F(\lambda)$ is expanded into a series

$$F(\lambda)=1-\lambda^2+\frac{\lambda^4}{5}+\frac{\lambda^6}{35}+\cdots. \tag{2·23}$$

Therefore,

$$\boldsymbol{\alpha}(\boldsymbol{\kappa})\cong-4\pi i\frac{[\boldsymbol{\kappa}\times\boldsymbol{M}]}{(1-4\pi\chi)\kappa^2},$$

where

$$\chi=-\frac{q^2}{16\pi k_0^2}$$

is Landau's diamagnetic susceptibility.[10] In the space coordinate

$$\boldsymbol{A}(\boldsymbol{x}_0)\cong\frac{1}{1-4\pi\chi}\boldsymbol{A}^0(\boldsymbol{x}_0)$$

with $\boldsymbol{A}^0(\boldsymbol{x}_0)$ defined by (2·4). Since χ is of order of 10^{-5} at most, we obtain an ordinary relation

$$\boldsymbol{A}(\boldsymbol{x}_0)\cong\mu\boldsymbol{A}^0(\boldsymbol{x}_0) \tag{2·24}$$

where μ is the permeability.

In Fig. 1, $F(\lambda)$ is plotted as a function of λ. In the limit of $\lambda\to0$, or in the limit of a homogeneous field, the screening effect is only weakly diamagnetic and practically negligible, but in the limit of $\lambda\to\infty$,* or in the limit of a violently spatially varying field, the screening comes out. This character is contrary to the case of the scalar potential, where the screening is the strongest in the limit of $\lambda\to0$, and tends to be ineffective as λ increases. One will return to this point in the next section.

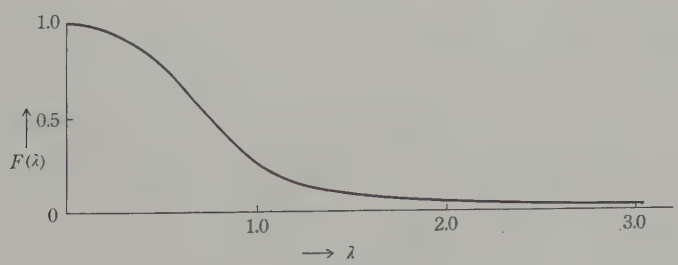

Fig. 1.

The result so far obtained depends more or less on the assumption introduced implicitly in the derivation. Some discussions were given of most of them when the present author studied the screening of the scalar potential. In the present

* Strictly speaking, such λ's will be out of the realm of the applicability of the method.

case another assumption is added. Namely, the point dipole is fixed in its direction; this assumption is peculiar to the vector case. Although it changes its direction, the method of the present section will still be applicable if this change is adiabatic, but breaks down otherwise. The distinction between adiabatic and non-adiabatic changes will depend on the dynamical properties of the system, but it is vague at present. In the next section, the same problem will be treated from a more dynamical point of view, and the above statement will become clearer. Quantum-mechanically, such a description is allowed only when an external magnetic field of suitable magnitude is present.

The present method can be applied to the self-consistent determination of other transverse electro-magnetic interaction in metals. Yosida[11] applied the method independently to the problem of the transverse electron-phonon interaction in metals.

§ 3. Dynamical method

The restriction that the imbedded point dipole is fixed in its direction is removed in this section. The transformation method in the field theory is applied to the problem. Some simplifying assumptions are found to be inevitable in the way of the calculation before one reaches a definite result. There might be some internal inconsistencies among these assumptions. However, detailed examination of them is beyond the scope of the present article and will not be made.

The problem is divided into two parts according as a flip motion of the point dipole occurs or not. In the first place, the method is applied to renormalize the interaction in which the flip does not occur. Obviously, the statistical method also is applicable to this problem, and the solution was already obtained in §2. Our belief is that the dynamical method should give the identical solution in this case. Of course, this should not be expected in a strict sense, and it is the object of the first problem to see what procedure the method undergoes to reach the result, or, in other words, what approximations and assumptions will inevitably be introduced in order for the method to have a common result with the statistical method. Thus a prescription, as it were, for the solution of the second problem is prepared. In the second place, the same procedure as in the first case is applied to the second problem, that is, to derive the self-consistent vector potential in the case where the flip does occur, a problem to which the statistical method is no more applicable.

Let the system be subject to an external magnetic field of magnitude H applied in the z-direction, then the Hamiltonian of the system, so far as one needs, is given by

$$\mathcal{H}=\mathcal{H}_0+\mathcal{H}_B+\mathcal{H}_M, \tag{3·1}$$

where

$$\mathcal{H}_0=\sum_i\left(\frac{\boldsymbol{p}_i^2}{2m}-\mu\boldsymbol{\sigma}_i\boldsymbol{H}\right)-\mu_I\boldsymbol{I}_z\boldsymbol{H}, \tag{3·2a}$$

$$\mathcal{H}_B = \sum_i \frac{2e}{mc} \{\boldsymbol{A}_i, \boldsymbol{p}_i\}, \tag{3·2b}$$

and

$$\mathcal{H}_M = -\frac{1}{2}\left(\frac{e}{mc}\right)^2 \sum_{ij}{}' \left\{\left(\boldsymbol{p}_i + \frac{e}{c}[\boldsymbol{A}_i]\right) \frac{1}{r_{ij}} \left(\boldsymbol{p}_j + \frac{e}{c}[\boldsymbol{A}_j]\right)\right\}. \tag{3·2c}$$

In the Hamiltonian (3·1), only the linear terms in μ_I are actually taken up, and the effect of the external magnetic field on the orbital motion is neglected. A rough estimation shows that this effect is of the order of $(\hbar\omega_0/\zeta)^2$, where ζ is the Fermi energy and ω_0 is the Larmour frequency of a free electron. It is easily shown that the Coulomb interaction discarded in $\mathcal{H}$ has no direct effect on the screening. Each term in $\mathcal{H}$ is explained in the following.

$\mathcal{H}_0$ is the unperturbed Hamiltonian consisting of the kinetic energy of the electrons, and of Zeemann energy of the electrons as well as of the point dipole. μ and σ are the Bohr magneton and the spin operator of the electron respectively. The point dipole is treated as having spin I and the effective Bohr magneton μ_I. $\mathcal{H}_B$ denotes the Biot-Savart interaction between the point dipole and the electrons, and $\boldsymbol{A}$ is the vector potential due to the point dipole

$$\boldsymbol{A}_i = \mu_I \frac{[\boldsymbol{I} \times \boldsymbol{x}_i]}{|\boldsymbol{x}_i|^3}. \tag{3·3}$$

It is convenient to divide $\boldsymbol{A}_i$ into three parts:

$$\boldsymbol{A}_i = \boldsymbol{A}_{i0} + \boldsymbol{A}_{i+} + \boldsymbol{A}_{i-}, \tag{3·4}$$

where

$$\boldsymbol{A}_{i0} = \mu_I \cdot \boldsymbol{I}_z \frac{[\boldsymbol{e}_z \times \boldsymbol{x}_i]}{|\boldsymbol{x}_i|^3} \tag{3·4a}$$

and

$$\boldsymbol{A}_{i\pm} = \mu_I \boldsymbol{I}^{\pm} \frac{[\boldsymbol{e}^{\mp} \times \boldsymbol{x}_i]}{|\boldsymbol{x}_i|^3}. \tag{3·4b}$$

In these expressions

$$\boldsymbol{I}^{\pm} = \boldsymbol{I}_x \pm i\boldsymbol{I}_y$$

and

$$\boldsymbol{e}^{\pm} = 1/2 \cdot (\boldsymbol{e}_x \pm i\boldsymbol{e}_y) \tag{3·5}$$

with $(\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z)$ denoting the three unit vectors in respective direction. $\mathcal{H}_M$ is the magnetic interaction among the electrons. In this expression, $(e/c)[\boldsymbol{A}_i]$ is the kinetic momentum of the i-th electron and is given by

$$[\boldsymbol{A}_i] = \boldsymbol{A}_i + \frac{e}{mc} \sum_k{}' \left\{\frac{1}{r_{ik}}, \left(\boldsymbol{p}_k + \frac{e}{c}[\boldsymbol{A}_k]\right)\right\}, \tag{3·6}$$

which consists of two parts: One is due to the point dipole, and the other to

the orbital motions of all electrons but itself. Note that, owing to the expression (3·6), $\mathcal{H}_M$ cannot be written in a closed form. (As for the choice of $\mathcal{H}_M$ in a form like (3·2c) with $[\boldsymbol{A}_i]$ given by (3·6), see the discussion in the later part of this section and of § 6, and also Appendix I.) $\mathcal{H}_M$ is divided into two parts:

$$\mathcal{H}_M=\mathcal{H}_M^{(1)}+\mathcal{H}_M^{(2)},$$

where

$$\mathcal{H}_M^{(1)}=-\frac{1}{2}\left(\frac{e}{mc}\right)^2\sum_{ij}{}'\left\{\boldsymbol{p}_i,\left\{\frac{1}{r_{ij}},\boldsymbol{p}_j\right\}\right\} \tag{3·7a}$$

and

$$\mathcal{H}_M^{(2)}=-\frac{1}{2}\left(\frac{e}{mc}\right)^2\sum_{ij}{}'\left(\left\{\boldsymbol{p}_i,\frac{1}{r_{ij}}\frac{e}{c}[\boldsymbol{A}_j]\right\}+\left\{\boldsymbol{p}_j,\frac{1}{r_{ij}}\frac{e}{c}[\boldsymbol{A}_i]\right\}\right). \tag{3·7b}$$

In all the above expressions, the curly brackets again mean the symmetrized products, and the sums are taken over the electrons with the usual exceptions in the double sums.

One needs the matrix elements of the operators in (3·2). These are written down in the following:

$$\left.\begin{aligned}&\langle m'|\boldsymbol{I}_z|m\rangle=\delta_{m,m'}m,\\&\langle m'|\boldsymbol{I}^{\pm}|m\rangle=\delta_{m',m\pm1}\sqrt{(\boldsymbol{I}\mp m)(\boldsymbol{I}\pm m+1)},\end{aligned}\right\} \tag{3·8}$$

where m is z-component of $\boldsymbol{I}$.

$$\left\langle\boldsymbol{k}+\boldsymbol{\kappa}\right|\left\{\frac{x_\mu}{|\boldsymbol{x}|^3},\boldsymbol{p}_\nu\right\}\left|\boldsymbol{k}\right\rangle=-\hbar\left(\boldsymbol{k}+\frac{\boldsymbol{\kappa}}{2}\right)_\nu\frac{4\pi i\kappa_\mu}{\kappa^2}, \tag{3·9}$$

$$\left\langle\boldsymbol{k}_1+\boldsymbol{\kappa},\boldsymbol{k}_2\right|\left\{\boldsymbol{p}_1,\left\{\frac{1}{r_{12}},\boldsymbol{p}_2\right\}\right\}\left|\boldsymbol{k}_1,\boldsymbol{k}_2+\boldsymbol{\kappa}\right\rangle=\frac{4\pi}{\kappa^2}\left(\boldsymbol{k}_1+\frac{\boldsymbol{\kappa}}{2}\right)\left(\boldsymbol{k}_2+\frac{\boldsymbol{\kappa}}{2}\right). \tag{3·10}$$

$$\begin{aligned}&\left\langle\boldsymbol{k}_1+\boldsymbol{\kappa},\boldsymbol{k}_2\right|\left\{\boldsymbol{p}_1,\frac{1}{r_{12}}\boldsymbol{A}_2\right\}+\left\{\boldsymbol{A}_1\frac{1}{r_{12}},\boldsymbol{p}_2\right\}\left|\boldsymbol{k}_1,\boldsymbol{k}_2+\boldsymbol{\kappa}'\right\rangle\\&\quad=\hbar\left\{\frac{4\pi}{\kappa^2}\left(\boldsymbol{k}_1+\frac{\boldsymbol{\kappa}}{2}\right)\boldsymbol{\alpha}(\boldsymbol{\kappa}'-\boldsymbol{\kappa})+\frac{4\pi}{\kappa'^2}\left(\boldsymbol{k}_2+\frac{\boldsymbol{\kappa}'}{2}\right)\boldsymbol{\alpha}(\boldsymbol{\kappa}'-\boldsymbol{\kappa})\right\},\end{aligned} \tag{3·11}$$

where $\boldsymbol{\alpha}(\boldsymbol{\kappa})$ is the Fourier transform of $\boldsymbol{A}$,

$$\boldsymbol{\alpha}(\boldsymbol{\kappa})=\int\boldsymbol{A}e^{i\boldsymbol{\kappa}\boldsymbol{x}}d\boldsymbol{x}=-4\pi i\mu_I\frac{[\boldsymbol{I}\times\boldsymbol{\kappa}]}{\kappa^2}. \tag{3·12}$$

From now on, the problem will be treated in the number representation. In the same notation as adopted in I, and using the expressions (3·9)–(3·12), one has

$$\mathcal{H}_0=\sum_{k\sigma}E_{k\sigma}n_{k\sigma}-\mu_I mH, \tag{3·13}$$

where

$$E_{k\sigma}=E_k-\sigma\mu H \tag{3·13a}$$

with σ denoting the spin quantum number of an electron.

$$\mathcal{H}_B=\mu\sum_{k\sigma}\sum_{\kappa}\{\boldsymbol{\alpha}_0(\boldsymbol{\kappa})+\boldsymbol{\alpha}_+(\boldsymbol{\kappa})+\boldsymbol{\alpha}_-(\boldsymbol{\kappa})\}(2\boldsymbol{k}+\boldsymbol{\kappa})c^*_{k+\kappa\sigma}c_{k\sigma}, \tag{3·14}$$

where $\boldsymbol{\alpha}_0(\boldsymbol{\kappa})$ and $\boldsymbol{\alpha}_\pm(\boldsymbol{\kappa})$ are the Fourier transforms of (3·4a) and (3·4b) respectively, and are given explicitly by (3·12) with due modifications.

$$\mathcal{H}_M^{(1)}=-\frac{1}{2}\mu^2\sum_{k_1\sigma}\sum_{k_2\sigma'}\sum_{\kappa}\frac{4\pi}{\kappa^2}(2\boldsymbol{k}_1+\boldsymbol{\kappa})(2\boldsymbol{k}_2+\boldsymbol{\kappa})c^*_{k_1+\kappa\sigma}c_{k_1\sigma}c^*_{k_2\sigma'}c_{k_2+\kappa\sigma'}. \tag{3·15}$$

$$\begin{aligned}\mathcal{H}_M^{(2)}=-\frac{1}{2}\mu\sum_{k_1\sigma}\sum_{k_2\sigma'}\sum_{\kappa\kappa'}\frac{e^2}{mc^2}\left\{\frac{4\pi}{\kappa^2}\left(\boldsymbol{k}_1+\frac{\boldsymbol{\kappa}}{2}\right)\boldsymbol{\alpha}(\boldsymbol{\kappa}'-\boldsymbol{\kappa})+\frac{4\pi}{\kappa'^2}\left(\boldsymbol{k}_2+\frac{\boldsymbol{\kappa}'}{2}\right)\boldsymbol{\alpha}(\boldsymbol{\kappa}'-\boldsymbol{\kappa})\right\}\\ \times c^*_{k_1+\kappa\sigma}c_{k_1\sigma}c^*_{k_2\sigma'}c_{k_2+\kappa'\sigma'}\\ +\text{terms of six order in } c^* \text{ and } c+\cdots.\end{aligned} \tag{3·16}$$

In deriving (3·15) and (3·16), certain terms are dropped out which are sure to vanish on account of a symmetry reasoning. (3·2c) and (3·6) show that there appear in $\mathcal{H}_M$ the interaction terms simultaneously containing more than three electron coordinates, so that the rigorous treatment will be prohibitively difficult. One is forced to truncate $\mathcal{H}_M$ half-way by making an assumption described later.

The program is now sketched the details of which are given in § 3 of I (Nakajima's method). A transformation is made

$$e^{-iS}\mathcal{H}e^{iS}=\mathcal{H}-i[S,\mathcal{H}]+\cdots,$$

with the generating function S to be determined so that the renormalized interaction $\mathcal{H}_B{}^0$—whose determination is our final goal—may be transformed away

$$-i[S,\mathcal{H}_0]+\mathcal{H}_B{}^0=0. \tag{3·17}$$

$\mathcal{H}_B{}^0$ is to be obtained by solving the equation of the self-consistency

$$-i[S,\mathcal{H}_M^{(1)}+\mathcal{H}_M^{(2)}]+\mathcal{H}_M^{(2)}+\mathcal{H}_B-\mathcal{H}_B{}^0=0. \tag{3·18}$$

Superscript 0 means the renormalized interaction, and the renormalization will be made on the vector potential $\boldsymbol{\alpha}(\boldsymbol{\kappa})$.

As mentioned at the beginning of this section, the procedure is required to be such that when it is applied to renormalize $\boldsymbol{\alpha}_0(\boldsymbol{\kappa})$ in (3·14), it leads to the identical result with that of § 2. For this, as one will see immediately, a further assumption is inevitable:

$$-i[S,\mathcal{H}_M^{(2)}]+\mathcal{H}_M^{(2)}-(\mathcal{H}_M^{(2)})^0=0, \tag{3·19}$$

where $(\mathcal{H}_M^{(2)})^0$ has the same form as $\mathcal{H}_M^{(2)}$ except that $\boldsymbol{\alpha}(\boldsymbol{\kappa})$ in $\mathcal{H}_M^{(2)}$ is replaced by $\boldsymbol{\alpha}^0(\boldsymbol{\kappa})$. Then (3·18) is simplified to

$$-i[S,\mathcal{H}_M^{(1)}]+(\mathcal{H}_M^{(2)})^0+\mathcal{H}_B-\mathcal{H}_B{}^0=0. \tag{3·20}$$

Assumption (3·19) is vital in this article. Some justification for it is tried in Appendix I, and it is found that the second term of the right side of (3·6)

is essential. Since (3·6) has a form like a power series in $(e^2/r)/(mc^2)$, the justification must be made step by step in each order, so that it cannot be done in a closed form just as the vector potential has no closed form. It is expected that we will not have bitterly mistaken with assumption (3·19).

To begin with, one renormalizes the interaction

$$\mathcal{H}_{B0}=\frac{2e}{mc}\sum_{k\sigma}\sum_{\kappa}\boldsymbol{\alpha}_0(\boldsymbol{\kappa})\,\hbar\left(\boldsymbol{k}+\frac{\boldsymbol{\kappa}}{2}\right)c^*_{\boldsymbol{k}+\boldsymbol{\kappa}\sigma}c_{k\sigma}.$$

(3·17) is easily solved to be given by

$$-iS_0=\sum_{k\sigma}\sum_{\kappa}\frac{\mu\boldsymbol{\alpha}_0^{\,0}(\boldsymbol{\kappa})\,(2\boldsymbol{k}+\boldsymbol{\kappa})}{E_{k+\kappa\sigma}-E_{k\sigma}}c^*_{\boldsymbol{k}+\boldsymbol{\kappa}\sigma}c_{k\sigma}, \tag{3·21}$$

where $\boldsymbol{\alpha}_0^{\,0}(\boldsymbol{\kappa})$ is the renormalized vector potential.* Assumption (3·19) gives

$$(\mathcal{H}_M^{(2)})^0=-\frac{1}{2}\mu\frac{e^2}{mc^2}\sum_{k_1\sigma}\sum_{k_2\sigma'}\sum_{\kappa\kappa'}\Big\{(2\boldsymbol{k}_1+\boldsymbol{\kappa})\,(\boldsymbol{\alpha}_0^{\,0}(\boldsymbol{\kappa}'-\boldsymbol{\kappa})+\boldsymbol{\alpha}_+^{\,0}(\boldsymbol{\kappa}'-\boldsymbol{\kappa})$$

$$+\boldsymbol{\alpha}_-^{\,0}(\boldsymbol{\kappa}'-\boldsymbol{\kappa}))\,\frac{4\pi}{\kappa^2}+(2\boldsymbol{k}_2+\boldsymbol{\kappa}')\,(\boldsymbol{\alpha}_0^{\,0}(\boldsymbol{\kappa}'-\boldsymbol{\kappa})+\boldsymbol{\alpha}_-^{\,0}(\boldsymbol{\kappa}'-\boldsymbol{\kappa})+\boldsymbol{\alpha}_-^{\,0}(\boldsymbol{\kappa}'-\boldsymbol{\kappa}))\,\frac{4\pi}{\kappa'^2}\Big\}$$

$$\times c^*_{\boldsymbol{k}_1+\boldsymbol{\kappa}\sigma}c_{k_1\sigma}c^*_{k_2\sigma'}c_{\boldsymbol{k}_2+\boldsymbol{\kappa}'\sigma'}. \tag{3·22}$$

From (3·15) and (3·21), using the random phase approximation (R.P.A.) and dropping out the exchange terms, one obtains

$$-i[S,\mathcal{H}_M^{(1)}]=-\mu^3\sum_{k\sigma}\sum_{\kappa}c^*_{\boldsymbol{k}+\boldsymbol{\kappa}\sigma}c_{k\sigma}\sum_{k'\sigma'}\frac{\boldsymbol{\alpha}_0^{\,0}(\boldsymbol{\kappa})\cdot(2\boldsymbol{k}'+\boldsymbol{\kappa})}{E_{k'+\kappa\sigma'}-E_{k'\sigma'}}$$

$$\times\frac{4\pi}{\kappa^2}(2\boldsymbol{k}+\boldsymbol{\kappa})\cdot(2\boldsymbol{k}'+\boldsymbol{\kappa})\,(n_{k'+\kappa\sigma'}-n_{k'\sigma'}). \tag{3·23}$$

Then (3·20) is explicitly written down:

$$(2\boldsymbol{k}+\boldsymbol{\kappa})_\mu\Big\{\boldsymbol{\alpha}_0^{\,0}(\boldsymbol{\kappa})_\mu-\boldsymbol{\alpha}_0(\boldsymbol{\kappa})_\mu+\frac{q^2}{\kappa^2}\boldsymbol{\alpha}_0^{\,0}(\boldsymbol{\kappa})_\mu$$

$$+\frac{4\pi\mu^2}{\kappa^2}\sum_{k'\sigma'}\frac{(2\boldsymbol{k}'+\boldsymbol{\kappa})_\mu(2\boldsymbol{k}'+\boldsymbol{\kappa})_\nu}{E_{k'+\kappa\sigma'}-E_{k'\sigma'}}(n_{k'+\kappa\sigma'}-n_{k'\sigma'})\,\boldsymbol{\alpha}_0^{\,0}(\boldsymbol{\kappa})_\nu\Big\}=0. \tag{3·24}$$

In deriving (3·24), the R.P.A. is used for $(\mathcal{H}_B^{(2)})^0$, too.**

When $\boldsymbol{\alpha}_0^{\,0}(\boldsymbol{\kappa})$ is so chosen that $\kappa_\nu\alpha_\nu^{\,0}(\boldsymbol{\kappa})=0$ holds, it is easily shown as before that the last term in the curly bracket has only diagonal elements. Then one has

$$\boldsymbol{\alpha}_0^{\,0}(\boldsymbol{\kappa})=\frac{\kappa^2\boldsymbol{\alpha}_0(\boldsymbol{\kappa})}{\kappa^2+q^2(1-F(\kappa,0))}, \tag{3·25}$$

where

* The principal value is taken. The same is the case in the following.

** Thereby, certain terms vanish on account of the symmetry.

$$F(\boldsymbol{\kappa}, 0)=-\frac{8\pi^2\mu^2}{q^2}\sum_{k\sigma}\frac{|\boldsymbol{k}_\perp|^2(n_{k+\kappa\sigma}-n_{k\sigma})}{E_{k+\kappa\sigma}-E_{k\sigma}}. \tag{3·26}$$

When the Fermi distribution $f(E_{k\sigma})$ is inserted into $n_{k\sigma}$, (3·25) agrees with (2·19). It is shown that the spin paramagnetism of the free electrons has little effect on the screening, being of the order of $(\mu H/\zeta)^2$, hence is negligible. This is due to the fact that throughout the interaction the spin of an electron is conserved and its effect is only through a slight modification of the distribution function. Thus, the same result as in § 2 follows. This is what is aimed at from the beginning.

Now, the same procedure is applied to renormalize the interaction which flips the spin of the point dipole. Instead of (3·17), one has

$$-i[S_\pm, \mathcal{H}_0]+(\mathcal{H}_{B\pm})^0=0, \tag{3·27}$$

where

$$(\mathcal{H}_{B\pm})^0=\mu\sum_{k\sigma}\sum_{\kappa}\boldsymbol{\alpha}_\pm^0(\boldsymbol{\kappa})\,(2\boldsymbol{k}+\boldsymbol{\kappa})\,c^*_{k+\kappa\sigma}c_{k\sigma}.$$

The solution is

$$-iS_\pm=\mu\sum_{k\sigma}\sum_{\kappa}\frac{\boldsymbol{\alpha}_\pm^0(\boldsymbol{\kappa})\cdot(2\boldsymbol{k}+\boldsymbol{\kappa})}{E_{k+\kappa\sigma}-E_{k\sigma}\pm\mu_I H}c^*_{k+\kappa\sigma}c_{k\sigma}. \tag{3·28}$$

It is a straightforward matter to obtain

$$\boldsymbol{\alpha}_\pm^0(\boldsymbol{\kappa})=\frac{\kappa^2\boldsymbol{\alpha}_\pm(\boldsymbol{\kappa})}{\kappa^2+q^2(1-F(\kappa, \Delta))}, \tag{3·29}$$

where

$$F(\kappa, \Delta)=-\frac{8\pi^2\mu^2}{q^2}\sum_{k\sigma}\frac{|\boldsymbol{k}_\perp|^2(n_{k+\kappa\sigma}-n_{k\sigma})}{E_{k+\kappa\sigma}-E_{k\sigma}+\Delta}=F(\kappa, -\Delta) \tag{3·30}$$

with

$$\Delta=\mu_I H. \tag{3·31}$$

In perfectly degenerate case, $F(\kappa, \Delta)$ is calculated to be given by

$$F(\kappa, \Delta)\equiv F(\lambda, \gamma)=\frac{1}{2\lambda}\{g(\lambda^+)+g(\lambda^-)\}, \tag{3·32}$$

where

$$g(x)=\frac{3}{16}\left\{(1-x^2)^2\log\left|\frac{x+1}{x-1}\right|+x\left(-2x^2+\frac{10}{3}\right)\right\}=xF(x), \tag{3·33}$$

$$\lambda^\pm=\lambda\pm\frac{\gamma}{\lambda}, \tag{3·34}$$

$$\gamma=\frac{\mu_I H}{2\zeta}, \tag{3·35}$$

and λ is defined by (2·21).

The function $F(\lambda, \gamma)$ is plotted in Fig. 2 against λ, and in Fig. 3 against γ

with all the other parameters having some typical values. The magnitude of this function signifies the extent to which the dynamical current cancels out the screening effect due to the kinematical current. Fig. 2 shows that this cancellation is imperfect both in the limit of $\lambda \to 0$ and $\lambda \to \infty$. In the former case, the presence of Δ in the denominator of (3·30) is responsible for this fact. From its expression, the presence of Δ is equivalent to saying that the perturbation is varying in time with a frequency $\omega = \Delta/\hbar$, or the Larmor frequency, because then $(q^2/\kappa^2)(1-F(\kappa, \omega))$ is nothing but the polarizability constant (the effect of displacement current being neglected) of the free electrons for a varying transversal electromagnetic field. Owing to the inertia of the electrons, there occurs a time lag and so a reduction in their response to the field. The dynamical current alone being affected by this, it is not sufficiently strong this time to cancel out the kinematical current.

An intuitive interpretation is offered. Associated with the flip motion of the point dipole, a pulsive force is generated around it. Some electrons cannot follow this change of force adiabatically, thus the magnitude of the polarization (i.e. the dynamical current) being somewhat diminished. Roughly speaking, each mode of the scattering distinguished by the parameter λ has its own screening electrons. The flip motion of the point dipole is adiabatic for some electrons (with large λ), while non-adiabatic for some other electrons (with small λ). In the first case, the screening effect is nearly the same as in the statistical method, but in the second case, the screening is governed by the kinematical current.

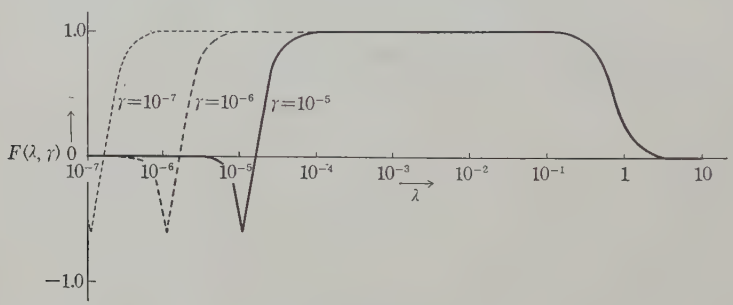

Fig. 2.

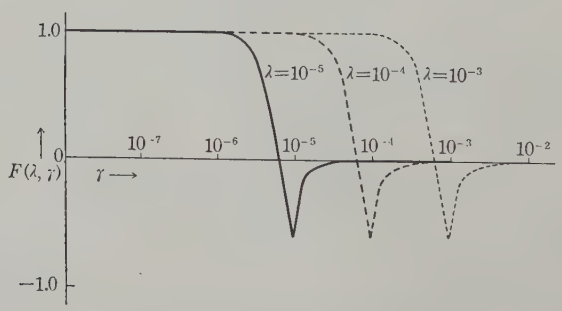

Fig. 3.

It is easily understood that, as Fig. 3 indicates, the number of screening electrons for which the flip motion is non-adiabatic increases as the external magnetic field, or the parameter γ, increases.*

In Figs. 2 and 3, the crosses of the curves with the abscissae remind us of the well-known dispersion curves of electro-magnetic polarization phenomena, and may be interpreted in the language of these phenomena. However, since one has assumed the adiabatic deformation of the electronic states independently of the motion of the point dipole (this assumption is inherent to the dynamical method), and this assumption seems most unjustifiable at these crossing points, too much attention should not be paid on this behaviour.

§ 4. Remarks on Bohm and Pines' method

In this section, we review the method of Bohm and Pines asking for the reason why their method, though applied in a little different connection from the present case, should have led to the result corresponding to the case in which, in our language, only the kinematical screening current is present.

They started from the Hamiltonian

$$H=\sum_i (\boldsymbol{p}_i+\frac{e}{c}\boldsymbol{A}(\boldsymbol{x}_i))^2/2m+\int([\boldsymbol{E}^2(\boldsymbol{x})+\boldsymbol{H}^2(\boldsymbol{x})]/8\pi)\,d\boldsymbol{x}, \qquad (4\cdot1)$$

with $\boldsymbol{E}$ and $\boldsymbol{H}$ denoting the transverse electromagnetic field. They introduced then canonical field variables in the classical treatment, or, quantized the field in the quantum mechanical treatment. After that, new variables were defined by a transformation determined so that the interaction between the electrons and the field, written in the new variables, should vanish in the lowest order. So electrons were dressed with the transversal photons. This gave rise to a new Hamiltonian consisting, in the lowest order, of the kinetic energy, the field energy with a renewed dispersion relation, and of the screened Biot-Savart interaction whose screening is just " superconductive ".

Such a transformation method is ordinary, and has been successful for the treatment of the longitudinal interactions. It has been employed both in the quantum electrodynamics where an elementary interaction between two elementary particles is the subject,[12] and in the many-particle problem where the correlation between the particles with a known or assumed interaction is the subject. In the longitudinal cases, the potential between two particles is determined (apart from a relativistic or similar correction) at their instantaneous positions. On the other hand, for the determination of the vector potential in question, the motion of the electrons (not necessarily limited to two) are essential. Therefore, it cannot in

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* One can define a sort of coherence length ξ_0 in such a way that the equality $\gamma/\lambda=1$ holds for the value $\kappa=\xi_0^{-1}$. From (3·35) and (2·21), one gets $\xi_0=v_0/\omega_0$, where v_0 is the Fermi velocity and ω_0 is the Larmor frequency ($\hbar\omega_0=\mu H$).

a microscopic sense be written as a function of the position and the (single) time only. It can be so only after proper averages are taken over space and time just like the field variables in classical electrodynamics.

If one supposes (indeed, one is forced to suppose) that the field variables in (4·1) are interpreted in this way, they must be correctly connected with the average motion of the electrons via some phenomenological equations. By introducing the canonical field variables (classical), or by quantizing the field (quantal), such connections are involved, which are Maxwell's equations.[13] One can derive from these an equation for the vector potential. Namely, Hamiltonian (4·1) with the field variables treated in the above way implies from the beginning the connection

$$\Delta \boldsymbol{A}(\boldsymbol{x}) - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = \frac{4\pi e}{mc}\sum_i \left\{\left(\boldsymbol{p}_i + \frac{e}{c}\boldsymbol{A}(\boldsymbol{x}_i)\right), \delta(\boldsymbol{x}-\boldsymbol{x}_i)\right\}. \tag{4·2}$$

Neglecting the retardation effect and putting

$$n = \sum_i \delta(\boldsymbol{x}-\boldsymbol{x}_i), \tag{4·3}$$

(4·2) is simplified to

$$\Delta \boldsymbol{A}(\boldsymbol{x}) - \frac{4\pi n e^2}{mc^2}\boldsymbol{A}(\boldsymbol{x}) = +\frac{4\pi e}{mc}\sum_i \{\boldsymbol{p}_i, \delta(\boldsymbol{x}-\boldsymbol{x}_i)\}. \tag{4·4}$$

From now on n is taken to be a constant, because an average is taken over the space coordinate. Use of the R.P.A. is equivalent to this averaging operation. This is shown in Appendix II. Then one obtains

$$\boldsymbol{A}(\boldsymbol{x}) = -\frac{e}{mc}\sum_i \left\{\boldsymbol{p}_i, \frac{e^{-q|\boldsymbol{x}-\boldsymbol{x}_i|}}{|\boldsymbol{x}-\boldsymbol{x}_i|}\right\}, \tag{4·5}$$

where q is the screening constant defined by (2·13). This is identical with Bohm and Pines' residual interaction.

However, it is clear that in the above treatment the average is not correctly taken over the electronic motion ($\boldsymbol{p}_i$), and that it is the kinematical current which screens the interaction. Actually, correct quantum statistical average must be taken, then one will return to § 2. The result there suggests that in normal cases the screening is weakly diamagnetic and practically negligible. However, the result of § 3 will impose on this statement some restrictive condition. That is, in order for the above statement to be precise, there must not be a non-adiabatic change in the interacting systems. Otherwise, one must employ the method of § 3 for this interaction in question. This will be a much more complicated problem than that of §§ 2 and 3 in which one of the interacting units is prescribed in its motion. Generally speaking, the result of § 2 seems valid for an elastic scattering of the magnetic origin.

As a conclusive remark of this section, the following can be said: A microscopic treatment cannot be executed by assuming the vector potential as a func-

tion of the position and the (single) time only, whereas a macroscopic treatment must be such that its basic phenomenological equations or the procedures leading to these equations are self-consistent.

§ 5. Spin-lattice relaxation in ferromagnetic metals

Now that the effective potential has a form different from the usually accepted one, it will be interesting to see its effect on relevant problems. One example of them is taken up here.

The mechanism which governs the spin-lattice relaxation in ferromagnetic metals has been obscure.[14] Abrahams carried out the calculation of T_1 with the Biot-Savart interaction screened "superconductively",* and concluded that T_1 obtained by using this interaction was too long to explain the line width of a ferromagnetic absorption line. On the other hand, Kasuya carried out the same calculation with the same interaction but without the screening and obtained a reasonable result. The result of the present article shows that neither is exactly true. The method of § 3 will be applied to this problem.

Suppose that the conduction electrons are described as before and the system responsible for the ferromagnetism is described in terms of spin waves. Then the Hamiltonian of the system is given by

$$\mathcal{H}=\mathcal{H}_0+\mathcal{H}_B+\mathcal{H}_M, \tag{5·1}$$

where

$$\mathcal{H}_0=\sum_{k\sigma} E_{k\sigma} n_{k\sigma}+\sum_{\kappa} \varepsilon_\kappa \left(N_\kappa+\frac{1}{2}\right), \tag{5·2a}$$

$$\mathcal{H}_B=\mu\sum_{k\kappa\sigma} \{\boldsymbol{\alpha}_+(\boldsymbol{\kappa})(2\boldsymbol{k}+\boldsymbol{\kappa}) c^*_{k\sigma} c_{k+\kappa\sigma}+\boldsymbol{\alpha}_-(\boldsymbol{\kappa})(2\boldsymbol{k}+\boldsymbol{\kappa}) c^*_{k+\kappa\sigma} c_{k\sigma}\}, \tag{5·2b}$$

and $\mathcal{H}_M$ is the same as (3·15) and (3·16). In (5·2a) ε_κ is the energy of a spin wave with a wave number $\boldsymbol{\kappa}$,

$$\varepsilon_\kappa=\frac{2\mu A\kappa^2}{M_S}+\mu H, \tag{5·3}$$

where A is the exchange stiffness introduced by Herring and Kittel,[15] and M_s is the saturation magnetization.

N_κ is the occupation operator of a spin wave $\boldsymbol{\kappa}$ and is defined by

$$N_\kappa=a_\kappa^* a_\kappa, \tag{5·4}$$

where a_κ^* and a_κ are respectively the creation and the annihilation operators of the spin wave with the commutation relations

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* His interaction was such that the original interaction (3·3) is multiplied simply by the screening factor $exp(-qr)$ of Bohm and Pines. Even if the screening were "superconductive", this would not be true. Instead, (2·12) should replace it.

$$\left.\begin{aligned}&[a_\kappa, a_{\kappa'}]=[a_\kappa^*, a_{\kappa'}^*]=0\\&[a_\kappa, a_{\kappa'}^*]=\delta_{\kappa,\kappa'}.\end{aligned}\right\} \tag{5·5}$$

$\mathcal{H}_B$ is the Biot-Savart interaction written in terms of the spin wave variables, and $\boldsymbol{\alpha}_\pm(\boldsymbol{\kappa})$ are given by

$$\left.\begin{aligned}\boldsymbol{\alpha}_+(\boldsymbol{\kappa})&=(2\mu M_S)^{1/2}\frac{4\pi i[\boldsymbol{e}^+\times\boldsymbol{\kappa}]}{\kappa^2}a_\kappa^*,\\ \boldsymbol{\alpha}_-(\boldsymbol{\kappa})&=-(2\mu M_S)^{1/2}\frac{4\pi i[\boldsymbol{e}^-\times\boldsymbol{\kappa}]}{\kappa^2}a_\kappa,\end{aligned}\right\} \tag{5·6}$$

where $\boldsymbol{e}^\pm$ are given by (3·5). $\mathcal{H}_B$ was written in such a form by previous authors, but due modification is made here to facilitate the comparison with § 3. Now the correspondence of (5·2) to (3·14) is obvious. ε_κ in (5·2a) does the job of $-\mu H$ in (3·2a). It is again a straightforward matter to obtain the renormalized vector potential

$$\boldsymbol{\alpha}_\pm^0(\boldsymbol{\kappa})=\frac{\kappa^2\boldsymbol{\alpha}_\pm(\boldsymbol{\kappa})}{\kappa^2+q^2[1-F(\lambda,\gamma)]}, \tag{5·7}$$

where $F(\lambda,\gamma)$ is defined by (3·26), or explicitly by (3·32). Instead of (3·34) one has now

$$\lambda^\pm=\lambda\pm\frac{\varepsilon_\kappa}{2\zeta}\frac{k_0}{\kappa}. \tag{5·8}$$

Comparing (5·7) with Abrahams' formula (7), it is seen that if his interaction is multiplied by a factor $S(\kappa)$:

$$S^{-1}(\kappa)=\left[1+\frac{q^2}{\kappa^2}(1-F(\lambda,\gamma))\right]\left[1-\frac{q}{\kappa}\tan^{-1}\frac{\kappa}{q}\right], \tag{5·6}$$

it coincides with (5·7). From now on, one can follow Abrahams' calculation, and the relaxation time $\tau(\kappa)$ of spin wave κ is given by

$$\tau(\kappa)=\tau_A(\kappa)\times|S^{-1}(\kappa)|^2, \tag{5·10}$$

where $\tau_A(\kappa)$ is Abrahams' value, or explicitly

$$\tau_\kappa^{-1}=\frac{32\pi me^2\mu M_S}{3c^2\hbar^5}\zeta\frac{\varepsilon_\kappa}{\kappa^3}\left[1+\frac{q^2(1-F(\lambda,\gamma))}{\kappa^2}\right]^{-2}. \tag{5·11}$$

For the values of practical interest, $F(\lambda,\gamma)$ is expanded as

$$F(\lambda,\gamma)=1-\frac{q^2}{3\kappa^2}\left(\frac{\gamma}{\lambda}\right)^2+\cdots. \tag{5·12}$$

If one accepts the argument that the phenomenon is governed by those spin waves whose wave number is the reciprocal of the skin depth δ, the second and the following terms in the expansion (5·12) are entirely negligible for actual values of $\kappa_0=\delta^{-1}$. This means that for this κ_0, the result of § 2 can be safely

applied and there is no practical screening in the interaction. Taking $M_s \sim 500$, $\kappa_0 \sim 7 \times 10^4$ corresponding to a room temperature value, $\varepsilon_\kappa \sim 2.4 \times 10^{-17}$ and $\zeta \sim 10^{-12}$, one obtains $\tau_\kappa \sim 4 \times 10^{-10}$, which is reasonable for explaining the line width. Though our estimation agrees numerically with that of Kasuya who made the estimation at $\kappa \sim 10^5$, his argument concerning this value together with its temperature independence is not clear.

Since δ depends on the temperature T through the electric conductivity, our mechanism fails to explain the experimentally observed temperature independence of the line width. However, the above argument needs some revision. Actually, microwave in a specimen has many other components which are the Fourier transforms of itself. Let a coordinate system be chosen in such a way that a static magnetic field is in the z-direction, an oscillating magnetic field $H_x(y, t)$ is in the x-direction and the oscillating field propagates in the y-direction which is also the inward normal of the surface of the specimen. Then one has

$$H_x(y, t) = \sum_\kappa H'(0) \varphi(\kappa, \omega) e^{i\kappa y - i\omega t}, \tag{5·13}$$

where $H'(0, t)$ is the value of $\partial/\partial y \; H_x(y, t)$ at the surface. Normal skin effect with permeability $\mu(\kappa, \omega)$ gives

$$\varphi(\kappa, \omega) = \frac{2}{\kappa^2 + (2i/\delta)\mu(\kappa, \omega)}. \tag{5·14}$$

Let the surface impedance due to single spin wave κ be $Z_\kappa(\omega)$, which is

$$Z_\kappa^{-1}(\omega) = \frac{2c}{\omega \delta^2} \varphi(\kappa, \omega), \tag{5·15}$$

then, the total impedance $Z(\omega)$ is given by

$$Z^{-1}(\omega) = \sum_\kappa Z_\kappa^{-1}(\omega). \tag{5·16}$$

If one neglects the exchange effect for simplicity, $\mu(\kappa, \omega)$ is given by

$$\mu(\kappa, \omega) = 1 + \frac{4\pi(H + 4\pi M_S) M_S (\mu/\hbar)^2}{(i\omega + 1/\tau_\kappa)^2 + \omega_0^2}, \tag{5·17}$$

where ω_0 is the resonance frequency

$$\omega_0 = \varepsilon_\kappa/\hbar = \frac{\mu}{\hbar} \sqrt{H(H + 4\pi M_S)}. \tag{5·18}$$

Brief derivation of Eqs. (5·14) to (5·18) is given in Appendix III. The sum over κ in (5·16) will yield a line width somewhat differing from the value estimated above and at the same time will render its temperature dependence a little weaker than is indicated by the single τ_{κ_0}. Yet the dependence seems still too strong to agree with the experimental fact. To be freed from this discrepancy, we introduce here the ineffectiveness concept known in the anomalous skin effect.[16]

In this phenomenon, the mean free path of conduction electrons is much larger than the apparent skin depth determined by the ordinary theory. Therefore, those electrons whose velocities have larger inclinations with the surface of a specimen are ineffective in absorbing the applied microwave. This is because these electrons spend most of their mean free time in the region where the microwave does not penetrate. The absorption is participated exclusively by the electrons with velocities nearly parallel to the surface. Let this concept be applied to the problem in hand.

From the facts that δ decreases and κ_0 increases as temperature decreases, and that τ_{κ_0} increases as κ_0 increases (see Eq. (5·11)), it can be said that at sufficiently low temperatures, the mean free path of the spin waves responsible for the absorption will become larger than the skin depth. Then, a spin wave with a small normal velocity component (i.e. with small κ) will become much more important for the absorption than a spin wave with a larger one. The most important κ's should be determined in a self-consistent way, a complicated problem which will not be treated here. However, the above qualitative argument is enough to suggest that the temperature dependence in question will be much weaker than is indicated by the single τ_{κ_0}. The reason is as follows. On the one hand the decrease of the skin depth makes the spin waves with larger wave vectors come into play in the absorption, and on the other hand the ineffectiveness concept shows that at the same time the spin waves with the smaller wave vectors are more effective for the absorption than those with the larger ones. These two opposite trends will compete to make the effect of the temperature indefinite.

A rough numerical estimation shows that this situation is quite probable. From Herring's expression for the velocity of the spin wave[17]

$$v_\kappa = \frac{\hbar}{m^*}\kappa, \tag{5·19}$$

where m^* is the effective mass of the magnetic electrons, the critical wave number is determined by

$$\frac{\hbar}{m^*}\cdot\kappa\tau_\kappa \gtrsim \delta,$$

or, since $m/\hbar \sim 1$,

$$\kappa\tau_\kappa \gtrsim \left(\frac{m^*}{m}\right)\delta. \tag{5·20}$$

At room temperature $\kappa_0 \sim 7\times10^4$, $\tau_{\kappa_0} \sim 4\times10^{10}$, so that $\kappa_0\tau_{\kappa_0} \sim 2.8\times10^{-5}$ and $\delta \sim 1.5\times10^{-5}$, so the situation will not be far from the anomalous region.

The discussion has so far been made on the supposition that the line width is determined only by the interaction in question. One cannot, however, discard the possibility that the inhomogeneous exchange interaction, or the dipole-dipole interaction in the magnetic system might change the spin wave structure so much

that there exists a temperature independent broadening of comparable or eventually larger order.[18]

Recently, Rado and Weertmann[19] suggested that the inhomogeneous exchange interaction is essential for determining the absorption line. Their argument was based on the following two facts. (I) $\mu_1^{(e)}-\mu_2^{(e)}$ diagram in which $\mu_1^{(e)}$ and $\mu_2^{(e)}$ are defined by the equation

$$\mu_1^{(e)}-i\mu_2^{(e)}=-2i(Z(\omega)c/\delta\omega)^2,$$

is not a sphere with H as a parameter as is the case if the absorption line would be of a relaxation type taking no accout of the exchange broadening (Eq. (5·17) of the present article with τ_κ replaced by some constant). (II) $\mu_2^{(e)}$ is negative for small values of H. However, it is remarked here that if the relaxation time has an H-dependence as is indicated by (5·11) together with (5·18), a case which was not examined by Rado et al., (5·17) yields a $\mu_1^{(e)}-\mu_2^{(e)}$ diagram very similar to their case though $\mu_2^{(e)}$ can never be negative. The actual situation is more complicated because of the κ-dependence of the relaxation time.

We can neglect the exchange interaction when the condition

$$\frac{\mu\kappa^2 A}{\hbar M_S} < \frac{1}{\tau_\kappa},$$

is satisfied for the broadening, and

$$\frac{2A\kappa^2}{M_S} < H,$$

for the eigen-frequency. At the numerical values introduced before and $A\sim 10^{-6}$, the two inequalities are found to hold. However, the decrease in T as well as in H makes the exchange effect dominant. The fact that the κ-dependence of the two broadening effects are nearly inverse to each other will weaken the T-dependence of the resultant line width still further. Anyhow, more rigorous treatment is required in the anomalous region before one deduces a quantitative conclusion.

§ 6. Summary and discussions

The statistical method was applied to derive the self-consistent vector potential due to a point dipole imbedded in a medium of degenerate free electrons. The derived potential depends on the mode of scattering in such a way that for a nearly forward scattering ($\lambda \ll 1$) the screening is weakly diamagnetic and practically negligible, but it increases as a momentum change throughout the scattering increases. This is just reverse to the case of the impurity potential.

The dynamical method was then applied to the same problem with particular emphasis on the scattering process by which the point dipole flips its direction in an applied external magnetic field. It was shown that for a scattering with a large momentum change, the result of the statistical method is approximately

valid. For a scattering with a small momentum change ($\lambda/\gamma \ll 1$), on the other hand, a screening arises. This screening was interpreted as due to the inability of some screening electrons to follow the induction force generated by the flip motion. In the intermediate case ($\lambda/\gamma \sim 1$), a behaviour like a magnetic dispersion was observed.

Bohm and Pines', and Abrahams' potential were shown to be such that only the spurious supercurrent was taken into account in the screening.

Estimation of T_1 in ferromagnetic metals gave a reasonable result as to its order of magnitude, but a difficulty arose as for its temperature dependence. Two discussions were given which might weaken the temperature dependence, but no decisive conclusion was obtained. It is required to treat the problem in the anomalous region more rigorously.

Our aim was first to be freed from the unacceptable results of the previous authors mentioned in § 1, and this was achieved by the statistical method in § 2. Our second aim was then to establish a method in a natural way and to apply it to the problem for which the statistical method is beyond its realm of applicability. This was achieved in a sense by a dynamical treatment in § 3 with results which allowed reasonable interpretations. However, in contrast to the statistical treatment where the calculation was rather straightforward, one encountered some mathematical difficulties this time, and the calculation was performed only after making some simplifying assumptions. These were: the random phase approximation, the neglect of the exchange terms, the convergence of the transformation leading to the renormalization of the potential, the assumption of adiabatic deformation of the electronic states in the perturbing field, and so on.

In addition to these, another more basic assumption was introduced with respect to the form of the magnetic interaction among the electrons. (See (3·2c).) On the one hand, there is no definite reasoning for supporting it except that the classical interaction between two prescribed electric currents is applied as it is to the many-electron system with each electron having no prescribed motion. On the other hand, there seems to be no decisive reasoning for refuting it at present. It will not be fruitful to discuss this point, because it has an intimate connection with the yet unsolved many-electron problem.

Under these circumstances, a conclusive remark on the content of the present article, especially of § 3, is necessarily confined in stating that a trial was offered of solving the problem set up in § 1, and that in the trial the dynamical method showed a practical usefulness over the statistical one.

The result of § 4 indicates that one must be careful in treating the problem of the transversal electromagnetic interaction lest one should deal only with the spurious supercurrent. The spurious supercurrent arises from the inconsistent use of the random phase approximation. At present, there is no definite rule for the use of the R.P.A., and we can only say that the legitimacy of its use will be manifested in the reasonableness of the result. There are some articles in which the inconsistencies of the treatment reveal themselves in the fact that some relations

therein are not gauge invariant. Detailed investigation will be made in a forthcoming article.

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Appendix I

Justification of (3·19)

The outline of the justification of (3·19) is sketched here. For the illustration, the space coordinate representation is found to be more convenient than the number representation. One must note that the higher order terms in (3·2c) and (3·6) have no correct numerical coefficients, which will be considered here. In this representation, Eq. (3·20) is written as

$$i\left[S, -\frac{1}{2}\sum_{ij}{}'\left\{\boldsymbol{p}_i, \frac{1}{r_{ij}}, \boldsymbol{p}_j\right\}\right]-\frac{1}{2}\sum_{ij}{}'\left\{\boldsymbol{p}_i, \frac{1}{r_{ij}}\boldsymbol{A}_j^0+\boldsymbol{p}_j, \frac{1}{r_{ij}}\boldsymbol{A}_i^0\right\}$$
$$+\sum_i\{\boldsymbol{A}_i, \boldsymbol{p}_i\}-\sum_i\{\boldsymbol{A}_i^0, \boldsymbol{p}_i\}=0. \qquad \text{(AI·1)}$$

For simplicity, one has put $e=c=m=1$, and only the interaction terms simultaneously containing two electron coordinates at most were retained. S is the solution of (3·17) and is a sum of functions of one electron coordinate $(\boldsymbol{x}_i, \boldsymbol{p}_i)$, whose matrix elements are the summand of (3·21) or (3·28) with the operators c^* and c removed. Namely,

$$S=\sum_i S_i, \qquad \text{(AI·2)}$$

where S_i is a function of $(\boldsymbol{x}_i, \boldsymbol{p}_i)$. Then (AI·1) is written as

$$i\frac{1}{2}\sum_{ij}{}'\left\{\left[S_i, \boldsymbol{p}_i\frac{1}{r_{ij}}\right]\boldsymbol{p}_j+\boldsymbol{p}_i\left[S_j, \frac{1}{r_{ij}}\boldsymbol{p}_j\right]\right\}-\frac{1}{2}\sum_{ij}{}'\left\{\boldsymbol{p}_i\frac{1}{r_{ij}}\boldsymbol{A}_j^0+\boldsymbol{A}_i^0\frac{1}{r_{ij}}\boldsymbol{p}_j\right\}$$
$$+\sum_i\{\boldsymbol{A}_i, \boldsymbol{p}_i\}-\sum_i\{\boldsymbol{A}_i^0, \boldsymbol{p}_i\}=0, \qquad \text{(AI·3)}$$

whence one gets

$$i\sum_i{}'\left\{\left[S_i, \boldsymbol{p}_i\frac{1}{r_{ij}}\right]\right\}-\sum_i{}'\frac{1}{r_{ij}}\boldsymbol{A}_i^0+\boldsymbol{A}_j-\boldsymbol{A}_j^0=0. \qquad \text{(AI·4)}$$

This separation of the equation is not strictly correct, but it may be so in the sense that by the summation an average is implied (a sort of the R.P.A.).

In order for (AI·1) to hold, (3·19) must hold, which is written down as

$$i\frac{1}{2}\left[S_i, \sum_{ijk}{}' \left\{\boldsymbol{p}_i \frac{1}{r_{ij}} \frac{1}{r_{jk}} \boldsymbol{p}_k\right\}\right] - \frac{1}{2}\sum_{ijk}{}' \left\{\boldsymbol{p}_i \frac{1}{r_{ij}} \frac{1}{r_{jk}} \boldsymbol{A}_k^0 + \boldsymbol{p}_k \frac{1}{r_{kj}} \frac{1}{r_{ji}} \boldsymbol{A}_i^0\right\}$$

$$-\frac{1}{2}\sum_{ij}{}' \left\{\boldsymbol{p}_i \frac{1}{r_{ij}} \boldsymbol{A}_j + \boldsymbol{p}_j \frac{1}{r_{ij}} \boldsymbol{A}_i\right\} + \frac{1}{2}\sum_{ij}{}' \left\{\boldsymbol{p}_i \frac{1}{r_{ij}} \boldsymbol{A}_j^0 + \boldsymbol{p}_j \frac{1}{r_{ij}} \boldsymbol{A}_i^0\right\} = 0. \quad \text{(AI·5)}$$

The first triple sum in the above expression follows from

$$\frac{1}{3!}\sum_{ijk}{}' \left\{\boldsymbol{p}_i \frac{1}{r_{ij}} \frac{1}{r_{jk}} \boldsymbol{p}_k + \boldsymbol{p}_j \frac{1}{r_{ij}} \frac{1}{r_{ik}} \boldsymbol{p}_k + \boldsymbol{p}_i \frac{1}{r_{ik}} \frac{1}{r_{kj}} \boldsymbol{p}_j\right\}$$

after a suitable permutation among the indices. The factor $1/3!$ is the correct numerical coefficient for the three-particle interaction. The other terms follow in a similar manner. For (AI·5), the following equality is assumed:

$$-i[S, \mathcal{H}_M^{(2)}] + \mathcal{H}_M^{(2)} - \frac{1}{2}\sum_{ijk}{}' \left\{\boldsymbol{p}_i \frac{1}{r_{ij}} \frac{1}{r_{jk}} \boldsymbol{A}_k + \boldsymbol{p}_k \frac{1}{r_{kj}} \frac{1}{r_{ji}} \boldsymbol{A}_i\right\}$$

$$+\frac{1}{2}\sum_{ijk}{}' \left\{\boldsymbol{p}_i \frac{1}{r_{ij}} \frac{1}{r_{jk}} \boldsymbol{A}_k^0 + \boldsymbol{p}_k \frac{1}{r_{kj}} \frac{1}{r_{ji}} \boldsymbol{A}_i^0\right\} = 0. \quad \text{(AI·6)}$$

In this expression, $\mathcal{H}_M^{(2)}$ includes the interaction terms simultaneously containing more than four electron coordinates. Eq. (AI·5) can be separated in the above sense

$$i\sum_{i}{}' \left\{\left[S_i, \boldsymbol{p}_i \frac{1}{r_{ij}}\right] \frac{1}{r_{jk}} \boldsymbol{p}_k\right\} - \sum_{i} \left\{\frac{1}{r_{ji}} \boldsymbol{A}_i^0 \frac{1}{r_{jk}} \boldsymbol{p}_k\right\} - \left\{\boldsymbol{A}_j \frac{1}{r_{jk}} \boldsymbol{p}_k\right\} + \left\{\boldsymbol{A}_j^0 \frac{1}{r_{jk}} \boldsymbol{p}_k\right\} = 0, \quad \text{(AI·7)}$$

which is a direct consequence of (AI·4). In writing (AI·7), the term

$$\sum_{i}{}' \left\{\boldsymbol{p}_i \left[S_j \frac{1}{r_{ij}} \frac{1}{r_{jk}}\right] \boldsymbol{p}_k\right\}$$

was ignored. The sum over i will reduce this to a negligible magnitude. The repeated use of the procedure will justify (AI·6), and so on.

Appendix II

Derivation of (4·5) by the R.P.A.

The Fourier transform of (4·2) gives

$$-\kappa^2 \boldsymbol{\alpha}(\boldsymbol{\kappa}) = \frac{4\pi e}{mc}\sum_{i} \{\boldsymbol{p}_i, e^{i\boldsymbol{\kappa}\boldsymbol{x}_i}\} + \frac{4\pi e^2}{mc^2}\sum_{i}\sum_{\boldsymbol{\kappa}'} \boldsymbol{\alpha}(\boldsymbol{\kappa}') e^{i\boldsymbol{\alpha}_i(\boldsymbol{\kappa}-\boldsymbol{\kappa}')}. \quad \text{(AII·1)}$$

Use of the R.P.A. in the double sum gives

$$-\kappa^2 \boldsymbol{\alpha}(\boldsymbol{\kappa}) = \frac{4\pi e}{mc}\sum_{i} \{\boldsymbol{p}_i, e^{i\boldsymbol{\kappa}\boldsymbol{k}_i}\} + q^2 \boldsymbol{\alpha}(\boldsymbol{\kappa}),$$

whence one gets

$$\boldsymbol{\alpha}(\boldsymbol{\kappa}) = \frac{-(4\pi e/mc)\sum_i \{\boldsymbol{p}_i, e^{i\boldsymbol{\kappa}\boldsymbol{x}_i}\}}{\kappa^2+q^2}, \qquad \text{(AII·2)}$$

which is the Fourier transform of (4·5). This use of the R.P.A. is not legitimate.

Appendix III

Derivation of the surface impedance

In the same coordinate system as that introduced in § 5, Maxwell's equations in a specimen are written as

$$\begin{aligned} \nabla\times\boldsymbol{e} &= -\frac{i\omega}{c}(\boldsymbol{h}+4\pi\boldsymbol{m}), \\ \nabla\times\boldsymbol{h} &= \frac{4\pi\sigma}{c}\boldsymbol{e}, \end{aligned} \qquad \text{(AIII·1)}$$

where $\boldsymbol{e}$, $\boldsymbol{h}$ and $\boldsymbol{m}$ are oscillating components of electromagnetic field and magnetization in the specimen (the displacement current is ignored), and σ is the electric conductivity. The equation of motion for the magnetization is

$$i\omega\boldsymbol{m} = \frac{\mu}{\hbar}\boldsymbol{i}_z\times\left[M_S\boldsymbol{h}-\left(H\boldsymbol{m}-\frac{2A}{\kappa^2}\nabla^2\boldsymbol{m}\right)\right]+\text{relaxation term},$$

where $\boldsymbol{i}_z$ is the unit vector in the z-direction.

In addition to the usual boundary conditions, one has

$$\left(\frac{\partial m}{\partial y}\right)_{y=0}=0, \qquad \text{(AIII·2)}$$

indicating that the magnetization vibrates freely at the surface. (It is assumed that no surface torque is present.) Taking account of (AIII·2), one obtains the Fourier transform of the equation of motion,

$$i\omega\boldsymbol{m}(\kappa)=\boldsymbol{i}_z\times\left[M_S\boldsymbol{h}(\kappa)-\left(H_z+\frac{2A\kappa^2}{M_S}\right)\boldsymbol{m}(\kappa)\right]\frac{\mu}{\hbar}-\frac{1}{\tau_\kappa}\boldsymbol{m}(\kappa), \qquad \text{(AIII·3)}$$

where $\boldsymbol{m}(\kappa)$ and $\boldsymbol{h}(\kappa)$ are defined by

$$(\boldsymbol{m}(\kappa), \boldsymbol{h}(\kappa)) = \int e^{i\kappa y}(\boldsymbol{m}, \boldsymbol{h})\,dy. \qquad \text{(AIII·4)}$$

Elimination of $\boldsymbol{e}$ from (AIII·1) gives

$$\text{grad div}\,\boldsymbol{h}-\Delta\boldsymbol{h}=-\frac{2i}{\delta^2}(\boldsymbol{h}+4\pi\boldsymbol{m}), \qquad \text{(AIII·5)}$$

where δ is the skin depth in the absence of the magnetization. y-component of this equation yields

$$(h+4\pi m)_y=0.$$

From this and (AIII·3), one obtains

$$m_x(\kappa)=\chi(\kappa)\cdot h_x(\kappa), \qquad \text{(AIII·6)}$$

where

$$\chi(\kappa)=\frac{(\mu/\hbar)^2(H+4\pi M_S)\cdot M_S}{(i\omega+1/\tau_\kappa)^2+\omega_0^2} \qquad \text{(AIII·7)}$$

with ω_0 given by (5·18), and the exchange term is dropped out. x-comporent of (AIII·5) together with (AIII·6 and 7) gives

$$h_x(\kappa)=\frac{2(\partial/\partial y\cdot h_x)_0}{\kappa^2+2i/\delta^2\cdot\mu(\kappa)}, \qquad \text{(AIII·8)}$$

where

$$\mu(\kappa)=1+4\pi\chi(\kappa). \qquad \text{(AIII·9)}$$

The inverse Fourier transform of (AIII·8) gives

$$h_x(y)=\sum_\kappa \frac{2(\partial/\partial y\cdot h_x)_0}{\kappa^2+2i/\delta^2\cdot\mu(\kappa)}e^{-i\kappa y}.$$

From the definition of the surface impedance, one finally obtains

$$Z(\omega)=\frac{e_z(0)}{h_x(0)}=\frac{\omega\delta^2}{4c}\frac{1}{\sum_\kappa 1/(\kappa^2+2i\mu(\kappa)/\delta^2)}. \qquad \text{(AIII·10)}$$

Note added in proof: It is shown that in the itinerant electron model of spin waves, there exists a cut-off wave number above which spin waves are dissolved into the individual excitations of Stoner type. This means a damping of the spin waves. The same situation was met in the plasma oscillations. The effect of this fact on the present problem depends on the magnitude of the cut-off wave number. Details will be published in future.

References

1) N. Takimoto, J. Phys. Soc. Japan **14** (1959), 1142.
2) M. Gell-Mann and K. A. Brueckner, Phys. Rev. **106** (1957), 364.
3) J. Bardeen, Phys. Rev. **52** (1937), 688.
H. Fröhlich, Proc. Roy. Soc. **A215** (1952), 291.
S. Nakajima, *Proc. of the International Conference of Theoretical Physics*, Kyoto and Tokyo (1954).
J. Bardeen and D. Pines, Phys. Rev. **99** (1955), 1140.
B. D. Silverman and P. R. Meiss, Phys. Rev. **114** (1959), 989.
4) D. Bohm and D. Pines, Phys. Rev. **82** (1951), 625.
5) T. Kasuya, Prog. Theor. Phys. **12** (1954), 803.
6) E. Abrahams, Phys. Rev. **98** (1955), 387.
7) F. London, *Superfluid* (John Willey & Sons, Inc., New York, 1950), vol. 1, p. 29.
8) M. R. Schafroth, Helv. Phys. Acta. **24** (1951), 645.
9) R. Ferrell, *Technical Report of Maryland Univ.* No. 129, Jan. (1959).
10) L. Landau, Z. Phys. **64** (1930), 629.
11) K. Yosida, Lecture at Research Institute for Fundamental Physics, Kyoto, Sept. 1959.
12) G. Wentzel, *Quantum Theory of Wave Field* (Interscience, New York, 1949), Chapters II and III.
13) Ref. 12), § 17.
14) C. Kittel and E. Abrahams, Revs. Modern Phys. **25** (1953), 233.
15) C. Herring and C. Kittel, Phys. Rev. **81** (1951), 869.
16) A. B. Pippard, *Advances in Electronics and Electron Physics*, vol. VI. (1954), p. 1.
17) C. Herring, Phys. Rev. **87** (1952), 60.
18) C. Kittel and C. Herring, Phys. Rev. **77** (1950), 725.
W. S. Ament and G. T. Rado, Phys. Rev. **97** (1955), 1558.
19) G. T. Rado and J. R. Weertmann, J. Phys. Chem. Solids **11** (1959), 315.

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Macroscopic Causality and Lower Limit for the Momentum Derivative of the Scattering Phase Shift

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Some investigations are made for a lower bound of the momentum derivative of the scattering phase shift on the basis of macroscopic causality in quantum field theory.

The aim of this paper is to derive Wigner's inequality[1] which is well known in quantum mechanics on the standpoint of quantum field theory. This purpose is achieved by using the macroscopic causality and wave-packet-formalism. Some further discussions will also be made on the validity of this inequality.

The use of the macroscopic causality and wave-packet-formalism in quantum field theory has been made first by K. Yamamoto,[2] by means of which he could derive the analytic properties of the scattering amplitude. Our method is on the basis of his formulation.

Suppose a wave packet initially localized at the position $\boldsymbol{x}$ which is separated by a sufficiently large distance from the scattering centre in comparison with the extention of the wave packet. If such a packet is approaching to or departing from the scatterer with an average momentum $\boldsymbol{p}$ at this instant, then we can correctly define the incident or scattered wave packet in terms of the outgoing or incoming solution of the Schrödinger equation, $|\boldsymbol{k}\rangle^+$ or $|\boldsymbol{k}\rangle^-$, as follows:

$$|\boldsymbol{p}, \boldsymbol{x}\rangle^{\pm} = \int d\boldsymbol{k} F(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{k}) |\boldsymbol{k}\rangle^{\pm}, \tag{1}$$

with

$$F(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{k}) \approx e^{-i(\boldsymbol{k}-\boldsymbol{p})\cdot\boldsymbol{x}} f(\boldsymbol{k}, \boldsymbol{p}), \tag{2}$$

where $F(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{k})$ is responsible for the shape of the packet, and $+$sign stands for the state of incident wave packet and $-$sign stands for the state of scattered wave packet. Here we have assumed that $f(\boldsymbol{k}, \boldsymbol{p})$ is nearly equal to zero except for $\boldsymbol{k} \approx \boldsymbol{p}$. Of course, the uncertainties of the momentum and position, $\Delta\boldsymbol{p}$ and $\Delta\boldsymbol{x}$, are related to each other by the uncertainty principle as $\Delta\boldsymbol{p} \cdot \Delta\boldsymbol{x} \approx 1$.

Using the definition (1) for the states of the incident and scattered wave packets, we can now write the transition matrix for the collision:

$${}^-\langle \boldsymbol{p}', \boldsymbol{x}' | e^{-iHt} | \boldsymbol{p}, \boldsymbol{x} \rangle^+ = \int d\boldsymbol{k}' \, d\boldsymbol{k} \, F^\dagger(\boldsymbol{p}', \boldsymbol{x}', \boldsymbol{k}') F(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{k}) e^{-iE'(k')t} \, {}^-\langle \boldsymbol{k}' | \boldsymbol{k} \rangle^+, \tag{3}$$

where $E'(\boldsymbol{k}')$ is the energy of the scattered particle and ${}^{-}\langle \boldsymbol{k}'|\boldsymbol{k}\rangle^{+}$ is the so-called S-matrix.[2] In general, the exponential factor in (3) gives rise to some deformation of wave packets while traveling, but as it is usual in the wave-packet-formalism we are concerned with the case in which this kind of deformation can be neglected. (Low and Chew[3] showed that such a case is actually realized in the condition $|\Delta \boldsymbol{x}| \gg |\boldsymbol{x}/p|^{1/2}$.) The S-matrix can be represented in terms of the phase shift $\delta_l(k)$ as

$$ {}^{-}\langle \boldsymbol{k}'|\boldsymbol{k}\rangle^{+} = \delta(E'-E)(1/k^2)(dE/dk)\sum_l (2l+1)/4\pi \cdot P_l(\cos\theta) e^{2i\delta_l(k')}, \quad (4) $$

where θ is the angle between $\boldsymbol{k}$ and $\boldsymbol{k}'$. In order that (3) be not zero, the variation with respect to $\boldsymbol{k}$ and $\boldsymbol{k}'$ of the phase of the integrand in (3) must be equal to zero apart from the error of $\Delta \boldsymbol{p}$ or $\Delta \boldsymbol{x}$*)

$$ \Delta \boldsymbol{k}' \cdot \boldsymbol{x}' - \Delta \boldsymbol{k} \cdot \boldsymbol{x} - \boldsymbol{V}'(\boldsymbol{p}')t \cdot \Delta \boldsymbol{k}' + 2(\boldsymbol{p}'/p)\delta_l'(p') \cdot \Delta \boldsymbol{k}' = 0, \quad (5) $$

where $\Delta \boldsymbol{k}$ and $\Delta \boldsymbol{k}'$ are restricted by virtue of the conservation law of total energy as follows:

$$ \Delta \boldsymbol{k} \cdot \boldsymbol{V}(\boldsymbol{p}) - \Delta \boldsymbol{k}' \cdot \boldsymbol{V}'(\boldsymbol{p}') = 0. \quad (6) $$

Here we have set $\boldsymbol{V}(\boldsymbol{p}) \equiv \boldsymbol{p}/m$ and $\delta_l'(p) \equiv [d\delta_l(k)/dk]_{k=p}$. Using the undeterminate multiplier T, we obtain from (5) and (6)

$$ \boldsymbol{x} + \boldsymbol{V}(\boldsymbol{p})T = 0, \quad (7) $$

$$ \boldsymbol{x}' - \boldsymbol{V}'(\boldsymbol{p}')t + \boldsymbol{V}'(\boldsymbol{p}')T + 2(\boldsymbol{p}'/p)\delta_l'(p') = 0. \quad (8) $$

Consider the forward scattering, $\boldsymbol{V}(\boldsymbol{p}) = \boldsymbol{V}'(\boldsymbol{p}') \equiv \boldsymbol{V}$ and $\boldsymbol{p} = \boldsymbol{p}'$ then we have

$$ \boldsymbol{x}' - \boldsymbol{x} = \boldsymbol{V}t - 2(\boldsymbol{p}/p)\delta_l'(p). \quad (9) $$

This means the following fact that the incident particle which is initially localized at the position $\boldsymbol{x}$ is scattered in the forward direction by the scattering centre and then arrive at the position $\boldsymbol{x}'$ after the time t with the retardation $2\delta_l'(p)$. It may be permissible that the causality is violated at most by the range $2a$ of the interaction between scattered particle and scatterer. However, the violation of more than the range $2a$ is unacceptable. Therefore, we get

$$ \boldsymbol{x}' - \boldsymbol{x} = \boldsymbol{V}t - 2(\boldsymbol{p}/p)\delta_l'(p) \leqq \boldsymbol{V}t + 2(\boldsymbol{p}/p)a, \quad (10) $$

so that

$$ \delta_l'(p) \geqq -a, \quad (10') $$

namely, $d\delta_l/dk$ cannot be less than $-a$, a being the radius of the interaction. This inequality is, of course, assured only in the case in which the extension of the wave packet can be neglected in comparison with the radius a. It should be

* We shall consider the case where the variation of the absolute value of (4) can be neglected. Such a case can be actually realized at high momentum.

noted that at low energy the inequality does not hold, since the wave packet can not be confined in a small region of space, besides the variation of the absolute value of (4) must be taken into consideration. Let us consider, as an illustration, the S-wave scattering at low energy. We know

$$\lim_{k\to 0}\delta_0'(k) = -l_0, \tag{11}$$

where l_0 is the scattering length. If the inequality (10′) also holds at low energy, we get $l_0 < a$. This is incorrect because l_0 must be greater than a, provided that the bound states exist.

As another example, we shall consider van Kampen's one.[4] He has given the S-matrix on the basis of the principle of causality as follows:*)

$$S(k) = e^{-2i\alpha k}\prod_m \frac{k+i\lambda_m}{k-i\lambda_m}\prod_n \frac{k-\nu_n}{k+\nu_n}\frac{k+\bar{\nu}_n}{k-\bar{\nu}_n}, \tag{12}$$

where α is a real number, $0 \leqq \alpha \leqq a$, λ_m are positive real numbers, and $\nu_n = \alpha_n + i\beta_n$ are complex numbers with $\alpha_n > 0$ and $\beta_n > 0$. The poles $i\lambda_m$ correspond to bound states, and $-\nu_n$ and $\bar{\nu}_n$ correspond to resonance states. From (12) the derivative of the S-wave phase shift $\delta_0'(k)$ is easily obtained as

$$\delta_0'(k) = -\sum_m \frac{\lambda_m}{k^2+\lambda_m^2} + \sum_n \left[\frac{\beta_n}{(k-\alpha_n)^2+\beta_n^2} + \frac{\beta_n}{(k+\alpha_n)^2+\beta_n^2}\right], \tag{13}$$

where $\delta_0'(\infty) = 0$ has been assumed. As is easily seen, if the inequality

$$\frac{M}{k} < 2a \tag{14}$$

holds for any value of λ_m, the following inequality is valid:

$$\delta_0'(k) \geqq -\sum_m \frac{\lambda_m}{k^2+\lambda_m^2} \geqq -\frac{M}{2k} > -a, \tag{15}$$

that is, $\delta_0'(k)$ cannot be less than $-a$ only in the momentum region $k > M/2a$, M being the number of bound states. It is a striking fact that if we consider M/k as the order of the extension of the wave packet, the inequality (14) is consistent with the assumption that we are concerned with the case in which the extension of the wave packet can be neglected in comparison with the interaction radius a, and that although the bound states contribute to violating the macroscopic causality (See (13)), the magnitude of these violations lies within the order of the extension of the wave packet, as is seen from the second inequality in (15).

The author would like to express his sincere thanks to Prof. R. Utiyama for his kind interest and encouragement in this work. Thanks are also due to Dr. K. Yamamoto for his constant help and guidance.

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* The principle of causality stated by van Kampen is as follows: At any time the total probability of finding the particle outside the scattering centre shall not be greater than 1, for every form of the incident wave packet. (See ref. 4)

References

1) E. P. Wigner, Phys. Rev. **98** (1955), 145.
2) K. Yamamoto, Prog. Theor. Phys. **23** (1960), 859; **20** (1958), 857.
3) F. Low and G. F. Chew, *Scattering Theory Seminar* (*Lecture note at University of Illinois*) (1953).
4) N. G. van Kampen, Phys. Rev. **91** (1953), 1267.

Note added in proof : Relativistic treatments on this problem, which involve the inelastic scattering, will be done in the next paper.

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Baryon Parity and Polarization Experiment

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Possible polarization experiments from which information about baryon parity can be gained are investigated using the reactions $\Sigma^-+p\to\Lambda^0+n$, $\Xi^-+p\to(\Lambda^0+\Lambda^0)$, $(\Sigma^0+\Lambda^0)$, $(\Sigma^++\Sigma^-)$, and $\Xi^-+\mathrm{He}^4\to{}_\Lambda\mathrm{H}^4+\Lambda^0$. In our experiments, the transversely polarized hyperon beam is scattered on the unpolarized target, p or He^4, and useful information is obtainable by measuring the transverse components of polarization of the final products. All experiments presented here appear to be possible in the near future.

§ 1. Introduction

Parity violation has been established in weak interactions, but parity conservation seems to be inherent in strong interactions. Thus it is important to determine the baryon parity as unambigously as possible within the present status of our knowledge about strongly interacting particles.

For this purpose, the experiments we have in mind here are the "polarization experiments" using the reactions:

$$\left.\begin{aligned} \Sigma^-+p&\to\Lambda^0+n, && \text{(a)}\\ \Xi^-+p&\to\Lambda^0+\Lambda^0, && \text{(b)}\\ &\to\Sigma^0+\Lambda^0, && \text{(c)}\\ &\to\Sigma^++\Sigma^-, && \text{(d)} \end{aligned}\right\} \qquad (1\cdot1)$$

$$\Xi^-+\mathrm{He}^4\to{}_\Lambda\mathrm{H}^4+\Lambda^0. \quad \text{(e)} \qquad (1\cdot2)$$

Of these reactions, (1·1b) has been investigated by S. B. Treiman[1] and L. B. Okun et al.[2] as follows. If the reaction (1·1b) takes place from a low-lying atomic orbit, the allowed nature of the final orbital state constitutes the particular polarization pattern of the Λ^0's. This polarization pattern might reflect on the directional correlation of emerging pions of the subsequent Λ^0-decay, hence useful information on the $(\Xi^-,\ p)$ parity is obtainable by measuring the pion-pion correlation. In their approach, however, they had to require additional information on the initial orbital state from which the capture reaction takes place, and their formulae are not applicable to the case of higher incident energy. Therefore, in this paper, we will adopt a different method of polarization experiment, and give the general formulae, applicable to the higher energy region, from which definite information on the baryon parity can be obtained.

In our experiments, the transversely polarized hyperon beam is scattered on the unpolarized target, p or He^4, and useful information is obtainable by measuring the transverse components of polarization of the final products as will be described in § 3 and § 5. We can express any quantity that can be obtained from such experiments in terms of the coefficients of the scattering matrix. The general form of this scattering matrix is written down based only upon the well-accepted assumptions (see § 2 and § 4). Except for these assumptions, there is no ambiguity in our phenomenological approaches.

Before the derivation of the formulae necessary for the proposed experiment, we touch upon the problem of the "polarizer" for the incident beam and the "analyzer" for the final polarization. We take the production processes of hyperons, such as

$$\begin{aligned} &K^- + p \to \Sigma^- + \pi^+, \\ &K^- + p \to \Xi^- + K^+. \end{aligned} \tag{1·3}$$

If we assume that K^- meson spin is zero and hyperon spin is $1/2$, these reactions provide polarized hyperons perpendicular to the production plane, irrespective of particle parities. After the main reaction, (1·1) or (1·2), we need to measure the transverse components of polarization of the final products. In such analyses, decay asymmetry is efficient for $\Lambda^0(\Sigma^0)$ or Σ^+, and the scattering by complex nuclei, such as carbon, are efficient for others (if they have sufficient kinetic energy).

According to these status of experiment, discussions on the examples of the reaction (1·1b) and (1·2) will follow since these two reactions appear to be possible in the nearest future. Other reactions, of course, are the same in principle.

§ 2. General formulae (I)

In this section we calculate the formulae necessary for the (1·1) type experiment.

Here we introduce two fundamental assumptions:

I) Λ, Σ, Ξ are spin $1/2$ particles.

II) P- and T-invariance are not violated in strong interactions.

The scattering matrix for the reaction (1·1) is given in the form of general representation in spin space:

$$\begin{aligned} M\,(\text{even}) = {} & A + BS + C(\boldsymbol{\sigma}_1 \cdot \boldsymbol{n}) + D(\boldsymbol{\sigma}_2 \cdot \boldsymbol{n}) \\ & + N(\boldsymbol{\sigma}_1 \cdot \boldsymbol{n})(\boldsymbol{\sigma}_2 \cdot \boldsymbol{n}) + F(\boldsymbol{\sigma}_1 \cdot \boldsymbol{k})(\boldsymbol{\sigma}_2 \cdot \boldsymbol{k}) + G(\boldsymbol{\sigma}_1 \cdot \boldsymbol{p})(\boldsymbol{\sigma}_2 \cdot \boldsymbol{p}), \\ & S = 1/4 \cdot [1 - (\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)] \end{aligned} \tag{2·1}$$

$$\begin{aligned} M\,(\text{odd}) = {} & a(\boldsymbol{\sigma}_1 \cdot \boldsymbol{p}) + b(\boldsymbol{\sigma}_2 \cdot \boldsymbol{p}) \\ & + f(\boldsymbol{\sigma}_1 \cdot \boldsymbol{p})(\boldsymbol{\sigma}_2 \cdot \boldsymbol{n}) + g(\boldsymbol{\sigma}_1 \cdot \boldsymbol{n})(\boldsymbol{\sigma}_2 \cdot \boldsymbol{p}) + h[(\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2) \cdot \boldsymbol{k}], \end{aligned} \tag{2·2}$$

where $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are the spin matrices relating to the incident particle and target particle respectively; (A, B, C, D, N, F, G) or (a, b, f, g, h) are energy dependent functions of the reaction angle in the center-of-mass system; and $\boldsymbol{n}$, $\boldsymbol{k}$, $\boldsymbol{p}$, unit vectors along the directions $\boldsymbol{k}_i \times \boldsymbol{k}_f$, $\boldsymbol{k}_i - \boldsymbol{k}_f$, and $\boldsymbol{k}_i + \boldsymbol{k}_f$, respectively. Here $\boldsymbol{k}_i$ and $\boldsymbol{k}_f$ are unit vectors along the center-of-mass momentum vectors for the initial and final states. Notation 〈even〉 (or 〈odd〉) means that the total intrinsic parity of the initial state is equal (or opposite) to that of the final state. If we take $\Xi^- + p \rightarrow \Lambda^0 + \Lambda^0$, for example, 〈odd〉 case means that Ξ^- parity is opposite to the proton parity.

In what follows baryons are considered to have non-relativistic energy, and $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are therefore treated as Pauli spin vectors. For the description of polarization experiment it is convenient to employ the formalism of the density matrix for spin 1/2 particles.[3),4)]

The density matrix before reaction is

$$\rho_i = \frac{1}{4}[1 + (\boldsymbol{\sigma}_1 \cdot \boldsymbol{\alpha})], \tag{2·3}$$

which is normalized to $\mathrm{Tr}(\rho_i) = 1$. The average value of $\boldsymbol{\sigma}_1$ in the initial state is

$$\langle \sigma_1 \rangle_i = \frac{\mathrm{Tr}\{\rho_i \boldsymbol{\sigma}_1\}}{\mathrm{Tr}(\rho_i)} = \boldsymbol{\alpha} \equiv p_i \boldsymbol{e}. \tag{2·4}$$

Accordingly $\boldsymbol{\alpha} \equiv p_i \boldsymbol{e}$ is the polarization vector of the incident hyperon beam. Unit vector $\boldsymbol{e}$ is fixed by the polarizer for the beam. p_i indicates the polarization degree (both magnitude and direction) along the vector $\boldsymbol{e}$. After the main reaction the density matrix has the form

$$\rho_f = M \rho_i M^+ = \frac{1}{4} M M^+ + \frac{1}{4} M (\boldsymbol{\sigma}_1 \cdot \boldsymbol{\alpha}) M^+. \tag{2·5}$$

The average value of any quantity related to spin space operator Ω, is given by

$$\langle \Omega \rangle \equiv \langle \Omega \rangle_f = \frac{\mathrm{Tr}\{\rho_f \Omega\}}{\mathrm{Tr}(\rho_f)}. \tag{2·6}$$

With this equation, we can easily calculate the average values of $(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\beta})$ and $(\boldsymbol{\sigma}_2 \cdot \boldsymbol{\beta})$. Here $\boldsymbol{\beta}$ is a unit vector, and will be chosen in accordance with the suitable experimental condition. $\mathrm{Tr}(\rho_f)\langle (\boldsymbol{\sigma}_1 \cdot \boldsymbol{\beta}) \rangle$ and $\mathrm{Tr}(\rho_f)\langle (\boldsymbol{\sigma}_2 \cdot \boldsymbol{\beta}) \rangle$ are given in the forms

$$\begin{aligned} \mathrm{Tr}(\rho_f)\langle (\boldsymbol{\sigma}_1 \cdot \boldsymbol{\beta}) \rangle = {} & (\boldsymbol{\beta} \cdot \boldsymbol{n}) A_1 + (\boldsymbol{\alpha} \cdot \boldsymbol{\beta}) A_2 + (\boldsymbol{\alpha} \cdot \boldsymbol{k})(\boldsymbol{\beta} \cdot \boldsymbol{k}) A_3 \\ & + (\boldsymbol{\alpha} \cdot \boldsymbol{p})(\boldsymbol{\beta} \cdot \boldsymbol{p}) A_4 + (\boldsymbol{\alpha} \cdot \boldsymbol{n})(\boldsymbol{\beta} \cdot \boldsymbol{n}) A_5 + (\boldsymbol{\alpha} \cdot \boldsymbol{k})(\boldsymbol{\beta} \cdot \boldsymbol{p}) A_6 \\ & + (\boldsymbol{\alpha} \cdot \boldsymbol{p})(\boldsymbol{\beta} \cdot \boldsymbol{k}) A_7 + (\boldsymbol{\alpha} \times \boldsymbol{\beta}) \cdot \boldsymbol{n} A_8, \end{aligned} \tag{2·7a}$$

$$\begin{aligned} \mathrm{Tr}(\rho_f)\langle (\boldsymbol{\sigma}_2 \cdot \boldsymbol{\beta}) \rangle = {} & (\boldsymbol{\beta} \cdot \boldsymbol{n}) A_1' + (\boldsymbol{\alpha} \cdot \boldsymbol{\beta}) A_2' + (\boldsymbol{\alpha} \cdot \boldsymbol{k})(\boldsymbol{\beta} \cdot \boldsymbol{k}) A_3' \\ & + (\boldsymbol{\alpha} \cdot \boldsymbol{p})(\boldsymbol{\beta} \cdot \boldsymbol{p}) A_4' + (\boldsymbol{\alpha} \cdot \boldsymbol{n})(\boldsymbol{\beta} \cdot \boldsymbol{n}) A_5' + (\boldsymbol{\alpha} \cdot \boldsymbol{k})(\boldsymbol{\beta} \cdot \boldsymbol{p}) A_6' \\ & + (\boldsymbol{\alpha} \cdot \boldsymbol{p})(\boldsymbol{\beta} \cdot \boldsymbol{k}) A_7' + (\boldsymbol{\alpha} \times \boldsymbol{\beta}) \cdot \boldsymbol{n} A_8'. \end{aligned} \tag{2·7b}$$

The coefficients, A_i and A_i' $(i=1, 2, \cdots, 8)$, are written explicitly in Tables I and II respectively.

Table I. $\mathrm{Tr}(\rho_f)\langle(\boldsymbol{\sigma}_1\cdot\boldsymbol{\beta})\rangle$

i		A_i 《even》	A_i 《odd》
1	$(\boldsymbol{\beta}\cdot\boldsymbol{n})$	$2\mathrm{Re}(AC^*+DN^*)+\frac{1}{2}\mathrm{Re}(BC^*-BD^*)$	$2\mathrm{Re}\{b^*(g-h)\}$
2	$(\boldsymbol{\alpha}\cdot\boldsymbol{\beta})$	$(\lvert A\rvert^2+\lvert D\rvert^2)-(\lvert C\rvert^2+\lvert N\rvert^2+\lvert F\rvert^2+\lvert G\rvert^2)$ $+\frac{1}{2}\mathrm{Re}\{B^*(A+N+F+G)\}$	$-(\lvert a\rvert^2-\lvert b\rvert^2+\lvert f\rvert^2+\lvert g\rvert^2)$
3	$(\boldsymbol{\alpha}\cdot\boldsymbol{k})(\boldsymbol{\beta}\cdot\boldsymbol{k})$	$2\lvert F\rvert^2-\mathrm{Re}(BF^*)$	$-2\lvert h\rvert^2-2\mathrm{Re}\{h^*(f-g)\}$
4	$(\boldsymbol{\alpha}\cdot\boldsymbol{p})(\boldsymbol{\beta}\cdot\boldsymbol{p})$	$2\lvert G\rvert^2-\mathrm{Re}(BG^*)$	$2(\lvert a\rvert^2+\lvert f\rvert^2)+2\mathrm{Re}\{h^*(f+g)\}$
5	$(\boldsymbol{\alpha}\cdot\boldsymbol{n})(\boldsymbol{\beta}\cdot\boldsymbol{n})$	$2(\lvert C\rvert^2+\lvert N\rvert^2)-\mathrm{Re}(BN^*)$	$2\lvert g\rvert^2-2\mathrm{Re}\{h^*(f+g)\}$
6	$(\boldsymbol{\alpha}\cdot\boldsymbol{k})(\boldsymbol{\beta}\cdot\boldsymbol{p})$		$-2\mathrm{Im}(bh^*)$
7	$(\boldsymbol{\alpha}\cdot\boldsymbol{p})(\boldsymbol{\beta}\cdot\boldsymbol{k})$		$2\mathrm{Im}(bh^*)$
8	$(\boldsymbol{\alpha}\times\boldsymbol{\beta})\cdot\boldsymbol{n}$	$2\mathrm{Im}(AC^*+DN^*)+\frac{1}{2}\mathrm{Im}(BC^*+BD^*)$	$2\mathrm{Im}(bg^*)$

Table II. $\mathrm{Tr}(\rho_f)\langle(\boldsymbol{\sigma}_2\cdot\boldsymbol{\beta})\rangle$

i		A_i' 《even》	A_i' 《odd》
1	$(\boldsymbol{\beta}\cdot\boldsymbol{n})$	$2\mathrm{Re}(AD^*+CN^*)+\frac{1}{2}\mathrm{Re}(BD^*-BC^*)$	$2\mathrm{Re}\{a^*(f+h)\}$
2	$(\boldsymbol{\alpha}\cdot\boldsymbol{\beta})$	$-\frac{1}{2}\mathrm{Re}\{B^*(A+N+F+G)\}$	
3	$(\boldsymbol{\alpha}\cdot\boldsymbol{k})(\boldsymbol{\beta}\cdot\boldsymbol{k})$	$\mathrm{Re}\{2(AF^*+NG^*)+BF^*\}$	$2\lvert h\rvert^2+2\mathrm{Re}(fh^*-gh^*-fg^*)$
4	$(\boldsymbol{\alpha}\cdot\boldsymbol{p})(\boldsymbol{\beta}\cdot\boldsymbol{p})$	$\mathrm{Re}(BG^*+NF^*)$	$2\mathrm{Re}(ab^*)$
5	$(\boldsymbol{\alpha}\cdot\boldsymbol{n})(\boldsymbol{\beta}\cdot\boldsymbol{n})$	$\mathrm{Re}\{2(AN^*+CD^*+FG^*)+BN^*\}$	
6	$(\boldsymbol{\alpha}\cdot\boldsymbol{k})(\boldsymbol{\beta}\cdot\boldsymbol{p})$	$-2\mathrm{Im}(DF^*+CG^*)$	$2\mathrm{Im}\{a(g^*-h^*)\}$
7	$(\boldsymbol{\alpha}\cdot\boldsymbol{p})(\boldsymbol{\beta}\cdot\boldsymbol{k})$	$2\mathrm{Re}(AG^*)+2\mathrm{Im}(DG^*+CF^*)$	$-2\mathrm{Im}\{b(f^*+h^*)\}$
8	$(\boldsymbol{\alpha}\times\boldsymbol{\beta})\cdot\boldsymbol{n}$	$-\frac{1}{2}\mathrm{Im}(BC^*+BD^*)$	

In Eq. (2·7), $\mathrm{Tr}(\rho_f)$ has the following form:

$$\begin{aligned}\mathrm{Tr}(\rho_f)\ \text{even}=&\lvert A\rvert^2+\lvert C\rvert^2+\lvert D\rvert^2+\lvert N\rvert^2+\lvert F\rvert^2+\lvert G\rvert^2\\&+\frac{1}{4}[\lvert B\rvert^2+2\mathrm{Re}\{B^*(A-F-G-N)\}]\\&+(\boldsymbol{\alpha}\cdot\boldsymbol{n})\left[2\mathrm{Re}(AC^*+DN^*)+\frac{1}{2}\mathrm{Re}(BC^*-BD^*)\right],\end{aligned}\tag{2·8a}$$

$$\begin{aligned}\mathrm{Tr}(\rho_f)\ \text{odd}=&\lvert a\rvert^2+\lvert b\rvert^2+\lvert f\rvert^2+\lvert g\rvert^2+2\lvert h\rvert^2\\&+2\mathrm{Re}\{h^*(f-g)\}+(\boldsymbol{\alpha}\cdot\boldsymbol{n})[2\mathrm{Re}\{b^*(g-h)\}].\end{aligned}\tag{2·8b}$$

In the next section, we shall use the conventional notation:

$$\begin{aligned}U\equiv&\lvert A\rvert^2+\lvert C\rvert^2+\lvert D\rvert^2+\lvert N\rvert^2+\lvert F\rvert^2+\lvert G\rvert^2\\&+\frac{1}{4}[\lvert B\rvert^2+2\mathrm{Re}\{B^*(A-F-G-N)\}],\\u\equiv&\lvert a\rvert^2+\lvert b\rvert^2+\lvert f\rvert^2+\lvert g\rvert^2+2\lvert h\rvert^2+2\mathrm{Re}\{h^*(f-g)\},\end{aligned}$$

$$V \equiv 2\mathrm{Re}(AC^*+DN^*)+\frac{1}{2}\mathrm{Re}(BC^*-BD^*),$$

$$v \equiv 2\mathrm{Re}\{b^*(g-h)\},$$

$$W \equiv 2\mathrm{Re}(AD^*+CN^*)+\frac{1}{2}\mathrm{Re}(BD^*-BC^*),$$

$$w \equiv 2\mathrm{Re}\{a^*(f+h)\},$$

$$X \equiv -\frac{1}{2}\mathrm{Re}\{B^*(A+F+G-N)\}+2\mathrm{Re}(AN^*+CD^*+FG^*),$$

$$x \equiv X/U.$$

Using the above notation, $\mathrm{Tr}(\rho_f)$, for example, has the following form:

$$\mathrm{Tr}(\rho_f)\ \text{even} = U+(\boldsymbol{\alpha}\cdot\boldsymbol{n})V, \tag{2·9a}$$

$$\mathrm{Tr}(\rho_f)\ \text{odd} = u+(\boldsymbol{\alpha}\cdot\boldsymbol{n})v. \tag{2·9b}$$

§ 3. Experiment (I)

In this section we propose an experiment with the reaction (1·1), from which definite information on baryon relative parity can be obtained. Detailed discussions will be made only on an example of the reaction (1·1b). Other (1·1) type reactions, of course, are the same in principle.

Now experimental conditions are as follows: The transversely polarized Ξ^- beam with definite kinetic energy is incident on the unpolarized hydrogen target, and two Λ^0 particles are produced with the reaction angle θ_1 and θ_2 to the direction of Ξ^- motion. The coordinate system o-xyz (see Fig. 1) is so chosen that oz coincides with the direction of the incoming Ξ^- beam, and ox is parallel to the polarization of Ξ^- beam, $\boldsymbol{\alpha} \equiv p_i \boldsymbol{e}$. The reaction plane $[(\boldsymbol{k}_i, \boldsymbol{k}_f) \equiv (\boldsymbol{k}, \boldsymbol{p})]$, which has the normal $\boldsymbol{n}$, is fixed by the angle ϕ which vector $\boldsymbol{n}$ forms with vector $\boldsymbol{e}$, the positive direction of the axis ox.

The reaction angles (θ_1 and θ_2) and incident energy are kept constant throughout our present consideration, hence we know the energies of the emerging particles. Thus we are to regard (A, B, C, D, N, F, G) or (a, b, f, g, h) as constants under the definite experimental condition. On the other hand, $\boldsymbol{\sigma}_1$ is the spin vector relating to both Ξ^- and $\Lambda^0(\theta_1)$, Λ^0 produced with the reaction angle θ_1, having the matrix element between $\Lambda^0(\theta_1)$ and Ξ^-. Similarly,

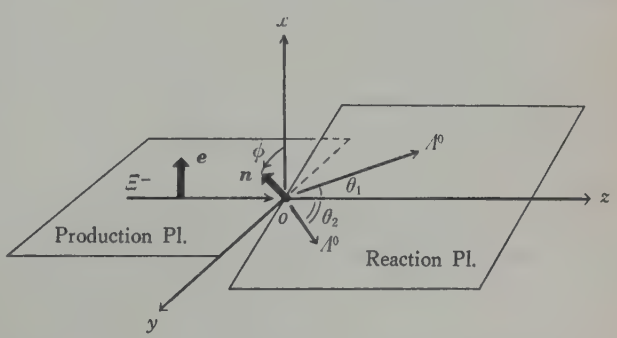

Fig. 1. Geometric condition of experiment.

$\boldsymbol{\sigma}_2$ has the matrix element between $A^0(\theta_2)$ and proton.

Now we are going to propose the polarization experiment. We must perform four kinds of experiments. In each experiment we measure the transverse component of polarization for one of the final products. All experiments can be made under the same incident energy and the definite reaction angles, θ_1 and θ_2. Therefore, we indicate the experimental condition only with the angle ϕ, which will be different for different experiments, and with the unit vector $\boldsymbol{\beta}$ along which we measure the polarization component of the final product; namely we are to observe the quantity, such as $\langle(\boldsymbol{\sigma}_1\cdot\boldsymbol{\beta})\rangle$ or $\langle(\boldsymbol{\sigma}_2\cdot\boldsymbol{\beta})\rangle$. In Table III we give a summary of the observing quantities, experimental conditions, and the theoretical information obtained from the measurement. For the expression of $p_\pm$, it should be noted that

$$p_\pm\ (\text{even})=\frac{W\pm p_i X}{U\pm p_i V}=\frac{\Gamma_2\pm p_i x}{1\pm p_i\Gamma_1}, \tag{3·1a}$$

$$p_\pm\ (\text{odd})=\frac{w}{u\pm p_i v}=\frac{\Gamma_2}{1\pm p_i\Gamma_1} \tag{3·1b}$$

where suffixes $(\pm)$ correspond to the initial polarization, $\boldsymbol{\alpha}=p_i(\pm\boldsymbol{n})$ respectively. The geometric conditions for various experiments are sketched in Fig. 2, where the bold arrow indicates the unit vector relating to the initial polarization or the direction in analyzing the polarization component.

Table III. Elements of polarization experiment (I)

No. of Exp.	Polarization Component or $\langle(\boldsymbol{\sigma}_1\cdot\boldsymbol{\beta})\rangle$ $\langle(\boldsymbol{\sigma}_2\cdot\boldsymbol{\beta})\rangle$	Geometric Conditions	Theoretical Information 《even》 case	《odd》 case
1	$\Gamma_1\equiv\langle(\boldsymbol{\sigma}_1\cdot\boldsymbol{n})\rangle$	$\phi=90°$ $\boldsymbol{\beta}=\boldsymbol{n}$	$\Gamma_1=\frac{V}{U}$	$\Gamma_1=\frac{v}{u}$
2	$\Gamma_2\equiv\langle(\boldsymbol{\sigma}_2\cdot\boldsymbol{n})\rangle$	$\phi=90°$ $\boldsymbol{\beta}=\boldsymbol{n}$	$\Gamma_2=\frac{W}{U}$	$\Gamma_2=\frac{w}{u}$
3	$p_+\equiv\langle(\boldsymbol{\sigma}_2\cdot\boldsymbol{n})\rangle$	$\phi=0°$ $\boldsymbol{\beta}=\boldsymbol{n}$	$p_+=\frac{\Gamma_2+p_i x}{1+p_i\Gamma_1}$	$p_+=\frac{\Gamma_2}{1+p_i\Gamma_1}$
4	$p_-\equiv\langle(\boldsymbol{\sigma}_2\cdot\boldsymbol{n})\rangle$	$\phi=180°$ $\boldsymbol{\beta}=\boldsymbol{n}$	$p_-=\frac{\Gamma_2-p_i x}{1-p_i\Gamma_1}$	$p_-=\frac{\Gamma_2}{1-p_i\Gamma_1}$
note	$p_\pm=\frac{\Gamma_2}{1\pm p_i\Gamma_1}$		no.	yes.

[Analysis] First, we assume the form of p_+ as

$$p_+=\frac{\Gamma_2}{1+p_i\Gamma_1},\ \text{i. e.}\quad p_i=\frac{\Gamma_2-p_+}{p_+\Gamma_1}. \tag{3·2}$$

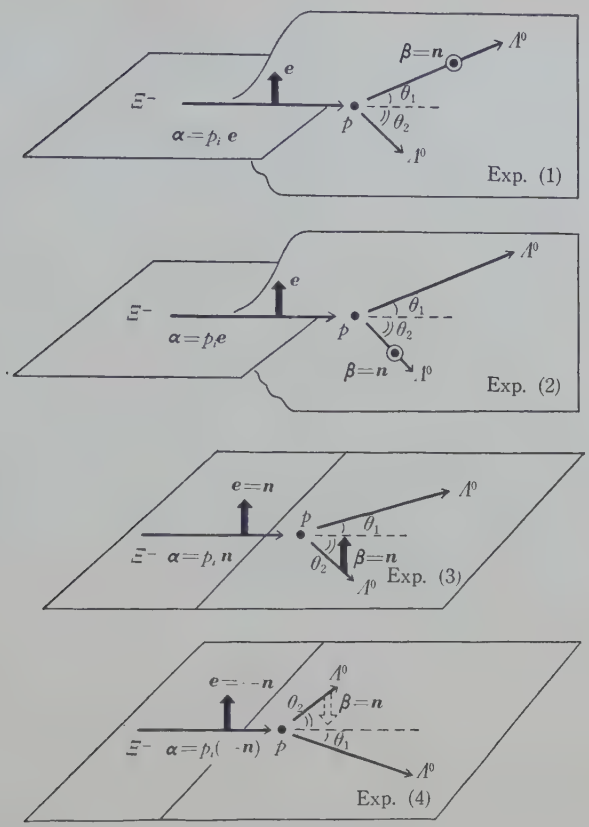

Fig. 2. Sketches of four experiments. The bold arrow indicates the unit vector relating to the initial polarization or the direction in analyzing the polarization component. The mark ⊙ shows the direction which does not lie on the plane of this figure.

Then, we caluculate the p_i using the observed values of Γ_1, Γ_2, and p_+. Next, we calculate the quantity, $\Gamma_2/(1-p_i\Gamma_1)$, with p_i thus obtained and Γ_1 and Γ_2 observed. If $\Gamma_2/(1-p_i\Gamma_1)$ coincides with the observed value of p_-,(Ξ^-, p) parity is ⟨odd⟩; if not, (Ξ^-, p) parity is ⟨even⟩:

$$\frac{\Gamma_2}{1-p_i\Gamma_1}\begin{cases}\neq p_- & \text{for ⟨even⟩ case,}\\ =p_- & \text{or ⟨odd⟩ case}\end{cases} \tag{3·3}$$

where p_i is given by Eq. (3·2).

Here it should be noted that the above analysis is applicable as long as $x\neq 0$.

§ 4. General formulae (II)

In this section, we will present an elementary discussion of polarization experiment with the reaction (1·2):

$$\Xi^- + \mathrm{He}^4 \rightarrow {}_\Lambda\mathrm{H}^4 + \Lambda^0. \tag{4·1}$$

Our investigation is based on the following fundamental assumptions:

I) Ξ^- and Λ^0 are spin 1/2 particles,

II) ${}_\Lambda\mathrm{H}^4$ is spin zero hyper-nucleus, and has no excited state,

III) P-invariance is guaranteed in the reaction (4·1).

Here we have not assumed the T-invariance in the reaction (4·1).

Without confusion we follow the notation which have been used in § 2. As He^4 and ${}_\Lambda\mathrm{H}^4$ are considered to be spin zero particles, the general matrix for the reaction (4·1) is in the following form:

$$M\ (\text{even}) = A + B(\boldsymbol{\sigma}\cdot\boldsymbol{n}), \tag{4·2a}$$

$$M\ (\text{odd}) = a(\boldsymbol{\sigma}\cdot\boldsymbol{k}) + b(\boldsymbol{\sigma}\cdot\boldsymbol{p}), \tag{4·2b}$$

where $\boldsymbol{\sigma}$ is the spin matrix relating to Ξ^- and Λ^0, and, of course, will be treated as the Pauli spin vector in the following discussions.

As in § 2, we can caluculate the expectation value $\langle(\boldsymbol{\sigma}\cdot\boldsymbol{\beta})\rangle$:

$$\begin{aligned}\mathrm{Tr}(\rho_f)\langle(\boldsymbol{\sigma}\cdot\boldsymbol{\beta})\rangle = {} & (\boldsymbol{\beta}\cdot\boldsymbol{n})A_1 + (\boldsymbol{\alpha}\cdot\boldsymbol{\beta})A_2 + (\boldsymbol{\alpha}\cdot\boldsymbol{k})(\boldsymbol{\beta}\cdot\boldsymbol{k})A_3 \\ & + (\boldsymbol{\alpha}\cdot\boldsymbol{p})(\boldsymbol{\beta}\cdot\boldsymbol{p})A_4 + (\boldsymbol{\alpha}\cdot\boldsymbol{n})(\boldsymbol{\beta}\cdot\boldsymbol{n})A_5 + (\boldsymbol{\alpha}\cdot\boldsymbol{k})(\boldsymbol{\beta}\cdot\boldsymbol{p})A_6 \\ & + (\boldsymbol{\alpha}\cdot\boldsymbol{p})(\boldsymbol{\beta}\cdot\boldsymbol{k})A_7 + (\boldsymbol{\alpha}\times\boldsymbol{\beta})\cdot\boldsymbol{n}A_8.\end{aligned} \tag{4·3}$$

A_i's ($i=1, 2, \cdots, 8$) and $\mathrm{Tr}(\rho_f)$ are given in Table IV:

Table IV. $\mathrm{Tr}(\rho_f)\langle(\boldsymbol{\sigma}\cdot\boldsymbol{\beta})\rangle$

i		A_i 《even》	A_i 《odd》
1	$(\boldsymbol{\beta}\cdot\boldsymbol{n})$	$2\,\mathrm{Re}\,(AB^*)$	$-2\,\mathrm{Im}\,(ab^*)$
2	$(\boldsymbol{\alpha}\cdot\boldsymbol{\beta})$	$\lvert A\rvert^2-\lvert B\rvert^2$	$-(\lvert a\rvert^2+\lvert b\rvert^2)$
3	$(\boldsymbol{\alpha}\cdot\boldsymbol{k})(\boldsymbol{\beta}\cdot\boldsymbol{k})$		$2\lvert a\rvert^2$
4	$(\boldsymbol{\alpha}\cdot\boldsymbol{p})(\boldsymbol{\beta}\cdot\boldsymbol{p})$		$2\lvert b\rvert^2$
5	$(\boldsymbol{\alpha}\cdot\boldsymbol{n})(\boldsymbol{\beta}\cdot\boldsymbol{n})$	$2\lvert B^2\rvert$	
6	$(\boldsymbol{\alpha}\cdot\boldsymbol{k})(\boldsymbol{\beta}\cdot\boldsymbol{p})$		$2\,\mathrm{Re}\,(ab^*)$
7	$(\boldsymbol{\alpha}\cdot\boldsymbol{p})(\boldsymbol{\beta}\cdot\boldsymbol{k})$		$2\,\mathrm{Re}\,(ab^*)$
8	$(\boldsymbol{\alpha}\times\boldsymbol{\beta})\cdot\boldsymbol{n}$	$2\,\mathrm{Im}\,(AB^*)$	
	$\mathrm{Tr}(\rho_f)$	$\lvert A\rvert^2+\lvert B\rvert^2+(\boldsymbol{\alpha}\cdot\boldsymbol{n})2\,\mathrm{Re}(AB^*)$	$\lvert a\rvert^2+\lvert b\rvert^2+(\boldsymbol{\alpha}\cdot\boldsymbol{n})2\,\mathrm{Im}(ab^*)$

§ 5. Experiment (II)

In this section we propose the polarization experiment with the reaction (4·1), from which definite information on (Ξ^-, p) parity is gained. Now we are to perform three experiments under the same incident energy and the definite emerging angle of Λ^0 particle.

As the experimental conditions and the analysis of these experiments are

similar to that of Experiment (I), we give a brief summary of the present experiment in Table V, and also sketch the geometric conditions in Fig. 3.

Table V. Elements of polarization experiment (II)

No. of Exp.	Polarization Component		Theoretical Information	
	$\langle(\boldsymbol{\sigma}\cdot\boldsymbol{\beta})\rangle$	Geometric Conditions	《even》 case	《odd》 case
1	$\Gamma\equiv\langle(\boldsymbol{\sigma}\cdot\boldsymbol{n})\rangle$	$\phi=90°$ $\boldsymbol{\beta}=\boldsymbol{n}$	$\Gamma=\dfrac{2\,\mathrm{Re}(AB^*)}{\lvert A\rvert^2+\lvert B\rvert^2}$	$\Gamma=\dfrac{-2\,\mathrm{Im}(ab^*)}{\lvert a\rvert^2+\lvert b\rvert^2}$
2	$p_+\equiv\langle(\boldsymbol{\sigma}\cdot\boldsymbol{n})\rangle$	$\phi=0°$ $\boldsymbol{\beta}=\boldsymbol{n}$	$p_+=\dfrac{\Gamma+p_i}{1+p_i\Gamma}$	$p_+=\dfrac{\Gamma-p_i}{1-p_i\Gamma}$
3	$p_-\equiv\langle(\boldsymbol{\sigma}\cdot\boldsymbol{n})\rangle$	$\phi=180°$ $\boldsymbol{\beta}=\boldsymbol{n}$	$p_-=\dfrac{\Gamma-p_i}{1-p_i\Gamma}$	$p_-=\dfrac{\Gamma+p_i}{1+p_i\Gamma}$

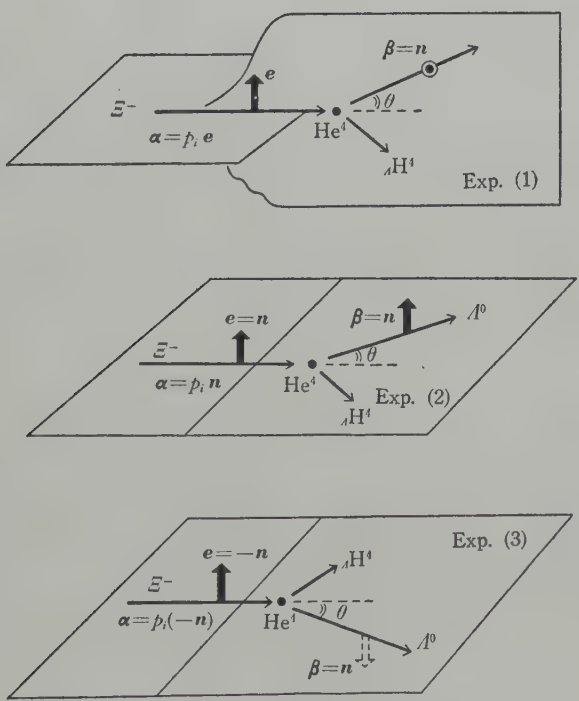

Fig. 3. Sketches of three experiments. The bold arrow indicates the unit vector relating to the initial polarization or the direction in analyzing the polarization component. The mark ⊙ shows the direction which does not lie on the plane of this figure.

§ 6. Discussions

The amount of information concerning the baryon relative parity that can be obtained from the above mentioned experiment (I) is as follows:

$$\Xi^- + p \to \Lambda^0 + \Lambda^0 : (\Xi, p) \text{ parity},$$

$$\Sigma^- + p \to \Lambda^0 + n : (\Sigma, \Lambda) \text{ parity},$$

$$\Xi^- + p \to \Sigma^0 + \Lambda^0 : (\Sigma, \Lambda) \text{ parity},$$

(if (Ξ, p) parity is known.)

$$\Xi^- + p \to \Sigma^+ + \Sigma^- : (\Xi, p) \text{ parity}.$$

As we have discussed in § 2 and § 3, definite information is obtainable as long as $x \neq 0$. This condition, $x \neq 0$, is satisfied if, for example, Re $(A^*B) \neq 0$.

In Experiment (II), from which (Ξ, p) parity is determined, it should be noted that

$$p_+ \text{ (even)} = 1, \quad p_+ \text{ (odd)} = -1$$

if $p_i = 1$, and

$$\Gamma \text{ (even)} \neq 0, \quad \Gamma \text{ (odd)} = 0,$$

if T-invariance $(a = 0)$ is assumed. In this type of experiment, the conclusion depends strongly upon the assumption II) in § 4. Validity of this assumption is not clear now.[5]

Our proposed experiments have not been carried out, largely because they require a hyperon beam (such as Ξ^--beam) and the analysing experiments are difficult except for Λ^0-polarization. It is desirable that these difficulties are overcome in the near future.

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References

1) S. B. Treiman, Phys. Rev. **113** (1959), 355.
2) L. B. Okun et al., Soviet Physics, JETP **7** (1958), 862.
3) L. Wolfenstein and J. Ashkin, Phys. Rev. **85** (1952), 947.
4) R. H. Dalitz, Proc. Phys. Soc. London **A65** (1957), 175.
5) See, also, R. H. Dalitz, Reports on Progress in Physics, Vol. **20** (1957), 163.

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On the Theory of Rearrangement Collisions*

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The formal theory of rearrangement collisions is developed from the standpoint of the S-matrix, and is formulated in such a way that the asymptotic Hamiltonian is the same for the final and initial states. The orthogonality of these states makes it possible to formulate the theory of rearrangement collisions in a way similar to that of single potential scattering. The "prior-post" discrepancy is discussed and the generalized optical theorem is derived.

§ 1. Introduction

Following Dyson's development of the theory of the S-matrix,[1] the formalism was extensively used in treating scattering problems. The validity of this theory, however, is restricted to the case in which the initial and final states are eigenstates of the same free Hamiltonian. In the case of rearrangement collisions, however, the initial and final states are eigenstates of different unperturbed Hamiltonians. Because of this fact, it becomes difficult for us to follow the argument of the usual S-matrix theory.

In order to clarify our problems, let us consider a system whose total Hamiltonian H can be divided in various ways such as

$$H=H_1+V_1=H_2+V_2=\cdots\cdots=H_n+V_n, \tag{1·1}$$

where H_i is a part which does not vanish even in the asymptotic region and V_i is a part which does. Let us assume that the initial state $\Phi_{1,i}$ is given by an eigenstate of the unperturbed Hamiltonian H_1, i.e., that

$$H_1\Phi_{1,i}=E_i\Phi_{1,i}.$$

Subscripts i and f will designate initial and final states, respectively. The final states which correspond to the various final channels are specified by the eigenstates $\Phi_{1,f}, \Phi_{2,f}, \ldots$ and $\Phi_{n,f}$ which satisfy

$$H_1\Phi_{1,f}=E_f\Phi_{1,f}$$
$$\vdots$$

and

$$H_n\Phi_{n,f}=E_f\Phi_{n,f}. \tag{1·2}$$

The outgoing (incoming) scattering state $\Psi^{(+)}_{1,f}$ ($\Psi^{(-)}_{n,f}$) generated by the interaction

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$V_1(V_n)$ from the state $\Phi_{1,i}$ ($\Phi_{n,f}$) satisfies the Lippmann-Schwinger equation[2]

$$\Psi_{1,i}^{(+)}=\Phi_{1,i}+\frac{1}{E_i-H_1+i\epsilon}V_1\Psi_{1,i}^{(+)} \tag{1·3}$$

or

$$\Psi_{n,f}^{(-)}=\Phi_{n,f}+\frac{1}{E_f-H_n-i\epsilon}V_n\Psi_{n,f}^{(-)}. \tag{1·4}$$

Now the extended S-matrix which is applicable to this case as well is defined by[3),4)]

$$S_{n,f;1,i}\equiv\langle\Psi_{n,f}^{(-)}|\Psi_{1,i}^{(+)}\rangle. \tag{1·5}$$

Contrary to the case of single-channel scattering, the final states $\Phi_1, \Phi_2, \cdots$ and Φ_n, which correspond to the respective channels, are in general not orthogonal to each other, because they are eigenstates of different unperturbed Hamiltonians.[5] Nevertheless every channel can be distinguished and is observable independently in experiments. This means that these channels should be orthogonal to each other.

The aim of the present paper is to resolve this dilemma in the case of a typical example of rearrangement collisions, and to investigate in detail the properties of the extended S-matrix defined by Eq. (1.5). In Sec. 2, the initial and final states will be modified[3),5)] so as to be eigenstates of the "same unperturbed Hamiltonian" and to be orthogonal to each other. By virtue of this orthogonality, it will become possible for us to treat our problem in a way similar to the Dyson theory of the S-matrix, and the correspondence of our S-matrix to Dyson's S-matrix will become very clear. In Sec. 3, we shall give the explicit expressions for the S-matrix elements for the various channels. By virtue of our knowledge of the general property of our S-matrix, the physical meaning of approximations to this theory can be easily understood. As an example, it will be shown that the so-called "prior-post discrepancy"[6] is a spurious one which is due to incorrect approximations. In Sec. 4, we shall investigate the unitary character of the S-matrix and, by use of the orthogonality and completeness of the modified initial and final states, derive the optical theorem for the case of rearrangement scattering. For brevity, our treatment will be confined to the case of distinguishable particles.

§ 2. Modification of the initial and final states

In order to explain our method lucidly, we consider the following typical rearrangement collisions as an example. A proton is bound to a fixed nucleus by a potential V_P and is bombarded by a neutron which interacts with the proton and the nucleus through the potentials V_{PN} and V_N, respectively. The total Hamiltonian H of this system is

$$H=K_P+V_P+K_N+V_N+V_{PN}, \tag{2·1}$$

where K_P and K_N are the kinetic energy of the proton and the neutron, respectively. This Hamiltonian can be divided in various ways corresponding to the different channels, as in Eq. (1·1).

In our case, the initial state $\Phi_{1,i}$ is given by

$$H_1\Phi_{1,i}=E_i\Phi_{1,i}, \tag{2·2}$$

where

$$H=H_1+V_1,$$

$$H_1=K_P+V_P+K_N \tag{2·3}$$

and

$$V_1=V_{PN}+V_N.$$

The state $\Phi_{1,i}$ is a product of the state of the bound proton ϕ_P^B (E_i^B) and that of the plane wave $u_N(E_i-E_i^B)$ of the free neutron, i. e.

$$\Phi_{1,i}=\phi_P^B(E_i^B)u_N(E_i-E_i^B), \tag{2·4}$$

where E_i^B is the binding energy of the proton.

If the initial energy is sufficient to produce them, the various final states after the collision can be produced.

(i) *Elastic and inelastic collisions*

In this case, the proton will remain bound to the nucleus and the neutron will come out after the collision. The Hamiltonian is divided in the same way as in Eq. (2·3) and the final state is given by

$$\Phi_{1,f}=\phi_P^B(E_f^B)u_N(E_f-E_f^B). \tag{2·5}$$

(ii) *Exchange scattering*

The bound proton is knocked out by the neutron which in turn becomes bound to the nucleus. Then the Hamiltonian should be divided as

$$H=H_2+V_2,$$

$$H_2=K_P+K_N+V_N \tag{2·6}$$

and

$$V_2=V_{PN}+V_P,$$

where the final state will be given by

$$H_2\Phi_{2,f}=E_f\Phi_{2,f}$$

and

$$\Phi_{2,f}=u_P(E_f-E_f^B)\phi_N^B(E_f^B). \tag{2·7}$$

(iii) *Ionization process*

The bound proton will be knocked out by the neutron and both the proton and the neutron will be scattered by their interaction with the nucleus and each other. The Hamiltonian will be split in the following way,

$$H=H_3+V_3,$$

$$H_3=K_P+K_N$$

and

$$V_3=V_{PN}+V_P+V_N, \tag{2·8}$$

and the final state is given by

$$H_3\Phi_{3,f}=E_f\Phi_{3,f}$$

and
$$\Phi_{3,f}=u_P(E_f^P)u_N(E_f-E_f^P), \tag{2·9}$$

where E_f^P is the energy of the final free proton.

(iv) *Pickup process*

The proton and the neutron form a deuteron in the final state, and we see that

$$\begin{aligned} H&=H_4+V_4,\\ H_4&=K_P+K_N+V_{PN},\\ V_4&=V_P+V_N,\\ H_4\Phi_{4,f}&=E_f\Phi_{4,f} \end{aligned} \tag{2·10}$$

and
$$\Phi_{4,f}=u_c(X,\ E_f-E_f^B)\phi_{PN}^B(r,\ E_f^B), \tag{2·11}$$

where X is the coordinate of the center of mass of the deuteron and r is the internal coordinate of the deuteron. For later convenience, we introduce the notations H_0 and V_0 which are defined respectively by

$$\begin{aligned} H&=H_0+V_0,\\ H_0&=K_P+V_P+K_N+V_N \end{aligned}$$

and
$$V_0=V_{PN}. \tag{2·12}$$

The final states given by Eqs. (2·5), (2·7), (2·9), and (2·11) are eigenstates of the different Hamiltonians H_i, so they are in general not orthogonal to each other. We shall modify the initial and final states into eigenstates of the same Hamiltonian H_0, which will be orthogonal to each other.

(A) *Initial state*

From Eqs. (2.2) and (2.3), we can obtain the outgoing wave solution in the form

$$\Psi_{1,i}^{(+)}=\Phi_{1,i}+\frac{1}{E_i-H+i\epsilon}(V_{PN}+V_N)\Phi_{1,i}. \tag{2·13}$$

This may be rewritten as

$$\begin{aligned} \Psi_{1,i}^{(+)}&=\Phi_{1,i}+\frac{1}{E_i-H+i\epsilon}V_{PN}\Phi_{1,i}\\ &\quad+\left(\frac{1}{E_i-H_0+i\epsilon}+\frac{1}{E_i-H+i\epsilon}V_{PN}\frac{1}{E_i-H_0+i\epsilon}\right)V_N\Phi_{1,i}\\ &=\psi_{1,i}^{(+)}+\frac{1}{E_i-H+i\epsilon}V_{PN}\psi_{1,i}^{(+)}, \end{aligned} \tag{2·14}$$

where

$$\psi_{1,i}^{(+)}=\Phi_{1,i}+\frac{1}{E_i-H_0+i\epsilon}V_N\Phi_{1,i}. \tag{2·15}$$

By using Eq. (2·4), we have

$$\begin{aligned}\psi_{1,i}^{(+)}&=\phi_P^B(E_i^B)u_N(E_i-E_i^B)\\&\quad+\frac{1}{E_i-(K_P+V_P+K_N+V_N)+i\epsilon}V_N\phi_P^B(E_i^B)u_N(E_i-E_i^B)\\&=\phi_P^B(E_i^B)\left[u_N(E_i-E_i^B)+\frac{1}{(E_i-E_i^B)-(K_N+V_N)+i\epsilon}V_Nu_N(E_i-E_i^B)\right]\\&=\phi_P^B(E_i^B)\phi_N^{(+)}(E_i-E_i^B).\end{aligned} \tag{2·16}$$

Therefore, the modified initial state $\psi_{1,i}^{(-)}$ is the product of the bound state of the proton and the state $\phi_N^{(+)}$ of the neutron scattered by the potential V_N.

(B) *Final states*

(i) *Elastic and inelastic collisions*

In this case, the modification is the same as above. The incoming wave solution $\Psi_{1,f}^{(-)}$ is given by

$$\Psi_{1,f}^{(-)}=\Phi_{1,f}+\frac{1}{E_f-H-i\epsilon}(V_{PN}+V_N)\Phi_{1,f}=\psi_{1,f}^{(-)}+\frac{1}{E_f-H-i\epsilon}V_{PN}\psi_{1,f}^{(-)}, \tag{2·17}$$

where

$$\psi_{1,f}^{(-)}=\Phi_{1,f}+\frac{1}{E_f-H_0-i\epsilon}V_N\Phi_{1,f}=\phi_P^B(E_f^B)\phi_N^{(-)}(E_f-E_f^B) \tag{2·18}$$

and $\phi_N^{(-)}$ is the incoming state of the neutron scattered by the potential V_N.

(ii) *Exchange scattering*

By using the same procedure as was employed in obtaining Eq. (2·14), the incoming scattering state $\Psi_{2,f}^{(-)}$, which is given by

$$\Psi_{2,f}^{(-)}=\Phi_{2,f}+\frac{1}{E_f-H-i\epsilon}(V_{PN}+V_P)\Phi_{2,f},$$

is deformed in such a way that

$$\Psi_{2,f}^{(-)}=\psi_{2,f}^{(-)}+\frac{1}{E_f-H-i\epsilon}V_{PN}\psi_{2,f}^{(-)}, \tag{2·19}$$

where

$$\psi_{2,f}^{(-)}=\Phi_{2,f}+\frac{1}{E_f-H_0-i\epsilon}V_P\Phi_{2,f}. \tag{2·20}$$

From Eq. (2·7), we see that

$$\psi_{2,f}^{(-)}=\phi_P^{(-)}(E_f-E_f^B)\phi_N^B(E_f^B) \tag{2·21}$$

and $\phi_P^{(-)}$ is the state of the proton scattered by the potential V_P.

(iii) *Ionization process*

The incoming wave solution $\Psi_{3,f}^{(-)}$ is written as

$$\Psi_{3,f}^{(-)}=\Phi_{3,f}+\frac{1}{E_f-H-i\epsilon}(V_{PN}+V_P+V_N)\Phi_{3,f}=\psi_{3,f}^{(-)}+\frac{1}{E_f-H-i\epsilon}V_{PN}\psi_{3,f}^{(-)}, \tag{2·22}$$

where

$$\psi_{3,f}^{(-)}=\Phi_{3,f}+\frac{1}{E_f-H_0-i\epsilon}(V_P+V_N)\Phi_{3,f}. \tag{2·23}$$

In order to obtain an expression similar to Eqs. (2·18) and (2·21), we rewrite Eq. (2·23) in the form:

$$\begin{aligned}\psi_{3,f}^{(-)}&=\Phi_{3,f}+\frac{1}{E_f-H_0-i\epsilon}V_N\Phi_{3,f}\\&\quad+\left(\frac{1}{E_f-H_1-i\epsilon}+\frac{1}{E_f-H_0-i\epsilon}V_N\frac{1}{E_f-H_1-i\epsilon}\right)V_P\Phi_{3,f}\\&=\left(1+\frac{1}{E_f-H_0-i\epsilon}V_N\right)\left(1+\frac{1}{E_f-H_1-i\epsilon}V_P\right)\Phi_{3,f}.\end{aligned} \tag{2·24}$$

Introducing Eq. (2·9) into Eq. (2·24), we have

$$\begin{aligned}\psi_{3,f}^{(-)}&=\left(1+\frac{1}{E_f-H_0-i\epsilon}V_N\right)u_N(E_f-E_f^P)\left(1+\frac{1}{E_f^P-(K_P+V_P)-i\epsilon}V_P\right)u_P(E_f^P)\\&=\left(1+\frac{1}{E_f-H_0-i\epsilon}V_N\right)u_N(E_f-E_f^P)\phi_P^{(-)}(E_f^P)\\&=\phi_P^{(-)}(E_f^P)\left(1+\frac{1}{(E_f-E_f^P)-(K_N+V_N)-i\epsilon}V_N\right)u_N(E_f-E_f^P)\\&=\phi_P^{(-)}(E_f^P)\phi_N^{(-)}(E_f-E_f^P).\end{aligned} \tag{2·25}$$

Therefore, $\psi_{3,f}^{(-)}$ is the product of the states of the proton and the neutron scattered by the potentials V_P and V_N, respectively.

(iv) *Pickup process*

As will be seen in the following discussion, this process has some different aspects from the other processes. We start from the integral equation

$$\Psi_{4,f}^{(-)}=\Phi_{4,f}+\frac{1}{E_f-H_4-i\epsilon}(V_P+V_N)\Psi_{4,f}^{(-)} \tag{2·26}$$

instead of its formal solution (like (2·13)). This equation can be rewritten as

$$\begin{aligned}\Psi_{4,f}^{(-)}=\Phi_{4,f}+\Bigg[&\frac{1}{E_f-H_0-i\epsilon}\\&-\frac{1}{E_f-H_0-i\epsilon}(V_P+V_N-V_{PN})\frac{1}{E_f-H_4-i\epsilon}\Bigg]\cdot(V_P+V_N)\Psi_{4,f}^{(-)}\end{aligned}$$

$$=\Phi_{4,f}+\frac{1}{E_f-H_0-i\epsilon}(V_P+V_N)\Psi_{4,f}^{(-)}$$

$$+\frac{1}{E_f-H_0-i\epsilon}(V_{PN}-V_P-V_N)[\Psi_{4,f}^{(-)}-\Phi_{4,f}]$$

$$=\psi_{4,f}^{(-)}+\frac{1}{E_f-H_0-i\epsilon}V_{PN}\Psi_{4,f}^{(-)}, \tag{2·27}$$

where

$$\psi_{4,f}^{(-)}=\Phi_{4,f}+\frac{1}{E_f-H_0-i\epsilon}(V_P+V_N-V_{PN})\Phi_{4,f}. \tag{2·28}$$

Now, let us inspect Eq. (2·28). Deforming it, we have

$$\psi_{4,f}^{(-)}=\Phi_{4,f}+\frac{1}{E_f-H_0-i\epsilon}(V_P+V_N)\Phi_{4,f}$$

$$-\left(\frac{1}{E_f-(K_P+K_N)-i\epsilon}+\frac{1}{E_f-H_0-i\epsilon}(V_P+V_N)\frac{1}{E_f-(K_P+K_N)-i\epsilon}\right)V_{PN}\Phi_{4,f}$$

$$=\left[1+\frac{1}{E_f-H_0-i\epsilon}(V_P+V_N)\right]\cdot\left[1-\frac{1}{E_f-(K_P+K_N)-i\epsilon}V_{PN}\right]\Phi_{4,f}.$$

Noting Eq. (2·11), we can see that

$$\left[1-\frac{1}{E_f-(K_P+K_N)-i\epsilon}V_{PN}\right]\Phi_{4,f}$$

$$=u_c(X,\ E_f-E_f^B)\left[1-\frac{1}{E_f^B-K(r)}V_{PN}\right]\phi_{PN}^B(r,\ E_f^B)=0, \tag{2·29}$$

where we used the fact that ϕ_{PN}^B is the bound state produced by the potential V_{PN}. The inhomogeneous term of Eq. (2·27) vanishes and we have for the pickup process the homogeneous equation

$$\Psi_{4,f}^{(-)}=\frac{1}{E_f-H_0-i\epsilon}V_{PN}\Psi_{4,f}^{(-)}. \tag{2·30}$$

If we write Eqs. (2·17), (2·19), and (2·22) as integral equations of the form

$$\Psi_{l,f}^{(-)}=\psi_{l,f}^{(-)}+\frac{1}{E_f-H_0-i\epsilon}V_{PN}\Psi_{l,f}^{(-)}, \quad (l=1,\ 2 \text{ and } 3)$$

these equations may be in contradiction with Eq. (2·30), as was shown by Foldy and Tobocman[7] and by Epstein.[8] Therefore, we always used the formal solution (2·17), (2·19) and (2·22) instead of the integral equations in order to avoid the superfluous solutions. Eq. (2·30) should be solved in such a way that it incorporates the boundary condition that in the asymptotic region $\Psi_{4,f}^{(-)}$ represents an incoming deuteron wave which is scattered by the potentials V_P and V_N.

So far we have considered only the case of the special initial channel 1. However, we can easily see that it is possible to apply our method to the cases of other initial channels 2, 3 and 4. Therefore, collecting our results, we have the following expressions

$$\begin{aligned}
&\Psi_{1,i}^{(\pm)}=U(V_{PN};\ 0,\ \mp\infty)\phi_P^{B}(E_i^{B})\phi_N^{(\pm)}(E_i-E_i^{B}),\\
&\Psi_{2,i}^{(\pm)}=U(V_{PN};\ 0,\ \mp\infty)\phi_P^{(\pm)}(E_i-E_i^{B})\phi_N^{B}(E_i^{B}) \qquad (2\cdot31)\\
\text{and}\quad &\Psi_{3,i}^{(\pm)}=U(V_{PN};\ 0,\ \mp\infty)\phi_P^{(\pm)}(E_i^{P})\phi_N^{(\pm)}(E_i-E_i^{P}),
\end{aligned}$$

where $U(V_{PN};\ 0,\ \pm\infty)$ is the Møller wave operator whose interaction term is V_{PN} and which is defined by[4]

$$U(V_{PN};\ 0,\ \mp\infty)=1-i\int_{\mp\infty}^{0}dt'\,e^{-\epsilon|t'|}\,U(0,\ t')\,V_{PN}(t') \qquad (2\cdot32)$$

and

$$U(V_{PN};\ \pm\infty,\ 0)=1-i\int_{0}^{\pm\infty}dt'\,e^{-\epsilon|t'|}\,V_{PN}(t')\,U(t',\ 0),$$

where

$$U(t,\ t')=e^{iH_0t}e^{-iH(t-t')}e^{-iH_0t'}$$

and

$$V_{PN}(t)=e^{iH_0t}\,V_{PN}e^{-iH_0t}.$$

Eqs. (2·31) correspond to the equation

$$\Psi_i^{(\pm)}=U(0,\ \mp\infty)\Phi_i \qquad (2\cdot33)$$

in the formal theory of single channel scattering. In Eqs. (2·31), we must note that the modified initial and final states $\phi_P\cdot\phi_N$ are eigenstates of the same Hamiltonian H_0 and are apparently orthogonal to each other, since the scattering state and the bound state are orthogonal. Therefore, it becomes possible for us to treat our problem in the same way as the usual formal theory of scattering as was mentioned in the introduction. From Eqs. (2·31) it follows that the scattering states $\Psi^{(\pm)}$ are orthogonal to each other, since

$$\begin{aligned}
\langle\Psi_{m,j}^{(\pm)}|\Psi_{l,i}^{(\pm)}\rangle&=\langle\psi_{m,j}^{(\pm)}|U(V_{PN};\ \mp\infty,\ 0)U(V_{PN};\ 0,\ \mp\infty)|\psi_{l,i}^{(\pm)}\rangle\\
&=\langle\psi_{m,j}^{(\pm)}|\psi_{l,i}^{(\pm)}\rangle=\delta_{m,l}\cdot\delta_{i,j}, \qquad (2\cdot34)
\end{aligned}$$

where m and l denote the channels and run from 1 to 3, and j and i mean the momenta and other variables which specify the state.

The state vector of the pickup process is given by the solution of the homogeneous Eq. (2·30); it can not be expressed in the form (2·31). This fact is very natural from the view-point of the usual formal theory of scattering, because the state $\Psi_{4,f}^{(-)}$ is a bound state (deuteron) resulting from the interaction V_{PN} and the Møller wave operator $U(V_{PN})$ cannot produce the bound state. The incoming wave solutions $\Psi_1^{(-)}$, $\Psi_2^{(-)}$ and $\Psi_3^{(-)}$ which are orthogonal to each other (see (2·34)) correspond to the scattering states resulting from the interaction V_{PN},

whereas $\Psi_4^{(-)}$ corresponds to the bound state produced by the interaction V_{PN}, so that the state $\Psi_4^{(-)}$ will be orthogonal to the scattering states $\Psi_m^{(-)}$ ($m=1, 2$ and 3). The proof of this orthogonality is as follows: With the aid of (2·31) and (2·32), we have

$$\begin{aligned}\langle \Psi_{m,i}^{(\pm)} | \Psi_{4,f}^{(\pm)} \rangle &= \langle \psi_{m,i}^{(\pm)} | U(V_{PN}; \mp\infty, 0) | \Psi_{4,f}^{(\pm)} \rangle \\ &= \langle \psi_{m,i}^{(\pm)} | \left(1 - \frac{1}{E_f - H_0 \pm i\epsilon} V_{PN}\right) \Psi_{4,f}^{(\pm)} \rangle = \langle \psi_{m,i}^{(\pm)} | \Psi_{4,f}^{(\pm)} \rangle - \langle \psi_{m,i}^{(\pm)} | \Psi_{4,f}^{(\pm)} \rangle = 0\end{aligned} \tag{2·35}$$

for $m=1$, 2 and 3, where $\Psi_{4,f}^{(+)}$ is the outgoing scattering state of a deuteron by the interactions V_P and V_N and satisfies the equation;

$$\Psi_{4,f}^{(+)} = \frac{1}{E_f - H_0 + i\epsilon} V_{PN} \Psi_{4,f}^{(+)}. \tag{2·36}$$

In the third equality of (2·35), we used Eqs. (2·30) and (2·36), respectively.

§ 3. Evaluation of the *S*-matrix

In this section, we shall discuss the properties of the *S*-matrix. The explicit expressions for the *S*-matrix corresponding to the transitions to the various final channels will be given and the properties of the approximations will be studied.

The *S*-matrix for the rearrangement collisions is defined, as (1·5), by

$$S_{n,f;\,m,i} = \langle \Psi_{n,f}^{(-)} | \Psi_{m,i}^{(+)} \rangle, \tag{3·1}$$

where f and i mean the momenta and other variables and n and m mean channels 1, 2, 3 and 4. In the simple scattering theory, we can define, of course, the *S*-matrix in the same form as (3·1), and have a single "*S*-operator" defined by

$$S = 1 - i \int_{-\infty}^{\infty} dt\, V(t)\, U(t, -\infty),$$

too. If we use the states $\Phi_{m,i}$ defined by (2·4), (2·5), (2·7), (2·9) and (2·11) as the initial and final states, Eq. (3·1) becomes

$$S_{n,f;\,m,i} = \langle \Phi_{n,f} | U(V_n; +\infty, 0)\, U(V_m; 0, -\infty) | \Phi_{m,i} \rangle,$$

where

$$\Psi_{m,i}^{(\pm)} = U(V_m; 0, \mp\infty) \Phi_{m,i},$$

and the Møller wave operators $U(V_m; 0, \mp\infty)$ depend on the corresponding channels m. Therefore, it is impossible to define a single *S*-operator which is common to every channel. However, our modification of the initial and final states will make it possible to define such an operator. By virtue of Eq. (2·31), the *S*-matrix (3·1) can be written as

$$\begin{aligned}S_{n,f;\,m,i} &= \langle \psi_{n,f}^{(-)} | U(V_{PN}; +\infty, 0)\, U(V_{PN}; 0, -\infty) | \psi_{m,i}^{(+)} \rangle \\ &= \langle \psi_{n,f}^{(-)} | S(V_{PN}) | \psi_{m,i}^{(+)} \rangle,\end{aligned} \tag{3·2}$$

where n and m run from 1 to 3 and the operator $S(V_{PN})$ is defined by

$$S(V_{PN})=1-i\int_{-\infty}^{\infty}dt\,V_{PN}(t)\,U(V_{PN};\,t,\,-\infty), \tag{3·3}$$

which is common to the channels 1, 2 and 3. In the case of channel 4 (pickup process), special consideration is necessary as will be discussed later, because the pickup process corresponds to the production of the bound state resulting from the potential V_{PN}.

Let us now calculate the elements of the S-matrix for the various channels.

(i) *Elastic and inelastic collisions*

From Eq. (3·3), one obtains the elements of the elastic and inelastic scattering matrix in the form

$$S_{1,f;1,i}=\langle\phi_{1,f}^{(-)}|S(V_{PN})\,|\phi_{1,i}^{(+)}\rangle=\langle\phi_{1,f}^{(-)}|\phi_{1,i}^{(+)}\rangle-2\pi i\delta(E_f-E_i)\langle\phi_{1,f}^{(-)}|\,V_{PN}|\Psi_{1,i}^{(+)}\rangle. \tag{3·4}$$

By using (2·14) and (2·17), the reciprocal form of Eq. (3·4) can be obtained in the following way:

$$\begin{aligned}S_{1,f;1,i}&=\langle\phi_{1,f}^{(-)}|\phi_{1,i}^{(+)}\rangle-2\pi i\delta(E_f-E_i)\left\langle\phi_{1,f}^{(-)}|\,V_{PN}\left(1+\frac{1}{E_i-H+i\epsilon}\,V_{PN}\right)|\phi_{1,i}^{(+)}\right\rangle\\&=\langle\phi_{1,f}^{(-)}|\phi_{1,i}^{(+)}\rangle-2\pi i\delta(E_f-E_i)\left\langle\left(1+\frac{1}{E_f-H-i\epsilon}\,V_{PN}\right)\phi_{1,f}^{(-)}|\,V_{PN}|\phi_{1,i}^{(+)}\right\rangle\\&=\langle\phi_{1,f}^{(-)}|\phi_{1,i}^{(+)}\rangle-2\pi i\delta(E_f-E_i)\langle\Psi_{1,f}^{(-)}|\,V_{PN}|\phi_{1,i}^{(+)}\rangle.\end{aligned} \tag{3·5}$$

With the aid of (2·15) and (2·18), we can rewrite Eqs. (3·4) and (3·5) in the familiar form, i. e.

$$\begin{aligned}S_{1,f;1,i}&=\langle\Phi_{1,f}|\Phi_{1,i}\rangle-2\pi i\delta(E_f-E_i)\left\{\langle\Phi_{1,f}|\,V_N|\phi_{1,i}^{(+)}\rangle+\langle\phi_{1,f}^{(-)}|\,V_{PN}|\Psi_{1,i}^{(+)}\rangle\right\}\\&=\delta_{f,i}-2\pi i\delta(E_f-E_i)\left\{\langle\Phi_{1,f}|\,V_N|\phi_{1,i}^{(+)}\rangle+\left\langle\Phi_{1,f}|\left(1+V_N\frac{1}{E_f-H_0+i\epsilon}\right)V_{PN}|\Psi_{1,i}^{(+)}\right\rangle\right\}\\&=\delta_{f,i}-2\pi i\delta(E_f-E_i)\left\{\left\langle\Phi_{1,f}|\,V_N|\phi_{1,i}^{(+)}+\frac{1}{E_i-H_0+i\epsilon}\,V_{PN}\Psi_{1,i}^{(+)}\right\rangle+\langle\Phi_{1,f}|\,V_{PN}|\Psi_{1,i}^{(+)}\rangle\right\}\\&=\delta_{f,i}-2\pi i\delta(E_f-E_i)\langle\Phi_{1,f}|\,V_N+V_{PN}|\Psi_{1,i}^{(+)}\rangle.\end{aligned} \tag{3·6}$$

Starting from (3·5), we get the reciprocal form of (3·5):

$$S_{1,f;1,i}=\delta_{f,i}-2\pi i\delta(E_f-E_i)\langle\Psi_{1,i}^{(-)}|\,V_N+V_{PN}|\Phi_{1,i}\rangle. \tag{3·7}$$

These results (3·6) and (3·7) coincide with those of the usual treatment.

(ii) *Exchange scattering*

The S-matrix element for the exchange scattering is given by

$$S_{2,f;1,i}=\langle\psi_{2,f}^{(-)}|S(V_{PN})|\psi_{1,i}^{(+)}\rangle=\langle\psi_{2,f}^{(-)}|\psi_{1,i}^{(+)}\rangle-2\pi i\delta(E_f-E_i)\langle\psi_{2,f}^{(-)}|V_{PN}|\Psi_{1,i}^{(+)}\rangle. \tag{3·8}$$

The first term of the right-hand side of Eq. (3·8) vanishes, i. e.

$$\langle\psi_{2,f}^{(-)}|\psi_{1,i}^{(+)}\rangle=\langle\phi_P^{(-)}|\phi_P^{B}\rangle\langle\phi_N^{B}|\phi_N^{(+)}\rangle=0$$

because of the orthogonality between the bound states and the scattering states. By the same procedure as used in the derivation of Eq. (3·5), we get the reciprocal form of (3·6); i. e.

$$S_{2,f;1,i}=-2\pi i\delta(E_f-E_i)\langle\Psi_{2,f}^{(-)}|V_{PN}|\psi_{1,i}^{(+)}\rangle. \tag{3·9}$$

Equation (3·8) can be rewritten in the following way. Remembering the existence of the δ-function in Eq. (3·8), we have

$$\langle\psi_{2,f}^{(-)}|V_{PN}|\Psi_{1,i}^{(+)}\rangle=\langle\Phi_{2,f}|V_{PN}|\Psi_{1,i}^{(+)}\rangle+\left\langle\Phi_{2,f}\middle|V_P\frac{1}{E_i-H_0+i\epsilon}V_{PN}\middle|\Psi_{1,i}^{(+)}\right\rangle$$

$$=\langle\Phi_{2,f}|V_{PN}|\Psi_{1,i}^{(+)}\rangle+\langle\Phi_{2,f}|V_P|\Psi_{1,i}^{(+)}-\psi_{1,i}^{(+)}\rangle=\langle\Phi_{2,f}|V_{PN}+V_P|\Psi_{1,i}^{(+)}\rangle-\langle\Phi_{2,f}|V_P|\psi_{1,i}^{(+)}\rangle.$$

Therefore, we obtain the result

$$S_{2,f;1,i}=-2\pi i\delta(E_f-E_i)\langle\Phi_{2,f}|V_{PN}+V_P|\Psi_{1,i}^{(+)}\rangle, \tag{3·10}$$

where we have used the important fact that

$$\langle\Phi_{2,f}|V_P|\psi_{1,i}^{(+)}\rangle=\langle u_P|V_P|\phi_P^{B}\rangle\langle\phi_N^{B}|\phi_N^{(+)}\rangle=0. \tag{3·11}$$

If we start from Eq. (3·9), we get the reciprocal form

$$S_{2,f;1,i}=-2\pi i\delta(E_f-E_i)\langle\Psi_{1,i}^{(+)}|V_{PN}+V_N|\Phi_{1,i}\rangle, \tag{3·12}$$

where we have used the fact that

$$\langle\psi_{2,f}^{(-)}|V_N|\Phi_{1,i}\rangle=\langle\phi_P^{(-)}|\phi_P^{B}\rangle\langle\phi_N^{B}|V_N|u_N\rangle=0. \tag{3·13}$$

From (3·11) and (3·13), we can obtain some information on approximation procedures. Namely, if we take the Born approximation directly in Eqs. (3·10) and (3·12), we have the results, respectively,

$$S_{2,f;1,i}^{\text{Born}}=-2\pi i\delta(E_f-E_i)\langle\Phi_{2,f}|V_{PN}+V_P|\Phi_{1,i}\rangle \tag{3·14}$$

$$=-2\pi i\delta(E_f-E_i)\langle\Phi_{2,f}|V_{PN}+V_N|\Phi_{1,i}\rangle. \tag{3·15}$$

The potential V_P in (3·14) is replaced by the potential V_N in Eq. (3·15). This situation is known as the "prior-post" discrepancy.*[6] Remembering the fact that

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* However, Professor W. Tobocman kindly informed me of the equality of (3·14) and (3·15). The proof is as follows: By using the equation $(E_i^{B}-K_P)\phi_P^{B}(E_i^{B})=V_P\phi_P^{B}(E_i^{B})$, we have

$$\langle\Phi_{2,f}|V_P|\Phi_{1,i}\rangle=\langle u_P(E_f-E_f^{B})|V_P|\phi_P^{B}(E_i^{B})\rangle\langle\phi_N^{B}(E_f^{B})|u_N(E_i-E_i^{B})\rangle$$

$$=\langle u_P(E_f-E_f^{B})|E_i^{B}-K_P|\phi_P^{B}(E_i^{B})\rangle\langle\phi_N^{B}(E_f^{B})|u_N(E_i-E_i^{B})\rangle.$$

(*Continued on back page*)

Φ_2 and Φ_1 do not involve the influence of the interaction V_{PN}, let us switch off the interaction V_{PN}. Even if V_{PN} becomes zero, however, Eqs. (3·14) and (3·15) do not vanish and the exchange scattering takes place. From the physical point of view, this result is not natural, because, if there exists no interaction between the proton and the neutron, the exchange process is not expected to occur. This difficulty, however, is a spurious one which appeared because of the inadequate approximation. This situation can be seen from the fact that the rigorous expressions (3·11) and (3·13) vanish, but that the approximations to them given by

$$\langle \Phi_{2,f}| V_P|\Phi_{1,i}\rangle=\langle u_P| V_P|\phi_P^B\rangle\langle\phi_N^B|u_N\rangle \neq 0 \tag{3·16}$$

and

$$\langle \Phi_{2,f}| V_N|\Phi_{1,i}\rangle=\langle u_P|\phi_P^B\rangle\langle\phi_N^B| V_N|u_N\rangle \neq 0 \tag{3·17}$$

do not vanish. Therefore, if we take into account the higher order effects of V_N and V_P in Eqs. (3·14) and (3·15) respectively, the second terms in (3·14) and (3·15) vanish. Therefore, the approximations in Eqs. (3·14) and (3·15) cannot be correct. At this place, we must emphasize that this situation became clear by virtue of our modification of the initial and final states. If we do not use them and start from the original initial and final states which are not orthogonal to each other, it will be very difficult to find this situation. The correct Born approximation can be obtained by using the expressions (3·8) and (3·9) without mistake. The first step is to take the term which is first order in the interaction V_{PN}, and we get

$$S^{(1)}_{2,f;1,i}=-2\pi i\delta(E_f-E_i)\langle\psi^{(-)}_{2,f}| V_{PN}|\psi^{(+)}_{1,i}\rangle. \tag{3·18}$$

This expression is still exact in the potentials V_P and V_N. The second step gives the correct Born approximation; i. e.

$$S^{\text{Born}}_{2,f;1,i}=-2\pi i\delta(E_f-E_i)\langle\Phi_{2,f}| V_{PN}|\Phi_{1,i}\rangle. \tag{3·19}$$

(iii) *Ionization process*

The S-matrix element for the ionization process becomes

$$S_{3,f;1,i}=\langle\psi^{(-)}_{3,f}|S(V_{PN})|\psi^{(+)}_{1,i}\rangle=\langle\psi^{(-)}_{3,f}|\psi^{(+)}_{1,i}\rangle-2\pi i\delta(E_f-E_i)\langle\psi^{(-)}_{3,f}| V_{PN}|\Psi^{(+)}_{1,i}\rangle \tag{3·20}$$

$$=\langle\psi^{(-)}_{3,f}|\psi^{(+)}_{1,i}\rangle-2\pi i\delta(E_f-E_i)\langle\Psi^{(-)}_{3,f}| V_{PN}|\psi^{(+)}_{1,i}\rangle. \tag{3·21}$$

Here we should note that K_P is *hermitian*, because the wave function $\phi_P{}^B$ is the bound state and the surface integral vanishes. Then we have

$$\langle\Phi_{2,f}|V_P|\Phi_{1,i}\rangle=(E_i{}^B-E_f+E_f{}^B)\langle u_P|\phi_P{}^B\rangle\langle\phi_N{}^B|u_N\rangle.$$

On the other hand, we have

$$\langle\Phi_{2,f}|V_N|\Phi_{1,i}\rangle=(E_f{}^B-E_i+E_i{}^B)\langle u_P|\phi_P{}^B\rangle\langle\phi_N{}^B|u_N\rangle,$$

in the same way as above. Noting the existence of the δ-function in(3·14) and (3·15), we have

$$\langle\Phi_{2,f}|V_P|\Phi_{1,i}\rangle=\langle\Phi_{2,f}|V_N|\Phi_{1,i}\rangle.$$

From Eqs. (2·16) and (2·25), the first terms of (3·20) and (3·21) vanish by virtue of the fact that the scattering state and bound state of the proton are orthogonal to each other. After a simple modification, Eq. (3·20) leads to

$$S_{3,f;1,i}=-2\pi i\delta(E_f-E_i)\{\langle\Phi_{3,f}|V_{PN}+V_P+V_N|\Psi_{1,i}^{(+)}\rangle-\langle\Phi_{3,f}|V_P+V_N|\psi_{1,i}^{(+)}\rangle\}, \tag{3·22}$$

and Eq. (3·21) becomes

$$S_{3,f;1,i}=-2\pi i\delta(E_f-E_i)\{\langle\Psi_{3,f}^{(-)}|V_{PN}+V_N|\Phi_{1,i}\rangle-\langle\psi_{3,f}^{(-)}|V_N|\Phi_{1,i}\rangle\}. \tag{3·23}$$

Now let us examine the last terms of the right-hand sides of Eqs. (3·22) and (3·23). With the aid of (2·25), we see that

$$\langle\psi_{3,f}^{(-)}|V_N|\Phi_{1,i}\rangle=\langle\phi_P^{(-)}|\phi_P^B\rangle\langle\phi_N^{(-)}|V_N|u_N\rangle=0, \tag{3·24}$$

because of the orthogonality of the proton bound state and the proton scattering state. Using Eqs. (2·9) and (2·16), we have

$$\langle\Phi_{3,f}|V_P+V_N|\psi_{1,i}^{(+)}\rangle=\langle u_P(E_f^P)|V_P|\phi_P^B(E_i^B)\rangle\langle u_N(E_f-E_f^P)|\phi_N^{(+)}(E_i-E_i^B)\rangle$$
$$+\langle u_P(E_f^P)|\phi_P^B(E_i^B)\rangle\langle u_N(E_f-E_f^P)|V_N|\phi_N^{(+)}(E_i-E_i^B)\rangle.$$

Introducing the equations

$$\phi_N^{(+)}(E_i-E_i^B)=u_N(E_i-E_i^B)+\frac{1}{(E_i-E_i^B)-K_N+i\epsilon}V_N\phi_N^{(+)}(E_i-E_i^B)$$

and

$$\phi_P^B(E_i^B)=\frac{1}{E_i^B-K_P}\cdot V_P\cdot\phi_P^B(E_i^B)$$

into the above expression, we have

$$\delta(E_i-E_f)\cdot\langle\Phi_{3,f}|V_P+V_N|\psi_{1,i}^{(+)}\rangle=\delta(E_i-E_f)\langle u_P(E_f^P)|V_P|\phi_P^B(E_i^B)\rangle$$
$$\times\left\{\langle u_N(E_f-E_f^P)|u_N(E_i-E_i^B)\rangle+\frac{1}{E_i^P-E_i^B}\langle u_N(E_f-E_f^P)|V_N|\phi_N^{(+)}(E_i-E_i^B)\rangle\right\}$$
$$+\delta(E_i-E_f)\frac{1}{E_i^B-E_i^P}\langle u_P(E_f^P)|V_P|\phi_P^B(E_i^B)\rangle\langle u_N(E_f-E_f^P)|V_N|\phi_N^{(+)}(E_i-E_i^B)\rangle.$$

The second term and the last term cancel and the first term vanishes because of the fact that $(E_f-E_f^P)\neq(E_i-E_i^B)$. Therefore

$$\delta(E_i-E_f)\langle\Phi_{3,f}|V_P+V_N|\psi_{1,i}^{(+)}\rangle=0, \tag{3·25}$$

and the S-matrix elements become

$$S_{3,f;1,i}=-2\pi i\delta(E_f-E_i)\langle\Phi_{3,f}|V_{PN}+V_P+V_N|\Psi_{1,i}^{(+)}\rangle \tag{3·26}$$

and

$$S_{3,f;1,i}=-2\pi i\delta(E_f-E_i)\langle\Psi_{3,f}^{(-)}|V_{PN}+V_N|\Phi_{1,i}\rangle. \tag{3·27}$$

If we take the Born approximations in Eqs. (3·26) and (3·27), we have the expressions, i. e.

$$S_{3,f;1,i}^{\text{Born}} = -2\pi i\delta(E_f - E_i)\langle \Phi_{3,f}| V_{PN} + V_P + V_N|\Phi_{1,i}\rangle \tag{3·28}$$

from (3·26) and

$$S_{3,f;1,i}^{\text{Born}} = -2\pi i\delta(E_f - E_i)\langle \Phi_{3,f}| V_{PN} + V_N|\Phi_{1,i}\rangle \tag{3·29}$$

from (3·27). However, by the same argument as in the case of exchange scattering, both of these are incorrect. The correct Born term is given by

$$S_{3,f;1,i}^{\text{Born}} = -2\pi i\delta(E_f - E_i)\langle \Phi_{3,f}| V_{PN}|\Phi_{1,i}\rangle \tag{3·30}$$

by starting from the expressions (3·20) and (3·21).

(iv) *Pickup process*

Finally let us consider the case of the pickup process. In this case, we cannot use the S-operator (3·3) and cannot express the S-matrix element in such a form as (3·2). This fact reflects the special situation of the pickup process which corresponds to the production of the bound state resulting from the interaction V_{PN}. Therefore, we must start directly from the original expression for the extended S-matrix (3·1). With the aid of Eqs. (2·14) and (2·30), we have

$$\begin{aligned}
S_{4,f;1,i} &\equiv \langle \Psi_{4,f}^{(-)}|\Psi_{1,i}^{(+)}\rangle \\
&= \langle \Psi_{4,f}^{(-)}|\phi_{1,i}^{(+)}\rangle + \left\langle \Psi_{4,f}^{(-)}\middle| \frac{1}{E_i - H + i\epsilon} V_{PN}|\phi_{1,i}^{(+)}\right\rangle \\
&= \left\langle \frac{1}{E_f - H_0 - i\epsilon} V_{PN}\Psi_{4,f}^{(-)}|\phi_{1,i}^{(+)}\right\rangle + \frac{1}{E_i - E_f + i\epsilon}\langle \Psi_{4,f}^{(-)}| V_{PN}|\phi_{1,i}^{(+)}\rangle \\
&= \left\{ \frac{1}{E_f - E_i + i\epsilon} + \frac{1}{E_i - E_f + i\epsilon}\right\}\langle \Psi_{4,f}^{(-)}| V_{PN}|\phi_{1,i}^{(+)}\rangle \\
&= -2\pi i\delta(E_f - E_i)\langle \Psi_{4,f}^{(-)}| V_{PN}|\phi_{1,i}^{(+)}\rangle,
\end{aligned} \tag{3·31}$$

which is the same form as those of other channels. In this case, it is impossible to get the reciprocal formula of (3·31) corresponding to Eqs. (3·4), (3·8) and (3·20), because $\phi_{4,f}^{(-)}$ vanishes. This expression, however, can be rewritten in more familiar forms in the following way. By using (2·15) and (2·30), we have

$$\begin{aligned}
S_{4,f;1,i} &= -2\pi i\delta(E_f - E_i)\left\langle \Psi_{4,f}^{(-)}| V_{PN}\left| \left(1 + \frac{1}{E_i - H_0 + i\epsilon} V_N\right)\Phi_{1,i}\right\rangle\right. \\
&= -2\pi i\delta(E_f - E_i)\left\{\langle \Psi_{4,f}^{(-)}| V_{PN}|\Phi_{1,i}\rangle + \left\langle \frac{1}{E_f - H_0 - i\epsilon} V_{PN}\Psi_{4,f}^{(-)}| V_N|\Phi_{1,i}\right\rangle\right\} \\
&= -2\pi i\delta(E_f - E_i)\langle \Psi_{4,f}^{(-)}| V_{PN} + V_N|\Phi_{1,i}\rangle.
\end{aligned} \tag{3·32}$$

Or, with the aid of (2·26), we obtain

$$S_{4,f;1,i}=-2\pi i\delta(E_f-E_i)\left\langle\left\{1+\frac{1}{E_f-H_0-i\epsilon}(V_P+V_N)\right\}\Phi_{4,f}|V_{PN}|\psi_{1,i}^{(+)}\right\rangle$$

$$=-2\pi i\delta(E_f-E_i)\left\{\langle\Phi_{4,f}|V_{PN}|\psi_{1,i}^{(+)}\rangle+\left\langle\Phi_{4,f}|(V_P+V_N)\frac{1}{E_i-H_0+i\epsilon}V_{PN}|\psi_{1,i}^{(+)}\right\rangle\right\}$$

$$=-2\pi i\delta(E_f-E_i)\{\langle\Phi_{4,f}|V_{PN}|\psi_{1,i}^{(+)}\rangle+\langle\Phi_{4,f}|(V_P+V_N)|\Psi_{1,i}^{(+)}-\psi_{1,i}^{(+)}\rangle\}$$

$$=-2\pi i\delta(E_f-E_i)\{\langle\Phi_{4,f}|V_P+V_N|\Psi_{1,i}^{(+)}\rangle+\langle\Phi_{4,f}|V_{PN}-V_P-V_N|\psi_{1,i}^{(+)}\rangle\}. \quad (3\cdot33)$$

Comparing Eq. (3·31) with Eqs. (3·32) and (3·33), we can see that the former is preferable to the latter ones from the theoretical point of view, since the treatment of V_P and V_N in Eq. (3·31) is more symmetric than those of the others. If we replace $\Psi_{4,f}^{(-)}$ in the above expressions by $\Phi_{4,f}$ as an approximation, we have the result

$$S_{4,f;1,i}\to-2\pi i\delta(E_i-E_f)\langle\Phi_{4,f}|V_{PN}|\psi_{1,i}^{(+)}\rangle \quad (3\cdot34)$$

from (3·31) and (3·33). However, Eq. (3·32) becomes

$$S_{4,f;1,i}\to-2\pi i\delta(E_i-E_f)\langle\Phi_{4,f}|V_{PN}+V_N|\Phi_{1,i}\rangle. \quad (3\cdot35)$$

The expression (3·34) includes the higher order effect of V_N, but Eq. (3·35) involves only the first order effect of V_N.

So far, we considered only the cases of the initial channel 1. However, our theory is applicable even in the case of other initial channels. Let us, for example, give the elements of the S-matrix for the following cases, and the reader will be able to get their expressions for other cases easily.

(a) *Initial channel 2 and final channel 1* (Exchangé scattering)
In this case, we have

$$S_{1,f;2,i}=\langle\psi_{1,f}^{(-)}|S(V_{PN})|\psi_{2,i}^{(+)}\rangle. \quad (3\cdot36)$$

(b) *Initial channel 2 and final channel 4* (Pickup process)
In this case, we get

$$S_{4,f;2,i}=\langle\Psi_{4,f}^{(-)}|\Psi_{2,i}^{(+)}\rangle=-2\pi i\delta(E_f-E_i)\langle\Psi_{4,f}^{(-)}|V_{PN}|\psi_{2,i}^{(+)}\rangle \quad (3\cdot37)$$

in the same way as (3·31).

(c) *Initial channel 4 and final channel 4* (Scattering of a deuteron by a nucleus)
The result is

$$S_{4,f;4,i}=\langle\Psi_{4,f}^{(-)}|\Psi_{4,i}^{(+)}\rangle=\langle\Phi_{4,f}|\Phi_{4,i}\rangle-2\pi i\delta(E_i-E_f)\langle\Phi_{4,f}|V_P+V_N|\Psi_{4,i}^{(+)}\rangle, \quad (3\cdot38)$$

where Eq. (2·26) was used.

(d) *Initial channel 4 and final channel 1* (Deuteron stripping)
The element of S-matrix becomes

$$S_{1,f;4,i}=\langle\Psi_{1,f}^{(-)}|\Psi_{4,i}^{(+)}\rangle=-2\pi i\delta(E_f-E_i)\langle\psi_{1,f}^{(-)}|V_{PN}|\Psi_{4,i}^{(+)}\rangle, \quad (3\cdot39)$$

after the similar calculation to Eq. (3·31). If we replace $\Psi_{4,i}^{(+)}$ by $\Phi_{4,i}$, we have a distorted wave approximation:

$$S_{1,f;4,i} \to -2\pi i\delta(E_f - E_i)\langle \phi_P^B \cdot \phi_N^{(-)} | V_{PN} | u_c \cdot \phi_{PN}^B \rangle \tag{3·40}$$

which is a symmetrical expression for the potentials V_P and V_N.

§ 4. The unitary character of the S-matrix

In this section, we shall discuss the unitary character of the S-matrix and derive the generalized optical theorem.

By using the fact that the state vectors

$$\begin{aligned} &\psi_0 \equiv \phi_P^B \cdot \phi_N^B, \\ &\psi_1^{(\pm)} = \phi_P^B \cdot \phi_N^{(\pm)}, \\ &\psi_2^{(\pm)} = \phi_P^{(\pm)} \cdot \phi_N^B \\ \text{and} \quad &\psi_3^{(\pm)} = \phi_P^{(\pm)} \cdot \phi_N^{(\pm)}, \end{aligned} \tag{4·1}$$

constitute complete sets in the Hilbert space, let us investigate the property of the S-operator $S(V_{PN})$. Inserting this complete set, we have

$$\begin{aligned} &\langle \psi_{m,j}^{(+)} | S^+(V_{PN}) \cdot S(V_{PN}) | \psi_{l,i}^{(+)} \rangle \\ &= \sum_f \sum_{n=0}^{3} \langle \psi_{m,j}^{(+)} | S^+(V_{PN}) | \psi_{n,f}^{(-)} \rangle \langle \psi_{n,f}^{(-)} | S(V_{PN}) | \psi_{l,i}^{(+)} \rangle \\ &= \sum_f \sum_{n=0}^{3} \langle \Psi_{m,j}^{(+)} | \Psi_{n,f}^{(-)} \rangle \langle \Psi_{n,f}^{(-)} | \Psi_{l,i}^{(+)} \rangle \\ &= \langle \Psi_{m,j}^{(+)} | \Psi_{l,i}^{(+)} \rangle - \sum_B \langle \Psi_{m,j}^{(+)} | \Psi_B \rangle \langle \Psi_B | \Psi_{l,i}^{(+)} \rangle, \end{aligned} \tag{4·2}$$

where the eigenstates Ψ_B of the total Hamiltonian are the bound states produced by the interaction V_{PN}, and are orthogonal to the scattering states $\Psi_{n,f}^{(-)}$. The summation over the bound states will have two terms; i. e. the first corresponds to the pickup process

$$\sum_f \langle \Psi_{m,j}^{(+)} | \Psi_{4,f}^{(-)} \rangle \langle \Psi_{4,f}^{(-)} | \Psi_{l,i}^{(+)} \rangle,$$

and the second is the bound state of the nuclear core and the deuteron, which do not contribute to the S-matrix by virtue of the energy conservation law. By using Eq. (2·34) in (4·2), we have

$$\begin{aligned} &\sum_f \sum_{n=1}^{3} \langle \psi_{m,j}^{(+)} | S^+(V_{PN}) | \psi_{n,f}^{(-)} \rangle \langle \psi_{n,f}^{(-)} | S(V_{PN}) | \psi_{l,i}^{(+)} \rangle + \sum_f \langle \Psi_{m,j}^{(+)} | \Psi_{4,f}^{(-)} \rangle \langle \Psi_{4,f}^{(-)} | \Psi_{l,i}^{(+)} \rangle \\ &= \delta_{m,l}\delta_{f,j}, \end{aligned} \tag{4·3}$$

where the contribution from the term $n=0$ vanishes because of the energy conservation law. From Eq. (4·3), we see that the S-matrix is unitary, and that the operator $S(V_{PN})$ is not unitary. This fact is quite natural from the physical point of view, because the formation of the bound state produced by the interaction V_{PN} corresponds to absorption of incident light in optics.

Inserting the expressions for the S-matrices (3·6), (3·10), (3·26), and (3·32) in Eq. (4·3) and putting $m=l=1$ and $j=i$, we obtain the generalized optical theorem:

$$\sum_f \{w_{f,i}\ (\text{elastic}) + \sum_{n=1}^{4} w_{n,f;1,i}\} = -2Im\, T_{i,i}, \tag{4·4}$$

where $T_{i,i}$ is the forward scattering amplitude of the elastic scattering and is given by

$$T_{i,i} = \langle \Phi_{1,i} | V_N + V_{PN} | \Psi_{1,i}^{(+)} \rangle \tag{4·5}$$

and $w_{f,i}$ are the transition probabilities defined by

$$\begin{aligned} w_{f,i}\ (\text{elastic}) &= 2\pi\delta(E_f - E_i)\, |\langle \Phi_{1,f} | V_N + V_{PN} | \Psi_{1,i}^{(+)} \rangle|^2 \\ \text{and} \qquad w_{n,f;1,i} &= 2\pi\delta(E_f - E_i)\, |\langle \Psi_{n,f}^{(-)} | V_{PN} | \psi_{1,i}^{(+)} \rangle|^2, \end{aligned} \tag{4·6}$$

where $w_{1,f;1,i}$ corresponds to inelastic scattering.

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References

1) F. J. Dyson, Phys. Rev. **75** (1949), 486, 1735.
2) B. A. Lippmann and J. Schwinger, Phys. Rev. **79** (1950), 489.
3) M. Gell-Mann and M. L. Goldberger, Phys. Rev. **91** (1953), 398.
4) S. Sunakawa, Prog. Theor. Phys. **14** (1955), 175.
5) P. A. M. Dirac, Can. J. Phys. **33** (1955), 709.
T.-Y. Wu, Can. J. Phys. **34** (1956), 179.
B. A. Lippmann, Phys. Rev. **102** (1956), 264.
S. Altshuler, Phys. Rev. **91** (1953), 1167; **92** (1953), 1157.
T. Fulton and G. E. Owen, Phys. Rev. **108** (1957), 789.
E. Gerjuoy, Ann. Phys. **5** (1958), 58.
G. Mohan, Nuovo Cimento **13** (1959), 1065.
6) Bates, Fundaminsky and Massey, Trans. Roy. Soc. (London) **243** (1951), 93.
Corinaldesi, Trainor and Wu, Nuovo Cimento **9** (1952), 436.
L. I. Schiff, *Quantum Mechanics* (McGraw-Hill, New York, 1949).
G. Takeda and K. M. Watson, Phys. Rev. **97** (1955), 1336.
7) L. L. Foldy and W. Tobocman, Phys. Rev. **105** (1957), 1099.
8) S. T. Epstein, Phys. Rev. **106** (1957), 598.

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Some Considerations on the Low-Energy Pion-Nucleon Scattering

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The unsubtracted dispersion relations of Chew et al. are reinterpreted as the approximate equations for low-energy pion-nucleon scattering with the additional function only of invariant momentum transfer (κ^2), in which the effect of the pion-pion interaction is partially included. Using this reinterpreted unsubtracted dispersion relations of Chew et al., we shall discuss the effect of the pion-pion interaction to the pion-nucleon scattering and the unsubtracted dispersion relations of Goldberger et al. for the forward pion-nucleon scattering in the laboratory system.

§ 1. Introduction

Previously, Chew, Goldberger, Low and Nambu (CGLN)[1] have derived the integral equations for the partial waves in the pion-nucleon scattering by using the unsubtracted dispersion relations in which the momentum transfer between the initial and final pions is held fixed. It is assumed there that the imaginary parts that appear in the dispersion integrals are dominated by the S- and P-waves at low energies, and that the undetermined function of only the invariant momentum transfer (κ^2) may be small. Thus it has been shown that the equations of P-waves agree with those obtained by the static theory in the lowest order of $1/M$ expansion, except the source function introduced in the static theory, and that, for the S-wave phase shifts α_1 and α_3, the theoretical $\alpha_1-\alpha_3$ agrees with experiment very well, and $\alpha_1+2\alpha_3$ is also not inconsistent with experiment. Further, Goldberger, Miyazawa and Oehme (GMO)[2] have shown that $\alpha_1-\alpha_3$ obtained from the unsubtracted dispersion relation for the forward scattering amplitude in the laboratory system agrees with experiment very well, and recently, Minami[3] has stressed that $\alpha_1+2\alpha_3$ obtained from another unsubtracted GMO's dispersion relation agrees with experiment, if one cuts off the dispersion integral properly. However, as the dispersion relations for CGLN's $A^{(+)}(\nu, \kappa^2)$ and GMO's $T^{(1)}(\omega)$ which are responsible for $\alpha_1+2\alpha_3$, may not hold without one-subtraction, it seems curious to us that the theoretical $\alpha_1+2\alpha_3$ is numerically not inconsistent with experiment.

Here, we shall first give an interpretation for the dispersion relations of CGLN, which will be mathematically consistent. We start with the consideration of $\mathrm{Im}A^{(+)}(\nu, \kappa^2)$, which may be approximated by S- and P-waves at low energies. Then, if one takes $\mathrm{Im}A_0^{(+)}(\nu, \kappa^2)$ approximated by S- and P-waves instead of actual $A^{(+)}(\nu, \kappa^2)$, CGLN's dispersion integral in the right-hand side of the equation will converge, and the remaining, such as the contributions from D- and higher partial

waves, etc., may be expressed by the one-subtracted dispersion integrals with the imaginary part $\mathrm{Im}(A^{(+)}(\nu, \kappa^2) - A_0^{(+)}(\nu, \kappa^2))$. However, it may be approximately expressed by the function only of κ^2, $A^{(+)}(\kappa^2)$, at low energies, owing to the fact that $\mathrm{Im}(A^{(+)}(\nu, \kappa^2) - A_0^{(+)}(\nu, \kappa^2))$ becomes appreciable at high energies. In the above mentioned meaning, CGLN's unsubtracted dispersion relations may be correct, except for the undetermined $A^{(+)}(\kappa^2)$ which may be interpreted as the high energy effects.

We shall next consider the effect of the pion-pion interaction to the pion-nucleon scattering. The importance of the pion-pion interaction in the additional pion production,[4] the electromagnetic structure of nucleon,[5] etc., has been stressed by many authors. The effect on the pion-nucleon scattering may be illustrated graphically as in Fig. 2. Now, if one can approximate the pion-pion interaction as $\lambda\phi^4$ with the renormalized coupling constant λ, the contribution to the pion-nucleon scattering amplitude is apparently expressed by the function of only κ^2, then it will be shown that it is completely included in $A^{(+)}(\kappa^2)$. CGLN have shown, however, that even if one takes $A^{(+)}(\kappa^2)$ to be zero, S-wave phase shifts are not inconsistent with experiment. Thus, it will be concluded that the effect of the pion-pion interaction $\lambda\phi^4$ on the S-wave pion-nucleon scattering may not be so large as to cancell the large Born term contribution as has been discussed by Kawarabayashi and Miyazawa.[6]

Finally, we shall derive the unsubtracted dispersion relations for the forward scattering amplitude in the laboratory system by using thus reinterpreted CGLN's unsubtracted dispersion relations. As is expected, they agree with GMO's unsubtracted dispersion relations except for an additional constant that appears in the relations of $T^{(1)}(\omega)$, but it will be estimated to be small. In this meaning, GMO's unsubtracted dispersion relations may hold at low energies.

In § 2 CGLN's unsubtracted dispersion relations will be discussed. In § 3 GMO's unsubtracted dispersion relations will be derived, and the effects of the pion-pion interaction on the pion-nucleon scattering will be discussed.

§ 2. CGLN's dispersion relations

Following Chew, Goldberger, Low and Nambu (CGLN),[1] one can write the invariant scattering matrix element of pion-nucleon system as

$$-A+i\gamma QB,$$

where

$$[A, B]=[A^{(+)}(\omega, \kappa^2), B^{(+)}(\omega, \kappa^2)]\delta_{\alpha\beta}+[A^{(-)}(\omega, \kappa^2), B^{(-)}(\omega, \kappa^2)][\tau_\alpha, \tau_\beta]/2. \tag{2·1}$$

In the above, α and β are the final and the initial pion's isotopic spin indexes, ω is the laboratory pion energy and κ^2 is the invariant momentum transfer between the initial and the final pions. Then, if $A(\omega, \kappa^2)$ and $B(\omega, \kappa^2)$ are assumed to

tend to zero sufficiently rapidly, the following unsubtracted dispersion relations may hold:

$$A^{(\pm)}(\omega, \kappa^2)=\frac{1}{\pi}\int_{\mu}^{\infty}\frac{\mathrm{Im}A^{(\pm)}(\omega', \kappa^2)}{\omega'-\omega}d\omega'\pm\frac{1}{\pi}\int_{\mu}^{\infty}\frac{\mathrm{Im}A^{(\pm)}(\omega', \kappa^2)}{\omega'+\omega-2\kappa^2/M}d\omega' \qquad (2\cdot2)$$

and

$$B^{(\pm)}(\omega, \kappa^2)=\frac{G^2}{-\mu^2-2M\omega}\mp\frac{G^2}{-\mu^2+2M\omega-4\kappa^2}$$
$$+\frac{1}{\pi}\int_{\mu}^{\infty}\frac{\mathrm{Im}B^{(\pm)}(\omega', \kappa^2)}{\omega'-\omega}d\omega'\mp\frac{1}{\pi}\int_{\mu}^{\infty}\frac{\mathrm{Im}B^{(\pm)}(\omega', \kappa^2)}{\omega'+\omega-2\kappa^2/M}d\omega', \qquad (2\cdot3)$$

where G is rationalized renormalized *Ps-Ps* coupling constant, M and μ are nucleon and pion masses respectively. However, at least $A^{(+)}(\omega, \kappa^2)$ may behave like constant when $\omega\to\infty$, thus it may be necessary to carry out a subtraction in the dispersion relation of $A^{(+)}(\omega, \kappa^2)$ once.[7] Here, we shall obtain the subtracted dispersion relation as follows, observing that $\mathrm{Im}A^{(+)}(\omega, \kappa^2)$ may well be approximated by S- and P-waves at low energies. Then, if one takes $\mathrm{Im}A_0^{(+)}(\omega, \kappa^2)$ expressed by finite partial waves S and P instead of actual $\mathrm{Im}A^{(+)}(\omega, \kappa^2)$, CGLN's dispersion integrals for $A^{(+)}(\omega, \kappa^2)$ in the right-hand side of (2·2) will be shown to converge, from the unitarity conditions for partial waves, thus this expression may be well defined mathematically; the remainder will be written as follows:

$$A^{(+)}(\omega, \kappa^2)-\frac{1}{\pi}\int_{\mu}^{\infty}\frac{\mathrm{Im}A_0^{(+)}(\omega', \kappa^2)}{\omega'-\omega}d\omega'-\frac{1}{\pi}\int_{\mu}^{\infty}\frac{\mathrm{Im}A_0^{(+)}(\omega', \kappa^2)}{\omega'+\omega-2\kappa^2/M}d\omega'$$
$$=A_0^{(+)}(\kappa^2)+\frac{1}{\pi}\int_{\mu}^{\infty}\left(\frac{\omega-\omega_0}{\omega'-\omega_0}\right)\frac{\mathrm{Im}A_R^{(+)}(\omega', \kappa^2)d\omega'}{\omega'-\omega}$$
$$-\frac{1}{\pi}\int_{\mu}^{\infty}\left(\frac{\omega-\omega_0}{\omega'+\omega_0-2\kappa^2/M}\right)\frac{\mathrm{Im}A_R^{(+)}(\omega', \kappa^2)}{(\omega'+\omega-2\kappa^2/M)}d\omega', \qquad (2\cdot4)$$

where $\mathrm{Im}A_R^{(+)}(\omega, \kappa^2)=\mathrm{Im}(A(\omega, \kappa^2)-A_0(\omega, \kappa^2))$, which will have appreciable amount only at high energies. Therfore the right-hand side of (2·4) may be approximated by the function of only κ^2 at low energies. Thus, (2·4) may be approximated as follows:

$$A^{(+)}(\omega, \kappa^2)=A^{(+)}(\kappa^2)+\frac{1}{\pi}\int_{\mu}^{\infty}\frac{\mathrm{Im}A_0^{(+)}(\omega', \kappa^2)d\omega'}{\omega'-\omega}+\frac{1}{\pi}\int_{\mu}^{\infty}\frac{\mathrm{Im}A_0^{(+)}(\omega', \kappa^2)d\omega'}{\omega'+\omega-2\kappa^2/M}, \qquad (2\cdot5)$$

which expresses just CGLN's unsubtracted dispersion relation for $A^{(+)}(\omega, \kappa^2)$, except the undetermined $A^{(+)}(\kappa^2)$. The same form may be derived by the following different way: As $A^{(+)}(\omega, \kappa^2)$ may be analytic function with respect to ω in the

complex upper half plane, Cauchy's integral representation of $A^{(+)}(\omega, \kappa^2)$ may be written as follows, by taking the contour of integration as in Fig. 1,

$$A^{(+)}(\omega, \kappa^2) = \frac{1}{\pi} \int_{-\Lambda}^{\Lambda} \frac{\mathrm{Im} A^{(+)}(\omega', \kappa^2)\, d\omega'}{\omega' - \omega} + \mathrm{Re}\left(\frac{1}{\pi i} \int \frac{A^{(+)}(\omega', \kappa^2)}{\omega' - \omega} d\omega' \right), \quad (2\cdot6)$$

where the last term is integrated on the semicircle of radius Λ. Now, when $\omega \ll \Lambda$, the last term is approximately independent of ω, hence is written as $A^{(+)}(\kappa^2)$, and if, for $-\Lambda < \omega' < \Lambda$, $\mathrm{Im} A^{(+)}(\omega', \kappa^2)$ is approximated by S- and P-waves, (2·6) becomes (2·5).

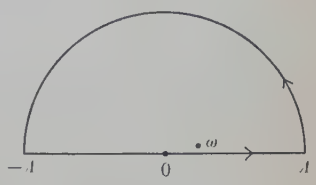

Fig. 1

Here, we consider the effect of the pion-pion interaction to the pion-nucleon scattering. It may be illustrated graphically as in Fig. 2. Now, if one can approximate the pion-pion interaction by $\lambda\phi^4$, with the renormalized coupling constant λ (or by S-wave pion-pion interaction), the contribution to the pion-nucleon scattering amplitude $A^{(+)}(\omega, \kappa^2)$ may be expressed as a real function of only κ^2 independent of ω, same as the vertex function that appears in the electromagnetic structure. In view of (2·6), this function may be included in $A^{(+)}(\kappa^2)$.

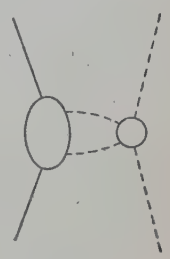
Fig. 2

Now, CGLN have shown that, using (2·2) and (2·3), S-wave phase shifts α_1 and α_3 are not inconsistent with experiment, assuming that the imaginary parts in the dispersion integrals of (2·2) and (2·3) are approximated by S- and P-waves (and especially dominated by 3-3 state). This means according to our interpretation that the large Born term contributions in (2·3) to S-waves may be almost cancelled out by the 3-3 state contribution, thus the contribution of the pion-pion interaction to S-waves, which is included in $A^{(+)}(\kappa^2)$ of (2·5), may not be so large as to cancell the Born term contributions as has been discussed by Kawarabayashi and Miyazawa.[6] In Appendix, we will more explicitly express the pion-pion effects on the pion-nucleon scattering by using Mandelstam representation for the pion-nucleon scattering.[7]

§ 3. GMO's unsubtracted dispersion relations. Discussion

Following the reinterpretation of CGLN's dispersion relations stated in § 2, we can derive the unsubtracted dispersion relations for the forward scattering amplitudes in the laboratory system,

$$T^{(1)}(\omega) = (1/4\pi)\,(A^{(+)}(\omega, 0) + \omega B^{(+)}(\omega, 0)) \quad (3\cdot1)$$

and

$$T^{(2)}(\omega) = (1/4\pi)\,(A^{(-)}(\omega, 0) + \omega B^{(-)}(\omega, 0)). \quad (3\cdot2)$$

Substituting (2·2) and (2·3) in (3·1) and (3·2), we have

$$T^{(1)}(\omega)=-\frac{2f^2\mu^2/2M}{\omega^2-(\mu^2/2M)^2}-\frac{G^2/4\pi}{M}+\frac{1}{\pi}\int_\mu^\infty 2\omega'\frac{\mathrm{Im}A^{(+)}(\omega')/4\pi}{\omega'^2-\omega^2}d\omega'$$

$$+\frac{1}{\pi}\int_\mu^\infty 2\omega^2\frac{\mathrm{Im}B^{(+)}(\omega')/4\pi}{\omega'^2-\omega^2}d\omega' \qquad (3\cdot3)$$

and

$$T^{(2)}(\omega)=\frac{2\omega f^2}{\omega^2-(\mu^2/2M)^2}+\frac{2\omega}{\pi}\int_\mu^\infty\frac{\mathrm{Im}T^{(2)}(\omega')}{\omega'^2-\omega^2}d\omega', \qquad (3\cdot4)$$

where imaginary parts in (3·3) are assumed to be approximated by S- and P-waves, and $f^2=(G\mu/2M)^2/4\pi\simeq 0.08$. (3·4) agrees with GMO's unsubtracted dispersion relation and has been estimated to be consistent with experiments by Haber-Schaim[8] and GMO.[2] The last term of the right-hand side of (3·3) may be formally rewritten as follows:

$$\frac{1}{\pi}\int_\mu^\infty 2\omega^2\frac{\mathrm{Im}B^{(+)}(\omega')/4\pi}{\omega'^2-\omega^2}d\omega'=\frac{1}{\pi}\int_\mu^\infty 2\omega'^2\frac{\mathrm{Im}B^{(+)}(\omega')/4\pi}{\omega'^2-\omega^2}d\omega'$$

$$-\frac{2}{\pi}\int_\mu^\infty(\mathrm{Im}B^{(+)}(\omega')/4\pi)\,d\omega'. \qquad (3\cdot5)$$

Substituting (3·5) in (3·3), we have

$$T^{(1)}(\omega)=-\frac{2f^2\mu^2/2M}{\omega^2-(\mu^2/2M)^2}+\frac{1}{\pi}\int_\mu^\infty 2\omega'\frac{\mathrm{Im}T^{(1)}(\omega')d\omega'}{\omega'^2-\omega^2}-\frac{(G^2/4\pi)}{M}$$

$$-\frac{2}{\pi}\int_\mu^\infty(\mathrm{Im}B^{(+)}(\omega')/4\pi)\,d\omega'. \qquad (3\cdot6)$$

As is expected, this agrees with GMO's unsubtracted dispersion relation, except the last two terms. But even if $\mathrm{Im}B^{(+)}(\omega)$ is expressed by partial waves, the dispersion integrals in (3·5) and in (3·6) may not be convergent, thus we must cut off the integrals in the spirit of (2·6) in § 2. Now, one can estimate with reasonable cutoff energy $\Lambda\simeq 4\mu$ that the last two terms in (3·6) cancel out each other,

$$G^2/4\pi\simeq-\frac{2M}{\pi}\int_\mu^\Lambda[\mathrm{Im}B^{(+)}(\omega')/4\pi]\,d\omega'. \qquad (3\cdot7)$$

In this meaning, we have the relation of $T^{(1)}(\omega)$, which agrees with GMO's unsubtracted dispersion relation:

$$T^{(1)}(\omega)\simeq-\frac{2f^2\mu^2/2M}{\omega^2-(\mu^2/2M)^2}+\frac{1}{\pi}\int_\mu^\Lambda 2\omega'\frac{\mathrm{Im}T^{(1)}(\omega')}{\omega'^2-\omega^2}d\omega'. \qquad (3\cdot8)$$

Thus, we can see that this approximate equation results from (3·7) which shows the large Born term contribution to S-waves cancelled out by the 3-3 state contribution as has been stated in § 2.

CGLN have shown that the contribution of 3-3 state to S-wave scattering amplitude $f_S^{(+)}(q)$ $(=(e^{i\alpha_1}\sin\alpha_1+2e^{i\alpha_3}\sin\alpha_3)/3q)$ does not show the resonance like energy behaviour and is almost constant over the energy range $\mu<\omega\lesssim M$; this may mean that this contribution gives the attractive S-wave interaction of the range of about $1/M$ same as the range of the Born term contribution, in terms of the pion-nucleon potential. This may be consistent from the observation that the S-wave contribution of 3-3 state may be graphically described as in Fig. 3a, as the relativistic dispersion relation (2·2) and (2·3) must include the graphs of Fig. 3a and of Fig. 3b equally in the sense of Feynman's overall space-time picture, and the latter may describe the 3-3 state resonance.

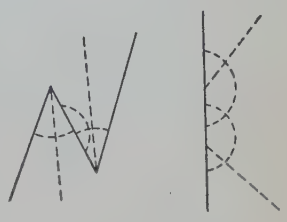

Fig. 3

Finally, we shall discuss the contribution of the pion-pion interaction to the pion-nucleon scattering. We have inferred in § 2 that it may give only higher order correction to S-wave pion-nucleon scattering. To estimate this effect actually seems, however, to be considerably difficult, because S-wave may penetrate deeply into the nucleon core and is affected by the small range interactions, and thus (in terms of momentum representation) it may be necessary to estimate the high energy pion-pion interaction and 4-pions exchange interaction, etc. The higher the angular momentum of the partial wave, however, the larger the relative importance of the long range (about $1/2\mu$) interaction from the pion-pion interaction may become compared with the short range (about $1/2M$) Born term interaction, and the treatment of the effect of the pion-pion interaction may become easier, for the higher partial wave goes through the farther distance from the nucleon core.

In fact, according to the preliminary estimates by I. Sato, A. Takahashi, Y. Ueda and the author,[3] the attractive pion-pion S-wave interaction with the scattering length of about $1/\mu$† gives an attractive P-wave pion-nucleon interaction and is rather consistent with the experimental P-wave scattering amplitudes. Therefore, we believe that the pion-pion interaction will give the more definite effect on the D-wave pion-nucleon scattering comparable with CGLN's result,[1] in which the pion-pion interaction has been neglected. Further investigation on this point is now in progress.

The author wishes to express his gratitude to Professor K. Nakabayasi for his

† It has been stated by Rodberg that this order of S-wave pion-pion interaction may be favorable in order to explain the cross section of additional pion production by pion-nucleon scattering near threshold, see ref. 4).

interest taken in this work. He also wishes to thank Professor I. Sato and the other members of Tohoku University for valuable discussions.

Appendix

Further consideration on the pion-pion effect

Here, we shall more explicitly express the pion-pion effect on the pion-nucleon scattering ("the pion-pion correction" to CGLN's dispersion relation) by the use of Mandelstam representation for the pion-nucleon scattering amplitude.[7] We first specify the momenta of the incoming and outgoing pions as q_1 and q_2, those of the incoming and outgoing nucleons as p_1 and p_2. Then we must simultaneously treat the reactions (I), (II) and (III), which are

$$\pi(q_1)+N(p_1)\to\pi(q_2)+N(p_2), \tag{I}$$

$$\pi(-q_2)+N(p_1)\to\pi(-q_1)+N(p_2), \tag{II}$$

$$N(p_1)+\bar{N}(-p_2)\to\pi(-q_1)+\pi(q_2), \tag{III}$$

respectively. Introducing the variables s, u and t, the squares of the center-of-mass energies for the reactions (I), (II) and (III), respectively,

$$\begin{aligned} s&=-(p_1+q_1)^2=M^2+\mu^2+2M\omega,\\ u&=-(p_1-q_2)^2,\\ t&=-(q_1-q_2)^2=-4\kappa^2, \end{aligned} \tag{A·1}$$

the invariant functions $A^{(\pm)}$ and $B^{(\pm)}$ defined in (2·1) are written as follows:[7]

$$\begin{aligned} A^{(\pm)}=&\frac{s-s_0}{\pi^2}\int_{(M+\mu)^2}^{\infty}ds'\int_{4\mu^2}^{\infty}dt'\frac{a_{13}^{(\pm)}(s',t')}{(s'-s_0)(s'-s)(t'-t)}\\ &+\frac{u-u_0}{\pi^2}\int_{(M+\mu)^2}^{\infty}du'\int_{4\mu^2}^{\infty}dt'\frac{a_{23}^{(\pm)}(u',t')}{(u'-u_0)(u'-u)(t'-t)}+\frac{1}{\pi}\int_{4\mu^2}^{\infty}dt'\frac{a_3^{(\pm)}(t',s_0,u_0)}{t'-t}\\ &+\frac{1}{\pi^2}\int_{(M+\mu)^2}^{\infty}ds'\int_{(M+\mu)^2}^{\infty}du'\frac{a_{12}^{(\pm)}(s',u')}{(u'-u)(s'-s)}+\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty}ds'\frac{a_1^{(\pm)}(s')}{s'-s}\\ &+\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty}du'\frac{a_2^{(\pm)}(u')}{u'-u}, \end{aligned} \tag{A·2}$$

where the first and second terms have been subtracted at the points $s=s_0$ and $u=u_0$, for the spectrum functions $a_{13}(s,t)$ and $a_{23}(u,t)$ will behave as like as constants when $s\to\infty$ and $u\to\infty$, respectively. (Similarly, $B^{(\pm)}$ is written without subtraction and adding the "Born terms".) By means of the symmetry relation

for the invariant functions $A^{(\pm)}(t, s, u)$ and $B^{(\pm)}(t, s, u)$, the spectrum functions must satisfy the following relations:

$$\begin{aligned}
&a_{13}^{(\pm)}(s, t) = \pm a_{23}^{(\pm)}(s, t), \quad && b_{13}^{(\pm)}(s, t) = \mp b_{23}(s, t), \\
&a_{12}^{(\pm)}(s, u) = \pm a_{12}^{(\pm)}(u, s), \quad && b_{12}^{(\pm)}(s, u) = \mp b_{12}(u, s), \\
&a_{1}^{(\pm)}(s) = \pm a_{2}^{(\pm)}(s), && b_{1}^{(\pm)}(s) = \mp b_{2}^{(\pm)}(s), \\
&a_{3}^{(\pm)}(t, s_0, u_0) = \pm a_{3}^{(\pm)}(t, u_0, s_0). && \qquad (A\cdot 3)
\end{aligned}$$

Then, we may derive the one-dimensional dispersion relations for which the momentum transfer, $t=-4\kappa^2$, is held fixed, but thus obtained expression for $A^{(\pm)}$ is not similar to the CGLN's dispersion relation owing to the fact that $A^{(\pm)}$ in (A·2) is once subtracted. Here, we shall first cut off the integrations with respect to s' and u' in (A·2) at the points $S_0 \gg s$ and $U_0 \gg u$. (Hereafter, we put $S_0 = U_0 \equiv \Lambda^2$ for simplicity.) Then, using the following relations,

$$\begin{aligned}
&\frac{s-s_0}{\pi^2}\int_{(M+\mu)^2}^{\infty} ds' \int_{4\mu^2}^{\infty} dt' \frac{a_{13}^{(\pm)}(s', t')}{(s'-s_0)(s'-s)(t'-t)} \\
&\simeq \frac{1}{\pi^2}\int_{(M+\mu)^2}^{\Lambda^2} ds' \int_{4\mu^2}^{\infty} dt' \frac{a_{13}^{(\pm)}(s', t')}{(s'-s)(t'-t)} - \frac{1}{\pi^2}\int_{(M+\mu)^2}^{\Lambda^2} ds' \int_{4\mu^2}^{\infty} dt' \frac{a_{13}^{(\pm)}(s', t')}{(s'-s_0)(t'-t)}, \text{ etc.,}
\end{aligned} \qquad (A\cdot 4)$$

(A·3) is rewritten as

$$\begin{aligned}
A^{(\pm)} \simeq\ & \frac{1}{\pi^2}\int_{(M+\mu)^2}^{\Lambda^2} ds' \int_{4\mu^2}^{\infty} dt' \frac{a_{12}^{(\pm)}(s', t')}{(s'-s)(t'-t)} + \frac{1}{\pi^2}\int_{(M+\mu)^2}^{\Lambda^2} du' \int_{4\mu^2}^{\infty} dt' \frac{a_{23}^{(\pm)}(u', t')}{(u'-u)(t'-t)} \\
&+ \frac{1}{\pi}\int_{4\mu^2}^{\infty} dt' \frac{a_{3}^{(\pm)}(t', \Lambda^2)}{t'-t} + \frac{1}{\pi^2}\int_{(M+\mu)^2}^{\Lambda^2} ds' \int_{(M+\mu)^2}^{\Lambda^2} du' \frac{a_{12}^{(\pm)}(s', u')}{(u'-u)(s'-s)} \\
&+ \frac{1}{\pi}\int_{(M+\mu)^2}^{\Lambda^2} ds' \frac{a_{1}^{(\pm)}(s')}{s'-s} + \frac{1}{\pi}\int_{(M+\mu)^2}^{\Lambda^2} du' \frac{a_{2}^{(\pm)}(u')}{u'-u}.
\end{aligned} \qquad (A\cdot 5)$$

In this expression, each of the first three terms might diverge when $\Lambda^2 \to \infty$, but as the whole they must converge when $\Lambda^2 \to \infty$, if the same cutoff Λ^2 for each term is used. The first three terms in (A·5) also can be interpreted, apart from the above derivation from (A·2), as the result of the application of Cauchy's integral representation, where the integrals in the variables s' and u' have been performed on the circles of radius Λ^2 with the branch cuts $(M+\mu)^2 < s < \Lambda^2$ and $(M+\mu)^2 < u < \Lambda^2$ on the complex s- and u-planes, and thus the third term is correspond to the integrals on the circles. Thus, we have been able to rewrite $A^{(\pm)}$ apparently in the unsubtracted form introducing the cutoff energy Λ. This expression is very useful for our purpose. We then can write down the dispersion

relations, when the momentum transfer t is held fixed, as follows:

$$A^{(\pm)} \cong \frac{1}{\pi}\int_{(M+\mu)^2}^{\Lambda^2} ds' \frac{A_1^{(\pm)}(s', t)}{s'-s} + \frac{1}{\pi}\int_{(M+\mu)^2}^{\Lambda^2} du' \frac{A_2^{(\pm)}(u', t)}{u'-u} + \frac{1}{\pi}\int_{4\mu^2}^{\infty} dt' \frac{a_3^{(\pm)}(t', \Lambda^2)}{t'-t}, \tag{A·6}$$

similarly,

$$B^{(\pm)} = \frac{G^2}{M^2-s} \mp \frac{G^2}{M^2-u} + \frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} ds' \frac{B_1^{(\pm)}(s', t)}{s'-s} + \frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} du' \frac{B_2^{(\pm)}(u', t)}{u'-u}, \tag{A·7}$$

where $(A, B)_1$ and $(A, B)_2$ are the imaginary parts of the reactions (I) and (II), and are expressed as follows:

$$A_1^{(\pm)}(s', t) \cong a_1^{(\pm)}(s') + \frac{1}{\pi}\int_{4\mu^2}^{\infty} dt' \frac{a_{13}^{(\pm)}(s', t')}{t'-t} + \frac{1}{\pi}\int_{(M+\mu)^2}^{\Lambda^2} du' \frac{a_{12}^{(\pm)}(s', u')}{u'-u(s', t)}, \text{ etc.} \tag{A·8}$$

Further, the $a_3^{(\pm)}(t, \Lambda^2)$ is related with the imaginary part of the reaction (III), $A_3^{(\pm)}(t, s, u)$, as

$$A_3^{(\pm)}(t, s, u) \cong a_3^{(\pm)}(t, \Lambda^2) + \frac{1}{\pi}\int_{(M+\mu)^2}^{\Lambda^2} ds' \frac{a_{13}^{(\pm)}(s', t)}{s'-s} + \frac{1}{\pi}\int_{(M+\mu)^2}^{\Lambda^2} du' \frac{a_{23}^{(\pm)}(u', t)}{u'-u}, \tag{A·9}$$

especially $a_3^{(-)}(t, \Lambda^2) = 0$ by means of Pauli principle for reaction (III). Therefore, CGLN's dispersion relations will hold for $A^{(-)}$ and $B^{(\pm)}$, when, in $A^{(-)}$, the same cutoff Λ^2 is introduced in the integrals in s' and u' as in (A·6).

Next, we will more carefully consider the low energy behaviour of the scattering amplitude than in Sec. 2. We first consider the $A_1(s, t)$, $A_2(u, t)$ and $A_3(t, s, u)$, which will be expressed as the sums of the contributions from the various intermediate states by means of the unitary conditions imposed on the reactions (I), (II) and (III), respectively. The lowest contributions to (I) and (II) come from the elastic pion-nucleon scattering. (We denote them as $A_{1(el)}$ and $A_{2(el)}$, respectively.) However, the related spectrum functions $a_{13}(s, t)$ and $a_{23}(u, t)$ in (A·8) do not include the lowest 2π-contribution to A_3 by means of Furry's theorem. We will then divide the regions of the integrals in s' and u' in (A·6) into the two parts $((M+\mu)^2, \Omega^2)$ and (Ω^2, Λ^2), taking Ω^2 so as to be able to approximate $(A_1(s', t), A_2(u', t))$ as $(A_{1(el)}(s', t), A_{2(el)}(u', t))$ in the former region. Then, we have

$$\int_{(M+\mu)^2}^{\Lambda^2} ds' \frac{A_1(s', t)}{s'-s} = \int_{(M+\mu)^2}^{\Omega^2} ds' \frac{A_{1(el)}(s', t)}{s'-s} + \int_{(M+2\mu)^2}^{\Omega^2} ds' \frac{A_{1(inel)}(s', t)}{s'-s} + \int_{\Omega^2}^{\Lambda^2} ds' \frac{A_1(s', t)}{s'-s}, \tag{A·10}$$

where the second term, which is the contribution from the inelastic scattering, will be the small correction. However, we will extract from this second term the lowest 2π-contribution to the reaction (III), which is written as

$$\frac{1}{\pi^2}\int_{4\mu^2}^{\infty} dt' \frac{1}{t'-t}\int_{(M+2\mu)^2}^{\Omega^2} ds' \frac{a_{13(2\pi)}(t', s')}{s'-s}.$$

Then, $A^{(\pm)}$ is rewritten as follows:

$$A \cong \frac{1}{\pi}\int_{(M+\mu)^2}^{\Omega^2} ds' \frac{A_{1(el)}(s', t)}{s'-s} + \frac{1}{\pi}\int_{(M+\mu)^2}^{\Omega^2} du' \frac{A_{2(el)}(u', t)}{u'-u}$$

$$+\frac{1}{\pi}\int_{4\mu^2}^{\infty} dt' \frac{1}{t'-t} A_3'(t', s, u, \Omega^2),$$

$$A_3'(t', s, u, \Omega^2) = a_3(t', \Lambda^2) + \frac{1}{\pi}\int_{\Omega^2}^{\Lambda^2} ds' \frac{a_{13}(s', t')}{s'-s} + \frac{1}{\pi}\int_{\Omega^2}^{\Lambda^2} du' \frac{a_{23}(u', t')}{u'-u}$$

$$+\frac{1}{\pi}\int_{(M+2\mu)^2}^{\Omega^2} ds' \frac{a_{13(2\pi)}(s', t')}{s'-s} + \frac{1}{\pi}\int_{(M+2\mu)^2}^{\Omega^2} du' \frac{a_{23(2\pi)}(u', t')}{u'-u}, \qquad (A\cdot 11)$$

where it is assumed that the upper limit Λ^2 of the integrals in the convergent last three terms in (A·5) will be replaced by Ω^2 at low energies. Comparing (A·11) with (A·9), we see that the 2π-contribution to A_3' is exactly equal to the 2π-contribution to A_3, which we denote as $A_{3(2\pi)}$. Then, our final convergent expression for $A^{(\pm)}$ is as follows:

$$A^{(\pm)} \cong \frac{1}{\pi}\int_{(M+\mu)^2}^{\Omega^2} ds' \frac{A^{(\pm)}_{1(el)}(s', t)}{s'-s} + \frac{1}{\pi}\int_{(M+\mu)^2}^{\Omega^2} du' \frac{A^{(\pm)}_{2(el)}(u', t)}{u'-u}$$

$$+\frac{1}{\pi}\int_{4\mu^2}^{\infty} dt' \frac{A^{(\pm)}_{3(2\pi)}(t', s, u)}{t'-t} + \frac{1}{\pi}\int_{16\mu^2}^{\infty} dt' \frac{A_3''^{(\pm)}(t', s, u)}{t'-t}, \qquad (A\cdot 12)$$

where the related spectrum functions a_{13} and a_{23} for A_3'' will be non zero only when $s, u > \Omega^2$. Similar expression will hold for $B^{(\pm)}$ by adding the Born terms. Thus, we have been able to express "the pion-pion correction" to CGLN's dispersion relation. The first two terms in (A·12) will be approximately equal to the last two terms in (2·5), because the most dominant contribution from the 3–3 state will be included in the region $((M+\mu)^2, \Omega^2)$, thus the $A(\kappa^2)$ in (2·5) will be equal to the last two terms of (A·12), which might be approximately independent of s and u at low energies. (A·12) are approximately equivalent to those used by Frazer and Fulco in the problem of the nucleon structure,[5] and also are approximately equivalent to those used by ours in ref. 9). Similar expressions

to (A·12) also have been derived by Cini and Fubini,* but they start with the unsubtracted dispersion relation, which will hold only for $B^{(\pm)}$.

References

1) G. F. Chew, M. L. Goldberger, F. E. Low and Y. Nambu, Phys. Rev. **106** (1957), 1337.
2) M. L. Goldberger, H. Miyazawa and R. Oehme, Phys. Rev. **96** (1955), 986.
3) S. Minami, Prog. Theor. Phys. **23** (1960), 887.
4) J. S. Kovacs, Phys. Rev. **93** (1954), 252.
   F. J. Dyson, Phys. Rev. **99** (1955), 1037.
   G. Takeda, Phys. Rev. **100** (1955), 440.
   D. Ito, M. Yamazaki, T. Kobayashi and K. Mori, Prog. Theor. Phys. **22** (1959), 448.
   L. S. Rodberg, Phys. Rev. Letters **3** (1959), 58.
5) P. Federbush, M. L. Goldberger and S. B. Treiman, Phys. Rev. **112** (1958), 642.
   W. R. Frazer and J. R. Fulco, Phys. Rev. Letters **2** (1959), 365.
6) K. Kawarabayashi and H. Miyazawa, Prog. Theor. Phys. **23** (1960), 490
7) S. Mandelstam, Phys. Rev. **112** (1958), 1344; **115** (1959), 1741, 1752.
8) U. Haber-Schaim, Phys. Rev. **104** (1956), 1113.
9) I. Sato, A. Takahashi and Y. Ueda, Prog. Theor. Phys. **22** (1959), 617.
   K. Ishida, A. Takahashi and Y. Ueda, Prog. Theor. Phys. **23** (1960), 731.

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Collective Representation of the Many-Particle System Interacting through Arbitrary Two-Body Potential*

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We develope a collective description of the many-particle system interacting through an arbitrary two-body potential. The description is given both in terms of the second quantized form and in terms of an auxiliary field analogous to the longitudinal electromagnetic field.

As Bohm-Pines theory of the collective description has some unsatisfactory features concerning the subsidiary condition imposed on the auxiliary field, we propose here a strict method of treatment on the subsidiary condition. To obtain the useful result of the collective representation, we have to take the three steps of the canonical transformations. Though the first two transformations are the same as those of B-P, the third transformation, which is not applied by B-P and is characteristic of our theory, is necessary for obtaining a meaningful result. By using both the strict subsidiary condition and the three canonical transformations, the collective field coordinate and the particle coordinate are separated not only in the total Hamiltonian but also in the subsidiary condition. Moreover, we are able to clarify the correct meaning of the subsidiary condition and clarify also the connections between our theory and the theory of Sawada-Brueckner-Fukuda-Brout.

§ 1. Introduction

The collective motion of a many-particle system has been known early. Among many examples of collective motions, a well-known example is the plasma oscillation of a system of electrons, as empirically observed first by Penning (1926) and by Langmuir (1929). The same was investigated theoretically by many authors,[1] among whom Bohm and Pines[2] have developed most interesting theories. Another example is, as is well known, the collective motion of the liquid helium. This was pointed out first by Landau,[3] who explained behaviors of He^4 in the very low temperature region based on quantum-hydrodynamics. Thereafter, the procedure of the quantum-hydrodynamics was investigated minutely by several authors.[4] Moreover, to give justification to Landau's theory starting from the Schrödinger equation of the particle system, various approaches have been attempted by several authors.[5] Most of the theories mentioned above are interesting not only as the proper method for each problem but also in suggestive aspects for a general method to solve the many-particle problem. Especially the Bohm-Pines theory seems important, though it deals only with a system of particles interacting through Coulomb

* This is the revised and enlarged one of the paper which was read at the annual meeting of the Physical Society of Japan held in Hiroshima, Oct., 1959.

force. On the other hand, Tomonaga[6] proposed a wide theory applicable to a system of particles interacting through non-Coulombian force.

The present work begins with the Hamiltonian which is essentially concerned with Tomonaga's "E"-field,[6] which is a generalization of the Bohm-Pines method. Although Tomonaga's and Bohm-Pines' theories were not, our formulation will be developed in the second-quantized form. This seems especially necessary for clarifying the differences in the collective representation according to the difference of statistical character of the particles.

In § 2 we shall give the fundamental Hamiltonian of the system and a strict definition of the subsidiary condition. The Hamiltonian taken at the start will be replaced by another equivalent Hamiltonian expressed in terms of an auxiliary field, i. e. the "E"-field. But here, the pivotal point is in the treatment of the subsidiary condition imposed upon the auxiliary field. In the theory of quantum-electrodynamics, the difficulty in the subsidiary condition has already been pointed out by Belinfante and others,[7] that is to say, the state vector Φ which satisfies the condition $(\nabla \cdot \boldsymbol{E} - 4\pi e\rho)\Phi = 0$ cannot be normalized. If we adopt such a type of condition for the present auxiliary field, as Bohm-Pines and Tomonaga did, the same difficulty as met in the case of quantum-electrodynamics will appear. The difficulty in the subsidiary condition of the Bohm-Pines theory was discussed by Adams and Kuper.[8] In this paper we shall propose a strict method of treatment of the subsidiary condition with special care.

In § 3, by using a canonical transformation we shall strictly verify the equivalence between the original Hamiltonian and the Hamiltonian expressed in terms of the auxiliary field. The meaning of the subsidiary condition will also be clarified there.

In § 4, we shall apply the first canonical transformation, which was applied by Bohm-Pines and by which a collectively oscillatory part of long wavelength will be separated from the Hamiltonian expressed in terms of the auxiliary field. The frequency of the collective oscillation thus obtained is, however, an expression at the lowest order in an approximation. At the same time, in this section, we shall explain the necessary condition for the existence of the collective oscillation.

In § 5, the second and the third canonical transformations will be applied, thus obtaining the collective representation in which the collective field variable and the particle variable are separated not only in the total Hamiltonian but also in the subsidiary condition. The second transformation is also the same as that of Bohm-Pines, but the third one is not applied by Bohm-Pines and is characteristic of our theory. When only the first two transformations are applied, the collective field variable and the particle variable are not separated either in the total Hamiltonian or in the subsidiary condition. This circumstance is caused by the new facts around the subsidiary condition which is correctly revised by us from the customary expression. Thus, to obtain a meaningful result, the third transformation is necessary.

In § 6, we shall clarify the meaning of the subsidiary condition in the collective representation, and reduce the total Hamiltonian by taking the subsidiary condition into account. The Hamiltonian thus obtained is independent of the subsidiary condition, and is applicable to the discussions not only of the ground state but also of the excited state. There, at the same time, the connection between our theory and the theory of Sawada-Brueckner-Fukuda-Brout[9] will be shown.

§ 2. Fundamental Hamiltonian of the system

We start from the following Hamiltonian of the second quantized form:

$$\overline{H}_0=\frac{1}{2m}\int\sum_{\sigma=1}^{2}\psi_\sigma^*\left(\frac{\hbar}{i}\nabla\right)^2\psi_\sigma\cdot dv$$
$$+\frac{1}{2}\iint\sum_{\sigma,\mu=1}^{2}\psi_\sigma^*(x)\psi_\mu^*(x')G(x-x')\psi_\mu(x')\psi_\sigma(x)\cdot dv\,dv'. \tag{1a}$$

Spin index σ or μ should be omitted for a system of Bose particles. In the case of Fermi particles, as only the nonrelativistic energy region is being considered, the negative energy state (or anti-particle) is neglected, and the potential $G(x-x')$ is conveniently assumed as spin-independent. The essential argument of the theory will not be affected, however, by these abbreviations. Commutation relations for the particle variables are well known as follows:

$$\left.\begin{aligned}&[\psi_\sigma^*(x),\ \psi_\mu(x')]_\mp=\mp\delta_{\sigma,\mu}\delta^3(x-x'),\\&[\psi_\sigma^*(x),\ \psi_\mu^*(x')]_\mp=[\psi_\sigma(x),\ \psi_\mu(x')]_\mp=0.\end{aligned}\right\} \tag{2}$$

In the double signs,* here and hereafter, the upper and lower signs respectively correspond to the Bose and Fermi particles.

If the density operator, $\rho=\sum_{\sigma=1}^{2}\psi_\sigma^*\psi_\sigma$, is used, (1a) may be rewritten as

$$\overline{H}_0=\frac{1}{2m}\int\sum_\sigma\psi_\sigma^*\left(\frac{\hbar}{i}\nabla\right)^2\psi_\sigma\cdot dv$$
$$+\frac{1}{2}\iint\rho(x)G(x-x')\rho(x')\cdot dv\,dv'-\frac{1}{2}G(0)\int\rho\cdot dv. \tag{1b}$$

Instead of working directly with (1a) or (1b), we introduce an equivalent Hamiltonian expressed in terms of an auxiliary field analogous to the longitudinal electromagnetic field. This is convenient for describing the longitudinal collective oscillation of the density and is the essence of the "E"-field method proposed by Tomonaga.[6] That is, we use the following Hamiltonian in place of (1b):

$$\mathcal{H}_0=\frac{1}{2m}\int\sum_\sigma\psi_\sigma^*\left(\frac{\hbar}{i}\nabla-\Lambda\right)^2\psi_\sigma\cdot dv$$
$$+\frac{1}{2}\iint(\nabla,\Pi)G(x-x')(\nabla',\Pi')\cdot dv\,dv'-\frac{1}{2}G(0)\int\rho\cdot dv. \tag{3}$$

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* $[A,B]_\mp=AB\mp BA$.

Here, the operators of the auxiliary field, $\boldsymbol{\Lambda}$ and $\boldsymbol{\Pi}$, have characters similar to those of the electromagnetic field, $\boldsymbol{A}$ and $\boldsymbol{E}$: $\boldsymbol{\Lambda}$ and $\boldsymbol{\Pi}$ correspond respectively to the longitudinal part of $\boldsymbol{A}$ and to the longitudinal part of $\boldsymbol{E}$, which is canonically conjugate to $\boldsymbol{A}$. When $G(x-x')$ is the Coulomb potential, there stands the equation $\Delta G(x-x')=-4\pi e^2\delta^3(x-x')$, and consequently, the second term of (3) becomes $1/8\pi\int \boldsymbol{E}^2\cdot dv$, provided $\boldsymbol{\Pi}=\boldsymbol{E}/4\pi e$. At the same time we should replace $\boldsymbol{\Lambda}$ by $e/c\cdot\boldsymbol{A}$. Thus (3) is the natural generalization of the Coulombian case. The commutation relation for the auxiliary field operators is given by

$$[\boldsymbol{\Pi}(x),\ \boldsymbol{\Lambda}(x')]_-=i\hbar\delta^3(x-x'). \tag{4}$$

To assure the equivalence of (3) and (1), we have to impose a subsidiary condition upon the auxiliary field. An expression of the subsidiary condition acting on the wave function of (3), $(\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}-\rho)\Phi=0$, is not, as pointed out by several authors, considered correct. Difficulties around this point were investigated already in the theory of quantum-electrodynamics by Gupta and others.[7] In the collective description of Bohm-Pines, essentially the same difficulty as this was pointed out by Adams and others.[8]

Comparing (3) and (1b), we can first assume the following condition in the form of the expectation value of the operator:

$$(\Phi^*(t)\,\Omega(x)\,\Phi(t))=0,\quad \Omega(x)=\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}-\rho. \tag{5}$$

As the Schrödinger function Φ is a function of time (t), relation (5) should be demanded for all (t). Fortunately, however, there stands the relationship $[\mathcal{H}_0,\ \Omega(x)]_-=0$; accordingly,

$$\frac{d}{dt}(\Phi^*(t)\,\Omega(x)\,\Phi(t))=\frac{i}{\hbar}(\Phi^*(t)[\mathcal{H}_0,\ \Omega(x)]_-\Phi(t))=0. \tag{6}$$

Therefore, condition (5) actually stands for all (t) when the following initial condition stands at the time $t=t_0$:

$$(\Phi^*(t_0)\,\Omega(x)\,\Phi(t_0))=0. \tag{5'}$$

Secondly, we can impose the following condition on the other auxiliary field operator $\boldsymbol{\Lambda}$:

$$(\Phi^*(t)\,\boldsymbol{\Lambda}(x)\,\Phi(t))=0. \tag{7}$$

At this time, as $\boldsymbol{\Lambda}(x)$ is not commutable with $\mathcal{H}_0$, there appears a difficulty. We shall find it possible, however, to remove this difficulty by adding an extra energy $\mathcal{H}_a$ upon (3), by which there stands the relationship,

$$[\mathcal{H},\ \boldsymbol{\Lambda}(x)]_-=0,\quad \mathcal{H}=\mathcal{H}_0+\mathcal{H}_a. \tag{8}$$

On the other hand, $\mathcal{H}_a$ should be selected so as to retain condition (6), say, so that $[\mathcal{H}_a,\ \Omega(x)]_-=0$. As a most simple form of $\mathcal{H}_a$ which satisfies the above conditions we may take

$$\mathcal{H}_a = -\frac{1}{2}\iint \Omega(x) G(x-x') \Omega(x') \cdot dv\, dv' - \iint \rho(x) G(x-x') \Omega(x') \cdot dv\, dv'. \tag{9}$$

Thus, we obtained the second subsidiary condition (7) for all (t) under the initial condition,

$$(\Phi^*(t_0) \mathbf{\Lambda}(x) \Phi(t_0)) = 0. \tag{7'}$$

When $\mathcal{H}_a$ is introduced into the Hamiltonian we should put the third condition

$$(\Phi^*(t) \Omega^2(x) \Phi(t)) = P, \tag{10}$$

where P is a c-number independent of any dynamical variable and should be determined so as to establish the equivalence $(\chi^* \bar{H}_0 \chi) = (\Phi^* \mathcal{H} \Phi)$. As the relationship $[\mathcal{H}, \Omega^2(x)]_- = 0$ is evident, (10) stands for all (t) under the initial condition

$$(\Phi^*(t_0) \Omega^2(x) \Phi(t_0)) = P. \tag{10'}$$

Furthermore, as the quantity $\mathbf{\Lambda}^2$ is contained in (3), it seems necessary to put the fourth condition

$$(\Phi^*(t) \mathbf{\Lambda}^2(x) \Phi(t)) = Q, \tag{11}$$

where Q is another c-number independent of any dynamical variable. At this time as well, as there stands the relation $[\mathcal{H}, \mathbf{\Lambda}^2(x)]_- = 0$, (11) stands for all (t) under the initial condition,

$$(\Phi^*(t_0) \mathbf{\Lambda}^2(x) \Phi(t_0)) = Q. \tag{11'}$$

Thus, we obtained the necessary and sufficient subsidiary conditions assuring the equivalence between the original Hamiltonian (1) and the Hamiltonian expressed in terms of the auxiliary field. We summarize here the results obtained: The fundamental Hamiltonian for future discussion is

$$\begin{aligned}\mathcal{H} = {} & \frac{1}{2m}\int \sum_\sigma \psi_\sigma^* \left(\frac{\hbar}{i}\nabla - \mathbf{\Lambda}\right)^2 \psi_\sigma \cdot dv \\ & + \frac{1}{2}\iint (\nabla, \mathbf{\Pi}) G(x-x') (\nabla', \mathbf{\Pi}') \cdot dv\, dv' - \frac{1}{2} G(0) \int \rho \cdot dv \\ & - \frac{1}{2}\iint \Omega G(x-x') \Omega' \cdot dv\, dv' - \iint \rho G(x-x') \Omega' \cdot dv\, dv',\end{aligned} \tag{12}$$

and it can be expected that this Hamiltonian be equivalent to (1) under the subsidiary conditions,

$$\left.\begin{aligned}(\Phi^* \Omega(x) \Phi) &= 0, \quad (\Phi^* \Omega^2(x) \Phi) = P, \\ (\Phi^* \mathbf{\Lambda}(x) \Phi) &= 0, \quad (\Phi^* \mathbf{\Lambda}^2(x) \Phi) = Q.\end{aligned}\right\} \tag{13}$$

If we pay attention here to the fact that $\mathbf{\Lambda}$ is not independent of Ω according to (4), we can easily understand that Q is not independent of P. In the coming

calculations we shall actually illustrate that Q is determined by P (or vice versa).

Further calculations will be worked out in the momentum space and more detailed circumstances on the subsidiary conditions will also be clarified.

§ 3. Equivalence between (1) and (12)

We hereafter consider the whole system to be in a cubic box of volume V and transform every operator into Fourier series, thus working out calculations in the momentum space. The fundamental formulae associated with the Fourier transformation are given by

$$\left.\begin{aligned} &\psi_\sigma(x)=V^{-1/2}\sum_k \psi_{\sigma,k}e^{ik\cdot x},\ \psi_\sigma^*(x)=V^{-1/2}\sum_k \psi_{\sigma,k}^* e^{-ik\cdot x},\\ &[\psi_{\sigma,k}^*,\ \psi_{\mu,k'}]_\mp=\mp\delta_{\sigma,\mu}\delta^3_{k,k'},\ [\psi^*,\psi^*]_\mp=[\psi,\psi]_\mp=0, \end{aligned}\right\} \tag{14}$$

$$\left.\begin{aligned} &\boldsymbol{\Lambda}(x)=V^{-1/2}\sum_k \boldsymbol{\varepsilon}_k q_k e^{ik\cdot x},\ \boldsymbol{\Pi}(x)=V^{-1/2}\sum_k \boldsymbol{\varepsilon}_k p_{-k}e^{ik\cdot x};\ \boldsymbol{\varepsilon}_k=\boldsymbol{k}/|\boldsymbol{k}|,\\ &[p_k,\ q_{k'}]_-=-i\hbar\delta^3_{k,k'}. \end{aligned}\right\} \tag{15}$$

The Fourier-analyzed form of the operator $\Omega(x)$ is

$$\left.\begin{aligned} &\Omega(x)=\nabla\cdot\boldsymbol{\Pi}-\rho=V^{-1/2}\sum_k i|\boldsymbol{k}|\Omega_k e^{ik\cdot x},\\ &\Omega_k=p_{-k}+(i/|\boldsymbol{k}|)\rho_k, \end{aligned}\right\} \tag{16}$$

where $\rho(x)=V^{-1/2}\sum_k \rho_k e^{ik\cdot x}$ and

$$\rho_k=V^{-1/2}\sum_{k'}^{\sigma}\psi_{\sigma,k'}^*\psi_{\sigma,k'+k}=V^{-1/2}\sum_{k'}^{\sigma}\psi_{\sigma,-k+k'}^*\psi_{\sigma,k'}. \tag{17}$$

If we assume the subsidiary condition to be $\Omega(x)\Phi=0$, the one in the momentum space becomes $\Omega_k\Phi=0$. We should not, however, adopt this subsidiary condition. The expression of our subsidiary condition (13) in the momentum space are given (for all k) by

$$\left.\begin{aligned} &(\Phi^*\Omega_k\Phi)=0,\ (\Phi^*\Omega_{k'}\Omega_k\Phi)=-f^2(k)\delta^3_{k',-k},\\ &(\Phi^* q_k\Phi)=0,\ (\Phi^* q_{k'}q_k\Phi)=-g^2(k)\delta^3_{k',-k}, \end{aligned}\right\} \tag{18}$$

where $f^2(k)$ and $g^2(k)$ are the c-number functions of k, to be determined later.

The total Hamiltonian (12) is expressed in the momentum space by

$$\begin{aligned} \mathcal{H}=&\sum_k^{\sigma}\frac{\hbar^2k^2}{2m}\psi_{\sigma,k}^*\psi_{\sigma,k}-\frac{\hbar}{m\sqrt{V}}\sum_{k,k'}^{\sigma}(\boldsymbol{k}-\boldsymbol{k}'/2)\cdot\boldsymbol{\varepsilon}_{k'}q_{k'}\psi_{\sigma,k}^*\psi_{\sigma,k-k'}\\ &+\frac{1}{2m\sqrt{V}}\sum_{k,k'}(\boldsymbol{\varepsilon}_k,\boldsymbol{\varepsilon}_{k'})q_kq_{k'}\rho_{-k-k'}-\frac{1}{2}\sum_k k^2J(k)(p_{-k}p_k-\Omega_{-k}\Omega_\kappa\\ &+2i/|\boldsymbol{k}|\cdot\rho_{-k}\Omega_\kappa)-\frac{N}{2V}\sum_\kappa J(k). \end{aligned} \tag{19}$$

Here, N is the total number of particles defined by

$$N=\sum_{k}^{\sigma}\psi^{*}_{\sigma,k}\psi_{\sigma,k}=\int\rho\cdot dv=\sqrt{V}\rho_{0}, \tag{20}$$

and $J(\boldsymbol{k})$ is the matrix element of the potential G corresponding to the momentum transfer of $\boldsymbol{k}$, and is a function of $|\boldsymbol{k}|$ only when $G(x-x')$ is spherically symmetric, that is,

$$G(x-x')=V^{-1}\sum_{k}J(k)\,e^{i\boldsymbol{k}\cdot(x-x')}. \tag{21}$$

Now we shall apply the canonical transformation $\Phi=U\Phi'=\exp(S)\Phi'$ by which the equivalence between (12) and (1) will be verified. Here the generating function S is given by

$$S=\sum_{k}\frac{1}{\hbar|\boldsymbol{k}|}q_{k}\rho_{-k}. \tag{22}$$

In the new representation where the state function is Φ', any operator transformed is given by the well-known formula,

$$e^{-S}Oe^{S}=O+[O,\,S]_{-}+\frac{1}{2!}[[O,\,S]_{-},\,S]_{-}+\cdots. \tag{23}$$

In the calculations of the transformed operators we use the following relationships,

$$\left.\begin{aligned}&[\psi^{*}_{\sigma,k}\psi_{\mu,l},\ \psi^{*}_{\sigma',k'}\psi_{\mu',l'}]_{-}=\delta_{\mu,\sigma'}\delta^{3}_{l,k'}\psi^{*}_{\sigma,k}\psi_{\mu',l'}-\delta_{\sigma,\mu'}\delta^{3}_{k,l'}\psi^{*}_{\sigma',k'}\psi_{\mu,l},\\&[\psi^{*}_{\sigma,k}\psi_{\mu,l},\ \rho_{k'}]_{-}=V^{-1/2}(\psi^{*}_{\sigma,k}\psi_{\mu,l+k'}-\psi^{*}_{\sigma,k-k'}\psi_{\mu,l}),\\&[\rho_{k},\ \rho_{k'}]_{-}=0,\end{aligned}\right\} \tag{24}$$

which stand both for the Bose and Fermi particles.

First, the new subsidiary conditions transformed from (18) are given (for all k) by

$$\left.\begin{aligned}&(\Phi^{*}p_{k}\Phi)=0,\quad(\Phi^{*}p_{k'}p_{k}\Phi')=-f^{2}(k)\,\delta^{3}_{k',-k},\\&(\Phi^{*}q_{k}\Phi)=0,\quad(\Phi^{*}q_{k'}q_{k}\Phi)=-g^{2}(k)\,\delta^{3}_{k',-k}.\end{aligned}\right\} \tag{25}$$

Secondly, when we put $\mathcal{H}$ of (19) for O in formula (23), the fourth term and the following terms vanish precisely, and the result is expressed by

$$U^{-1}\mathcal{H}U=\sum_{k}^{\sigma}\frac{\hbar^{2}k^{2}}{2m}\psi^{*}_{\sigma,k}\psi_{\sigma,k}+\frac{1}{2}\sum_{k}J(k)\rho_{-k}\rho_{k}-\frac{N}{2V}\sum_{k}J(k). \tag{26}$$

This is exactly the momentum representation of $\bar{H}_{0}$ of (1).

Now, in order to clarify the meaning of subsidiary condition (25), we transform $(q_{k},\,p_{k})$ into $(a_{k},\,a_{k}^{*})$ by

$$q_{k}=\frac{\hbar}{2f(k)}(a_{k}-a^{*}_{-k}),\quad p_{k}=if(k)\,(a_{-k}+a_{k}^{*}), \tag{27}$$

where $f(k)$ is the same as that of (18) or (25), and the new operators, a_{k} and a_{k}^{*}, obey the well-known commutation relations,

$$[a_k, a_{k'}^*]_- = \delta^3_{k,k'}, \quad [a_k, a_{k'}]_- = [a_k^*, a_{k'}^*]_- = 0. \tag{28}$$

Then, we can easily verify that the solution satisfying the four conditions of (25) is uniquely given by the following simple form, provided $f^2(k) = \hbar^2/4f^2(k)$:

$$a_k \Phi' = 0. \tag{29}$$

The relationship $f^2(k) = \hbar^2/4f^2(k)$ means that Q is not independent of P (and vice versa) in (13). Hereafter we will always adopt this relationship.

Φ' which satisfies (29) is the state vector of the vacuum (or the ground state) for the auxiliary field and accordingly, in this state, no auxiliary field quanta exist. This is a very natural result; thus, the equivalence between (1) and (12) is verified.

§ 4. Separation of collective parts from the Hamiltonian

In this section we shall proceed to a representation in which the collective mode of the motion will be explicitly presented, though it will be an expression at the lowest order in an approximation. And we shall illustrate in this representation as well that (12) is equivalent to the original Hamiltonian (1) under the subsidiary conditions. The expected representation may be obtained by applying the canonical transformation $\Phi = T_1 \Phi' = \exp(S_1)\Phi'$, where

$$S_1 = \sum_{k>k_c} \frac{1}{\hbar|\boldsymbol{k}|} q_k \rho_{-k}. \tag{30}$$

At this time, there is the limitation $k > k_c$ in the generating function; thus accordingly, (22) is considered as the special case, $k_c = 0$, from the viewpoint of (30).

The new subsidiary conditions transformed from (18) are divided into two groups for $k < k_c$ and $k > k_c$. The first group for $k < k_c$ is essentially the same as (18). Thus, for $k < k_c$,

$$\left.\begin{aligned} &(\Phi'^* \Omega_k \Phi') = 0 \quad (1), \quad (\Phi'^* \Omega_{k'} \Omega_k \Phi') = -f^2(k)\delta^3_{k',-k} \quad (2), \\ &(\Phi'^* q_k \Phi') = 0 \quad (3), \quad (\Phi'^* q_{k'} q_k \Phi') = -\frac{\hbar^2}{4f^2(k)} \delta^3_{k',-k} \quad (4), \end{aligned}\right\} \tag{31}$$

where Ω_k is the same form as (16). The second group for $k > k_c$ is essentially the same as (25) :

$$\left.\begin{aligned} &(\Phi'^* p_k \Phi') = 0, \quad (\Phi'^* p_{k'} p_k \Phi') = -f^2(k)\delta^3_{k',-k}, \\ &(\Phi'^* q_k \Phi') = 0, \quad (\Phi'^* q_{k'} q_k \Phi') = -\frac{\hbar^2}{4f^2(k)} \delta^3_{k',-k}. \end{aligned}\right\} \tag{32}$$

As is easily seen from the discussion of (25), the state vector satisfying (32) is given by

$$a_k \Phi' = 0 \quad (\text{for } k > k_c). \tag{33}$$

That is to say, the state vector for $k>k_c$ represents the vacuum for the auxiliary field and accordingly, not any auxiliary field quantum of $k>k_c$ is excited.

On the other hand, in order to consider the meaning of (31), we shall find it convenient to decompose Ω_k as $\Omega_k=\Omega_k^{(-)}+\Omega_{-k}^{(+)}$, where

$$\left.\begin{aligned}\Omega_k^{(-)}&=if(k)\{a_k+(1/2|\boldsymbol{k}|f(k))\rho_k\},\\ \Omega_k^{(+)}&=if(k)\{a_k^*+(1/2|\boldsymbol{k}|f(k))\rho_{-k}\},\end{aligned}\right\} \tag{34}$$

and there stand the relationships

$$\left.\begin{aligned}&[\Omega_k^{(+)},\ \Omega_{k'}^{(-)}]_-=f^2(k)[a_{k'},\ a_k^*]_-=f^2(k)\delta^3_{k,k'},\\ &(\Omega_k^{(+)})^*=-\Omega_k^{(-)},\ \ (\Omega_k^{(-)})^*=-\Omega_k^{(+)}.\end{aligned}\right\} \tag{35}$$

Then we can easily obtain the relations $(\Phi'^*\Omega_k^{(-)}\Phi')=0$ and $(\Phi'^*\Omega_k^{(+)}\Phi')=0$ by taking both (31–1) and (31–3) into consideration. As $(\Phi'^*\Omega_k^{(+)}\Phi')=0$ is the Hermite conjugate expression of $(\Phi'^*\Omega_k^{(-)}\Phi')=0$, it is sufficient to take only $(\Phi'^*\Omega_k^{(-)}\Phi')=0$ as the subsidiary condition which is equivalent to (31–1) and (31–3), that is,

$$(\Phi'^*\Omega_k^{(-)}\Phi')=0\ \ (\text{for } k<k_c). \tag{36}$$

Now, we can easily verify that the solution satisfying (36), (31–2) and (31–4) is uniquely determined by the simple expression,

$$\Omega_k^{(-)}\Phi'=0\ \ (\text{for } k<k_c). \tag{37}$$

This is an expression corresponding to (33).

The state vector which satisfies the subsidiary conditions (33) and (37) is given by

$$\Phi'=\exp\left[-\frac{1}{2}\sum_{k<k_c}{}'\left(\frac{1}{2|\boldsymbol{k}|f(k)}\right)^2\rho_{-k}\rho_k-\sum_{k<k_c}\left(\frac{1}{2|\boldsymbol{k}|f(k)}\right)\rho_k a_k^*\right]\Phi_0\cdot\chi(\psi), \tag{38}$$

where Φ_0 is the eigen-vector of $a_k^*a_k$ with eigen-value zero, say, the eigen-vector of the vacuum for the auxiliary field. Φ' of (38) explicitly illustrates that the auxiliary field quanta are excited in the region of $k<k_c$, though not in $k>k_c$.

Now, the total Hamiltonian (19) is transformed as follows by the canonical transformation (30) (cf. Appendix I) :

$$\begin{aligned}\mathcal{H}'=T_1^{-1}\mathcal{H}T_1=&\sum_k^{\sigma}\frac{\hbar^2k^2}{2m}\psi^*_{\sigma,k}\psi_{\sigma,k}-\frac{\hbar}{m\sqrt{V}}\sum_{k'<k_c}^{k}(\boldsymbol{k}-\boldsymbol{k}'/2)\cdot\boldsymbol{\varepsilon}_{k'}q_{k'}\psi^*_{\sigma,k}\psi_{\sigma,k-k'}\\ &+\frac{1}{2m\sqrt{V}}\sum_{k,k'<k_c}(\boldsymbol{\varepsilon}_k,\boldsymbol{\varepsilon}_{k'})q_kq_{k'}\rho_{-k-k'}-\frac{1}{2}\sum_{k<k_c}k^2J(k)p_{-k}p_k\\ &+\frac{1}{2}\sum_{k<k_c}k^2J(k)(\Omega_{-k}\Omega_k-2i/|\boldsymbol{k}|\cdot\rho_{-k}\Omega_k)\\ &+\frac{1}{2}\sum_{k>k_c}J(k)\rho_{-k}\rho_k-\frac{N}{2V}\sum_kJ(k).\end{aligned} \tag{39}$$

Here, it should be noticed that this expression is not approximated and is independent of the auxiliary field variables for $k > k_c$. If we calculate the expectation value of (39), $(\Psi'^* \mathcal{H}' \Psi')$, by using (38), we can show that this coincides with the momentum representation of (1). However, at this time, we do not necessarily have to emphasize the equivalence, because the equivalence between (19) (or (12)) and (1) was already verified in the previous section.

We easily find that the part derived by putting $\boldsymbol{k}' = -\boldsymbol{k}$ in the third term of (39) is independent of the particle coordinates, and that those plus the fourth term of (39) are the energies of a harmonic oscillator defined by

$$-\frac{1}{2m\sqrt{V}} \sum_{k<k_c} \rho_0 q_{-k} q_k - \frac{1}{2} \sum_{k<k_c} k^2 J(k) p_{-k} p_k . \tag{40}$$

Here, when the relationship $\rho_0 = N/\sqrt{V}$ is taken into account, the frequency of this harmonic oscillation is found to be

$$\nu_k = (Nk^2 J(k)/mV)^{1/2}, \tag{41}$$

which is the characteristic frequency of the collective oscillation.

Now, we shall find it convenient for further study to rewrite Hamiltonian (39) in the following form by using the transformation of (27) and the definition $f^2(k) = \hbar\omega_k/2k^2 J(k) = N\hbar\omega_k/2Vm\nu_k^2$, the definition of which is conveniently taken to draw out the higher-order collective oscillation of frequency ω_k, though the lowest-order frequency is ν_k as given by (41). The rewritten form of the Hamiltonian is

$$\mathcal{H}' = T + \mathcal{H}_{coll} + \mathcal{H}_{sr} + \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_c' + W - \frac{N}{2V} \sum_{k<k_c} J(k), \tag{42a}$$

where,

$$\begin{aligned}
T &= \sum_k^{\sigma} \frac{\hbar^2 k^2}{2m} \psi^*_{\sigma,k} \psi'_{\sigma,k}, \\
\mathcal{H}_{coll} &= \sum_{l<k_c} \hbar\omega_l \left(a_l^* a_l + \frac{1}{2} \right), \\
\mathcal{H}_{sr} &= \frac{1}{2} \sum_{p>k_c} J(p) \left(\rho_{-p} \rho_p - \frac{N}{V} \right), \\
\mathcal{H}_1 &= \frac{1}{2} \sum_{l<k_c} \tau(l) \hbar\omega_l (a_{-l} - a_l^*) \sum_k^{\sigma} \alpha(l, k) \psi^*_{\sigma,k} \psi_{\sigma,k+l}, \\
\tau(l) &= \left(\frac{2m\nu_l^2}{N\hbar l^2 \omega_l} \right)^{1/2}, \quad \alpha(l, k) = \frac{\hbar(\boldsymbol{l}, \boldsymbol{k} + \boldsymbol{l}/2)}{m\omega_l}, \\
\mathcal{H}_2 &= \frac{1}{2} \sum_{l<k_c} l^2 J(l) \left\{ Q_{-l} Q_l - \frac{2i}{|\boldsymbol{l}|} \rho_{-l} Q_l \right\}, \\
\mathcal{H}_c' &= \sum_{l<k_c} \frac{\hbar(\nu_l^2 - \omega_l^2)}{4\omega_l} (a_l^* a_l + a_l a_l^* - a_l a_{-l} - a_{-l}^* a_l^*),
\end{aligned} \tag{42b}$$

$$W=\frac{\hbar\sqrt{V}}{4N}\sum_{\substack{l,\,l'<k_c\\ l'\neq -l}}\frac{\nu_l\nu_{l'}}{\sqrt{\omega_l\omega_{l'}}}(\boldsymbol{\varepsilon}_l,\boldsymbol{\varepsilon}_{l'})(a_l-a^*_{-l})(a_{l'}-a^*_{-l'})\rho_{-l-l'}.$$

At the end of this section, we refer to the necessary condition for the existence of the collective oscillation. As is easily seen from the expression of ν_k, (41), the collective oscillation does not exist unless $J(k)$ is positive-definite for the region $k<k_c$, although it is not necessary for $J(k)$ to be positive for the region $k>k_c$. The case where G is wholly attractive should be excluded from the present consideration. Even though G contains an attractive part, if it contains a strongly repulsive part at the same time, the case where $J(k)$ is positive-definite for the region $k<k_c$ may exist. Where $J(k)$ can be expanded in powers of k for the region $k<k_c$, $J(k)=C_0+C_2k^2+C_4k^4+\cdots(|C_0|\gg|C_2|\gg|C_4|\gg\cdots)$, the condition $C_0>0$ should be satisfied at least for the above circumstances. Though we developed here the discussions based on the expressions of ν_k, the essence of the present discussion will be retained also in the coming collective representation, as will be seen in expression (47) for the higher-order frequency ω_k.

§ 5. Collective representation

In this section, the collective representation will be investigated, in which $\mathcal{H}_1$ of (42) will be eliminated and the total Hamiltonian will be separated into two independent parts, one for a set of particles moving at random in a field of screened potential, and the other for the collective phonons of frequency ω_k. At the same time in this representation, the subsidiary condition will also be separated with respect to the particle variables and the phonon variables.

The canonical transformation eliminating $\mathcal{H}_1$ from (42) is given by $\Phi'=T_2\Psi'=\exp(S_2)\Psi'$, where the generating function S_2 is

$$S_2=\frac{1}{2}\sum_{l<k_c}\tau(l)\sum_k^{\sigma}\left\{\frac{\alpha(l,k)}{1+\alpha(l,k)}a_{-l}+\frac{\alpha(l,k)}{1-\alpha(l,k)}a_l^*\right\}\psi^*_{\sigma,k}\psi_{\sigma,k+l}. \tag{43}$$

The transformation T_2 is the Bohm-Pines transformation itself, and there stands the relationship

$$[T+\mathcal{H}_{coll},\,S_2]_-=-\mathcal{H}_1. \tag{44}$$

For this generating function, the expansion parameter for the perturbation series derived by T_2 is given by

$$\bar{\alpha}^2=\left\langle\left(\frac{\hbar\boldsymbol{l}\cdot\boldsymbol{k}}{m\omega_l}\right)^2\right\rangle_{av}, \tag{45}$$

where the average should be taken over the particle momenta $\hbar\boldsymbol{k}$ and the collective field wave vectors $\boldsymbol{l}$. In the practical calculations, the condition $\bar{\alpha}^2\ll 1$ is assumed and the effects higher than the order of $\bar{\alpha}^3$ will be neglected. (Hereafter, we call this the $\bar{\alpha}^2$-approximation.)

First of all, to derive from (31), the new expressions of the subsidiary condition for $l<k_c$, we have to calculate the quantities $(T_2^{-1}\Omega_l T_2)$ and $(T_2^{-1}q_l T_2)$. After straightforward calculations we shall obtain

$$T_2^{-1}\Omega_l T_2=i\sqrt{2}f(l)\xi_l,\quad T_2^{-1}q_l T_2=\frac{\hbar}{2f(l)}(a_l-a_{-l}^*+i\sqrt{2}\pi_{-l}), \tag{46a}$$

where

$$\left.\begin{aligned}\xi_l&=\frac{\tau(l)}{\sqrt{2}}\sum_k^{\sigma}\frac{1}{1-\alpha^2(l,k)}\psi_{\sigma,k}^*\psi_{\sigma,k+l},\\ \pi_l&=\frac{\tau(l)}{i\sqrt{2}}\sum_k^{\sigma}\frac{\alpha(-l,k)}{1-\alpha^2(-l,k)}\psi_{\sigma,k}^*\psi_{\sigma,k-l}.\end{aligned}\right\} \tag{46b}$$

(46a) are expressions in the $\bar{\alpha}^2$-approximation, and are derived by using the following dispersion relation in the process of the calculations (cf. Appendix II) :

$$1=\frac{\nu_l^2}{N}\sum_k^{\sigma}\frac{\psi_{\sigma,k}^*\psi_{\sigma,k}}{\{\omega_l-(\hbar/m)(\boldsymbol{l},\boldsymbol{k})\}^2-(\hbar l^2/2m)^2}\quad(\text{for } l<k_c). \tag{47}$$

Now we introduce the new operators $\Gamma_l^{(-)}$ and $\Gamma_l^{(+)}(=(\Gamma_l^{(-)})^*)$ in the definition,

$$\left.\begin{aligned}\Gamma_l^{(-)}&=\frac{1}{\sqrt{2}}(i\pi_{-l}+\xi_l)=\frac{\tau(l)}{2}\sum_k^{\sigma}\frac{\psi_{\sigma,k}^*\psi_{\sigma,k+l}}{1-\alpha(l,k)},\\ \Gamma_l^{(+)}&=\frac{1}{\sqrt{2}}(-i\pi_l+\xi_{-l})=\frac{\tau(l)}{2}\sum_k^{\sigma}\frac{\psi_{\sigma,k}^*\psi_{\sigma,k-l}}{1+\alpha(-l,k)}.\end{aligned}\right\} \tag{48}$$

Then we can show that there stand the relationships

$$[\Gamma_l^{(-)},\Gamma_{l'}^{(-)}]_-=\delta_{l,l'}^3,\ [\Gamma_l^{(-)},\Gamma_{l'}^{(-)}]_-=[\Gamma_l^{(+)},\Gamma_{l'}^{(-)}]_-=0, \tag{49a}$$

$$[\pi_l,\xi_{l'}]_-=-i\delta_{l,l'}^3,\ [\pi_l,\pi_{l'}]_-=[\xi_l,\xi_{l'}]_-=0. \tag{49b}$$

These commutation relations approximately stand under the random phase approximation; for example, we can show for an arbitrary function $F(l)$ the following relationship,

$$\begin{aligned}\sum_{l'}F(l')[\Gamma_{l'}^{(-)},\Gamma_l^{(+)}]_-&=\frac{1}{4}\sum_{l'}\tau(l)\tau(l')F(l')\sum_k^{\sigma}\psi_{\sigma,k}^*\psi_{\sigma,k+l'-l}\\ &\times\left[\frac{1}{\{1+\alpha(-l,l'+k)\}\{1-\alpha(l',k)\}}-\frac{1}{\{1+\alpha(-l,k)\}\{1-\alpha(l',-l+k)\}}\right]\\ &\simeq F(l)\ (\simeq\sum_{l'}[\Gamma_l^{(-)},\Gamma_{l'}^{(+)}]_-F(l')).\end{aligned} \tag{50}$$

Thus, by using (46) and (49), we can verify that the solution of the new subsidiary conditions transformed from (31) by T_2 is given uniquely by

$$\left[\frac{1}{2}(a_l-a_{-l}^*)+\Gamma_l^{(-)}\right]\Psi'=0\quad(\text{for } l<k_c). \tag{51}$$

On the other hand, this is the same as the new expression which is directly transformed from (37) by T_2.

Kanazawa[10] has said that there is no normalizable state vector Ψ' which satisfies (51), but we can actually obtain the following normalizable Ψ' satisfying (51) :

$$\left.\begin{aligned} &\Psi'=T_3\Psi=\exp(S_3)\Psi, \\ &S_3=\frac{1}{\sqrt{2}}\sum_{l<k_c}(a_l^*-a_{-l})\xi_l, \quad (S_3^*=-S_3), \end{aligned}\right\} \tag{52}$$

where Ψ satisfies the equation

$$\Gamma_l^{(-)}\Psi=\frac{1}{\sqrt{2}}(i\pi_{-l}+\xi_l)\Psi=0 \quad (\text{for } l<k_c). \tag{53}$$

The expressions (51), (52) and (53) show that the subsidiary condition in the state $\Psi'(=T_2^{-1}\Phi')$, namely in the Bohm-Pines representation, is not separated with respect to the particle variables and the collective field variables, but the one in the state $\Psi(=T_3^{-1}\Psi'=T_3^{-1}T_2^{-1}\Phi')$ can be separated as given by (53). This indicates at the same time that we should add the third canonical transformation T_3 defined by (52) upon the Bohm-Pines transformation T_2 in order to obtain satisfactory results. In other words, we can show that the subsidiary condition (53) is obtained as the unique solution of the following conditions transformed from (31) by $\Phi'=T_2T_3\Psi$,

$$\left.\begin{aligned} &(\Psi^*\xi_l\Psi)=0, \quad (\Psi^*\xi_{l'}\xi_l\Psi)=\frac{1}{2}\delta^3_{l',-l}, \\ &(\Psi^*\pi_l\Psi)=0, \quad (\Psi^*\pi_{l'}\pi_l\Psi)=\frac{1}{2}\delta^3_{l',-l}. \end{aligned}\right\} \tag{54}$$

These expressions are evident from (46) and (49), that is, we can easily obtain the relations

$$T_3^{-1}T_2^{-1}\Omega_lT_2T_3=i\sqrt{2f(l)}\xi_l, \quad T_3^{-1}T_2^{-1}q_lT_2T_3=\frac{i\hbar}{\sqrt{2f(l)}}\pi_{-l}. \tag{55}$$

On the other hand, the subsidiary condition (33) for $k>k_c$ is retained also in the state Ψ, because a_k for $k>k_c$ is commutable with both the generating operators S_2 and S_3, that is

$$a_k\Psi=0 \quad (\text{for } k>k_c). \tag{56}$$

Now, when we obtain the expressions of the new Hamiltonian transformed from $\mathcal{H}'$ of (42), we first neglect the term W in $\mathcal{H}'$ as very small, after the random phase approximation. After straightforward but tedious calculations (cf. Appendix II), we arrive at the following results,

$$H=T_3^{-1}T_2^{-1}\mathcal{H}'T_2T_3=H_{coll}+H_p-N/2V\sum_{l<k_c}J(l), \tag{57a}$$

where

$$H_{coll}=\sum_{l<k_c}\hbar\omega_l(a_l^* a_l+1/2),$$

$$H_p=T+H_p^c+H_{sr},$$

$$T=\sum_k^{\sigma}\frac{\hbar^2k^2}{2m}\psi^*_{\sigma,k}\psi_{\sigma,k}, \tag{57b}$$

$$H_p^c=\frac{1}{4}\sum_{l<k_c}\tau^2(l)\hbar\omega_l\sum_{k,k'}^{\sigma,\mu}\frac{\alpha(-l,k)\alpha(l,k')}{1-\alpha^2(l,k')}\psi^*_{\sigma,k}\psi_{\sigma,k-l}\psi^*_{\mu,k'}\psi_{\mu,k'+l}$$

$$-\sum_{l<k_c}\hbar\omega_l\{\hat{\xi}_{-l}-\sqrt{V/2}\,\tau(l)\rho_{-l}\}\hat{\xi}_l,$$

$$H_{sr}=\frac{1}{2}\sum_{p>k_c}J(p)(\rho_{-p}\rho_p-N/V).$$

These expressions were obtained by the $\bar{\alpha}^2$- and random phase approximation. And moreover, the dispersion relation (47) was used in the process of the calculations. (The subsidiary condition has not yet been taken into account for the expression of this Hamiltonian.) As is easily seen from the result, H_{coll} is expressed only by the collective field variables, and H_p only by the particle variables.

§ 6. Meaning of the subsidiary condition

In this section, we first consider the meaning of the subsidiary condition (53), and then reduce the total Hamiltonian (57) by taking the subsidiary condition into account.

The commutation relations (49) mean that $\hat{\xi}_l$ and π_l are the canonically conjugate variables of a Bose field, and that $\Gamma_l^{(-)}$ and $\Gamma_l^{(+)}$ are respectively the annihilation and the creation operators of the Bose field quanta. Then (53) (or (54)) indicates that the Bose field is in the ground state (or the vacuum state); namely the state vector satisfying (53) is given by the following form in the representation where $\hat{\xi}_l$ is diagonal:

$$\Gamma_l^{(-)}\Xi_0=\frac{1}{\sqrt{2}}\left(\frac{\delta}{\delta\hat{\xi}_{-l}}+\hat{\xi}_l\right)\Xi_0=0,\quad \Xi_0=C\cdot\exp\left[-\frac{1}{2}\sum_{l<k_c}\hat{\xi}_{-l}\hat{\xi}_l\right], \tag{58}$$

where C is the normalization constant.

Here, to clarify the meaning of the subsidiary condition (58), it is convenient to notice that there stand the following relationships for the original Hamiltonian $\bar{H}_0$ of (1);

$$[\bar{H}_0,\Gamma_l^{(-)}]_-=-\hbar\omega_l\Gamma_l^{(-)},\quad [\bar{H}_0,\Gamma_l^{(+)}]_-=\hbar\omega_l\Gamma_l^{(+)}. \tag{59a}$$

Accordingly, for the original Hamiltonian, we can immediately derive the equation of motion,

$$\ddot{\hat{\xi}}_l+\omega_l^2\hat{\xi}_l=0\ \ (\text{for } l<k_c). \tag{59b}$$

That is, $\tilde{\xi}_l(l<k_c)$ is exactly the collective coordinate in the original representation where the auxiliary field variables are not introduced. In the present theories utilizing the auxiliary field variables, however, the collective oscillation is described by the other collective coordinate $q_l(l<k_c)$; in other words, q_l stands for $\tilde{\xi}_l$(or, a_l and a_l^* stand respectively for $\Gamma_l'^{(-)}$ and $\Gamma_l'^{(+)}$). These circumstances are expressed mathematically by the following relationship for Hamiltonian (57) ;

$$[H, a_l]_-=-\hbar\omega_l a_l,\ [H, a_l^*]_-=\hbar\omega_l a_l^*, \tag{60a}$$

accordingly

$$\ddot{q}_l+\omega_l^2 q_l=0 \ \text{(for } l<k_c). \tag{60b}$$

In the collective representation in which the collective coordinate q_l is introduced and the collective energy is expressed stably by H_{coll}, the freedom of the collective coordinate expressed directly in terms of the original particle variables, say $\tilde{\xi}_l$, should be eliminated. The elimination of $\tilde{\xi}_l$ is assured by the subsidiary condition (58). Thus subsidiary condition (58) does not indicate that the collective oscillation is always in the vacuum state, but only that one of the double freedoms for the collective coordinates is eliminated. This is a very natural result. It should be noticed here that our operators $I'^{(-)}$ and $I'^{(+)}$ correspond respectively to the operators A and A^* defined by Sawada-Brueckner-Fukuda-Brout[9] in their theory of the electron gas; A and A^* are respectively the annihilation and creation operator for a plasmon.

Now, the state vector Ψ of the collective representation is given by the following separated form;

$$\Psi=\Xi_0 ZQ, \tag{61}$$

where Q is the function of only the collective field variable $q_l(l<k_c)$. When we define the internal coordinate of the individual particles by ζ, Z is a function of ζ. Though Z may as well depend upon the $\tilde{\xi}$-coordinate, we shall be able to show later that Hamiltonian (57) does not contain the intersecting terms between $\tilde{\xi}$ and ζ. Thus Z is a function of only the internal coordinate ζ.

As $\Xi_0(\tilde{\xi})$ is explicitly given by (58), we first take the expectation value of Hamiltonian (57) with respect to Ξ_0. In the calculations of $(\Xi_0^* H\Xi_0)$ we can naturally take the subsidiary condition into account.

First of all, it is evident that $(\Xi_0^* H_{coll}\Xi_0)=H_{coll}$. Secondly we can easily see that $(\Xi_0^* H_{sr}\Xi_0)=H_{sr}$, because there stand the following relations under the same circumstance as (49) :

$$[\Gamma_l^{(-)}, \rho_{-k}]_-=[\rho_k, \Gamma_l^{(+)}]_-=1/\sqrt{V}\tau(l)\cdot\delta^s_{l,k}, \tag{62a}$$

accordingly

$$[\tilde{\xi}_l, \rho_k]_-=0 \ \text{and}\ [i\pi_l, \rho_k]_-=1/\sqrt{V/2}\,\tau(l)\cdot\delta^s_{l,k}. \tag{62b}$$

Thirdly, we can show that

$$(\Xi_0^* H_p^c \Xi_0) = -\frac{1}{4}\sum_{l<k_c}\hbar\omega_l, \tag{63}$$

because the terms contained in H_p^c can be rewritten as follows:

$$\frac{1}{4}\sum_{l<k_c}\tau^2(l)\hbar\omega_l\sum_{k,k'}^{\sigma,\mu}\frac{\alpha(-l,k)\alpha(l,k')}{1-\alpha^2(l,k')}\psi^*_{\sigma,k}\psi_{\sigma,k-l}\psi^*_{\mu,k'}\psi_{\mu,k'+l}$$

$$=\frac{1}{4}\sum_{l<k_c}\tau(l)\hbar\omega_l\sum_{k}^{\sigma}\alpha(-l,k)\psi^*_{\sigma,k}\psi_{\sigma,k-l}(\Gamma_l^{(-)}-\Gamma_{-l}^{(+)})$$

$$=\frac{1}{4}\sum_{l<k_c}\tau(l)\hbar\omega_l\{\sum_{k}^{\sigma}\alpha(-l,k)\psi^*_{\sigma,k}\psi_{\sigma,k-l}\Gamma_l^{(-)}-\Gamma_{-l}^{(+)}\sum_{k}^{\sigma}\alpha(-l,k)\psi^*_{\sigma,k}\psi_{\sigma,k-l}\}$$

$$-\frac{1}{4}\sum_{l<k_c}\hbar\omega_l,$$

$$\sum_{l<k_c}\hbar\omega_l\{\xi_{-l}-\sqrt{V/2}\,\tau(l)\rho_{-l}\}\xi_l=\sum_{l<k_c}\hbar\omega_l\xi_{-l}\xi_l$$

$$-\frac{\sqrt{V}}{2}\sum_{l<k_c}\tau(l)\hbar\omega_l\{\rho_{-l}\Gamma_l^{(-)}+\Gamma_{-l}^{(+)}\rho_{-l}\}-\frac{1}{2}\sum_{l<k_c}\hbar\omega_l.$$

Finally, we have to calculate the expectation value of the kinetic energy T, $(\Xi_0^* T \Xi_0)$. Though this is rather more complicated than the above, we can obtain the explicit result by using Tomonaga's pretty treatment.[6] That is, we expand T in a power series of π_l and ξ_l. When we notice the relationship

$$[\xi_l, T]_- = i\hbar\omega_l\pi_{-l}, \tag{64}$$

we can express T as

$$T=T_{in}+\frac{1}{2}\sum_{l<k_c}\hbar\omega_l\pi_{-l}\pi_l, \quad T_{in}=T-\frac{1}{2}\sum_{l<k_c}\hbar\omega_l\pi_{-l}\pi_l. \tag{65}$$

Here, T_{in} no longer contains π_l, but still contains ξ_l, because we have

$$[i\pi_l, T_{in}]_- = [i\pi_l, T]_- = \hbar\omega_l\{\xi_{-l}-\sqrt{V/2}\,\tau(l)\rho_{-l}\}. \tag{66}$$

Then we expand T_{in} in a power series of ξ_l as follows, noticing the relation $[i\pi_l, T_{in}]_- = \delta T_{in}/\delta\xi_l$,

$$T_{in}(\xi) = T_{in}|_{\xi=0} + \sum_{l<k_c}[i\pi_l, T_{in}]_-|_{\xi=0}\xi_l + \cdots. \tag{67}$$

Here, $T_{in}|_{\xi=0}$ and $[i\pi_l, T_{in}]_-|_{\xi=0}$, etc., depend only on the internal coordinate ζ. When we consider $[i\pi_{l'}, [i\pi_l, T_{in}]_-]_- = 0$, the terms above the third term in the right-hand side of (67) vanish and $T_{in}|_{\xi=0}$ is given by

$$T_{in}|_{\xi=0} = T_{in}(\xi) - \sum_{l<k_c}[i\pi_l, T_{in}]_-\xi_l$$

$$= T-\frac{1}{2}\sum_{l<k_c}\hbar\omega_l\pi_{-l}\pi_l-\sum_{l<k_c}\hbar\omega_l\{\xi_{-l}-\sqrt{V/2}\,\tau(l)\rho_{-l}\}\xi_l, \tag{68}$$

which, when expressed in terms of the original particle variables, depends only on the internal coordinate; that is,

$$T_{in}|_{\xi=0}=\sum_{k}^{\sigma}\frac{\hbar^2k^2}{2m}\psi^*_{\sigma,k}\psi_{\sigma,k}+H_{lr}, \tag{69}$$

$$H_{lr}=-\frac{1}{4}\sum_{l<k_c}\tau^2(l)\hbar\omega_l\sum_{k,k'}^{\sigma,\mu}\frac{\alpha(l,k')\alpha(l,k+2k')}{\{1-\alpha^2(-l,k)\}\{1-\alpha^2(l,k')\}}\psi^*_{\sigma,k}\psi_{\sigma,k-l}\psi^*_{\mu,k'}\psi_{\mu,k'+l}.$$

Thus, the expectation value $(\Xi_0{}^*T\Xi_0)$ is given by

$$(\Xi_0{}^*T\Xi_0)=T_{in}|_{\xi=0}+\frac{1}{4}\sum_{l<k_c}\hbar\omega_l, \tag{70}$$

because the expectation value of the second term of (67) vanishes according to the subsidiary conditions, which means that there is no intersection between ξ and ζ. Here, on the other hand, we may notice that the second term of (67) cancels with the second term of H_p^c (57b), and that the same results as the above are obtained.

From the above considerations, we obtain the following final expressions,

$$(\Xi_0{}^*H\Xi_0)=H_{eff}=H_{coll}+H_{pair}, \tag{71a}$$

where

$$\begin{aligned}
H_{coll}&=\sum_{l<k_c}\hbar\omega_l(a_l{}^*a_l+1/2),\\
H_{part}&=T+H_{lr}+H_{sr},\\
T&=\sum_{k}^{\sigma}\frac{\hbar^2k^2}{2m}\psi^*_{\sigma,k}\psi_{\sigma,k},\\
H_{lr}&=-\frac{1}{4}\sum_{l<k_c}\tau^2(l)\hbar\omega_l\sum_{k,k'}^{\sigma,\mu}\frac{\alpha(l,k')\alpha(l,k+2k')}{\{1-\alpha^2(-l,k)\}\{1-\alpha^2(l,k')\}}\\
&\quad\times\psi^*_{\sigma,k}\psi_{\sigma,k-l}\psi^*_{\mu,k'}\psi_{\mu,k'+l}-\frac{N}{2V}\sum_{l<k_c}J(l),\\
H_{sr}&=\frac{1}{2V}\sum_{p>k_c}J(p)\sum_{k,k'}^{\sigma,\mu}\psi^*_{\sigma,k}\psi_{\sigma,k-p}\psi^*_{\mu,k'}\psi_{\mu,k'+p}-\frac{N}{2V}\sum_{p>k_c}J(p).
\end{aligned} \tag{71b}$$

The subsidiary condition is entirely taken into account in these expressions, and H_{part} is described in terms of the internal coordinate only. The latter circumstance is evident also from the fact that $[H_{part},\xi_l]_-=[H_{part},\pi_l]_-=0$. H_{lr} is the screened long-range correlation energy and H_{sr} the screened short-range correlation energy. The state vector for Hamiltonian (71) is now given by $Z(\zeta)Q(q)$. Here, we should not forget the dispersion relation (47), which is independent of the subsidiary condition, as the operator of the right-hand side commutes approximately with ξ_l and π_l.

§ 7. Concluding remarks

Heretofore we were able to obtain a desirable result of the collective representation using the auxiliary field variables, though the result was under $\bar{\alpha}^2$-approximation. The important points of our results are the following: (1) A strict method of treatment of the subsidiary condition for the auxiliary field was proposed; accordingly, the meaning of the subsidiary condition was clarified, and the state vector which satisfies the subsidiary condition was obtained explicitly. (2) As the subsidiary condition was entirely taken into account in the total Hamiltonian, we could investigate behaviours of the system not only in the ground state but also in an excited state. (3) The result can be applied for the many-particle system interacting through not only the Coulomb force but also an arbitrary two-body force. Moreover, the result is applicable to both the Bose and Fermi particle system.

It is our future problem to solve the practical problem by using the obtained Hamiltonian (71), which itself will be a fairly laborious work.

Finally, we should like to notice the following possibility; that is, we can develope the collective description as well, starting from the original Hamiltonian (1) and using ξ_l from the beginning as the collective coordinate, though usually ρ_l is used as the collective coordinate.

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Appendix I

We illustrate the process of deriving expression (39) from expression (19). Hamiltonian (19) is

$$\mathcal{H}=T+\mathcal{H}_1+\mathcal{H}_2+\mathcal{H}_3+\mathcal{H}_4,$$

where

$$\begin{aligned}
T&=\sum_{k}^{\sigma}\frac{\hbar^2k^2}{2m}\psi^*_{\sigma,k}\psi_{\sigma,k},\\
\mathcal{H}_1&=-\frac{\hbar}{m\sqrt{V}}\sum_{k,k'}^{\sigma}(\boldsymbol{k}-\boldsymbol{k}'/2)\cdot\boldsymbol{\varepsilon}_{k'}q_{k'}\psi^*_{\sigma,k}\psi_{\sigma,k-k'},\\
\mathcal{H}_2&=\frac{1}{2m\sqrt{V}}\sum_{k,k'}(\boldsymbol{\varepsilon}_k,\boldsymbol{\varepsilon}_{k'})q_kq_{k'}\rho_{-k-k'},\qquad\text{(A·1)}\\
\mathcal{H}_3&=-\frac{1}{2}\sum_{k}k^2J(k)\left(p_{-k}p_k-\Omega_{-k}\Omega_k+\frac{2i}{|\boldsymbol{k}|}\rho_{-k}\Omega_k\right),\\
\mathcal{H}_4&=-\frac{N}{2V}\sum_{k}J(k).
\end{aligned}$$

Applying the canonical transformation (30) to this Hamiltonian, we obtain the following relations by using formula (24) :

$$[T, S_1]_- = \frac{\hbar}{2m} \sum_{\substack{k \\ k' > k_c}}^{\sigma} \frac{k^2}{|k'|} q_{k'} [\psi^*_{\sigma,k} \psi_{\sigma,k}, \rho_{-k'}]_-$$

$$= \frac{\hbar}{m\sqrt{V}} \sum_{\substack{k \\ k' > k_c}}^{\sigma} (k - k'/2) \cdot \varepsilon_{k'} q_{k'} \psi^*_{\sigma,k} \psi_{\sigma,k-k'}, \qquad (A \cdot 2)$$

$$[\mathcal{H}_1, S_1]_- = -\frac{1}{m\sqrt{V}} \sum_{\substack{k, k' \\ k' > k_c}}^{\sigma} (k - k'/2) \cdot \varepsilon_{k'} \frac{q_{k'} q_{k''}}{|k''|} [\psi^*_{\sigma,k} \psi_{\sigma,k-k'}, \rho_{-k''}]_-$$

$$= -\frac{1}{m\sqrt{V}} \sum_{\substack{k \\ k' > k_c}} (\varepsilon_k, \varepsilon_{k'}) q_k q_{k'} \rho_{-k-k'}, \qquad (A \cdot 3)$$

$$[\mathcal{H}_2, S_1]_- = 0,$$

$$[\mathcal{H}_3, S_1]_- = \frac{1}{2\hbar} \sum_{\substack{k \\ k' > k_c}} \frac{k^2}{|k'|} J(k) \rho_{-k'} \left[q_{k'}, \left(p_{-k} p_k - \Omega_{-k} \Omega_k + \frac{2i}{|k|} \rho_{-k} \Omega_k \right) \right]_-$$

$$= i \sum_{k > k_c} |k| J(k) \rho_k \left(p_k - \Omega_{-k} + \frac{i}{|k|} \rho_{-k} \right) = 0, \qquad (A \cdot 4)$$

$$[\mathcal{H}_4, S_1]_- = 0,$$

$$\frac{1}{2!} [[T, S_1]_-, S_1]_- = \frac{1}{2m\sqrt{V}} \sum_{k, k' > k_c} (\varepsilon_k, \varepsilon_{k'}) q_k q_{k'} \rho_{-k-k'}, \qquad (A \cdot 5)$$

and all other commutators $= 0$.

When we sum up the expressions $(A \cdot 1) \sim (A \cdot 5)$ to obtain the new Hamiltonian (39), we use in $(A \cdot 1)$ the following re-arranged form of $\mathcal{H}_3$,

$$\mathcal{H}_3 = \frac{1}{2} \sum_{k > k_c} J(k) \rho_{-k} \rho_k - \frac{1}{2} \sum_{k < k_c} k^2 J(k) \left(p_{-k} p_k - \Omega_{-k} \Omega_k + \frac{2i}{|k|} \rho_{-k} \Omega_k \right). \qquad (A \cdot 6)$$

Appendix II

Here, we obtain the expressions of the subsidiary condition and the total Hamiltonian in the collective representation, which are transformed by T_2 and T_3.

First of all, under the $\tilde{\alpha}^2$-approximation, we obtain

$$T_2^{-1} a_l T_2 = a_l + [a_l, S_2]_- = a_l + \frac{\tau(l)}{2} \sum_k^{\sigma} \frac{\alpha(l, k)}{1 - \alpha(l, k)} \psi^*_{\sigma,k} \psi_{\sigma,k+l}, \qquad (A \cdot 7)$$

accordingly,

$$T_2^{-1} q_l T_2 = \frac{\hbar}{2f(l)} T_2^{-1} (a_l - a^*_{-l}) T_2 = \frac{\hbar}{2f(l)} (a_l - a^*_{-l} + i\sqrt{2}\pi_{-l}), \qquad (A \cdot 8)$$

where π_{-l} is defined by (46b). On the other hand, there stands

$$T_2^{-1}\rho_k T_2=\rho_k+[\rho_k, S_2]_-=\begin{cases}\rho_k & (k>k_c),\\ \rho_k-|\boldsymbol{k}|f(k)(a_k+a_{-k}^*) & (k<k_c),\end{cases} \tag{A·9}$$

where the expression for $k<k_c$ is obtained by using dispersion relation (47). Then we obtain (for $l<k_l$)

$$T_2^{-1}\Omega_l T_2=if(l)T_2^{-1}\left[a_l+a_{-l}^*+\frac{1}{|\boldsymbol{l}|f(l)}\rho_l\right]T_2=i\sqrt{2}f(l)\xi_l, \tag{A·10}$$

where ξ_l is defined by (46b). Next, by using the commutation relations for (ξ, π) and (a, a^*), the following expressions are obtained precisely:

$$\left.\begin{aligned}T_3^{-1}T_2^{-1}q_l T_2 T_3&=\frac{i\hbar}{\sqrt{2}f(l)}\pi_{-l},\\ T_3^{-1}T_2^{-1}\Omega_l T_2 T_3&=i\sqrt{2}f(l)\xi_l.\end{aligned}\right\} \tag{A·11}$$

Thus expression (54) is obtained.

Now, let us proceed to the calculations of the transformed Hamiltonian. The total Hamiltonian after transformation T_1 is given by (42). When it is transformed by T_2T_3, the term W is neglected from the beginning as very small after the random phase approximation.

From (A·9) and (A·10) we can easily obtain the following expressions:

$$\begin{aligned}&T_2^{-1}\mathcal{H}_{sr}T_2=\mathcal{H}_{sr},\\ &T_2^{-1}\mathcal{H}_2T_2=-\frac{1}{2}\sum_{l<k_c}\hbar\omega_l\xi_{-l}\xi_l+\sqrt{V/2}\sum_{l<k_c}\tau(l)\hbar\omega_l\rho_{-l}\xi_l\\ &\qquad-\frac{1}{\sqrt{2}}\sum_{l<k_c}\hbar\omega_l(a_{-l}+a_l^*)\xi_l.\end{aligned} \tag{A·12}$$

As there stands relationship (44), we have the following in the $\bar{\alpha}^2$-approximation:

$$T_2^{-1}(T+\mathcal{H}_{coll}+\mathcal{H}_1+\mathcal{H}_c')T_2=T+\mathcal{H}_{coll}+\mathcal{H}_c'+\frac{1}{2}[\mathcal{H}_1, S_2]_-, \tag{A·13}$$

where the term $[\mathcal{H}_c, S_2]_-$ is neglected, because it gives the higher-order effect as will be seen later. In (A·13) the important term $1/2[\mathcal{H}_1, S_2]_-=Q_1+Q_2$ is given by

$$Q_1=\frac{1}{4}\sum_{\substack{k,k'\\ l<k_c}}^{\sigma,\mu}\tau^2(l)\hbar\omega_l\frac{\alpha(-l,k)\alpha(l,k')}{1-\alpha^2(l,k')}\psi_{\sigma,k}^*\psi_{\sigma,k-l}\psi_{\mu,k'}^*\psi_{\mu,k'+l}, \tag{A·14}$$

$$\begin{aligned}Q_2=&-\frac{\hbar}{8}\sum_{\substack{k\\ l,l'<k_c}}\tau(l)\tau(l')\alpha(-l,k)\alpha(-l',k-l)\psi_{\sigma,k}^*\psi_{\sigma,k-l-l'}\\ &\times\left[\frac{\omega_l(a_{-l}^*-a_l)a_{l'}}{1+\alpha(-l',k-l)}+\frac{\omega_l(a_{-l}^*-a_l)a_{l'}^*}{1-\alpha(-l',k-l)}\right.\\ &\left.-\frac{\omega_{l'}(a_{-l'}^*-a_{l'})a_l}{1+\alpha(-l,k)}-\frac{\omega_{l'}(a_{-l'}^*-a_{l'})a_l^*}{1-\alpha(-l,k)}\right].\end{aligned} \tag{A·15}$$

Here, Q_1 is derived from the commutator between $(a^*_{-l}-a_l)$ in $\mathcal{H}_1$ and $(a_{l'}+a^*_{-l'})$ in S_2. Q_2 is derived from the commutator between the particle variables contained both in $\mathcal{H}_1$ and S_2. In Q_2 the part of $\boldsymbol{l}'=-\boldsymbol{l}$ is considered largest after the random phase approximation, and accordingly it is reduced to

$$Q_2=\sum_{\substack{k\\ l<k_c}}\frac{\hbar\nu_l^2}{4N\omega_l}\psi^*_{\sigma,k}\psi_{\sigma,k}\left[\frac{\omega_l^2}{\{\omega_l-(\hbar/m)(\boldsymbol{l},\boldsymbol{k})\}^2-(\hbar l^2/2m)^2}-1\right]$$
$$\times(a_l^*a_l+a_la_l^*-a_la_{-l}-a^*_{-l}a_l^*). \tag{A16}$$

Then, dispersion relation (47) will be obtained by the condition $\mathcal{H}_c'+Q_2=0$.

The calculations accompanying transformation T_3 will be given as follows. As there stands $[\tilde{\xi}_l, \rho_k]_-=0$ given by (62b), we precisely obtain from (A·12) the following:

$$T_3^{-1}T_2^{-1}\mathcal{H}_{sr}T_2T_3=\mathcal{H}_{sr},$$
$$T_3^{-1}T_2^{-1}\mathcal{H}_2T_2T_3=-\frac{3}{2}\sum_{l<k_c}\hbar\omega_l\tilde{\xi}_{-l}\tilde{\xi}_l+\sqrt{V/2}\sum_{l<k_c}\tau(l)\hbar\omega_l\rho_{-l}\tilde{\xi}_l$$
$$-\frac{1}{\sqrt{2}}\sum_{l<k_c}\hbar\omega_l(a_{-l}+a_l^*)\tilde{\xi}_l. \tag{A·17}$$

Another quantity to be calculated is

$$T_3^{-1}T_2^{-1}(T+\mathcal{H}_{coll}+\mathcal{H}_1+\mathcal{H}_c')T_2T_3=T_3^{-1}(T+\mathcal{H}_{coll}+Q_1)T_3.$$

Here $(T_3^{-1}\mathcal{H}_{coll}T_3)$ is exactly given by

$$T_3^{-1}\mathcal{H}_{coll}T_3=\mathcal{H}_{coll}+\frac{1}{\sqrt{2}}\sum_{l<k_c}\hbar\omega_l(a_{-l}+a_l^*)\xi_l+\frac{1}{2}\sum_{l<k_c}\hbar\omega_l\tilde{\xi}_{-l}\tilde{\xi}_l. \tag{A·18}$$

In the calculations of the terms higher than $[Q_1, S_3]_-$ of the expression $(T_3^{-1}Q_1T_3)=Q_1+[Q_1, S_3]_-+\cdots$, we may put $Q_1\simeq-1/2\sum_{l<k_c}\hbar\omega_l\pi_{-l}\pi_l$ consistently with the $\bar{\alpha}^2$-approximation. Then we obtain the results, $[Q_1, S_3]_-=-i/\sqrt{2}\sum_{l<k_c}\hbar\omega_l(a_{-l}-a_l^*)\pi_{-l}$, $1/2[[Q_1, S_3]_-, S_3]_-=-1/4\sum_{l<k_c}\hbar\omega_l(a_l^*a_l+a_la_l^*-a_la_{-l}-a^*_{-l}a_l^*)$ etc. On the other hand, in the calculations of $(T_3^{-1}TT_3)=T+[T, S_3]_-+\cdots$, we see that $[T, S_3]_-=i/\sqrt{2}\sum_{l<k_c}\hbar\omega_l(a_{-l}-a_l^*)\pi_{-l}=-[Q_1, S_3]_-$. Thus, under the $\bar{\alpha}^2$-approximation, we obtain

$$T_3^{-1}(T+Q_1)T_3=T+Q_1. \tag{A·19}$$

Thus, by combining (A·17)$\sim$(A·19), we obtain expression (57) of the total Hamiltonian in the collective representation.

References

1) R. Kronig and J. Korringa, Physica **10** (1943), 406.
H. A. Kramers, Physica **13** (1947), 401.
A. Bohr, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **24** (1948), No. 19.
R. Kronig, Physica **14** (1949), 667.

2) D. Pines and D. Bohm, Phys. Rev. **85** (1952), 338.
D. Bohm and D. Pines, Phys. Rev. **92** (1953), 609.
3) L. Landau, J. Phys. USSR **5** (1941), 71.
L. Landau, and I. Khlatonikov, J. Exp. Theor. Phys. USSR **19** (1949), 637.
4) R. Kronig and A. Thellung, Physica **18** (1952), 749; **19** (1953), 217.
J. M. Ziman, Proc. Roy. Soc. **219** (1953), 257.
H. Ito, Prog. Theor. Phys. **9** (1953), 117.
5) F. London, Rev. Mod. Phys. **17** (1945), 310.
N. N. Bogoliubov, J. Phys. USSR **11** (1947), 23.
T. Nishiyama, Prog. Theor. Phys. **6** (1951), 366, 897; **8** (1952), 655; **9** (1953), 245; **12** (1954), 265; **17** (1957), 711.
N. N. Bogoliubov and D. N. Zubarev, J. Exp. Theor. Phys. USSR **28** (1955), 129.
R. P. Feynman, Phys. Rev. **94** (1954), 262.
R. P. Feynman and M. Cohen, Phys. Rev. **102** (1956), 1189.
W. Brenig, Zeit. f. phys. **144** (1956), 488.
J. M. Blatt and S. T. Butler, Phys. Rev. **100** (1955), 476, 495.
J. M. Blatt, S. T. Butler and M. R. Schafroth, Phys. Rev. **100** (1955), 481.
6) S. Tomonaga, Prog. Theor. Phys. **13** (1955), 467, 482.
7) F. J. Belinfante, Physica **12** (1946), 17.
S. N. Gupta, Proc. Phys. Soc. **A 63** (1950), 681.
K. Bleuler, Helv. Phys. Acta **23** (1950), 567.
R. Utiyama, T. Imamura, S. Sunakawa and T. Dodo, Prog. Theor. Phys. **6** (1953), 587.
J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons*, (1955), p. 100.
8) E. N. Adams, Phys. Rev. **98** (1955), 1130.
C. G. Kuper, Proc. Phys. Soc. **A 69** (1956), 492.
9) K. Sawada, K. A. Brueckner, N. Fukuda and R. Brout, Phys. Rev. **108** (1957), 507.
10) H. Kanazawa, Prog. Theor. Phys. **18** (1957), 287.

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Isobar Model and the S-Wave Pion-Nucleon Scattering

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For the scattering lengths of the S-wave pion-nucleon scattering, Chew, Goldberger, Low and Nambu have shown that the result of their dispersion relation without subtraction exhibits an excellent agreement with the observations, as long as the integrations are carried out over the 3-3 resonance alone. Correspondence of their result to the Hamiltonian formalism is examined by replacing the 3-3 resonant state by an isobaric particle of spin 3/2, which is described by the Rarita-Schwinger theory. The scattering amplitudes are calculated by means of the lowest order perturbation theory. By requiring further that the amplitudes should decrease to zero in the high energy limit, it is found necessary to introduce the interaction terms other than the conventional one. These terms are left undetermined from the location and the width of the 3-3 resonance. By the appropriate choice of the parameters involved in these terms, CGLN's result can be reproduced in the narrow width approximation.

§ 1. Introduction

Many works have been devoted to explain the curious character of the S-wave scattering lengths in the pion-nucleon scattering. Among them the dispersion relation approach has obtained considerable successes.[1),2)] In particular, this method gives a fairly close agreement with experiment for the difference of the two scattering lengths $\lambda^{(-)}=(1/3)\,(M/\mu)\,(a_1-a_3)$. In this case, the integrand in the dispersion integral, $\sigma(\pi^+p)-\sigma(\pi^-p)$, has been conjectured to vanish in the extremely high energy and consequently the 3-3 resonance gives a major contribution. On the other hand, for the sum of the scattering lengths $\lambda^{(+)}=-(1/6)\,(a_1+2a_3)$, the dispersion relation without subtraction is considered to fail to hold because the integrand $\sigma(\pi^+p)+\sigma(\pi^-p)$ seems to tend to a finite value in the extremely high energy. If this is really the case, at least one subtraction is necessary and we cannot "calculate" $\lambda^{(+)}$.[2)] On the other hand, Chew, Goldberger, Low and Nambu[3)] (CGLN) have shown that their dispersion relation without subtraction yields a close agreement with the observed value for $\lambda^{(+)}$ also, so long as the integration is carried out only over the 3-3 resonance. Their results may be reproduced as follows:

$$\left.\begin{aligned}\lambda^{(+)}&=\frac{G^2}{2M}-\frac{M}{6\pi^2}\int_{\mu}^{\infty}\frac{d\omega}{q}\left(1+\frac{2\omega}{M}\right)\sigma_{33}(\omega)=1.14-1.17=-0.03,\\ \lambda^{(-)}&=\frac{G^2}{2M}-\frac{M}{6\pi^2}\int_{\mu}^{\infty}\frac{d\omega}{q}\sigma_{33}(\omega)=1.14-0.67=0.47.\end{aligned}\right\}\qquad(1\cdot1)$$

The observed values are

$$\lambda^{(+)}=0.01, \quad \lambda^{(-)}=0.6.$$

In Eqs. (1·1) we observe two features.

First, (1·1) shows that the scattering in the P-state affects that in the S-state. Such a correlation between different partial wave states is not resulted in the usual Tamm-Dancoff method.[4] G. Takeda has given an intuitive argument, basing on the causality condition in the scattering process, that different partial wave states are always correlated in the dispersion relation approach.[5] On the other hand, it is shown that each perturbation term in the scattering amplitude satisfies dispersion relation,[6] so that it is expected to reproduce the above correlation in terms of the Hamiltonian method.

Second, the contribution from the 3-3 resonance is "large". More specifically, there appears a large factor M in the second terms in the right-hand sides of (1·1), though it is only fictitious in the second equation in (1·1), since $\lambda^{(-)}$ has been conveniently defined by multiplying M/μ to the true scattering amplitude. It seems rather peculiar, since the cross section in the P-states is at most of the order of $f^2=(\mu/2M)^2G^2$, which is smaller by the factor μ/M compared with the Born approximation term $G^2/2M$.* In any case we would understand these features better, if we could reproduce Eqs. (1·1) by means of the Hamiltonian method.

Now we shall discuss about the contributions from high energy region and the problem of subtraction. In CGLN's formalism, the imaginary parts, ImA and ImB, cannot be simply expressed in terms of the total cross section. As a tentative estimation of the high energy contribution to $\lambda^{(\pm)}$, we shall evaluate the dispersion integrals over the second, and the third resonances in the pion-nucleon scattering, which appear around 700 Mev and $I=1/2$.[7] The results are shown in Table I.** Though the non-resonant cross sections contribute appreciably, we find that the usual Born approximation term and the 3-3 resonance dominate the effect to the S-wave scattering lengths, if we take only contributions from the sub-Bev region into account. We are tempted to conjecture that the dispersion relations can be written without subtraction even if we have to give up the rigorous validity of the causality condition for this purpose.*** In this article, however, without entering into this problem further, we shall assume tentatively that the dispersion relations without subtraction of the form (1·1) are valid as far as the effect of the 3-3 resonance to the S-wave scattering lengths is concerned, and inquire into

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* In fact, in the previous formalism,[1] the contribution from the 3-3 resonance to $\lambda^{(+)}$ amounts only to a half of the second term in the integral in the first equation of (1·1).

** Here we cannot use the approximate expression as in (1·1). Instead, we have calculated the values in Table I from the original relativistic formulas.

*** Bogoliubov et al.[8] and Oehme[9] have shown that, if the causality condition breaks down inside the small region ($r<a$), the dispersion relations can be written in which the high energy contributions ($E>1/a$) are suppressed.

Table I

	2nd resonance		3rd resonance	
Assumed state	$P_{3/2}$	$D_{3/2}$	$D_{5/2}$	$F_{5/2}$
$\lambda^{(+)}$	−0.05	+0.08	−0.06	+0.08
$\lambda^{(-)}$	+0.06	+0.03	+0.04	+0.02

the physical meaning of this effect by comparing the results implied in (1·1) with the Hamiltonian formalism. In the last section we shall briefly discuss about the case in which the above assumption is not justified.

It is difficult to treat a resonant state in the perturbational method. We shall, therefore, make an approximation in which the width of the resonance Γ tends to an infinitesimal (the narrow width approximation). Dispersion relations (1·1) are still valid in this limiting case. Namely, substituting the expression

$$\sigma_{33}(\omega)=\frac{8\pi}{q^2}\frac{\Gamma^2}{(\omega-\omega_1)^2+\Gamma^2}\xrightarrow[(\Gamma\to 0)]{}\frac{8\pi^2}{q_1^2}\Gamma\delta(\omega-\omega_1),$$

into (1·1), we obtain

$$\left.\begin{aligned}\lambda^{(+)}&=\frac{G^2}{2M}-\frac{4}{3}M\frac{\Gamma}{q_1^3}\left(1+\frac{2\omega_1}{M}\right),\\ \lambda^{(-)}&=\frac{G^2}{2M}-\frac{4}{3}M\frac{\Gamma}{q_1^3},\end{aligned}\right\}\qquad(1\cdot2)$$

where $\omega_1\approx 2\mu$ is the resonance energy and $q_1\cong\sqrt{\omega_1^2-\mu^2}$ is the momentum in the center of mass system. According to CGLN, we can determine the width Γ from the effective range theory of the static P-wave equation as follows:

$$\Gamma=\frac{4}{3}f^2\frac{q_1^3}{\mu^2}.\qquad(1\cdot3)$$

Using (1·3), we obtain from (1·2)

$$\left.\begin{aligned}\lambda^{(+)}&=\frac{G^2}{2M}\left[1-\frac{8}{9}\left(1+2\,\frac{\omega_1}{M}\right)\right]=-0.45,\\ \lambda^{(-)}&=\frac{G^2}{2M}\left(1-\frac{8}{9}\right)=0.13.\end{aligned}\right\}\qquad(1\cdot4)$$

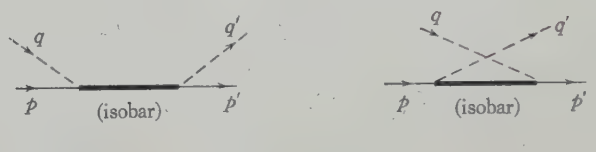
Numerical agreement is obviously not so satisfactory as in (1·1), yet the results (1·4) still exhibit the qualitative feature of the results (1·1) that the 3-3 resonance contributes as large as the Born approximation term and with the opposite sign.

Now we notice that a sufficiently sharp resonant state resembles an actual particle in many respects. We, thus, approximate the 3-3 resonant state by a particle of spin 3/2, iso-spin 3/2, and nucleon number 1. We shall describe this isobaric particle in Rarita-Schwinger formalism.[10),11)] The isobar model of this line

has been considered by Minami et al.[12] They have considered mainly the scattering in the P-state. Since then the problems in the pion-nucleon scattering have been rearranged and it seems worthwhile to discuss this model in a new light.

§ 2. Perturbation calculation

We shall calculate the scattering amplitude for the graphs in Fig. 1 by the lowest order perturbation theory.



(a) (b)

Fig. 1

The propagation function of a Rarita-Schwinger particle has been given by[11,12]

$$S_{F\mu\nu}(k) = \frac{\mathcal{S}_{\mu\nu}(k)}{k^2+m^2-i\varepsilon},$$

$$\mathcal{S}_{\mu\nu}(k) = (m-ik\gamma)\left[\delta_{\mu\nu}-\frac{1}{3}\gamma_\mu\gamma_\nu+\frac{i}{3m}(\gamma_\mu k_\nu-k_\mu\gamma_\nu)+\frac{2}{3m^2}k_\mu k_\nu\right]$$
$$+\frac{2}{3m^2}(k^2+m^2)[i(\gamma_\mu k_\nu-k_\mu\gamma_\nu)-(m-ik\gamma)\gamma_\mu\gamma_\nu]. \qquad (2\cdot1)$$

Next, we have to determine the form of the interaction Lagrangian. For the interaction between nucleon, pion and isobar, the most general expression including up to the first derivatives can be written down as follows:

$$(4\pi)^{-1/2}L_{int}=(g/M)[\xi(\partial_\mu\bar{\psi})\phi^*-\eta\bar{\psi}(\partial_\mu\phi^*)]\varphi_\mu$$
$$-g_1\bar{\psi}\gamma_\mu\varphi_\mu\phi^*-(g_2/M)\bar{\psi}\gamma_\nu\gamma_\mu\varphi_\mu(\partial_\nu\phi^*)-(g_3/M)(\partial_\nu\bar{\psi})\gamma_\nu\gamma_\mu\varphi_\mu\phi^*$$
$$-(g_4/M)\bar{\psi}\gamma_\mu\gamma_\nu\varphi_\mu(\partial_\nu\phi^*)-(g_5/M)(\partial_\nu\bar{\psi})\gamma_\mu\gamma_\nu\varphi_\mu\phi^*+\text{h. c.}, \qquad (2\cdot2)$$
$$\xi+\eta=1, \quad (\xi,\ \eta \text{ real}).$$

In this expression charge state specification is suppressed. Instead it is understood that ψ describes the proton field, ϕ the positive pion field, and φ_μ the doubly positive charged isobar ($I_3=+3/2$). Charge independence consideration will be made later.

Using the identity

$$\gamma_\mu\gamma_\nu=2\delta_{\mu\nu}-\gamma_\nu\gamma_\mu,$$

we can drop the last two terms in (2·2) (terms with g_4 and g_5) provided we redefine other coupling constants as follows:

$$\xi g \to \xi g - 2g_5,$$
$$\eta g \to \eta g - 2g_4,$$
$$g_2 \to g_2 - g_4,$$
$$g_3 \to g_3 - g_5.$$

If the nucleon is in the free state, further simplification is possible. Namely, we can drop the term with g_3 by virtue of the equation

$$\partial_\nu \bar{\psi}\gamma_\nu - M\bar{\psi} = 0,$$

with g_1 redefined as

$$g_1 \to g_1 + g_3.$$

In the following we shall confine ourselves to the lowest order perturbation calculation for the pion-nucleon scattering where nucleons can always be considered to be free. We, thus, arrive at the relevant interaction Lagrangian

$$(4\pi)^{-1/2} L_{int} = (g/M)[\bar{\xi}(\partial_\mu \bar{\psi})\phi^* - \eta\bar{\psi}(\partial_\mu \phi^*)]\varphi_\mu$$
$$- g_1 \bar{\psi}\gamma_\mu \varphi_\mu \phi^* - (g_2/M)\bar{\psi}\gamma_\nu\gamma_\mu\varphi_\mu(\partial_\nu\phi^*) + \text{h. c.} \qquad (2\cdot3)$$

From the requirement of charge conjugation invariance we can conclude that g, g_1 and g_2 should have the same phase factors. In the following we shall take all g's as real.

In (2·3) we have included the term with the first derivative of the nucleon ($g\bar{\xi}$-term) which is known to allow no consistent method of quantization.[13] But we have already given up consistent formalism in the introduction of the particle of spin 3/2.[13] We have introduced it rather phenomenologically as a simplifying substitute for the 3-3 resonant state, which, we consider, is resulted as a higher order effect of pion-nucleon system. In the pion-nucleon scattering, therefore, we should not consider higher order contributions beyond the second order approximation with respect to g's (pion-nucleon-isobar coupling constants), because otherwise we would calculate the same graph repeatedly. In this point our method differs from that by Minami et al.[12] in which the damping has been taken into account as higher order effects.

Further we cannot drop either $g\bar{\xi}$-term or $g\eta$-term by the partial integration and the application of the equation

$$\partial_\mu \varphi_\mu = 0, \qquad (2\cdot4)$$

because this equation is valid only when the isobar is free. From the same reasoning we cannot drop the last two terms (terms with g_1 and with g_2) in (2·3) by virtue of the equation

$$\gamma_\mu \varphi_\mu = 0, \qquad (2\cdot5)$$

which is also valid only when the isobar is free.

g_1 and g_2 may not be considered to be "coupling constants" in the usual sense, which are defined as extrapolated values of the strength of interactions between "free" particles. But we have introduced them as some parameters contributing to the interaction.

We have chosen a pseudospinor field as an isobar field. This choice is necessary for an isobar to stand for a resonance in the $P_{3/2}$-state (see § 3). A spinor isobar (γ_5 must be inserted in (2·3)) is found to stand for a resonance in the $D_{3/2}$-state.

From (2·1) and (2·3) we can calculate the amplitude for the scattering (Fig. 1(a)) in the lowest order perturbation theory.

According to CGLN, we divide the T-matrix into two parts:

$$T=-A+iQ\gamma B.$$

The following notation will be used:

$$Q=(1/2)(q+q'),\quad \kappa=(1/2)(q-q'),$$
$$\nu=-(1/4M)(p+p')\cdot(q+q'),\quad \nu=\nu_L-\kappa^2/M.$$

Choosing ν_L(meson energy in the lab. system) and κ^2(square of the invariant momentum transfer) as independent variables, we can write A and B as follows:

$$\left.\begin{aligned} A(\nu_L,\kappa^2)&=-\frac{g_r^2}{2M(\nu_1-\nu_L-i\varepsilon)}a(\nu_L,\kappa^2),\\ B(\nu_L,\kappa^2)&=-\frac{g_r^2}{2M(\nu_1-\nu_L-i\varepsilon)}b(\nu_L,\kappa^2),\end{aligned}\right\}\qquad (2\cdot6)^{*,**}$$

where ν_1 is the resonance energy in the lab. system given by

$$\nu_1=\frac{m^2-M^2-\mu^2}{2M}.$$

$a(\nu_L,\kappa^2)$ and $b(\nu_L,\kappa^2)$ can be expanded into the power series with respect to $\nu_1-\nu_L$:

$$\left.\begin{aligned} a(\nu_L,\kappa^2)&=a_0(\kappa^2)+(\nu_1-\nu_L)a_1(\kappa^2)+(\nu_1-\nu_L)^2a_2(\kappa^2),\\ b(\nu_L,\kappa^2)&=b_0(\kappa^2)+(\nu_1-\nu_L)b_1(\kappa^2)+(\nu_1-\nu_L)^2b_2(\kappa^2).\end{aligned}\right\}\qquad (2\cdot7)$$

In the present case there appear terms up to the second power with respect to ν_L.

Explicit calculation yields

$$a_0(\kappa^2)=2(m+M)\frac{\kappa^2}{M^2}+\frac{2}{3}(m+M)\frac{\mu^2}{m^2}$$
$$-\frac{2}{3}\frac{M^2}{m^2}\nu_1\left(1+\frac{m}{M}-\frac{\mu^2}{M^2}\right)-2\frac{M}{m^2}\nu_1^2\left(1+\frac{2}{3}\frac{m}{M}\right),\qquad (2\cdot8a)$$

---

* $(p+q)^2+m^2=2M(\nu_1-\nu_L)$, $g_r^2/4\pi=g^2$.

** Hereafter usual Born approximation terms will not be written explicitly.

$$b_0(\kappa^2)=2\frac{\kappa^2}{M^2}+\frac{2}{3}\frac{M^2}{m^2}\left[1+\frac{m}{M}+2\frac{\mu^2}{M^2}+\frac{\nu_1}{M}\left(2+\frac{m}{M}\right)-\frac{\nu_1^2}{M^2}\right], \tag{2·8b}$$

$$\begin{aligned}a_1(\kappa^2)=a_1=&\frac{1}{3}\frac{M^2}{m^2}\left\{\left(1+\frac{m}{M}\right)^2+\frac{\mu^2}{M^2}\right.\\&+2\eta\left[2-5\frac{m}{M}-2\frac{m^2}{M^2}+5\frac{m^3}{M^3}-\frac{\mu^2}{M^2}\left(2-5\frac{m}{M}\right)\right]+8\eta^2\frac{\nu_1}{M}\left(1-4\frac{m}{M}\right)\Big\}\\&+4\frac{M^2}{m^2}\left\{-x\left(1-3\frac{m}{M}-\frac{m^2}{M^2}+\frac{\mu^2}{M^2}\right)+2y\left(1-\frac{m^2}{M^2}\right)\right.\\&\left.+3\eta\frac{m}{M}\left[-x+y\left(1-\frac{m^2}{M^2}\right)\right]\right\}\\&+12\frac{M^2}{m^2}\left\{x^2\left(1-2\frac{m}{M}\right)+y^2\left(1+2\frac{m}{M}\right)\left(1-\frac{m^2}{M^2}\right)-2xy\left(1-\frac{m^2}{M^2}\right)\right\},\end{aligned} \tag{2·9a}$$

$$\begin{aligned}b_1(\kappa^2)=b_1=&\frac{2}{3}\frac{M}{m^2}\left(3-\frac{m}{M}+\frac{m^2}{M^2}-\frac{\mu^2}{M^2}+6\eta\frac{m}{M}\right)\\&+4\frac{M}{m^2}\left[-3\eta\frac{m}{M}(x-y)-2x+y\left(3+3\frac{m}{M}+\frac{m^2}{M^2}-\frac{\mu^2}{M^2}\right)\right]\\&+12\frac{M}{m^2}\left[x^2+y^2\left(1+4\frac{m}{M}+\frac{m^2}{M^2}\right)-2xy\left(1+2\frac{m}{M}\right)\right],\end{aligned} \tag{2·9b}$$

$$\begin{aligned}a_2(\kappa^2)=a_2=&\frac{2}{3}\frac{M}{m^2}\left(3-2\frac{m}{M}+6\eta\frac{m}{M}\right)\\&+8\frac{M}{m^2}\left[-x+y\left(2+3\eta\frac{m}{M}\right)\right]+24\frac{M}{m^2}\left[-2xy+y^2\left(1+2\frac{m}{M}\right)\right],\end{aligned} \tag{2·10a}$$

$$b_2(\kappa^2)=b_2=-\frac{1}{m^2}\left(\frac{2}{3}+8y+24y^2\right), \tag{2·10b}$$

where abbreviations

$$x=g_1/g,\quad y=g_2/g$$

were used and ξ has been eliminated by means of $\xi=1-\eta$.

Complete scattering amplitudes with proper consideration of charge independence and crossing symmetry are given by

$$\left.\begin{aligned}A^{(\pm)}(\nu_L,\kappa^2)&=\frac{(2,-1)}{3}[A(\nu,\kappa^2)\pm A(-\nu,\kappa^2)],\\B^{(\pm)}(\nu_L,\kappa^2)&=\frac{(2,-1)}{3}[B(\nu,\kappa^2)\mp B(-\nu,\kappa^2)],\end{aligned}\right\} \tag{2·11}$$

where $(\pm)$ amplitudes are defined in CGLN in terms of the amplitudes in the total isotopic spin by

$$\left.\begin{aligned} &A^{(+)}=(1/3)\,(A^{(1/2)}+2A^{(3/2)}),\\ &A^{(-)}=(1/3)\,(A^{(1/2)}-A^{(3/2)}),\\ &\text{and similar equations for } B^{(\pm)}. \end{aligned}\right\} \tag{2·12}$$

Recalling that A and B in (2·6) are associated with π^+-p scattering, or the scattering purely in the total isotopic spin 3/2, we have substituted them for $A^{(3/2)}$ and $B^{(3/2)}$ in (2·12) to obtain the first terms in (2·11). The second terms in (2·11) come from the crossing symmetric graph (Fig. 1(b)).

Decomposing $A^{(\pm)}$ and $B^{(\pm)}$ in accordance with (2·7) as

$$A^{(\pm)}=A_0^{(\pm)}+A_1^{(\pm)}+A_2^{(\pm)},$$
$$B^{(\pm)}=B_0^{(\pm)}+B_1^{(\pm)}+B_2^{(\pm)},$$

we can write as follows:

$$A_0^{(\pm)}=-\frac{(2,-1)}{3}\frac{g_r^2}{2M}\left(\frac{1}{\nu_1-\nu_L}\pm\frac{1}{\nu_1+\nu_L-2\kappa^2/M}\right)a_0(\kappa^2),$$

$$B_0^{(\pm)}=-\frac{(2,-1)}{3}\frac{g_r^2}{2M}\left(\frac{1}{\nu_1-\nu_L}\mp\frac{1}{\nu_1+\nu_L-2\kappa^2/M}\right)b_0(\kappa^2),$$

$$A_1^{(+)}=-\frac{4}{3}\frac{g_r^2}{2M}a_1,\quad B_1^{(+)}=0,$$

$$A_1^{(-)}=0,\quad B_1^{(-)}=\frac{2}{3}\frac{g_r^2}{2M}b_1,$$

$$A_2^{(+)}=-\frac{4}{3}\frac{g_r^2}{2M}\left(\nu_1-\frac{\kappa^2}{M}\right)a_2,\quad B_2^{(+)}=\frac{4}{3}\frac{g_r^2}{2M}\left(\nu_L-\frac{\kappa^2}{M}\right)b_2,$$

$$A_2^{(-)}=-\frac{2}{3}\frac{g_r^2}{2M}\left(\nu_L-\frac{\kappa^2}{M}\right)a_2,\quad B_2^{(-)}=\frac{2}{3}\frac{g_r^2}{2M}\left(\nu_1-\frac{\kappa^2}{M}\right)b_2. \tag{2·13}$$

§ 3. Imaginary parts

From (2·6) and (2·7) we have

$$\left.\begin{aligned} &\mathrm{Im}\,A(\nu_L,\kappa^2)=-\frac{g_r^2}{2M}\pi\delta(\nu_1-\nu_L)\,a(\nu_L,\kappa^2)=-\pi\frac{g_r^2}{2M}\delta(\nu_1-\nu_L)\,a_0(\kappa^2),\\ &\mathrm{Im}\,B(\nu_L,\kappa^2)=-\frac{g_r^2}{2M}\pi\delta(\nu_1-\nu_L)\,b(\nu_L,\kappa^2)=-\pi\frac{g_r^2}{2M}\delta(\nu_1-\nu_L)\,b_0(\kappa^2). \end{aligned}\right\} \tag{3·1}$$

It should be noted that the imaginary parts of the scattering amplitudes involve only $a_0(\kappa^2)$ and $b_0(\kappa^2)$ which are the values of $a(\nu_L,\kappa^2)$ and $b(\nu_L,\kappa^2)$ evaluated at $\nu_L=\nu_1$. These are also independent of η, x and y. This is reasonably

expected if we recall that the isobar is free and Eqs. (2·4) and (2·5) are applied in the evaluation of the imaginary parts of the scattering amplitudes.

Introducing the spin-flip- and spin-nonflip-amplitudes in the center of mass system according to CGLN, we can calculate $\mathrm{Im}f_1^{(\pm)}$ and $\mathrm{Im}f_2^{(\pm)}$ from (3·1) and (2·8) as follows:

$$\left.\begin{aligned}
\mathrm{Im}f_1^{(\pm)} &\cong (1/4\pi)[\mathrm{Im}A^{(\pm)}+\omega\,\mathrm{Im}B^{(\pm)}] \\
&\cong \frac{(2,\,-1)}{3}\pi\delta(\nu_1-\nu_L)\,\frac{g^2}{M^2}q_1^2\cos\theta, \\
\mathrm{Im}f_2^{(\pm)} &\cong (1/4\pi)\,(q^2/4M^2)[-\mathrm{Im}A^{(\pm)}+2M\,\mathrm{Im}B^{(\pm)}] \\
&\cong -\frac{(2,\,-1)}{3}\pi\delta(\nu_1-\nu_L)\,\frac{g^2}{M^2}\,\frac{1}{3}q_1^2.
\end{aligned}\right\} \tag{3·2}$$

Here we have neglected higher order terms with respect to $1/M$.

Comparing the equations in (3·2) with (3·7) and (3·8) in CGLN,* we can confirm that the imaginary parts of the scattering amplitudes here considered describe the scattering in the $P_{3/2}$-state. In fact, contributions from the 3-3 resonance to $\mathrm{Im}f_1$ and $\mathrm{Im}f_2$ are given by

$$\left.\begin{aligned}
\mathrm{Im}f_1^{(\pm)} &= \frac{(2,\,-1)}{3}3\,\mathrm{Im}f_{33}(\omega)\cos\theta, \\
\mathrm{Im}f_2^{(\pm)} &= -\frac{(2,\,-1)}{3}\mathrm{Im}f_{33}(\omega),
\end{aligned}\right\} \tag{3·3}$$

where $\mathrm{Im}f_{33}$ is expressed in terms of the cross section,

$$\mathrm{Im}f_{33}(\omega)=\frac{q}{8\pi}\sigma_{33}(\omega)=\frac{1}{q}\,\frac{\Gamma^2}{(\omega-\omega_1)^2+\Gamma^2}. \tag{3·4}$$

In the limit where the width Γ tends to an infinitesimal, (3·4) can be put into the form

$$\mathrm{Im}f_{33}(\omega)=\pi\frac{\Gamma}{q_1}\delta(\omega-\omega_1). \tag{3·5}$$

Substituting this in (3·3) and comparing them with (3·2) we find

$$\frac{g^2}{M^2}=3\frac{\Gamma}{q_1^3}\,\frac{m}{M}, \tag{3·6}$$

where the factor (m/M) comes from

$$\delta(\nu_1-\nu_L)=(M/m)\,\delta(\omega_1-\omega).$$

Here we shall reproduce (1·3),

* $f_1=f_S-f_{D3/2}+3\cos\theta f_{P3/2}+\frac{1}{2}(15\cos^2\theta-3)f_{D5/2}+\cdots$, (3·7)
$f_2=f_{P1/2}-f_{P3/2}+3\cos\theta(f_{D3/2}-f_{D5/2})+\cdots$. (3·8)

$$\Gamma=\frac{4}{3}f^2\frac{q_1^3}{\mu^2}.$$

Substituting this expression in (3·6) we obtain

$$g^2=\frac{m}{M}G^2. \tag{3·7}$$

We have thus determined the isobar mass m and one of the coupling constants g^2 from the location and the width of the resonance in the $P_{3/2}$-state. It should be emphasized that the other constants η, g_1 and g_2 are left completely undetermined.

The imaginary parts (3·2) can be obtained in another way. By the standard method we have

$$\bar{u}_{p'}\,\mathrm{Im}\,T^{(\pm)}(\nu_L,\kappa^2)u_p=-\frac{(2,-1)}{3}\frac{m}{M}\pi\delta(\nu_1-\nu_L)$$
$$\times\sum_{spin}\langle p'|j|k\rangle\langle k|j^*|p\rangle,\quad (k=p+q) \tag{3·8}$$

for $\nu_L>0$. Here j is the source function of the positive pion whose relevant part is given by

$$j=-\delta L_{int}/\delta\phi^*\doteq-(g_r/M)[\bar{\xi}(\partial_\mu\bar{\psi})\varphi_\mu+\eta\partial_\mu(\bar{\psi}\varphi_\mu)]$$
$$+\sqrt{4\pi}g_1\bar{\psi}\gamma_\mu\varphi_\mu-\sqrt{4\pi}(g_2/M)\partial_\nu(\bar{\psi}\gamma_\nu\gamma_\mu\varphi_\mu).$$

In the evaluation of the matrix element $\langle p'|j|k\rangle$ in the rest frame of the isobar, Eqs. (2·4) and (2·5) are applied to give

$$\langle p'|j|J_z\rangle=ip_\mu'(g_r/M)\bar{u}_{p'}\varphi_\mu(J_z)=ip_i'(g_r/M)\bar{u}_{p'}\varphi_i(J_z), \tag{3·9}$$
$$(i=1,\,2,\,3)$$

where J_z represents the z-component of the isobar spin. Together with the similar expression for $\langle J_z|j^*|p\rangle$, we have

$$\sum_{J_z}\langle p'|j|J_z\rangle\langle J_z|j^*|p\rangle=(g_r^2/M^2)p_i'p_j\bar{u}_{p'}\sum_{J_z}U_i(J_z)U_j^*(J_z)u_p, \tag{3·10}$$

where $U_i(J_z)$ is the positive energy "6-component Pauli spinor" for an isobar. This is subject to the supplementary condition

$$\sigma_i U_i(J_z)=0,$$

from (2·5). This condition is guaranteed by the projection operator

$$\Lambda_{ij}=\delta_{ij}-\frac{1}{3}\sigma_i\sigma_j.$$

By inserting this operator, the summation in (3·10) can be carried out as

$$\sum_{J_z}U_i(J_z)U_j^*(J_z)=\Lambda_{ik}\sum_{J_z}U_k(J_z)U_j^*(J_z)=\Lambda_{ik}\delta_{kj}=\Lambda_{ij},$$

from which we have

$$(3\cdot10)=(g_r^2/M^2)\left[\boldsymbol{q}'\boldsymbol{q}-\frac{1}{3}(\boldsymbol{q}'\boldsymbol{\sigma})(\boldsymbol{q}\boldsymbol{\sigma})\right].$$

Substituting this in (3·8) and decomposing into the spin-flip- and spin-nonflip-amplitudes we obtain

$$\left.\begin{aligned}\mathrm{Im}f_1^{(\pm)}&=\frac{(2,-1)}{3}\pi\delta(\nu_1-\nu_L)\frac{m}{M}\frac{g^2}{M^2}q_1^2\cos\theta,\\ \mathrm{Im}f_2^{(\pm)}&=-\frac{(2,-1)}{3}\pi\delta(\nu_1-\nu_L)\frac{m}{M}\frac{g^2}{M^2}\frac{1}{3}q_1^2,\end{aligned}\right\}\tag{3·11}$$

which are identical with (3·2).

It should be noted that the imaginary parts can always be evaluated in the rest frame of the isobar so that any dynamical properties of the isobar—propagation character and the structure of the interaction—is irrelevant. In fact we have obtained (3·11) only from the kinematical knowledge that the isobar has an angular momentum of the magnitude 3/2 in its rest system.

In this place we show that a spinor isobar contributes to the scattering in the $D_{3/2}$-state. If we take

$$\langle p'|j|J_z\rangle=ip_i'(g_r/M)\bar{u}_{p'}\gamma_5\varphi_i(J_z),\tag{3·12}$$

in place of (3·9), the small component of $\bar{u}_{p'}$ survives in (3·12) and we have

$$\langle p'|j|J_z\rangle\cong i(1/2M)(g_r/M)\chi^*p_i'(\boldsymbol{p}'\boldsymbol{\sigma})U_i(J_z).$$

Analogous calculations as in the preceding case leads to

$$\left.\begin{aligned}\mathrm{Im}f_1&=-\frac{m}{M}\pi\delta(\nu_1-\nu_L)\frac{1}{4M^2}\frac{g^2}{M^2}\frac{1}{3}(q_1^2)^2,\\ \mathrm{Im}f_2&=\frac{m}{M}\pi\delta(\nu_1-\nu_L)\frac{1}{4M^2}\frac{g^2}{M^2}(q_1^2)^2\cos\theta.\end{aligned}\right\}\tag{3·13}$$

These are found to be the amplitudes in the $D_{3/2}$-state in comparison with (3·7) and (3·8) in CGLN.

§ 4. Real parts

From (2·6) and (2·7) we have

$$\left.\begin{aligned}\mathrm{Re}A(\nu_L,\kappa^2)&=-\frac{g_r^2}{2M}\left[\frac{P}{\nu_1-\nu_L}a_0(\kappa^2)+a_1+(\nu_1-\nu_L)a_2\right],\\ \mathrm{Re}B(\nu_L,\kappa^2)&=-\frac{g_r^2}{2M}\left[\frac{P}{\nu_1-\nu_L}b_0(\kappa^2)+b_1+(\nu_1-\nu_L)b_2\right].\end{aligned}\right\}\tag{4·1}$$

In CGLN, the following dispersion relations were assumed:

$$\left.\begin{aligned}\mathrm{Re}A(\nu_L,\kappa^2)&=\frac{P}{\pi}\int\frac{\mathrm{Im}A(\nu_L',\kappa^2)}{\nu_L'-\nu_L}d\nu_L',\\ \mathrm{Re}B(\nu_L,\kappa^2)&=\frac{P}{\pi}\int\frac{\mathrm{Im}B(\nu_L',\kappa^2)}{\nu_L'-\nu_L}d\nu_L'.\end{aligned}\right\}\qquad(4\cdot2)$$

Substituting (3·1) into ImA and ImB in (4·2), we obtain

$$\left.\begin{aligned}\mathrm{Re}A(\nu_L,\kappa^2)&=-\frac{g_r^2}{2M}\frac{P}{\nu_1-\nu_L}a_0(\kappa^2),\\ \mathrm{Re}B(\nu_L,\kappa^2)&=-\frac{g_r^2}{2M}\frac{P}{\nu_1-\nu_L}b_0(\kappa^2),\end{aligned}\right\}\qquad(4\cdot3)$$

which is different from (4·1). The reason for this discrepancy is obvious.

The amplitudes $A(\nu_L,\kappa^2)$ and $B(\nu_L,\kappa^2)$ given by (2·6) and (2·7) increase like$\sim\nu_L$ as ν_L tends to infinity. In this case, according to Bogoliubov,[8] we can apply Cauchy's theorem only to $A(\nu_L,\kappa^2)/(\nu_L-\bar{\nu})^n$ and $B(\nu_L,\kappa^2)/(\nu_L-\bar{\nu})^n$, where n is larger than 2, and $\bar{\nu}$ is some constant energy. Choosing $n=2$, $\bar{\nu}=\nu_1$, the dispersion relations are written down as follows:

$$\begin{aligned}\mathrm{Re}A(\nu_L,\kappa^2)=&\mathrm{Re}A(\nu_1,\kappa^2)+(\nu_1-\nu_L)\frac{\partial}{\partial\nu_L}\mathrm{Re}A(\nu_L,\kappa^2)|_{\nu_L=\nu_1}\\ &+(\nu_1-\nu_L)^2\frac{1}{\pi}\int_\mu^\infty\frac{\mathrm{Im}A(\nu_L',\kappa^2)}{(\nu_L'-\nu_L)(\nu_1-\nu_L')^2}d\nu_L',\end{aligned}\qquad(4\cdot4)$$

and a similar equation for B. These are easily found to be identical with (4·1). On the other hand, dispersion relations (4·2) can be written down under the assumption that the amplitudes behave like $\lesssim 1/\nu_L$ as ν_L tends to infinity. This assumption has not been justified *theoretically*. It may be suggestive that the *observed* scattering cross sections will tend to some constant values for very high energy. In order to write down dispersion relations, however, it is not sufficient to know the behavior of the imaginary part alone. In the present example, Im$A(\nu_L,\kappa^2)$, and Im$B(\nu_L,\kappa^2)$ vanish for $\nu_L\neq\nu_1$, and of course for $\nu_L\to\infty$. Nevertheless, additional polynomials were required from the behavior of Re$A(\nu_L,\kappa^2)$ and Re$B(\nu_L,\kappa^2)$.

We may also conjecture that no subtractions or no polynomials are required if the theory is renormalizable. In fact it should be remembered that the Rarita-Schwinger theory with derivative couplings is not renormalizable.[11]* It may be the case that the 3-3 resonant state cannot be described even approximately by Rarita-Schwinger theory if the original interaction between pion and nucleon is renormalizable. In the present consideration, however, we shall explore another

* We cannot write down the simplest form of the dispersion relation if we choose the pseudovector coupling for the elementary interaction between pion and nucleon, in which case the theory is also not renormalizable. See Appendix I.

possibility. Namely, we shall try to make a_1, a_2, b_1 and b_2 vanish by an appropriate choice of η, g_1 and g_2 which are still undetermined.

§ 5. Choice of η, g_1 and g_2

We shall begin with observing that b_2 given by (2·10b) is the function of $y(=g_2/g)$ alone. Putting

$$b_2=0,$$

and solving the quadratic equation for y, we obtain only a single solution

$$y=-1/6. \tag{5·1}$$

Next, substituting (5·1) in the expression for a_2 in (2·10a), we find

$$a_2=0,$$

irrespective of η and x.

If we use (5·1), a_1 and b_1 given by (2·9) are now the functions of x and η. General expressions for x and η that give $a_1=b_1=0$ are rather cumbersome to write down. We shall first consider the simplified case in which $m=M$ and $\mu=0$ are assumed. From (2·9) and (5·1) we have

$$\left.\begin{aligned} a_1&=\frac{4}{3}+12x(1-\eta)-12x^2,\\ Mb_1&=-\frac{2}{3}+2\eta+4(1-3\eta)x+12x^2. \end{aligned}\right\} \tag{5·2}$$

Putting

$$a_1=b_1=0,$$

we can graphically represent the solutions in Fig. 2.

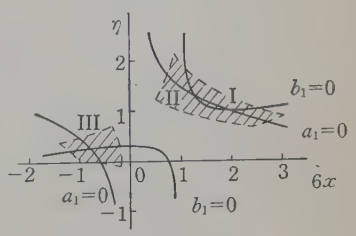


Fig. 2

Either curve representing $a_1=0$ or $b_1=0$ respectively forms a pair of hyperbola. There are three solutions indicated by I, II and III in the graph. The solution I is given by

$$x=1/3,\quad \eta=1. \tag{5·3}$$

The solution given by (5·1) and (5·3) looks very simple and attractive.

But this solution is no longer a solution when the difference between m and M and the finiteness of μ are taken into account. In fact, to the next order in $1/M$, we find

$$a_1\cong-(2/3)(\omega_1/M),\quad b_1\cong 0.$$

Similarly other solutions II and III are not the exact solutions.

In order to obtain some information about the exact solutions, we shall take

$$m=1.3M,\quad \mu=0.15M,$$

and substitute them in (2·9), together with $y=-1/6$, and numerically draw a similar graph with Fig. 2. The curve representing $a_1(x,\eta)=0$ forms an ellipse, different from the preceding approximation, and the curve representing $b_1(x,\eta)=0$ forms a pair of hyperbola. These are shown in Fig. 3.

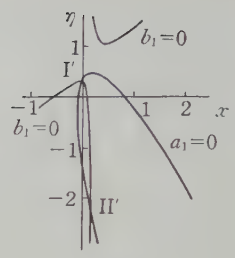


Fig. 3

There are two solutions indicated by I′ and II′, which are given, together with the values of other parameters, by

$$\left.\begin{aligned}&\text{I}':\ \xi=0.68,\ \eta=0.32,\ g_1=-0.00g,\ g_2=-(1/6)g,\\&\text{II}':\ \xi=3.19,\ \eta=-2.19,\ g_1=0.16g,\ g_2=-(1/6)g.\end{aligned}\right\}\qquad(5\cdot4)$$

Choosing either set of values, we have

$$a_1=b_1=a_2=b_2=0,$$

to arrive at Eqs. (4·3).

From (4·3) and (2·8) we obtain

$$\left.\begin{aligned}\lambda^{(+)}&\cong-\frac{1}{4\pi}\frac{1}{2}[\mathrm{Re}A^{(+)}(\mu,0)+\mu\,\mathrm{Re}B^{(+)}(\mu,0)]\cong-\frac{8}{9}\frac{g^2}{2M}\left(1+\frac{\omega_1}{M}\right),\\\lambda^{(-)}&\cong\frac{1}{4\pi}\frac{M}{\mu}[\mathrm{Re}A^{(-)}(\mu,0)+\mu\,\mathrm{Re}B^{(-)}(\mu,0)]\cong-\frac{8}{9}\frac{g^2}{2M},\end{aligned}\right\}$$

or substituting from (3·7),

$$\left.\begin{aligned}\lambda^{(+)}&\cong-\frac{8}{9}\frac{G^2}{2M}\left(1+2\frac{\omega_1}{M}\right),\\\lambda^{(-)}&\cong-\frac{8}{9}\frac{G^2}{2M}.\end{aligned}\right\}\qquad(5\cdot5)$$

Adding the usual Born approximation terms, we have thus reproduced the results (1·4).

§ 6. Summary and discussion

The foregoing discussion can be summarized as follows: When we intend to substitute an isobaric particle of $I=J=3/2$ for the 3-3 resonant state, we cannot obtain complete information about this particle from the knowledge of the imaginary part of the scattering amplitude, namely, from the location and the width of the resonance (in the narrow width approximation). We can determine only the mass, the parity and one of the coupling constants. It is even undetermined to what equation the isobaric particle should obey. The result in the lowest order perturbation theory is found to be equivalent, in general, to that obtained from

the application of the dispersion relation with subtraction, or with the additional polynomial to the first order with respect to the laboratory energy. It is possible to make the polynomial part vanish by an appropriate choice of the other parameters, namely, three coupling constants, γ, g_1 and g_2, so as to give the results equivalent to CGLN's, where the dispersion relation without subtraction is assumed.

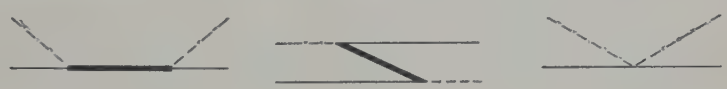
We shall now re-examine the contents of calculation and clarify why we have been able to reproduce the features mentioned in § 1, namely the correlation between S- and P-waves and its large amount.

At first we shall consider a simple example where a sharp resonance in the $P_{1/2}$-state with $I=1/2$ affects the scattering in the $S_{1/2}$-state. These two states are different in the space parity. Replacing the resonant state by a spinor particle of the mass m, which we shall call a "heavy nucleon", we obtain the amplitude very similar to the usual Born approximation term:

$$A=-\frac{g_r^2}{2M}\frac{m-M}{\nu_1-\nu_L-i\varepsilon}, \quad B=\frac{g_r^2}{2M}\frac{1}{\nu_1-\nu_L-i\varepsilon}.$$

These contribute obviously to the scattering lengths in the S-state.*

It is well known that these contributions come from the "pair graph" in Fig. 4(b). More specifically, we shall consider the "scattering" in which pion and nucleon are both at rest. The incident pion creates a pair which involves the



(a) (b) (c)

Fig. 4

final nucleon and the heavy antinucleon to form the ${}^1S_0$(odd parity)-state. The heavy antinucleon succeedingly forms a similar pair with the incident nucleon to annihilate into the final pion. In this example, therefore, the correlation between different partial wave states is a natural consequence of the relativistic Hamiltonian formalism.

The correlation between $P_{3/2}$-state and $S_{1/2}$-state is, however, somewhat different. The pair graph in Fig. 4(b) with the heavy nucleon replaced by an isobaric particle of spin 3/2, does not contribute to the $S_{1/2}$-state, since a nucleon and an antiisobar forms a pair in the state either with $J=1$ or $J=2$, which cannot be created by or annihilated into a pion.

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* Explicit calculation yields

$$\lambda^{(+)}=\frac{1}{6}\frac{g^2}{2M}\frac{1}{\nu^2{}_1-\mu^2}\frac{1}{M}[m^3-m^2M-m(M^2+\mu^2)+M^3-M\mu^2],$$

$$\lambda^{(-)}=\frac{1}{3}\frac{g^2}{2M}\frac{1}{\nu_1{}^2-\mu^2}[(m-M)^2-\mu^2].$$

In fact the amplitudes with nucleon and pion both at rest can be calculated to give

$$\lambda^{(\pm)}=0,$$

if g_1- and g_2-terms are omitted in the interaction Lagrangian (2·3). Though this consequence is not a very general one but seems peculiar to the case here considered (See Appendix II), it was one of the motivations from which we have introduced g_1- and g_2-terms. There was another motivation which we shall state as follows.

The amplitudes calculated without g_1- and g_2-terms behave like $\sim\nu_L$ as $\nu_L\to\infty$, as is easily found from (2·8)−(2·10). This behavior can be considered to be connected with unrenormalizability of the particle with spin 3/2.[11] On the other hand we may conjecture that the pion-nucleon scattering amplitudes behave like $\sim 1/\nu_L$ as $\nu_L\to\infty$, if the original pion-nucleon interaction is renormalizable and the direct pion-pion interaction ($\lambda\phi^4$-term) is neglected. If this conjecture is found to be true, the Rarita-Schwinger particle with the conventional form of the interaction (g_1- and g_2-terms omitted) cannot be considered to be a proper substitute for the 3-3 resonance, as long as the resonance is assumed to be resulted from the (renormalizable) pion-nucleon interaction at all. In the present consideration we have supplemented the conventional Rarita-Schwinger theory by g_1- and g_2-terms so that the resultant amplitudes behave like $\sim 1/\nu_L$ as $\nu_L\to\infty$.

We can show that the above procedure is equivalent to the introduction of the appropriate direct terms corresponding to the graph in Fig. 4(c).

In evaluating the amplitudes we have employed the method of P^*-symbol.[11] In order to see the physical processes involved, however, the original method of P-symbol is more suitable. In the latter method the interaction Hamiltonian must be calculated up to the second order in the coupling constants as follows:[11]

$$\left.\begin{aligned} &H_{int}(x)=-L_{int}(x)+W(x),\\ &\int W(x)dx=\frac{1}{2}\int dx\int dx'\bar{\eta}_\mu(x)[\not{\Delta}_{\mu\nu}(\partial),\ \varepsilon(x-x')]\Delta(x-x';m)\eta_\nu(x'),\end{aligned}\right\}$$

where $\eta_\mu(x)$ is the source function of the isobar. We have omitted the terms contributing to the scattering of pion and isobar, and of isobar and antinucleon. The "shrinked" term $W(x)$, which corresponds obviously to the graph in Fig. 4(c), can be expected to cause the scattering in the S-state. Substituting (2·1) for $\not{\Delta}_{\mu\nu}$, we obtain

$$\begin{aligned} W(x)=(1/3m)\{&-\bar{\eta}_\mu\gamma_4\gamma_\mu\eta_4+\bar{\eta}_4\gamma_4\gamma_\mu\eta_\mu-2\bar{\eta}_4\eta_4+i(4/m)\bar{\eta}_4\gamma_4\eta_4\\ &-(2/m)\bar{\eta}_\mu[\gamma_\mu\partial_\nu-\gamma_\nu\partial_\mu+(\gamma\partial-m)\gamma_\mu\gamma_\nu]\eta_\nu\}.\end{aligned}$$

Though $W(x)$ involves not only normal independent terms but also normal dependent terms, we can obtain definite results in the case in which pion and nucleon are both at rest. $\lambda^{(\pm)}$ thus obtained are found to be identical with those calculated in the previous sections.

We can summarize the above discussions as follows: We have introduced an isobaric particle and its interaction from two requirements: (i) that the 3-3 resonance should be reproduced in the narrow width approximation, (ii) that the resultant scattering amplitudes should behave like $\sim 1/\nu_L$ as $\nu_L \to \infty$. Then there appears necessarily the interaction contributing to the S-wave (as well as the P-wave), from which we can obtain the result (1·4) by means of the perturbation theory.

So far, we have discussed on the assumption that the scattering amplitudes behave asymptotically like $\sim 1/\nu_L$, or dispersion relations can be written down without subtraction or additional polynomial. This assumption is, however, only one of the most simple assumptions that we can expect at present. If the above assumption is not true, then the dispersion relation approach becomes less useful, since the polynomial parts cannot be calculated from the imaginary parts. As an example, we shall consider the case in which one subtraction is necessary.

In the isobar model this case is realized by putting $y=-1/6$ to give $a_2=b_2=0$. x and η are left undetermined. If we assume that the contributions to $\lambda^{(\pm)}$ come mainly from the usual Born approximation term and the 3-3 resonance, then the values of a_1 and b_1 will be restricted.

Contributions from a_1 and b_1 are given by

$$\lambda^{(+)} \simeq \frac{2}{3}\frac{G^2}{2M} a_1, \quad \lambda^{(-)} \simeq \frac{2}{3}\frac{G^2}{2M} M b_1.$$

Adding these expressions to (5·5) or (1·4) and comparing with the observed values, we may conclude

$$|a_1| \lesssim 1, \quad |Mb_1| \lesssim 1. \tag{6·1}$$

These limitations restrict the values of x and η. In the approximation where we take $M=m$ and $\mu=0$, the allowed regions are shown in Fig. 2 by the shaded areas. We can also show that, if the conditions (6·1) are fulfilled, the contributions to the P-wave amplitudes are small compared with the Born approximation term and the terms from a_0 and b_0.

For the actual case where the 3-3 resonance has a finite width, we shall only make a following discussion:

Dispersion relations can be written as

$$\left.\begin{aligned} \mathrm{Re}A(\nu_L, \kappa^2) = \mathrm{Re}A(\bar{\nu}, \kappa^2) + \frac{\nu_L - \bar{\nu}}{\pi}\int d\nu_L' \frac{\mathrm{Im}A(\nu_L', \kappa^2)}{(\nu_L' - \nu_L)(\nu_L' - \bar{\nu})}, \\ \text{and a similar equation for } B(\nu_L, \kappa^2). \end{aligned}\right\} \tag{6·2}$$

Here we must calculate $\mathrm{Re}A(\bar{\nu}, \kappa^2)$ in addition to $\mathrm{Im}A$. Eq. (6·2) is, therefore, not very useful unless we can find a particular $\bar{\nu}$ at which the real part can be calculated more easily than at ν_L at which we wish to obtain the final result. Such a special circumstance may arise when the amplitude behaves as

$$\left.\begin{aligned}&\mathrm{Re}A(\nu_L,\ \kappa^2)\to\text{const},\\&\mathrm{Im}A(\nu_L,\ \kappa^2)\to 1/\nu_L,\end{aligned}\right\}\quad(\nu_L\to\infty)\tag{6·3}$$

and the amplitude at $\nu_L\to\infty$ is estimated in some method, since under the condition (6·3) we can put $\bar{\nu}$ in (6·2) into infinity, to give

$$\mathrm{Re}A(\nu_L,\ \kappa^2)=\mathrm{Re}A(\infty,\ \kappa^2)+\frac{1}{\pi}\int d\nu_L'\ \frac{\mathrm{Im}A(\nu_L',\ \kappa^2)}{\nu_L'-\nu_L}\,.$$

Acknowledgement

The author wishes to express his sincere thanks to Prof. H. Suura, Prof. S. Goto, Dr. T. Goto and Dr. S. Furuichi for their valuable discussions.

Appendix I

Pseudovector theory for the pion-nucleon interaction

Perturbation calculation yields the result which is, in general, different from that in the pseudoscalar theory. On the other hand, if we assume the same form for the dispersion relation as assumed in the pseudoscalar theory, the calculated real parts are the same as those in the pseudoscalar theory, since in the evaluation of the imaginary parts the equivalence theorem can always be applied. This apparent discrepancy is merely a result of a wrong assumption for the form of the dispersion relation.

In order to see this circumstance quickly, we shall consider the Born approximation term for the pion-nucleon scattering.

Assuming the interaction Lagrangian,

$$(4\pi)^{-1/2}L_{int}=-i(G/2M)\bar{\psi}\gamma_5\gamma_\mu\tau_i\psi\partial_\mu\phi_i,$$

we can calculate the scattering amplitudes in the lowest order perturbation theory as follows:

$$\left.\begin{aligned}A^{(+)}&=\frac{G_r^2}{M},\\A^{(-)}&=0,\\B^{(+)}&=\frac{G_r^2}{2M}\left(\frac{1}{\nu_0-\nu_L-i\varepsilon}-\frac{1}{\nu_0+\nu_L-2\kappa^2/M-i\varepsilon}\right),\\B^{(-)}&=-2\frac{f_r^2}{\mu^2}+\frac{G_r^2}{2M}\left(\frac{1}{\nu_0-\nu_L-i\varepsilon}+\frac{1}{\nu_0+\nu_L-2\kappa^2/M-i\varepsilon}\right),\end{aligned}\right\}\tag{A·1}$$

where

$$\nu_0=-\mu^2/2M.$$

As is easily seen, $A^{(-)}$ and $B^{(+)}$ are identical with those obtained from the

pseudoscalar theory, while $A^{(+)}$ and $B^{(-)}$ differ by additional constants G_r^2/M and $-2f_r^2/\mu^2$ respectively. $A^{(+)}$ and $B^{(-)}$ are found to remain finite as ν_L tends to infinity, so that the dispersion relation without subtraction or polynomial cannot be justified. The analogous procedure as in (4·4) is necessary and resultant additive constants can be adjusted to reproduce the perturbation result (A·1) exactly.

Appendix II

An example in which the scalar pion is scattered by the scalar nucleon through the intermediary vector isobar

At first sight the scattering in the P-state (through the isobar) does not seem to affect the scattering in the S-state. The reason is the same as that described in the case of the 3-3 resonance. Namely, in the pair graph Fig. 4(b), a nucleon and an anti-isobar forms a pair in the state with $J=1$, which cannot be created by or annihilated into a pion. We can show, however, that the "shrinked" term appears to give the scattering in the S-state without introducing any special interaction.

Assuming the interaction Lagrangean

$$(4\pi)^{-1/2}L_{int}=-g\varphi_\mu^*\psi(\partial_\mu\phi)-g\phi^*\varphi_\mu(\partial_\mu\psi^*),$$

we can calculate the scattering amplitude by the method of P^*-symbol to obtain

$$T=-g^2\frac{q_\mu R_{\mu\nu}(p+q)q_\nu}{(p+q)^2+m^2}=g^2\frac{\mu^2}{m^2}, \tag{B·1}$$

where both pion and nucleon are assumed to be at rest.

On the other hand, the "shrinked" term $W(x)$ responsible for the pion-nucleon scattering is obtained from

$$\int W(x)dx=\frac{1}{2}\int dx\int dx'\eta_\mu^*(x)[R_{\mu\nu}(\partial),\varepsilon(x-x')]\Delta(x-x';m)\eta_\nu(x'),$$

where

$$R_{\mu\nu}(\partial)=\delta_{\mu\nu}-(1/m^2)\partial_\mu\partial_\nu,$$

and

$$\eta_\mu(x)=-\delta L_{int}/\delta\varphi_\mu^*=g_r\psi(\partial_\mu\phi).$$

Observing

$$[R_{\mu\nu}(\partial),\varepsilon(x-x')]\Delta(x-x';m)=-(2/m^2)\delta_{4\mu}\delta_{4\nu}\delta(x-x'),$$

we obtain

$$W(x)=(g^2/m^2)\psi^*\psi\dot{\phi}^*\dot{\phi}, \tag{B·2}$$

from which we can easily obtain the same result as (B·1).

In the case of the Rarita-Schwinger particle the shrinked term similar to (B·2) exists even without g_1- and g_2-terms. But it gives zero when it is evaluated between the states where both pion and nucleon are at rest.

References

1) For example, M. L. Goldberger, H. Miyazawa and R. Oehme, Phys. Rev. **99** (1955), 986.
2) K. Kawarabayashi and H. Miyazawa, Prog. Theor. Phys. **23** (1960), 490.
3) G. F. Chew, M. L. Goldberger, F. E. Low and Y. Nambu, Phys. Rev. **106** (1957), 1337.
4) N. Fukuda, S. Goto, S. Okubo and K. Sawada, Prog. Theor. Phys. **12** (1954), 79.
   S. Chiba, M. Yamazaki and N. Fukuda, Prog. Theor. Phys. **12** (1954), 767.
5) G. Takeda, "Buturi" **15** (1960), 298 (Physical Society of Japan).
6) N. Nakanishi, Prog. Theor. Phys. **22** (1959), 128.
   S. Mandelstam, Phys. Rev. **115** (1959), 1741, 1752.
7) J. C. Brisson, J. Detoef, P. Falk-Vairant, L. van Rossum, G. Valladas and Luke C. L. Yuan, Phys. Rev. Letters **3** (1959), 561.
8) Bogoliubov, Medvedev and Polivanov, lecture note (Princeton).
9) R. Oehme, Phys. Rev. **100** (1955), 1503.
10) W. Rarita and J. Schwinger, Phys. Rev. **60** (1941), 61.
11) H. Umezawa, *Quantum Field Theory* (North Holland Publishing Company, 1956).
12) S. Minami, T. Nakano, K. Nishijima, H. Okonogi and E. Yamada, Prog. Theor. Phys. **8** (1952), 531.
13) For example, Y. Katayama, Prog. Theor. Phys. **10** (1953), 31.

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A Semiphenomenological Proton-Proton Potential*

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An energy independent potential is constructed which reproduces all available p-p data up to 310 Mev. At 310 Mev the potential predicts the phase shift solution 1 of MacGregor et al. The potential includes the central, tensor, linear and quadratic LS potentials. The quadratic LS potential is manifestly required in the singlet even parity state where the linear LS potential vanishes. The linear LS potential turns out to be more singular but of shorter range than previously thought. It appears now that the p-p data below 310 Mev can be understood in terms of a potential consistent in all respects with the pion theory of nuclear forces.

§ 1. Introduction

There is no *a priori* reason for the existence of the energy independent two-nucleon potential in terms of which the two-nucleon data can be understood over the wide energy range. It is nevertheless useful to have a potential picture which reflects, as close as possible, the over-all features of the two-nucleon interaction. Not only does such a picture give a basis on which various two-nucleon date can be understood systematically and in simple ways, but also it provides a lead in more fundamental field theoretical studies of nuclear forces. Such a picture can also find useful applications in other branches of nuclear physics.

The aim of the present paper is to report on an attempt to interpret the available proton-proton data up to 310 Mev in terms of a potential of the form

$$V=V_C+V_T S_{12}+V_{LS}(\boldsymbol{L}\cdot\boldsymbol{S})+V_Q Q_{12}. \tag{1}$$

Here, the subscripts C, T, LS, and Q stand for the central, tensor, linear LS, and quadratic LS potential, respectively. Q_{12} is the operator

$$Q_{12}=1/2\{(\boldsymbol{L}\cdot\boldsymbol{\sigma}_1)(\boldsymbol{L}\cdot\boldsymbol{\sigma}_2)+(\boldsymbol{L}\cdot\boldsymbol{\sigma}_2)(\boldsymbol{L}\cdot\boldsymbol{\sigma}_1)\}, \tag{2}$$

with eigenvalues

$$\begin{aligned} Q_{12}=2(\boldsymbol{L}\cdot\boldsymbol{S})^2+(\boldsymbol{L}\cdot\boldsymbol{S})-\boldsymbol{L}^2=(1/4)[&J(J^3+2J^2+3J+2)\\ &-L(L^3+2L^2+7L+6)-S(S^3+2S^2+3S+2)]. \end{aligned} \tag{3}$$

* A preliminary account has been published : Prog. Theor. Phys. **24** (1960), 220. The potential reported in the present paper is a slightly improved version of the one reported in the preliminary account.

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The functions V_i ($i=C, T, LS, Q$) depend on the spin and parity of the two-nucleon system. The neutron-proton data will be studied in a separate paper.

It is known that the two-nucleon potential of the form (1) is the most general one consistent with the usual invariance requirements.[1),2)] In general, the functions V_i may depend on the relative momentum p and the angular momentum L as well as on the inter-nucleon distance. In the present work, however, we assume that V_i are the energy independent functions only of the internucleon distance.

Potentials of the form (1) with $V_Q \equiv 0$ have been studied by several authors. Gammel and Thaler[3)] considered a purely phenomenological V_i of Yukawa type. The lack of the pion theoretical foundation in their potential has been criticised elsewhere.[4)] It is remarkable, however, that they have succeeded in reproducing most of the qualitative features of the p-p data up to 310 Mev with such a simple form of V_i. In fact, their work has provided a considerable lead in the course of the present study.

Signell and others[5),6)] have chosen the Gartenhaus potential[7)] together with a phenomenological V_{LS}. Their potential can fit the date up to 150 Mev reasonably well but above this energy only very poorly. It is likely that this shortcoming is due to the unnecessary inflexibility introduced into the potential at small distances by the use of the Gartenhaus potential. No pion theoretical potential can claim its validity in the core region.

The Japanese group, on the other hand, studied the potential (1) with $V_{LS}=V_Q \equiv 0$.[9),10),11)] Following the method proposed by Taketani et al.,[12)] they considered phenomenological inner potentials together with the one-pion-exchange tail. They have been able to reproduce the experimental differential cross section and the polarization below 150 Mev. However, it is very unlikely that this type of potential can be applied successfully to higher energy scattering with only minor modifications so as not to destroy the fit attained at lower energies. Further, the prediction of the potential on the parameter R is in qualitative disagreement with the recent measurement at 150 Mev.[13)] This and perhaps the large negative D (90°) predicted by the potential[11)] indicate that the deviation from the potential expected on the basis of the static pion theory is already significant at 150 Mev. This fact is one of the motivations for the present work.

In its spirit the present work follows the method of Taketani et al.[12)] This is reflected, in particular, in choosing the functional form of V_i. In order to determine parameters specifying thus chosen V_i, we first study the 310 Mev p-p data. We shall find a potential which gives the phase shifts close to the solution 1 of MacGregor et al.[14)] Using this potential we then calculate the phase shifts and the observable quantities at various energies. Wherever the comparison with the experimental data is possible, the agreement is satisfactory.

After the completion of the present work, we were shown a preprint of a recent work by Bryan.[8)] His results as well as the method of approach to the

problem are very similar to ours. The only qualitative difference is in the singlet even parity potential, where we include a certain non-static effect whereas Bryan does not.

§ 2. Functional form of V_i

For the V_i in (1) we have chosen the following forms:

singlet even:

$$\begin{aligned} {}^1V_C^+(x) &= -(g^2/4\pi)\,\mu(e^{-x}/x)\,(1+{}^1a_C^+(e^{-x}/x)+{}^1b_C^+(e^{-x}/x)^2), \\ {}^1V_Q^+(x) &= {}^1G_Q^+\,\mu(e^{-x}/x)\,(1+{}^1a_Q^+(e^{-x}/x)+{}^1b_Q^+(e^{-x}/x)^2), \end{aligned} \tag{4}$$

triplet odd:

$$\begin{aligned} {}^3V_C^-(x) &= (\mu/3)\,(g^2/4\pi)\,(e^{-x}/x)\,(1+{}^3a_C^-(e^{-x}/x)+{}^3b_C^-(e^{-x}/x)^2), \\ {}^3V_T^-(x) &= (\mu/3)\,(g^2/4\pi)\,(e^{-x}/x)\,(1+3/x+3/x^2) \\ &\quad\times(1+{}^3a_T^-(e^{-x}/x)+{}^3b_T^-(e^{-x}/x)^2), \\ {}^3V_{LS}^-(x) &= {}^3G_{LS}^-\mu(e^{-x}/x)^2\,(1+{}^3b_{LS}^-(e^{-x}/x)), \\ {}^3V_Q^-(x) &= {}^3G_Q^-\,\mu(e^{-x}/x)\,(1+{}^3a_Q^-(e^{-x}/x)+{}^3b_Q^-(e^{-x}/x)^2). \end{aligned} \tag{5}$$

Here μ is the pion mass and $x=\mu r$. We have also assumed a hard core of radius ${}^1x_0^+$ and ${}^3x_0^-$ for (4) and (5), respectively.

As far as V_C and V_T are concerned, the potential reduces, for large x, to the well-known one-pion-exchange potential (OPEP) with the effective pion-nucleon coupling constant $g^2/4\pi$. V_Q has the same range μ^{-1} as V_C and V_T. We shall see later, however, that $|{}^1G_Q^+|, |{}^3G_Q^-| \ll g^2/4\pi$. From this and the eigenvalues of Q_{12} given in (3), it is seen that only higher partial waves are appreciably affected by V_Q. At lower energies, where impact parameters of those higher partial waves are large, the tail of the potential (4) and (5) is effectively that of OPEP. This point will be discussed later in more detail.

The parameters a_C and a_T in (4) and (5) determine the deviation of the potential from the OPEP around the pion range $x=1$. In this region we consider the pion theory of nuclear force to be qualitatively reliable.[9),15),16)] Then we expect

$$ {}^1a_C^+ \gg 0, \quad {}^3a_C^- \ll 0, \quad {}^3a_T^- < 0. \tag{6}$$

The parameters b_C and b_T will be determined purely phenomenologically by analysing the 310 Mev p-p data.

All the available pion theoretical calculations predict negative ${}^3V_{LS}^-(x)$ with the range $(2\mu)^{-1}$. This is reflected in (5). Since we cannot expect the quantitative validity of the pion theoretical prediction regarding V_{LS} at present, the parameters appearing in ${}^3V_{LS}^-(x)$ will be determined purely phenomenologically.

§ 3. Choice of the phase shift solution at 310 Mev

In order to determine the parameters in (4) and (5), we first study the p-p data at 310 Mev. This will be a natural starting point for the following reasons. First, the experimental data are most abundant at this energy.[17] Consequently, an extensive phase shift analysis has been possible yielding only two solutions.[14]* Second, 310 Mev is sufficiently high so that the scattering, particularly P-wave scattering, is affected by the interaction in the inner-most region. This is necessary for the determination of the parameters b and the hard core radius in (4) and (5). Yet, it may be expected that the non-relativistic potential picture still retains its usefulness at this energy.

We have tried to fit MMS 1. We can enumerate a few reasons for this choice, although none of them is quite convincing. First, apart from the interpretation of phase shifts in terms of potential pictures, MacGregor et al.[14] found in their analysis that MMS 1 is slightly favoured over MMS 2.

Second, it appears that the singlet even phase shifts** of MMS 2 are very hard, if not impossible, to understand in terms of the energy independent potential picture. From Table I we see that, as far as the singlet even phase shifts are

Table I. Singlet even parity phase shifts at 310 Mev. The entries are the nuclear Blatt-Biedenharn phase shifts in radians.

	MMS 1	MMS 2	Bryan[8]	calculated from (4) and (11) with ${}^1V_Q^+ \equiv 0$	calculated from (4), (11) and (12)
${}^1S_0$	−0.156	−0.506	−0.058	−0.150	−0.150
${}^1D_2$	0.207	0.083	0.216	0.264	0.228
${}^1G_4$	0.013	0.015	0.021	0.035	0.021
${}^1I_6$				0.009	0.002

concerned, MMS 2 is characterised by the large negative ${}^1S_0$ and small positive ${}^1D_2$ phase shift as compared with MMS 1. ${}^1G_4$ phase shifts are much the same for the two solutions as expected from the way the phase shift analysis has been performed.[14] Unlike in GT, the smallness of ${}^1D_2$ phase shift itself does not cause any difficulty here since it is always possible to adjust ${}^1V_Q^+$ to reduce the ${}^1D_2$ phase shift by a suitable amount without affecting the ${}^1S_0$ phase shift. The difficulty lies in the relative magnitudes of ${}^1D_2$ and ${}^1G_4$ phase shifts of MMS 2. We could not find the ${}^1V_Q^+$ which reduces ${}^1D_2$ phase shift to that of MMS 2 without at the same time reducing ${}^1G_4$ phase shift to substantially negative values.

* We shall, in the following, refer to these two solutions as MMS 1 and MMS 2.

** Throughout the present paper the word phase shift refers to the nuclear Blatt-Biedenharn phase shift. More specifically, it is the Blatt-Biedenharn n-p phase shift in the sense that only the nuclear interaction is included in Schrödinger equations and the phase shift is calculated by comparing the solutions with Bessel functions (not with Coulomb functions).

Further, we did not find it possible to reconcile the large negative $^1S_0$ phase shift of MMS 2 with the shape independent parameters known experimentally at low energies.

Finally, as we shall see in §6, the potential which fits MMS 1 can reproduce all other lower energy data remarkably well. This fact may be taken as an indirect evidence in favour of MMS 1. It seems unlikely that the similar situation holds for MMS 2, although such a possibility cannot be entirely excluded.

§ 4. Triplet odd parity potential

In order to facilitate the qualitative understanding of the relation between the potential and phase shifts, we first write down the Schrödinger equation in the triplet odd parity states. Writing

$$\kappa^2=ME_{CM}/\hbar^2 \quad \text{and} \quad U_i(x)=(M/\hbar^2)\,V_i(x),$$

with M=nucleon mass, they read

$$\begin{aligned}&[d^2/dx^2+\kappa^2-J(J+1)/x^2-U_C(x)-2U_T(x)+U_{LS}(x)\\&\qquad+\{J(J+1)-1\}\,U_Q(x)]v_J(x)=0,\end{aligned}\tag{7}$$

for the uncoupled ($L=J$) states, and

$$\begin{aligned}&[d^2/dx^2+\kappa^2-J(J-1)/x^2-U_C(x)+2(J-1)/(2J+1)\,U_T(x)\\&\qquad-(J-1)\,U_{LS}(x)-(J-1)^2\,U_Q(x)]u_J(x)\\&\qquad-6\sqrt{J(J+1)}/(2J+1)\,U_T(x)w_J(x)=0,\\&[d^2/dx^2+\kappa^2-(J+1)\,(J+2)/x^2-U_C(x)+2(J+2)/(2J+1)\,U_T(x)\\&\qquad+(J+2)\,U_{LS}(x)-(J+2)^2\,U_Q(x)]w_J(x)\\&\qquad-6\sqrt{J(J+1)}/(2J+1)\,U_T(x)u_J(x)=0,\end{aligned}\tag{8}$$

for the coupled ($L=J\pm1$) states. Here appropriate superscripts on $U_i(x)$ are understood (see (5)).

The necessity of the forces other than the central and tensor is most clearly seen in the behaviour of $^3P_0$ and $^3P_1$ phase shifts at 310 Mev. From (5), (7) and (8), it may be seen that V_T is much more important than V_C in these states. The strong positive V_T in the OPEP gives rise to a large positive $^3P_0$ phase shift as seen in the second column of Table II. This is in drastic disagreement with MMS 1. On the other hand, the same V_T yields a large negative $^3P_1$ phase shift consistent with MMS 1. If one wants to retain the OPEP tail, as we do here, the V_T has to become strongly negative for $x\lesssim1$ in order to reduce $^3P_0$ phase shift. This modification in V_T is indeed effective as seen in the third column of Table II. In doing so, however, the agreement of $^3P_1$ phase shift with MMS 1 is destroyed.

Table II. Triplet odd parity phase shifts at 310 Mev. Entries are nuclear Blatt-Biedenharn phase shifts in radians.

	MMS 1	OPEP*	from (5) and (9) but ${}^3V_{LS}^-={}^3V_Q^-\equiv 0$	from (5) and (9) but ${}^3V_Q^-\equiv 0$	from (5) and (9)
${}^3P_0$	−0.197	0.753	0.158	−0.206	−0.213
${}^3P_1$	−0.480	−0.488	−0.262	−0.452	−0.450
${}^3F_3$	−0.062	−0.086	−0.061	−0.068	−0.063
${}^3H_5$	−0.020	−0.025	−0.022	−0.022	−0.019
${}^3P_2$	0.291	0.154	0.071	0.321	0.316
${}^3F_2$	0.020	−0.101	−0.100	0.013	0.007
ϵ_2	−0.051	−0.825	−1.067	−0.091	−0.090
${}^3F_4$	0.056	0.050	0.047	0.078	0.070
${}^3H_4$	0.006	−0.016	−0.009	0.002	−0.002
ϵ_4	−0.215	−0.675	−0.637	−0.391	−0.413
${}^3H_6$	0.004	0.015	0.015	0.015	0.012
${}^3L_6$		−0.007	−0.006	−0.006	−0.008
ϵ_6		−0.733	−0.711	−0.779	−0.788

* $g^2/4\pi=0.08$ and ${}^3x_0^-=0.32$.

Again referring to (7) and (8), it is now easy to see that a strong negative V_{LS} can meet this difficulty. Thus, roughly speaking, the role played by the strong negative V_{LS} is the following. It cooperates with the negative V_T at $x\lesssim 1$ to reduce the ${}^3P_0$ phase shift further down to the MMS 1 value. On the other hand, it also cancels the increase in the ${}^3P_1$ phase shift caused by the negative V_T at $x\lesssim 1$.

The effect of V_{LS} in the coupled states is not so easy to see. Perhaps the most important effect here is the large increase in ϵ_2. This increase is of particular importance in fitting the polarization data.[10]

In Table II we note that the phase shifts for $L\gtrsim 3$ calculated without V_{LS} are in substantial agreement with those of MMS 1. This indicates that V_{LS} must be short ranged.

Based on these qualitative understanding of the effects of various forces, we tried some 150 combinations of parameters appearing in (5). The condition (6) has been imposed on all of them. For some of the phase shifts thus obtained the observable quantities have been calculated and directly compared with the available data. We have finally decided on the set

$$\begin{aligned} &g^2/4\pi=0.08,\ {}^3G_{LS}^-=0.1541,\ {}^3G_Q^-=0.00045,\\ &{}^3x_0^-=0.32,\ {}^3a_C^-=-9,\ {}^3b_C^-=4.6,\ {}^3a_T^-=-1.14,\\ &{}^3b_T^-=0.2,\ {}^3b_{LS}^-=-8.09,\ {}^3a_Q^-=10,\ {}^3b_Q^-=6. \end{aligned} \tag{9}$$

The potential is plotted in Fig. 1 and the phase shifts are given in the last column of Table II.

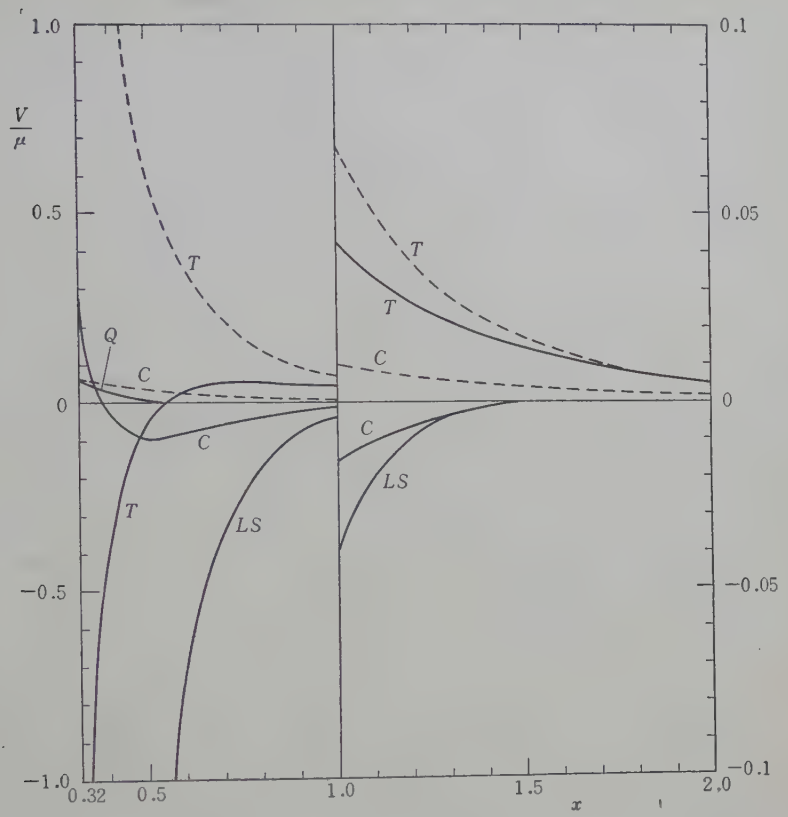

Fig. 1. The triplet odd parity potential given by (5) and (9). The dashed curves represent the OPEP with $g^2/4\pi=0.08$.

Among other things, (9) gives an extremely weak ${}^3V_Q^-$. Its effect on the phase shifts are also quite small. Nevertheless, we have decided to include ${}^3V_Q^-$ because it was found that the fit to the experimental differential cross section at 310 Mev is noticeably improved by doing so. The decrease in the ${}^3F_4$ phase shift, although very slight, appears to be mainly responsible for the improvement. At lower energies, the effect of ${}^3V_Q^-$ is entirely negligible.

(9) shows that our ${}^3V_{LS}^-$ is composed of a weak repulsion of range $(2\mu)^{-1}$ and a strong attraction of range $(3\mu)^{-1}$. The main function of the former is to weaken the latter at $x\gtrsim 1$. This was found necessary mainly in order not to produce too large a ${}^3F_4$ phase shift.

The weak repulsive part of ${}^3V_{LS}^-$ should not be taken literally, since the argument here depends on the particular functional form assumed for ${}^3V_{LS}^-$. Our finding is simply that the strongly attractive part of ${}^3V_{LS}^-$ should not extend too far beyond $x\simeq 1$.

In this connection, we often felt during the course of calculations that the functional form of ${}^3V_{LS}^-$ assumed in (5) might not be quite adequate. Indeed,

Table III. Comparison of various proposed ${}^3V_{LS}^-$. Entries are the ${}^3V_{LS}^-(x)$ in Mev at several internucleon distances $x=\mu r$.

x	GT[3]	SM[5]	present work	Bryan[8]
0.5	−198.5	−88.1	−278.8	−550.0
0.8	−25.8	−17.9	−24.0	−14.4
1.0	−7.53	−7.95	−5.72	−1.62
1.2	−2.12	−3.94	−1.39	−0.25
1.5	−0.35	−1.51	−0.10	−0.02
2.0	−0.03	−0.42	+0.04	−0.00

Bryan[8] finds a monotonic ${}^3V_{LS}^-$ of the form e^{-2x}/x^8 with a straight cutoff at $x=0.54$ in his attempt to fit MMS 1. As may be seen in Table III, Bryan's ${}^3V_{LS}^-$ is substantially weaker for $x\gtrsim 1$ but stronger for $x\lesssim 0.7$ than our ${}^3V_{LS}^-$. For his ${}^3V_{LS}^-$ it is clear from what has been said above that no ${}^3V_Q^-$ is required even at 310 Mev.

Summarizing the discussions on ${}^3V_{LS}^-$, it appears from Bryan's work and what we have learned during the calculations that the ${}^3V_{LS}^-$ should be substantially weaker for $x\gtrsim 1$ but stronger for $x\lesssim 0.7$ than GT, SM or SM1. It is satisfying that this conclusion is in better agreement with the pion theoretical predictions on V_{LS}[10),15),16)] than previously thought.

In Tables IV and V we give the triplet odd parity phase shifts calculated

Table IV. Triplet odd parity uncoupled phase shifts. Entries are the nuclear Blatt-Biedenharn phase shifts in radians calculated from (5) and (9).

E (Mev)	${}^3P_0$	${}^3P_1$	${}^3F_3$	${}^3H_5$
10	0.068	−0.039	−0.001	...
18.2	0.117	−0.068	−0.002	...
19.8	0.124	−0.073	−0.003	...
25.63	0.147	−0.090	−0.004	...
30.14	0.160	−0.102	−0.005	...
39.4	0.177	−0.124	−0.008	−0.001
50	0.184	−0.146	−0.012	−0.001
66	0.179	−0.174	−0.017	−0.002
80	0.166	−0.196	−0.021	−0.003
95	0.147	−0.218	−0.025	−0.005
120	0.108	−0.251	−0.031	−0.006
145	0.065	−0.282	−0.037	−0.008
180	0.003	−0.321	−0.044	−0.011
200	−0.032	−0.343	−0.047	−0.012
240	−0.100	−0.384	−0.054	−0.015
270	−0.150	−0.413	−0.058	−0.017
310	−0.213	−0.450	−0.063	−0.019

Table V. Triplet odd parity coupled phase shifts. Entries are the nuclear Blatt-Biedenharn phase shifts in radians calculated from (5) and (9).

E (Mev)	$^3P_2$	$^3F_2$	ϵ_2	$^3F_4$	$^3H_4$	ϵ_4	$^3H_6$	$^3L_6$	ϵ_6
10	0.011	−0.001	−0.319	...	...	...	...	...	...
18.2	0.028	−0.002	−0.336	...	...	...	...	...	...
19.8	0.032	−0.003	−0.336	...	...	...	...	...	...
25.63	0.045	−0.003	−0.332	0.001	−0.001	−0.759	...	...	...
30.14	0.056	−0.004	−0.326	0.001	−0.001	−0.755	...	...	...
39.4	0.078	−0.004	−0.313	0.003	−0.002	−0.748	...	...	...
50	0.103	−0.003	−0.298	0.004	−0.003	−0.741	...	...	...
66	0.139	−0.002	−0.275	0.007	−0.004	−0.730	...	...	...
80	0.168	0.002	−0.257	0.009	−0.005	−0.720	0.001	−0.001	−0.804
95	0.196	0.003	−0.239	0.012	−0.006	−0.708	0.002	−0.002	−0.792
120	0.234	0.006	−0.213	0.017	−0.008	−0.684	0.003	−0.003	−0.792
145	0.264	0.010	−0.191	0.022	−0.008	−0.656	0.004	−0.004	−0.793
180	0.293	0.013	−0.164	0.031	−0.008	−0.609	0.006	−0.005	−0.795
200	0.304	0.014	−0.151	0.036	−0.008	−0.579	0.007	−0.006	−0.795
240	0.317	0.013	−0.126	0.047	−0.006	−0.514	0.009	−0.007	−0.795
270	0.319	0.011	−0.110	0.057	−0.004	−0.464	0.010	−0.008	−0.793
310	0.316	0.007	−0.090	0.070	−0.002	−0.413	0.012	−0.008	−0.788

from (5) and (9) at various energies. We notice that our potential predicts the phase shift solution type *b* of MacGregor and Moravcsik at 210 Mev.[23]

§ 5. Singlet even parity potential

The Schrödinger equation for the singlet even parity state reads

$$[d^2/dx^2+\kappa^2-J(J+1)/x^2-U_C(x)+J(J+1)\,U_Q(x)]u(x)=0. \qquad (10)$$

Previous calculations on the zero energy *n-p* scattering parameters[18] have provided a convenient basis for the determination of $^1a_C^+$, $^1b_C^+$ and $^1x_0^+$ in (4). The combination

$$g^2/4\pi=0.08,\ \ ^1x_0^+=0.337,\ \ ^1a_C^+=10,\ \ ^1b_C^+=8, \qquad (11)$$

was found to give the zero energy scattering length -17.5×10^{-13} cm, the effective range 2.85×10^{-13} cm and at the same time the $^1S_0$ phase shift of −0.150 at 310 Mev (see Table I).

The $^1D_2$, $^1G_4$ and $^1I_6$ phase shifts calculated from (11) (with $^1V_Q^+\equiv0$) are shown in Table I. It is clear from (10) that a weak attractive (negative) $^1V_Q^+(x)$ can improve the fit to MMS 1. Indeed, it was easy to fit the $^1D_2$ phase shift of MMS 1 exactly. It was found, however, that the $^1D_2$ phase shifts calculated at lower energies, particularly at 145 Mev where accurate data are available, from

such a potential turned out too small.* After some readjustment we decided on the set

$$ {}^1G_Q^+ = -0.00155, \quad {}^1a_Q^+ = 8 \quad \text{and} \quad {}^1b_Q^+ = 6. \tag{12} $$

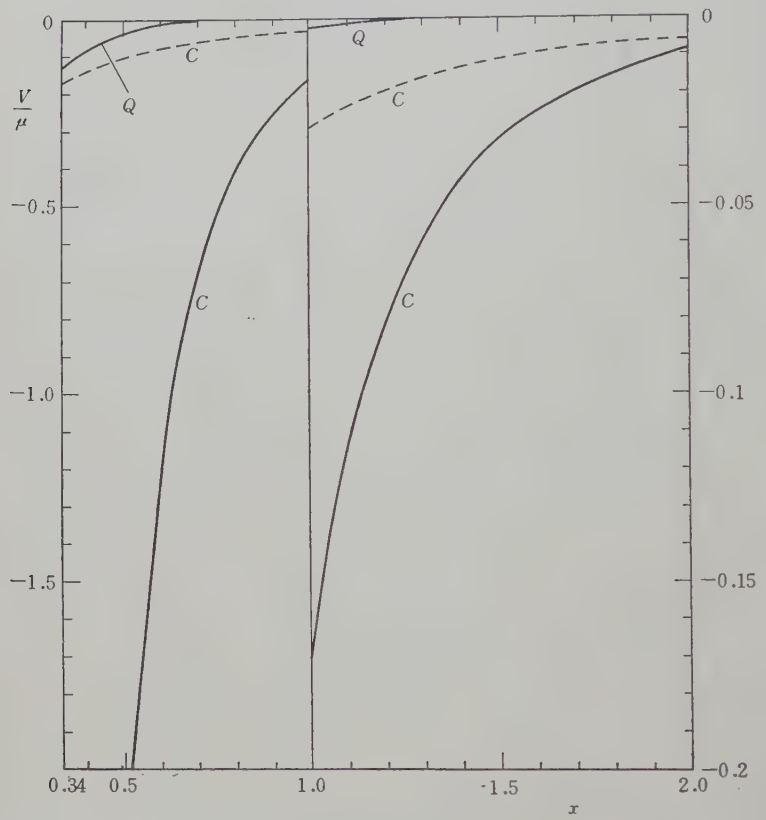

Fig. 2. The singlet even parity potential given by (4), (11) and (12). The dashed curve represents the OPEP with $g^2/4\pi = 0.08$.

The potential determined by (11) and (12) is shown in Fig. 2. The phase shifts are given in Table VI. We again note that our phase shifts at 210 Mev are of type b of MacGregor et al.[23]

Unlike in the triplet odd states, the V_Q is really needed in the singlet even state. As already noticed by GT, the ${}^1D_2$ phase shift at 310 Mev is largely determined by the shape independent parameters at zero energy and the 310 Mev ${}^1S_0$ phase shift, and must be about 0.26 as long as the interaction is assumed to be the same in ${}^1S_0$ and ${}^1D_2$ states. Such a large ${}^1D_2$ phase shift gives rise to an appreciable forward peak in the differential cross section at 310 Mev. Similar arguments hold for the ${}^1G_4$ phase shift.

* It is likely that this difficulty is due to an inadequate functional form of ${}^1V_Q^+$ assumed in (4).

Table VI. Singlet even parity phase shifts. Entries are the Blatt-Biedenharn phase shifts in radians calculated from (4), (11) and (12).

E (Mev)	$^1S_0$	$^1D_2$	$^1G_4$	$^1I_6$
10	1.000	0.003	...	...
18.2	0.912	0.008	...	...
19.8	0.897	0.009	...	...
25.63	0.844	0.014	...	...
30.14	0.806	0.017	...	...
39.4	0.739	0.024	0.001	...
50	0.670	0.032	0.002	...
66	0.582	0.045	0.003	...
80	0.514	0.057	0.004	...
95	0.448	0.069	0.005	...
120	0.350	0.090	0.007	...
145	0.265	0.114	0.010	0.001
180	0.159	0.139	0.012	0.001
200	0.104	0.154	0.013	0.001
240	0.004	0.184	0.017	0.001
270	−0.065	0.204	0.019	0.001
310	−0.150	0.228	0.021	0.002

In this connection it is interesting to note a remark by Signell[19] regarding the non-static effect in the nuclear forces. Using the formulation in momentum space, he noticed that one of the non-static effects is to reduce (in magnitude) the OPEP tail at high energies. Sugawara[20] argues to the same effect in connection with his recent calculations with Okubo.[16] We see that the introduction of our weak negative $^1V_Q^+$ has just such an effect. At low energies when the S-wave scattering predominates and all the higher partial waves have large impact parameters, $^1V_Q^+$ has no effect. As the energy increases, $^1V_Q^+$ becomes effective but the effect is felt more strongly by higher partial waves (see (10)) which are scattered by the tail of the potential. The Signell effect must also be felt by the S-wave at high energies. However, since the S-wave already feels very strong attraction in the core region (see (11)), the small reduction in the OPEP tail should be relatively unimportant.

We did not find a conclusive evidence for $^3V_Q^-$. This probably means that the Signell effect is masked by the strong $^3V_{LS}^-$ in the triplet odd parity state. It is quite natural that we found $^1V_Q^+$ is really needed since here $V_{LS}\equiv 0$.

Bryan[8] has not considered any non-static potential in the singlet even parity states. We believe that an appropriate $^1V_Q^+$ added to his potential will improve the fit to the data considerably.

§ 6. Observable quantities

Using the phase shifts given in Tables IV, V and VI, we have calculated

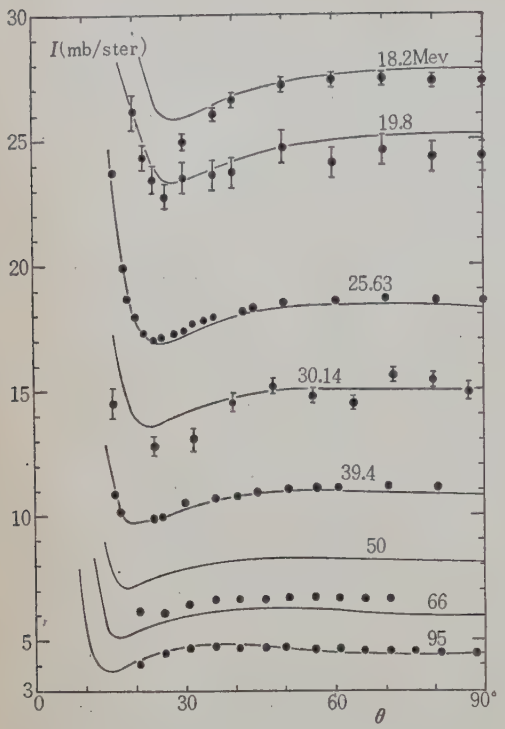

Fig. 3. p-p differential cross sections below 100 Mev. Experimental points are taken from references 24, 25, 26 and 27.

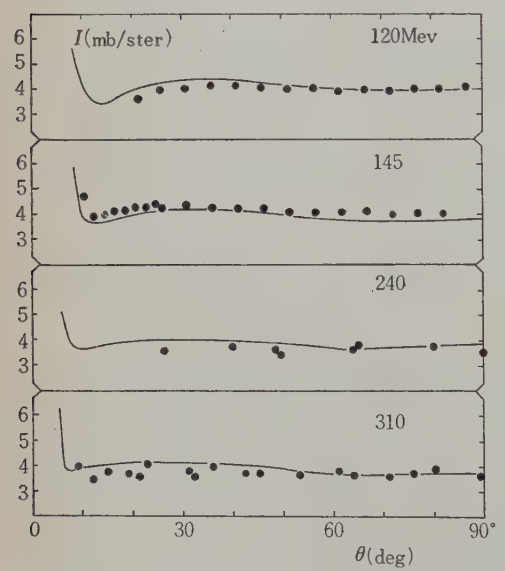

Fig. 4. p-p differential cross sections above 100 Mev. Experimental points are taken from references 17, 24, and 25.

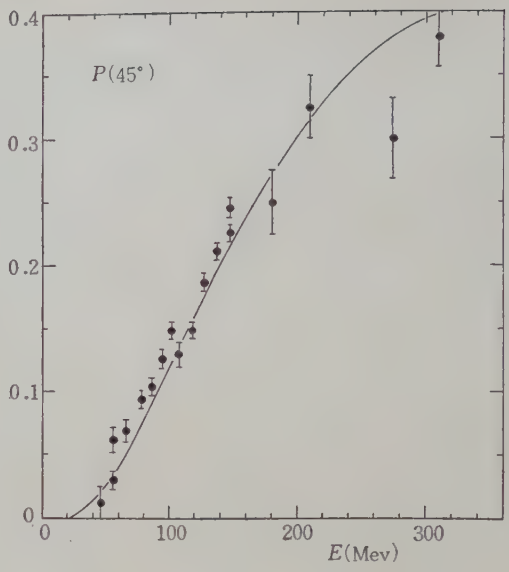

Fig. 5. Plot of $P(45°)$ *vs* energy. Experimental points are taken from references 17, 24, and 25.

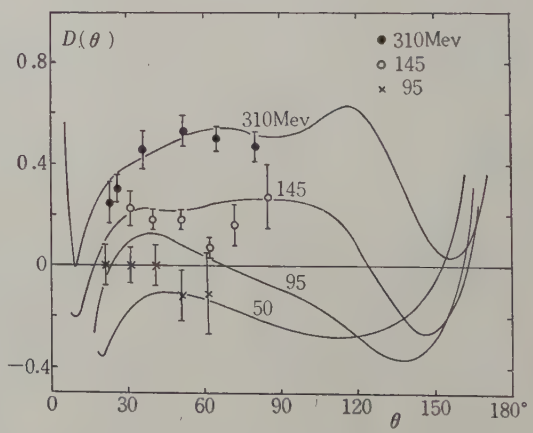

Fig. 6. Plot of D *vs* θ at several energies. Experimental points are taken from references 17, 21, and 28.

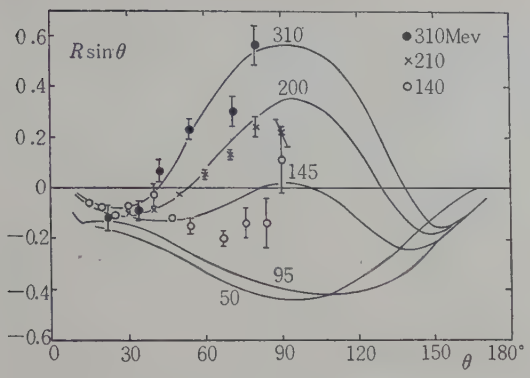

Fig. 7. Plot of $R\sin\theta$ *vs* θ at several energies. Experimental points are taken from references 13, 17, and 29.

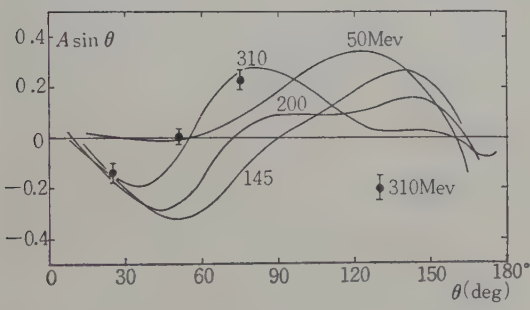

Fig. 8. Plot of $A\sin\theta$ *vs* θ at several energies. Experimental points are taken from reference 17.

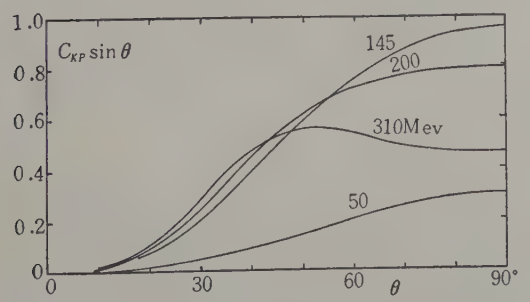

Fig. 9. Plot of $C_{kp}\sin\theta$ *vs* θ at several energies.

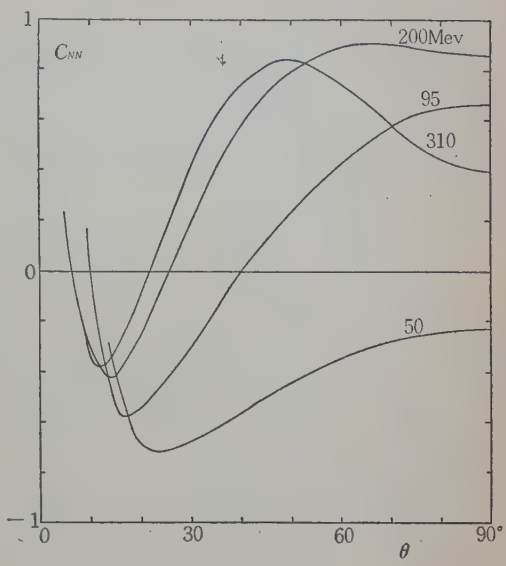

Fig. 10. Plot of C_{nn} *vs* θ. Relevant experimental data are[29]: C_{nn}(315 Mev, 90°) $=0.52\pm0.20$ (Dubna) and C_{nn}(320 Mev, 90°) $=0.75\pm0.11$ (Liverpool).

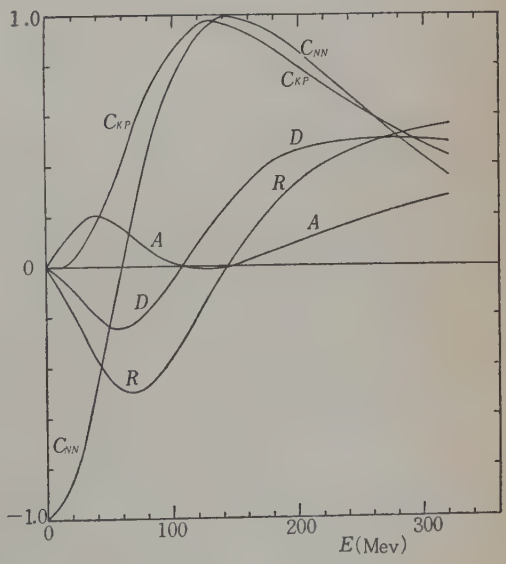

Fig. 11. Plot of triple scattering and spin correlation parameters at 90° *vs* E.

the complete experiment parameters. Some of them are shown in Figs. 3 to 11. In view of the wide energy range covered the fit to the data is remarkable.

Regarding the polarization, we have only shown $P(45°)$ as the function of energy. The angular distribution of P is in general in good agreement with the experimental data. The largest deviation from the dada occurs at 145 Mev where the calculated P is smaller than the observed by about 0.04 in the $15°\sim35°$ region.

At 145 Mev, the calculated $D(\theta)$ agrees with the Harvard data.[21] According to Nigam,[22] this is due to our small ${}^1P_0$ phase shift which, in turn, is a consequence of the strong negative ${}^3V_{\bar{L}S}$ as discussed in § 4.

An interesting feature of our potential can be seen from Fig. 11. Our potential predicts $C_{nn}(90°)\simeq 1$ at about 140 Mev indicating that there is no contribution to the scattering from the singlet state at this energy and angle.* Consequently, $R(90°)\simeq A(90°)\simeq 0$ at the same energy. $R(90°, 140\text{ Mev})=0$ is consistent with the recent measurements.[13] Fortunately, it appears that the Harvard and Harwell groups are both working on triple scattering measurements at the energy close to 140 Mev. It will be very interesting to see whether further measurements realize our prediction. If they do, we should be able to get valuable information on the singlet even parity phase shifts at this energy. It may become possible to exclude one or the other of MMS solutions on that ground.

§ 7. Conclusions

We have seen that the singlet even and triplet odd parity potentials defined by (1), (4), (5), (9), (11) and (12) can reproduce all available p-p data up to 310 Mev. At 310 Mev our potential predicts the phase shifts solution 1 of MacGregor et al.[14]

As far as the tail is concerned, our potential has the OPEP tail at low energies. At higher energies it appears that the OPEP tail has to be reduced by small amount. This effect is particularly manifest in the singlet even parity states. Such an effect can be incorporated into the energy independent potential picture in the form of a weak but long ranged quadratic spin-orbit potential.

The LS potential required in the precise fit at 310 Mev is more singular but of shorter range than those of GT, SM and SM1.** It is satisfying that this feature of the LS potential is in better agreement with the predictions of the pion theory of nuclear forces.

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* This can be seen from the expressions for observable quantities in terms of various scattering amplitudes. See reference 30).

** In this connection we feel that, if any substantial improvement is to be attempted over our potential, the functional form of ${}^3V_{\bar{L}S}$ will have to be modified. The modification must be in such a way to make the ${}^3V_{\bar{L}S}$ considerably weaker at $x\gtrsim 1$ without having recourse to the repulsive part in it. Such a modification may well lead to the ${}^3V_{\bar{L}S}$ considered by Bryan.[8]

Our central and tensor potentials are consistent with the pion theory of nuclear forces in the sense that they satisfy the conditions (6). The LS potential now appears to be not inconsistent with the pion theory. Further, there is some pion theoretical evidence for the weakening of the OPEP tail at higher energy.[19] Thus we are led to the conclusion that the proton-proton scattering up to 310 Mev can be understood in terms of the energy independent potential which is consistent with the pion theory of nuclear forces in every respect. We shall consider the same question in the neutron-proton scattering in a separate paper.

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I am grateful to Professor H. Messel for making excellent research facilities available for the present work. I would also like to thank Professor S. T. Butler for his interest shown in this work and continual encouragement. Professor J. M. Blatt led me into the use of SILLIAC three years ago through the numerical solution of Schrödinger equations. Many of his advices received at that time proved indispensable for the present work. I record here my indebtedness to him. Frequent communications with the Japanese nuclear force group through Dr. W. Watari have been most stimulating. Finally, I thank the members of the Basser Computing Department for their assistance with the use of SILLIAC.

References

1) L. Puzikov, R. Rydin and J. Smorodinsky, Nucl. Phys. **3** (1957), 436.
2) S. Okubo and R. E. Marshak, Ann. Phys. **4** (1958), 166.
3) J. L. Gammel and R. M. Thaler, Phys. Rev. **107** (1957), 291.
This will be referred to as GT.
4) S. Otsuki, Prog. Theor. Phys. **20** (1958), 171.
R. Tamagaki, Prog. Theor. Phys. **20** (1958), 505.
5) P. S. Signell and R. E. Marshak, Phys. Rev. **109** (1958), 1229.
This will be referred to as SM.
6) P. S. Signell R. Zinn and R. E. Marshak, Phys. Rev. Letters **1** (1958), 416.
This will be referred to as SM1.
7) S. Gartenhaus, Phys. Rev. **100** (1955), 900.
8) R. A. Bryan, Nuovo Cimento **XVI** (1960), 895.
9) J. Iwadare, S. Otsuki, R. Tamagaki and W. Watari, Suppl. Prog. Theor. Phys. No. **3** (1956), 32.
Also see reference 4) and W. Watari, Prog. Theor. Phys. **20** (1958), 181.
10) T. Hamada, J. Iwadare, S. Otsuki, R. Tamagaki and W. Watari, Prog. Theor. Phys. **22** (1959), 566.
11) T. Hamada, J. Iwadare, S. Otsuki, R. Tamagaki, and W. Watari, Prog. Theor. Phys. **23** (1960), 366.
12) M. Taketani, S. Nakamura and M. Sasaki, Prog. Theor. Phys. **6** (1951), 581.
M. Taketani, Suppl. Prog. Theor. Phys. No. **3** (1956), 1.
13) L. Bird, D. N. Edwards, B. Rose, A. E. Taylor and E. Wood, Phys. Rev. Letters. **4** (1960), 302.
E. Thorndike, J. Lefrancois and R. Wilson, Bull. Am. Phys. Soc. **5** (1960), 282.
14) M. H. MacGregor, M. J. Moravcsik and H. P. Stapp, Phys. Rev. **116** (1959), 1284.
15) R. J. N. Phillips, Reports on Progress in Physics **XXII** (1959), 562.
(Physical Society, London).

16) M. Sugawara and S. Okubo, Phys. Rev. **117** (1960), 605, 611.
17) O. Chamberlain, E. Segre, R. D. Tripp, C. Wiegand and T. Ypsilantis, Phys. Rev. **105** (1957), 288.
18) T. Hamada, Prog. Theor. Phys. **24** (1960), 126.
19) P. S. Signell, Prog. Theor. Phys. **22** (1959), 492.
20) M. Sugawara, private communication.
21) C. F. Hwang, T. R. Ophel, E. H. Thorndike, R. Wilson and N. F. Ramsey, Phys. Rev. Letters **2** (1959), 310.
22) B. P. Nigam, Prog. Theor. Phys. **23** (1960), 61.
23) M. H. MacGregor and M. J. Moravcsik, Phys. Rev. Letters **4** (1960), 524.
24) W. N. Hess, Rev. Mod. Phys. **30** (1958), 368.
25) J. N. Palmieri, A. M. Cormak, N. F. Ramsey and R. Wilson, Ann. Phys. **5** (1958), 299.
26) T. H. Jeong, L. H. Johnston and D. E. Young, Phys. Rev. **118** (1960), 1080.
27) L. H. Johnston and D. A. Swanson, Phys. Rev. **111** (1958), 212.
28) E. H. Thorndike and T. R. Ophel, Phys. Rev. **119** (1960), 362.
29) Y. Smorodinsky, Kiev Conference, 1959.
30) H. P. Stapp, T. Ypsilantis and N. Metropolis, Phys. Rev. **105** (1957), 302.

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Scattering of Electrons by the Thermal Motion of Impurity Ions. II

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The change of electrical resistance of dilute alloys due to the thermal motion of impurity ions at low temperatures is calculated with the use of a variational method. The obtained change of electrical resistance depends on T^2 at the limit of low temperature and its magnitude is larger than that of the normal phonon-scattering resistance when the temperature is sufficiently low.

§ 1. Introduction

We discussed the change of electrical resistance of dilute alloys due to the thermal motion of impurity ions at high temperature in the previous paper[1] (referred hereafter as Part I). It was obtained there that the change of electrical resistance depends linearly on the temperature and also on the concentration of impurity, and that its magnitude is comparable to the experimental deviation of electrical resistance from Matthiessen's rule.

In the present paper we shall calculate the change of electrical resistance due to the thermal motion of impurity ions at low temperature in the cases of impure monovalent metals. As will be shown in the next section, it depends on T^2 which is due to the lack of the conservation of total wave number in the scattering process. The ratio of the resistance due to the thermal motion of impurity ions to the normal phonon-scattering resistance is therefore dependent on the temperature and the former becomes predominant as the temperature decreases.

In the next section the electrical resistance will be calculated with the use of the transition probability obtained in Part I and the variational method which is due to Kohler[2] and Sondheimer.[3] In § 3 discussions will be given on the result obtained in § 2.

As is well known,[4] a minimum of electrical resistance has been observed at low temperatures in many kinds of impure noble metals and also a maximum has been observed in some of them. These facts show that the effect of the thermal motion of impurity ions cannot be detected in these noble metals. Careful observations on alkali metals may explain the thermal motion of impurity ions experimentally.

§ 2. Low temperature resistance

It is convenient for the calculation of the low temperature electrical resistance

to make use of the variational method which is due to Kohler[2] and Sondheimer.[3] When only an electric field is present in the direction of x-axis, the Bloch equation may be written in the form

$$-\frac{df_0}{dE}=\Lambda\varphi(E) \tag{1}$$

in the first order approximation, where

$$\Lambda\varphi(E)=\frac{1}{8\pi^3\kappa T}\int P(\boldsymbol{k}'\boldsymbol{k})f_0(E)\{1-f_0(E')\}\cdot\left\{\varphi(E)-\frac{(\boldsymbol{k}'\cdot\boldsymbol{k})}{k^2}\varphi(E')\right\}d\boldsymbol{k}', \tag{2}$$

and f_0 is the equilibrium distribution function and φ is defined by the expansion of the distribution $f(\boldsymbol{k})$ as follows:

$$f(\boldsymbol{k})=f_0(E)-eFk_x\frac{\hbar}{m}\varphi(E)\frac{df_0}{dE}. \tag{3}$$

Other quantities in the above equations have been defined in Part I. The transition probability $P(\boldsymbol{k}'\boldsymbol{k})$ has been obtained in Part I as

$$P(\boldsymbol{k}'\boldsymbol{k})=Nx\sum_q\frac{16\pi^3Z^2e^4}{d\omega}\frac{|\boldsymbol{k}-\boldsymbol{k}'|^2}{\{p^2+|\boldsymbol{k}-\boldsymbol{k}'|^2\}^2}$$
$$\times\{n\delta(E'-E-\hbar\omega)+(n+1)\delta(E'-E+\hbar\omega)\}, \tag{4}$$

provided that frequencies of phonons are independent of the direction of polarization, where p is the screening constant of impurity potential. The electrical conductivity σ is given by

$$\sigma=e^2L/T, \tag{5}$$

where L is defined by

$$L=-(\hbar^2T/4\pi^3m^2)\int k_x^2\varphi\frac{df_0}{dE}d\boldsymbol{k} \tag{6}$$

in terms of the solution of Eq. (1).

Inserting the formal solution of the transport equation (1), which is obtained with the use of the variational method,[3] into Eq. (6), we obtain

$$L=(\hbar^2T/4\pi^3m^2)(a_0b_0+a_1b_1), \tag{7}$$

where

$$a_0=(b_1d_{01}-b_0d_{11})/(d_{10}^2-d_{11}d_{00}), \tag{8}$$

$$a_1=(b_0d_{10}-b_1d_{00})/(d_{10}^2-d_{11}d_{00}), \tag{9}$$

$$b_n=-\int k_x^2\eta^n\frac{df_0}{dE}d\boldsymbol{k} \tag{10}$$

and

$$d_{nm}=\int k_x^2\eta^n\Lambda\eta^m d\boldsymbol{k}. \tag{11}$$

In the above definitions, η is the dimensionless parameter defined by $\eta=(E-\zeta)/\kappa T$, in which ζ is the Fermi energy. When f_0 is the Fermi distribution, Eq. (10) is reduced to

$$b_0=(2\pi/3)\,(2m/\hbar^2)^{5/2}\zeta^{3/2} \tag{12}$$

and

$$b_1=(\pi^2/2)\beta b_0, \tag{13}$$

where $\beta=\kappa T/\zeta$. From (2) and (4), Eq. (11) is reduced to

$$d_{mn}=A\int_{-\infty}^{\infty}\eta^m\,d\eta\int_{-\theta/T}^{\theta/T}\frac{z\,dz}{(1+e^{-\eta})\,(1+e^{\eta+z})\,(1-e^{-z})}$$
$$\times\left[\frac{1}{2}\left\{(1+\beta\eta)\,\eta^n-\left(1+\beta\eta+\frac{1}{2}\beta z\right)(\eta+z)^n\right\}G+\left(1+\beta\eta+\frac{1}{2}\beta z\right)(\eta+z)^nF\right], \tag{14}$$

where $A=16Z^2e^4m^3(\kappa T)^2\zeta\ Nx$, $z=\hbar\omega/\kappa T$, and F and G are defined by

$$F=1-(p^2/2w^2)\log(1+4w^2/p^2)+p^2/(p^2+4w^2) \tag{15}$$

and

$$G=\log(1+4w^2/p^2)-4w^2/(p^2+4w^2), \tag{16}$$

respectively, in which w is the wave number at the top of the Fermi sphere. Neglecting terms which are of the order of β^2 or higher orders, we obtain

$$d_{00}=2AFJ_2, \tag{17}$$

$$d_{10}=d_{01}=A\beta\left\{\frac{1}{6}\,(4\pi^2J_2+J_4)F+\frac{1}{4}J_4G\right\} \tag{18}$$

and

$$d_{11}=A\left\{\frac{1}{3}\,(2\pi^2J_2-J_4)F+\frac{1}{2}J_4G\right\}, \tag{19}$$

where

$$J_n=\int_0^{\theta/T}\frac{z^n\,dz}{(e^z-1)\,(1-e^{-z})}. \tag{20}$$

As is seen from (17), (18) and (19), d_{10}^2 may be neglected in comparison with $d_{11}d_{00}$ since the ratio of the former to the latter is of the order of β^2. Similarly b_1d_{01} may be neglected compared with b_0d_{11}. Eq. (8) is therefore reduced to

$$a_0=b_0/d_{00}. \tag{21}$$

Eq. (9) is also reduced to

$$a_1=(b_0d_{10}-b_1d_{00})/d_{11}d_{00}, \tag{22}$$

which is the first order quantity with respect to β. Since both of a_1 and b_1 are first order quantities with respect to β, while both of a_0 and b_0 are zeroth order quantities, a_1b_1 may be neglected compared with a_0b_0. Eq. (7) is therefore reduced to

$$L=(\hbar^2T/4\pi^3m^2)\,a_0b_0. \tag{23}$$

Inserting (23) into (5), we obtain the electrical resistance

$$\rho=\frac{2\sqrt{2}\,Z^2e^2m^{3/2}F(\kappa T)^2xJ_2}{\pi d\hbar^3v^3\zeta^{1/2}}. \tag{24}$$

The integral J_2 defined by Eq. (20) can be easily calculated at each limit of temperature and we obtain

$$J_2=\begin{cases}2.4 & (T\to 0),\\ \theta/T & (T/\theta\gg 1).\end{cases} \tag{25}$$

With the replacement of J_2 by θ/T Eq. (24) is reduced to the high temperature resistance which has been obtained in Part I. At the limit of low temperature, on the other hand, Eq. (24) is reduced to

$$\rho=\frac{4.8\sqrt{2}\,Z^2e^2m^{3/2}Fx\kappa^2T^2}{\pi d\hbar^3v^3\zeta^{1/2}}. \tag{26}$$

It is noted that the resistance (26) depends on T^2, which is due to the lack of the conservation of total wave number in the scattering process discussed in Part I.

§ 3. Discussions

To estimate our result (26) we first compare it with the normal phonon-scattering resistance ρ_{ph} which depends on T^5 at the limit of low temperature. With the use of the well-known low temperature value of ρ_{ph},[5] ρ/ρ_{ph} is obtained as

$$\rho/\rho_{ph}=B^3Z^2x/T^3, \tag{27}$$

where $B^3=6.4\times10^{-5}\;AF\;\zeta\theta^6/dv^3C^2$ in which A is the atomic weight of solvent atom. From (27) the temperature T_c at which $\rho/\rho_{ph}=1$ is obtained as

$$T_c=B(Z^2x)^{1/3}. \tag{28}$$

Eq. (27) shows that the electrical resistance due to the scattering by the thermal motion of impurity ions is larger than the normal phonon-scattering resistance when the temperature is lower than T_c. Though Eq. (28) is only an approximate estimation of the value of T_c, it is useful as a rough estimation of T_c when the value given by it is sufficiently smaller than θ. Values of B are listed, in the Table, for monovalent metals. From Eq. (28) and the Table it is obtained, for example, that T_c of a Cu-based alloy which contains 0.1% of divalent element is 7°K.

Table. Values of B

	Li	Na	K	Cu	Ag	Au
B	1.3×10^2	70	45	46	40	27

Next, let us compare the result (26) with the residual resistance. In order to estimate the order of magnitude of the residual resistance we make use of Mott's calculation.[6] The ratio of (26) to the residual resistance given by Mott[6] is then obtained as

$$\rho/\rho_{res}=9.2\times10^{22}\frac{\zeta}{dv^3}\frac{F}{G}T^2, \tag{29}$$

$$\simeq 9\times10^{-7}\frac{F}{G}T^2 \quad \text{for Cu,} \tag{30}$$

where F and G have been defined by (15) and (16), respectively. Since it is easily shown that the magnitude of G is usually comparable with that of F, it is obtained that the magnitude of ρ/ρ_{res} is of the order of $10^{-6}T^2$ for Cu-based alloys. From the above estimations it is concluded that the depndence of temperature-dependent part of resistance on the temperature becomes slow and approaches T^2 as the temperature decreases beyond the critical temperature T_c which is given roughly by (28) if the resistance due to all other mechanisms is negligible. This behaviour, however, will be difficult to observe unless the concentration of impurity is sufficiently large, since the temperature-dependent resistance is only of the order of $10^{-6}T^2$ times of the residual resistance.

It is experimentally well known[4] that the electrical resistance of impure noble metals has usually marked anomalies at low temperatures. Since it is obvious from the above discussions that the effect of the thermal motion of impurity ions does not give rise to such anomalies, it is probable that other mechanisms are predominant in noble metals. For alkali metals, on the other hand, no sufficiently detailed experiment on the electrical resistance of impure samples has been made at very low temperatures. On highly pure samples it has been made by MacDonald and Mendelssohn.[7] Since their samples contain impurities of the order smaller than 10^{-3}, it is expected that the effect of the thermal motion of impurity ions is very difficult to be observed. In fact they have found no anomaly for Na. But for Li and Rb they observed that the temperature-dependence of electrical resistance is lower than that expected by Gruneisen's law. Since the value of the Debye temperature decreases with the temperature in these metals and the effect of the decrease has the tendency to lower the temperature dependence of the electrical resistance, more experimental data are necessary in order to decide whether the thermal motion of impurity ions contributes to MacDonald and Mendelssohn's observation. The essential difference of the effect of the thermal motion of impurity ions from that of the decrease of the Debye temperature is the dependence on the concentration of impurity since the temperature dependence of the Debye temperature is expected to depend slightly on the concentration of impurity.

Finally, it is noted that the effect of the temperature-dependent part of the screening constant which has been obtained in Part I is smaller. In fact its ratio to the effect of the thermal motion of impurity ions is of the order of

$$10^6(\kappa/\zeta)^2 \simeq 3\times 10^{-4} \quad \text{for Cu,}$$

which is much smaller than unity.

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References

1) S. Koshino, Prog. Theor. Phys. **24** (1960), 484.
2) M. Kohler, Z. Physik **124** (1948), 772; **125** (1949), 679.
3) E. H. Sondheimer, Proc. Roy. Soc. **A203** (1950), 75.
4) D. K. C. MacDonald, *Handbuch der Physik* (Springer, Berlin, 1956), Vol. 14.
5) A. Sommerfeld and H. Bethe, *Handbuch der Physik* (Springer, Berlin, 1933), Vol. 24/2.
6) N. F. Mott, Proc. Cam. Phil. Soc. **32** (1936), 281.
7) D. K. C. MacDonald and K. Mendelssohn, Proc. Roy. Soc. **A202** (1950), 103.

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Impurity Induced Infrared Lattice Vibration Absorption

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Impurities are known to affect vibrational properties of crystals by modifying the distribution of normal mode frequencies and altering the nature of the atomic displacements in the neighborhood of the impurities. We have calculated the effects of isotopic impurities on the lattice vibrational optical absorption of both monatomic and diatomic linear chains of alternately charged particles. It is found that even with the harmonic approximation and the use of the cyclic boundary condition, the presence of impurities leads to a broad absorption of the low frequency side of the main maximum. This is in contrast with the delta-function type of absorption at the optical frequency predicted by these models in the absence of impurities. For those cases in which discrete frequencies associated with localized vibrational modes occur, absorption at these isolated frequencies also occurs. This latter absorption can take place at frequencies higher than that of the main maximum. A discussion will be given of the relation between the results of our calculations and available experimental data.

§ 1. Introduction

Various examples are known of optical absorption lines in solids which are due to the effect of impurities on the lattice vibrations. For example silicon containing oxygen impurities exhibits[1] an infrared absorption line which has been qualitatively associated with a Si-O stretching vibration. Pick[2] has observed an infrared absorption line in each of several alkali halide crystals containing hydride ions and has attributed these lines to vibrations localized at the hydride ions. Recently Braunstein[3] has observed infrared impurity absorption lines in a series of germanium-silicon alloys.

In the present paper a theoretical investigation based on lattice dynamics is given of the effect of impurities on the infrared lattice-vibrational optical absorption. Qualitatively, introduction of an impurity can lead to a change in mass and a change in force constants associated with a given site. In addition, polarizabilities and effective charges may be modified. Of these changes only the change in mass

can be controlled precisely and predictably, so we have restricted our attention to effects due to the change in mass of the impurity.

In a series of papers Montroll, Potts, Maradudin and collaborators[4),5),6),7)] have investigated the effect of impurities on the normal vibrational modes of crystal lattices. We summarize some of their qualitative results which have special interest. First let us consider a monatomic linear chain of atoms with mass M and coupled by nearest neighbor Hooke's law forces.[4)] If one of the atoms is replaced by an atom of mass $M'=(1-\varepsilon)M$ with the force constants unchanged, the normal mode frequencies can be described schematically as in Fig. 1. For a lighter impurity atom, $0<\varepsilon<1$, a discrete frequency associated with a mode localized about the impurity separates from the top of the allowed band of frequencies. For a heavier impurity atom, $\varepsilon<0$, no discrete frequency exists. In both cases the distribution function for the in-band frequencies is modified.[5)]

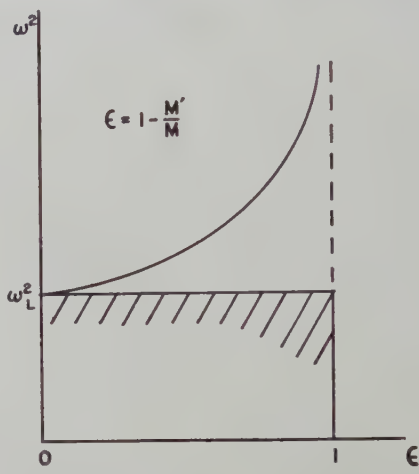

Fig. 1. The variation of the impurity frequency associated with a light isotope defect in a monatomic linear chain is plotted as a function of the impurity mass.

Turning now to the diatomic linear chain[6)] the introduction of an isotopic impurity atom may lead to one or two discrete frequencies associated with localized modes depending on which host atom is replaced and on the relative change in mass. A diagram of the situation is shown in Fig. 2. Of particular interest is the case in which one of the heavier host atoms is replaced by a lighter atom. One then gets a discrete frequency above the top of the optical branch and a second discrete frequency in the forbidden gap between the acoustical and optical branches. As in the monatomic case the frequency distribution function for the in-band modes is modified by the presence of impurity atoms.

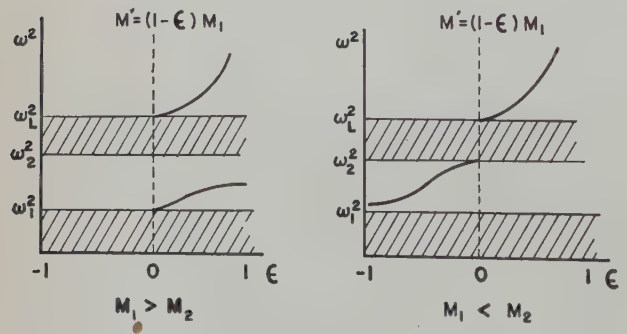

Fig. 2. The dependence on the impurity mass of the frequencies of the localized vibration modes in a linear diatomic chain when a heavy mass and a light mass atom are replaced by an isotopic impurity, respectively.

In this paper we study the effects of isotopic impurities on optical absorption associated both with localized modes and with the ordinary in-band modes. Monatomic and diatomic one and three dimensional cubic lattices of

the sodium chloride type are considered.

§ 2. The monatomic lattice, cyclic boundary conditions

We study the following simple model of a one-dimensional ionic lattice. We have a linear chain of N ions of alternating charge each of whose mass is M, and each of which interacts only with its nearest neighbors through Hooke's law forces with force constant γ. The time-independent equation of motion for the nth ion is

$$Lu_n = M\omega^2 u_n + \gamma(u_{n+1} - 2u_n + u_{n-1}) = 0, \tag{2·1}$$

whose general solution is

$$u_n = A\cos n\theta + B\sin n\theta, \tag{2·2}$$

provided that

$$M\omega^2 = 2\gamma(1-\cos\theta). \tag{2·3}$$

If we impose the cyclic boundary condition on the displacements,

$$u_{n+N} = u_n, \tag{2·4}$$

we find that

$$\theta = \frac{2\pi s}{N}\ ;\ s = 1, 2, 3, \cdots, N. \tag{2·5}$$

When defects are introduced into the lattice, the time-independent equations assume in general the form

$$Lu_n = \sum_p c_{np} u_p. \tag{2·6}$$

Equation (2·6) can be solved formally if we introduce the Green's function $g(n)$ which satisfies the equation

$$Lg(n) = \delta_{n0} \tag{2·7}$$

and also the cyclic boundary condition

$$g(n+N) = g(n). \tag{2·8}$$

With the aid of this function the solution to Eq. (2·6) is

$$u_n = \sum_{m,p} g(n-m)\, c_{mp} u_p. \tag{2·9}$$

If the ion at $n=0$ is replaced by one of the same charge but whose mass is $M' = (1-\varepsilon)M$ the coefficients c_{np} in Eq. (2·6) become

$$c_{np} = \varepsilon M\omega^2 \delta_{np}\delta_{n0}, \tag{2·10}$$

so that Eq. (2·9) becomes

$$u_n = \varepsilon M\omega^2 g(n) u_0. \tag{2·11}$$

Since Eq. (2·11) holds for all n, it must hold for $n=0$, and we thus obtain the eigenvalue equation for the normal mode frequencies of the perturbed lattice:

$$1=\varepsilon M\omega^2 g(0). \tag{2·12}$$

An expression for $g(n)$ which satisfies Eqs. (2·7) and (2·8) is

$$g(n)=\frac{1}{N}\sum_{s=1}^{N}\frac{\exp(2\pi isn/N)}{M\omega^2-2\gamma+2\gamma\cos(2\pi s/N)}=g(-n). \tag{2·13}$$

The sum in Eq. (2·13) is evaluated in the Appendix with the result that

$$g(n)=g(0)\cos n\varphi+\frac{\sin|n|\varphi}{2\gamma\sin\varphi}\;;\; n=0, 1, 2, \cdots, N-1 \tag{2·14}$$

$$g(0)=\frac{\cot(N\varphi/2)}{2\gamma\sin\varphi} \tag{2·15}$$

where we have put

$$\frac{\omega^2}{\omega_L{}^2}=f^2=\sin^2(\varphi/2),\quad 0<\varphi<2\pi. \tag{2·16}$$

The eigenvalue equation (2·12) becomes

$$\tan(N\varphi/2)=\varepsilon\tan(\varphi/2). \tag{2·17}$$

From Eqs. (2·11) and (2·13) we see that the isotope defect affects only the symmetric modes, i. e., those for which $u_n=u_{-n}$, while the antisymmetric modes, for which $u_{-n}=-u_n$, are unaffected by the presence of the defect. Thus we can classify the solutions to the perturbed equations as follows:

$$\text{symmetric}:\quad u_n=\varepsilon M\omega^2 g(n)u_0,\quad \omega=\omega_L\sin(\varphi/2) \tag{2·18a}$$

$$\tan(N\varphi/2)=\varepsilon\tan(\phi/2) \tag{2·18b}$$

$$\text{antisymmetric}:\quad u_n=B\sin(2\pi ns/N),\quad \omega=\omega_L\sin(\varphi/2) \tag{2·19a}$$

$$\varphi=\frac{2\pi s}{N}. \tag{2·19b}$$

In each of these cases we can restrict φ to the interval $(0, \pi)$ since we gain no new normal modes from the interval $(\pi, 2\pi)$, and this restriction gives the correct number of normal mode vibrations.

We now turn to the calculation of the dipole moment of the lattice associated with each normal mode. This is defined by

$$M(s)=\sum_{n=0}^{N-1}e_n u_n(s) \tag{2·20}$$

where $u_n(s)$ is the displacement of the nth ion in the sth normal mode of vibration and e_n is the charge on the nth ion. In the present case $e_n=e(-1)^n$, and we find for the antisymmetric modes that

$$M(s)=eB\sum_{n=0}^{N-1}(-1)^n\sin\frac{2\pi ns}{N}=0. \tag{2·21}$$

Thus we see that the antisymmetric modes do not contribute to the optical absorption of the lattice.

To calculate the contribution of the symmetric modes to the dipole moment it is convenient to proceed as follows. From Eqs. (2·18a) (2·20) and (2·13) we find that

$$M(s)=\varepsilon eM\omega_s^2u_0\sum_{n=0}^{N-1}(-1)^ng(n)=\frac{\varepsilon e\omega_s^2}{\omega_s^2-\omega_L^2}u_0 \tag{2·22}$$

where ω_s is the sth root of Eq. (2·18b).

The displacement u_0 is obtained from the normalization condition

$$\sum_{n=0}^{N-1}M_n u_n^2(s)=M\sum_{n=0}^{N-1}u_n^2(s)-\varepsilon Mu_0^2(s)=1. \tag{2·23}$$

This equation can be rewritten as

$$\begin{aligned}1+\varepsilon Mu_0^2(s)&=\varepsilon^2M^3\omega_s^4u_0^2(s)\sum_{n=0}^{N-1}g^2(n)\\&=\frac{MN}{2}\varepsilon^2u_0^2(s)\tan^2\frac{\varphi_s}{2}\csc^2\frac{N\varphi_s}{2}\left\{1+\frac{1}{N}\cot\varphi_s\sin N\varphi_s\right\}\end{aligned} \tag{2·24}$$

where this result is obtained from Eqs. (2·13) and (2·15) by a differentiation with respect to φ. With the aid of the eigenvalue equation, Eq. (2·17), this expression can be simplified to

$$\frac{MN}{2}u_0^2(s)\left\{1+\varepsilon^2\tan^2\frac{\varphi_s}{2}-\frac{\varepsilon}{N}\left(1+\tan^2\frac{\varphi_s}{2}\right)\right\}=1. \tag{2·25}$$

Combining the results expressed by Eqs. (2·22) and (2·25) we finally obtain for the dipole moment associated with the sth normal mode

$$M(s)=-\frac{\sqrt{2}\,\varepsilon e}{\sqrt{MN}}\frac{\tan^2(\varphi_s/2)}{\sqrt{1+\varepsilon^2\tan^2(\varphi_s/2)-(\varepsilon/N)[1+\tan^2(\varphi_s/2)]}} \tag{2·26a}$$

$$=-\frac{\sqrt{2}\,\varepsilon e}{\sqrt{MN}}\frac{f_s^2}{\sqrt{(1-f_s^2)[1-(1-\varepsilon^2)f_s^2-\varepsilon/N]}}. \tag{2·26b}$$

Ordinarily, the term of $O(1/N)$ in the denominator of Eq. (2·26b) is negligible compared to the rest of the denominator, and we have the approximate result that

$$M(s)\simeq-\frac{\sqrt{2}\,\varepsilon e}{\sqrt{MN}}\frac{f_s^2}{\sqrt{(1-f_s^2)[1-(1-\varepsilon^2)f_s^2]}}. \tag{2·27}$$

We have so far not considered explicitly the contribution to the dipole moment due to localized impurity modes of vibration. It is known from the work of Montroll and Potts[4] that such a mode exists if $0<\varepsilon<1$ but not if $\varepsilon<0$. In the

former case, which is the more interesting one, there is a solution to Eq. (2·17) with $f>1$. This special case is included in the general treatment just presented, and is obtained by replacing φ by

$$\varphi=\pi+iz \tag{2·28}$$

so that now

$$f^2=\cosh^2(z/2)>1. \tag{2·29}$$

If Eq. (2·28) is substituted into Eq. (2·17) the eigenvalue equation for the impurity frequency becomes

$$\varepsilon\coth(z/2)=\tanh(Nz/2)\to 1 \text{ as } N\to\infty. \tag{2·30}$$

In the large N limit the solution of Eq. (2·30) is

$$z=\ln\frac{1+\varepsilon}{1-\varepsilon} \tag{2·31}$$

which together with Eq. (2·29) gives the Montroll-Potts result

$$f_{imp}^2=\frac{1}{1-\varepsilon^2}. \tag{2·32}$$

If this result is substituted into Eq. (2·26b), or, alternatively, if Eqs. (2·28) and (2·31) are substituted into Eq. (2·26a), we obtain for the dipole moment associated with the impurity mode

$$M_{imp}=\frac{\sqrt{2}\,e}{\sqrt{M}}\frac{1}{\sqrt{\varepsilon}\sqrt{1-\varepsilon^2}}. \tag{2·33}$$

The absorption coefficient is proportional to $M^2(f)g(f)$ where $g(f)$ is the frequency distribution function for the perturbed lattice. It has the form

$$g(f)=g_0(f)+\Delta g(f) \tag{2·34}$$

where $g_0(f)$ is the distribution function for the perfect lattice,

$$g_0(f)=\frac{2}{\pi}\frac{1}{\omega_L\sqrt{1-f^2}}, \tag{2·35}$$

and $\Delta g(f)$ is the change in $g_0(f)$ due to an isotope defect[5]:

$$\Delta g(f)=-\frac{\varepsilon}{2N}\frac{g_0(f)}{[1+(\varepsilon^2-1)f^2]}-\frac{1}{2N\omega_L}\delta(f-1)+\frac{1}{N\omega_L}\delta\left(f-\frac{1}{\sqrt{1-\varepsilon^2}}\right),$$

$$0<\varepsilon<1, \tag{2·36a}$$

$$=\frac{|\varepsilon|}{2N}\frac{g_0(f)}{[1+(\varepsilon^2-1)f^2]}-\frac{1}{2N\omega_L}\delta(f-1), \quad \varepsilon<0. \tag{2·36b}$$

For the more interesting case of $0<\varepsilon<1$ we find that

$$M^2(f)g(f)=\frac{2e^2}{MN}\frac{\varepsilon^2 f^4}{(1-f^2)[1-(1-\varepsilon^2)f^2]}$$
$$\times\left\{\frac{2}{\pi\omega_L}\frac{1}{\sqrt{1-f^2}}-\frac{\varepsilon}{N\pi\omega_L}\frac{1}{\sqrt{1-f^2}[1-(1-\varepsilon^2)f^2]}-\frac{1}{2N\omega_L}\delta(f-1)\right\}$$
$$+\frac{2e^2}{M}\frac{1}{\varepsilon(1-\varepsilon^2)}\frac{1}{N\omega_L}\delta\left(f-\frac{1}{\sqrt{1-\varepsilon^2}}\right). \qquad (2\cdot37a)$$

For the case $\varepsilon<0$, we have

$$M^2(f)g(f)=\frac{2e^2}{MN}\frac{\varepsilon^2 f^4}{(1-f^2)[1-(1-\varepsilon^2)f^2]}$$
$$\times\left\{\frac{2}{\pi\omega_L}\frac{1}{\sqrt{1-f^2}}+\frac{|\varepsilon|}{N\pi\omega_L}\frac{1}{[1+(\varepsilon^2-1)f^2]\sqrt{1-f^2}}-\frac{1}{2N\omega_L}\delta(f-1)\right\}. \qquad (2\cdot37b)$$

These results are exact.

The physically most interesting case is one where the crystal contains a finite number n of defects, and not just one. In this case we can expand $M^2(f)g(f)$ in powers of the concentration of impurities,

$$c=\frac{n}{N} \qquad (2\cdot38)$$

in the form

$$M^2(f)g(f)=I(f)=I_0(f)+cI_1(f)+\cdots. \qquad (2\cdot39)$$

In this expansion $I_0(f)$ is the value of $I(f)$ in the absence of impurities, I_1, is the change in $I(f)$ due to the addition of a single impurity, and so on[7]. In the present case $I_0(f)$ is just a delta-function centered at $f=1$, and we will neglect it here since it contributes at one frequency only. This means that to terms linear in the concentration of impurities we obtain

$$I(f)\simeq c\cdot\frac{2e^2}{M\omega_L}\left\{\frac{2}{\pi}\frac{\varepsilon^2 f^4}{(1-f^2)^{3/2}[1-(1-\varepsilon^2)f^2]}\right.$$
$$\left.+\frac{1}{\varepsilon(1-\varepsilon^2)}\delta\left(f-\frac{1}{\sqrt{1-\varepsilon^2}}\right)\right\},\quad 0<\varepsilon<1, \qquad (2\cdot40a)$$
$$\simeq c\cdot\frac{2e^2}{M\omega_L}\cdot\frac{2}{\pi}\frac{\varepsilon^2 f^4}{(1-f^2)^{3/2}[1-(1-\varepsilon^2)f^2]},\quad \varepsilon<0. \qquad (2\cdot40b)$$

Thus in this approximation there is no formal difference between the in-band absorption in the two cases $0<\varepsilon<1$, and $\varepsilon<0$.

In Fig. 3 a plot is given of the integrated absorption coefficient in arbitrary units as a function of the fractional change in mass ε for the localized mode of vibration. One sees that the absorption is strong both when ε is close to zero

and to unity. In Fig. 4 the absorption coefficient in arbitrary units is plotted as a function of frequnency for the in-band modes when $\varepsilon=1/2$. For the perfect lattice the absorption is zero except at ω_L. The presence of impurities leads to an increase in absorption at in-band frequencies when cyclic boundary conditions are employed.

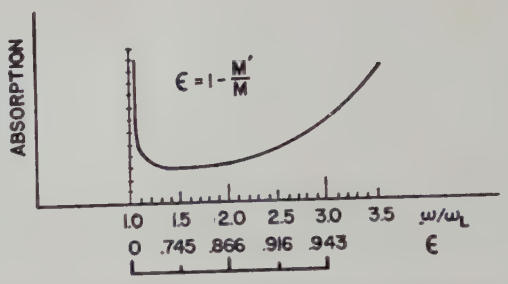

Fig. 3. Absorption (in arbitrary units) due to the localized vibration mode associated with a light isotope impurity in a monatomic linear chain plotted as a function of the impurity mass.

§ 3. A pair of isotope defects in a monatomic chain

It is of interest to investigate the effect of increasing concentration of impurities on the optical absorption. One would like to carry out a calculation for a system with n impurities, but the computational problems become rather severe. We therefore consider the case in which two ions of the same kind are replaced by heavier or lighter impurities. The time independent equations of motion become

$$Lu_n=\varepsilon M\omega^2(u_{-m}\delta_{n,-m}+u_m\delta_{n,m}). \tag{3·1}$$

The solution to this equation is

$$u_n=\varepsilon M\omega^2[g(n+m)u_{-m} +g(n-m)u_m]. \tag{3·2}$$

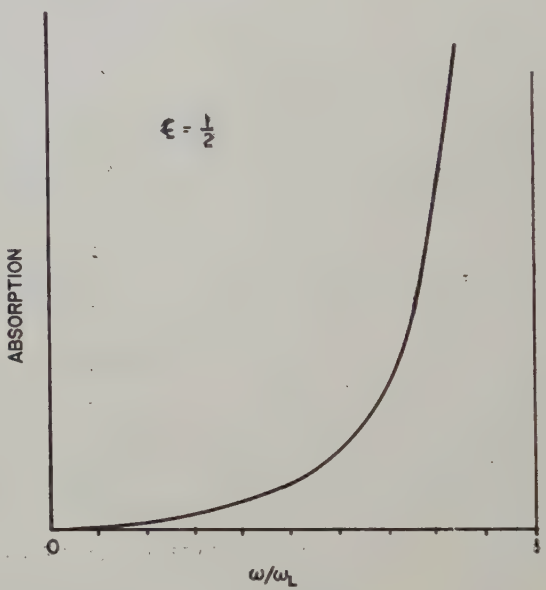

Fig. 4. The in-band absorption (in arbitrary units) due to a single light isotope impurity in a monatomic linear chain plotted as a function of the frequency.

Setting n equal to $-m$ and m respectively, we obtain the following set of equations for the amplitudes u_{-m} and u_m:

$$\begin{pmatrix} \varepsilon M\omega^2 g(0)-1 & \varepsilon M\omega^2 g(2m) \\ \varepsilon M\omega^2 g(2m) & \varepsilon M\omega^2 g(0)-1 \end{pmatrix} \begin{pmatrix} u_{-m} \\ u_m \end{pmatrix} = 0. \tag{3·3}$$

The solubility condition leads to the eigenvalue equations for the normal mode frequencies

$$\varepsilon M\omega^2 g(0)=1\pm\varepsilon M\omega^2 g(2m). \tag{3·4}$$

If we substitute the solution corresponding to the plus sign in Eq. (3·4) back into Eq. (3·3) we find that

$$u_{-m}+u_m=0, \tag{3·5}$$

i.e. this choice corresponds to antisymmetric modes of vibration. Similarly, the solution of Eq. (3·4) with the minus sign leads to the condition

$$u_{-m}-u_m=0, \tag{3·6}$$

and hence to symmetric vibration modes. The displacement components in each case are given by

$$\text{antisymmetric}: \quad u_n=\varepsilon M\omega^2[g(n-m)-g(n+m)]u_m \tag{3·7a}$$

$$\text{symmetric}: \quad u_n=\varepsilon M\omega^2[g(n-m)+g(n+m)]u_m. \tag{3·7b}$$

It is straightforward to show that for the antisymmetric modes the dipole moment of the lattice vanishes, so that we need to consider only the symmetric modes in computing the absorption spectrum of the lattice. Furthermore, we shall restrict our attention to the localized modes, because the in-band modes lead to extremely complicated expressions for the absorption coefficient from which it is difficult to extract simple physical results.

The eigenvalue equation (3·4) for symmetric modes has a solution corresponding to a localized vibration mode when $0<\varepsilon<1$. Replacing ϕ by $\pi+iz$ in the usual way, we obtain the equation which determines the frequency of the localized mode:

$$\varepsilon\coth\frac{z}{2}=1-\varepsilon\coth\frac{z}{2}e^{-2mz}. \tag{3·8}$$

This equation can be solved by iteration, using the solution for $m\to\infty$ as the starting point. The result is

$$f^2=\frac{1}{1-\varepsilon^2}\left\{1+\frac{2\varepsilon^2}{1-\varepsilon^2}\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{2m}+\cdots\right\}, \tag{3·9}$$

and corresponds to a value of z given by

$$z=\ln\frac{1+\varepsilon}{1-\varepsilon}+\frac{2\varepsilon}{1-\varepsilon^2}\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{2m}+\cdots. \tag{3·10}$$

The displacement amplitudes become

$$u_n=\varepsilon\coth\frac{z}{2}(-1)^{n+m}u_m[e^{-|n-m|z}+e^{-|n+m|z}], \tag{3·11}$$

so that the expression for the dipole moment associated with this mode becomes

$$M(z)=(-1)^m eu_m\varepsilon\coth\frac{z}{2}\sum_{n=-\infty}^{\infty}[e^{-|n-m|z}+e^{-|n+m|z}]=2(-1)^m e\varepsilon u_m\coth^2\frac{z}{2}. \tag{3·12}$$

The normalization condition, which determines the amplitude u_m, is

$$M\sum_n u_n^2+2(M'-M)u_m^2=1. \tag{3·13}$$

If we substitute Eq. (3·11) into Eq. (3·13), we can rewrite the latter equation as

$$M\varepsilon^2\coth^2\frac{z}{2}u_m^2\sum[e^{-2|n-m|z}+2e^{-|n-m|z-|n+m|z}+e^{-2|n+m|z}]-2\varepsilon Mu_m^2=1. \tag{3·14}$$

The sums are readily evaluated and we find that

$$2M\varepsilon u_m^2\left\{[\varepsilon\coth z(1+e^{-2mz})+2\varepsilon me^{-2mz}]\coth^2\frac{z}{2}-1\right\}=1. \tag{3·15}$$

However, from the eigenvalue equation we see that

$$1+e^{-2mz}=\frac{1}{\varepsilon}\tanh\frac{z}{2}, \tag{3·16}$$

so that

$$2M\varepsilon u_m^2\left\{\frac{\coth z}{\tanh(z/2)}+2m\coth\frac{z}{2}-2m\varepsilon\coth^2\frac{z}{2}-1\right\}=1. \tag{3·17}$$

If we substitute for z the expression given in Eq. (3·10), and transpose terms we obtain

$$u_m=\frac{\sqrt{\varepsilon}}{\sqrt{M}\sqrt{1-\varepsilon^2}}\left[1-\frac{(2m\varepsilon-1)}{1-\varepsilon^2}\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{2m}+\cdots\right]. \tag{3·18}$$

This result together with Eq. (3·12) gives us finally that

$$M_{imp}=\frac{2e(-1)^m}{\sqrt{M}\sqrt{\varepsilon}\sqrt{1-\varepsilon^2}}\left[1-\frac{2m\varepsilon+1-2\varepsilon^2}{1-\varepsilon^2}\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{2m}+\cdots\right]. \tag{3·19}$$

If we compare the results given by Eqs. (2·33) and (3·19) and recall that the absorption is proportional to M^2, we see that although only one of the two impurity modes in the present case is optically active it contributes twice as much to the absorption at the impurity frequency as does the single impurity mode which is associated with a single, isolated impurity. This tends to confirm our statement in the preceding section that if we have n impurities in our lattice, where $n\ll N$ so that they are widely separated, then to the first approximation the absorption at the impurity frequency is n times the absorption due to a single, isolated impurity.

§ 4. A single isotope defect in a monatomic lattice with free ends

In all cases discussed so far the unperturbed lattice has an absorption spectrum which consists only of a single δ-function peak at $\omega=\omega_L$. This result is a direct consequence of the assumption of the cyclic boundary condition on the lattice. It is known from the results of Rosenstock[8] that if more realistic boundary conditions

are imposed on the lattice, e. g., if the ends of the lattice are allowed to be free, a non-vanishing absorption in the frequency range $(0, \omega_L)$ results. Since in the cyclic boundary case all in-band absorption is due to the impurity, it is of some interest to see how the presence of the impurity affects the in-band absorption in a lattice where it is non-vanishing to begin with.

We choose for this purpose a lattice of (an even number) N atoms which we number from 1 to N. We place an isotope defect whose mass is $M'=(1-\varepsilon)M$ at $p=N/2$ in the lattice. The equations of motion for this lattice become

$$M\omega^2 u_n+\gamma(u_{n+1}-u_n)-\gamma(u_n-u_{n-1})=0, \quad n\neq p \tag{4·1a}$$

$$M'\omega^2 u_p+\gamma(u_{p+1}-u_p)-\gamma(u_p-u_{p-1})=0, \quad n=p. \tag{4·1b}$$

The boundary conditions are

$$\begin{aligned} u_1-u_0&=0 \\ u_{N+1}-u_N&=0. \end{aligned} \tag{4·2}$$

A solution satisfying the boundary condition $u_{N+1}-u_N=0$ and satisfying the equations of motion for $n\geq p$ is

$$u_n=A\cos\left(N-n+\frac{1}{2}\right)\varphi, \quad n\geqq p, \tag{4·3}$$

provided that $\omega^2=\omega_L^2\sin^2\varphi/2$, where $\omega_L=\sqrt{4\gamma/M}$ is the largest unperturbed frequency of the lattice. A solution satisfying the boundary condition $u_1-u_0=0$ and satisfying the equations of motion for $n<p$ is

$$u_n=B\cos\left(n-\frac{1}{2}\right)\varphi, \quad n<p, \tag{4·4}$$

provided again that $\omega^2=\omega_L^2\sin^2\varphi/2$. The coefficients A and B are determined by substituting the solutions (4·3) and (4·4) into the equations for the displacement components which are linked by the defect, u_p and u_{p-1}. This leads to the pair of equations

$$A\left\{[2(1-\varepsilon)\sin^2(\varphi/2)-1]\cos\left(\frac{N+1}{2}\right)\varphi+\frac{1}{2}\cos\left(\frac{N-1}{2}\right)\varphi\right\} + \frac{1}{2}B\cos\left(\frac{N-3}{2}\right)\varphi=0 \tag{4·5a}$$

$$\frac{1}{2}A\cos\left(\frac{N+1}{2}\right)\varphi+B\left\{[2\sin^2(\varphi/2)-1]\cos\left(\frac{N-3}{2}\right)\varphi + \frac{1}{2}\cos\left(\frac{N-5}{2}\right)\varphi\right\}=0. \tag{4·5b}$$

The condition that this pair of equations have non-trivial solutions for A and B is that the determinant of the coefficients vanishes. After some manipulations this condition reduces to

$$1=\varepsilon\tan(\varphi/2)\left[\cot N\varphi+\frac{\cos\varphi}{\sin N\varphi}\right], \tag{4·6}$$

and we also find that

$$B=A\frac{\cos\left(\frac{N+1}{2}\right)\varphi}{\cos\left(\frac{N-1}{2}\right)\varphi}. \tag{4·7}$$

The dipole moment for the sth normal mode is

$$M(s)=\sum_{n=1}^{N}e(-1)^n u_n(s)=\frac{eB}{2\cos(\varphi/2)}[-1+(-1)^{p-1}\cos(p-1)\varphi]$$

$$+\frac{eA}{2\cos(\varphi/2)}[(-1)^p\cos(N-p+1)\varphi+(-1)^N] \tag{4·8}$$

$$=\frac{2eA\sin\frac{\varphi}{2}\sin\frac{N\varphi}{2}}{\cos\frac{\varphi}{2}\cos\frac{N-1}{2}\varphi}\times\begin{cases}\sin^2\frac{N}{4}\varphi, & p \text{ even}\\ \cos^2\frac{N}{4}\varphi, & p \text{ odd.}\end{cases} \tag{4·9}$$

The constant A is determined by the normalization condition

$$\sum_{n=1}^{N}Mu_n^2+(M'-M)u_p^2=1. \tag{4·10}$$

The first term on the left-hand side of Eq. (4·10) is summed as follows:

$$\sum_{n=1}^{N}Mu_n^2=M\left\{\sum_{n=1}^{p-1}B^2\cos^2\left(n-\frac{1}{2}\right)\varphi+\sum_{n=p}^{N}A^2\cos^2\left(N-n+\frac{1}{2}\right)\varphi\right\}$$

$$=\frac{1}{2}MB^2\left[\frac{N}{2}-1+\frac{\sin(N-2)\varphi}{2\sin\varphi}\right]$$

$$+\frac{1}{2}MA^2\left[\frac{N}{2}+1+\frac{\sin(N+2)\varphi}{2\sin\varphi}\right]. \tag{4·11}$$

The normalization condition, Eq. (4·10) becomes

$$1=\frac{1}{2}MA^2\left\{\frac{\cos^2\left(\frac{N+1}{2}\right)\varphi}{\cos^2\left(\frac{N-1}{2}\right)\varphi}\left[\frac{N}{2}-1+\frac{\sin(N-2)\varphi}{2\sin\varphi}\right]+\frac{N}{2}+1+\frac{\sin(N+2)\varphi}{2\sin\varphi}\right\}$$

$$-\varepsilon MA^2\cos^2\left(\frac{N+1}{2}\right)\varphi$$

$$\simeq\frac{MNA^2}{4\cos^2\left(\frac{N-1}{2}\right)\varphi}[1+\cos N\varphi\cos\varphi] \tag{4·12}$$

where we have retained only the terms proportional to N. It can be shown that this approximation is permissible for all φ in $(0, \pi)$, i. e. for the in-band modes. We finally obtain

$$A=\frac{2\cos\left(\frac{N-1}{2}\right)\varphi}{\sqrt{NM}}\,\frac{1}{\sqrt{1+\cos N\varphi\cos\varphi}}. \tag{4·13}$$

The dipole moment can be rewritten as

$$M(s)=\frac{2e}{\sqrt{NM}}\,\frac{\sin\frac{\varphi}{2}\sin\frac{N}{2}\varphi}{\cos\frac{\varphi}{2}\sqrt{1+\cos N\varphi\cos\varphi}}\left[1\pm\cos\frac{N\varphi}{2}\right],\begin{cases}p\text{ odd}\\ p\text{ even.}\end{cases} \tag{4·14}$$

It is a rather difficult matter to eliminate φ from Eq. (4·14) in favor of the frequency ω. We have therefore evaluated $M(s)$ numerically for the particular case in which $N=20$ and $\varepsilon=1/2$. One finds that as s increases, $M(s)$ undergoes oscillations which become extremely rapid when N is large. In any physically resolvable frequency interval, however, these oscillations will be averaged out so that the absorption is a smoothly varying function of ω. We have therefore averaged $M(s)$ over small ranges of s to smooth out these physically unimportant oscillations. The absorption coefficient is plotted as a function of frequency in Fig. 5. One sees that when a lighter impurity atom is present, the in-band absorption is less than that for the perfect lattice. This result is the opposite of that obtained with cyclic boundary conditions. We believe that the free boundary condition gives the more reliable result since it seems physically more realistic than the cyclic boundary condition.

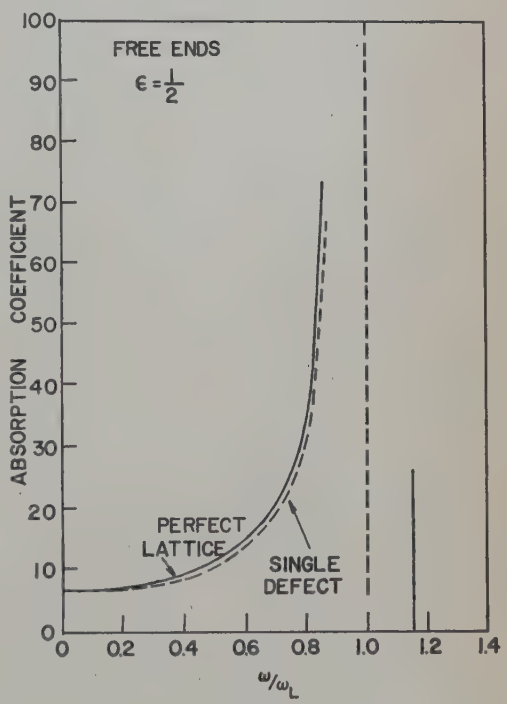

Fig. 5. The absorption in a monatomic linear chain with free ends is plotted both for the case of no impurity and for the case of a single light mass impurity.

In the case that $0<\varepsilon<1$, the eigenvalue equation (4·6) has a solution which corresponds to a mode of vibration which is localized about the defect. This solution is obtained by replacing φ by $\pi+iz$ in Eq. (4·6) :

$$1=\varepsilon \coth\frac{z}{2}\left(\coth Nz-\frac{\cosh z}{\cos N\pi \sinh Nz}\right). \tag{4·15}$$

In the limit as $N\to\infty$ this equation reduces to the eigenvalue equation for the impurity frequency in a lattice satisfying the cyclic boundary condition, Eq. (2·30).

The expressions for the displacement components u_n associated with the localized mode become

$$\begin{aligned} u_n&=iA(-1)^n \sinh\left(n-N-\frac{1}{2}\right)z, \quad n\geq p \\ &=iB(-1)^n \sinh\left(n-\frac{1}{2}\right)z, \quad n<p \end{aligned} \tag{4·16}$$

provided that

$$\frac{\omega^2}{\omega_L^2}=\cosh^2\frac{z}{2}. \tag{4·17}$$

In the limit as $N\to\infty$ we find that

$$B=-Ae^z. \tag{4·18}$$

The expression for the dipole moment associated with the localized mode is

$$M(z)=-i\frac{eA}{2\varepsilon}e^{(N+1)z/2}. \tag{4·19}$$

The normalization condition for the localized mode becomes (in the limit of large N)

$$\sum_n M|u_n|^2+(M'-M)|u_p^2|=-\frac{1}{8}MA^2e^{(N+1)z}\left(\frac{1-\varepsilon^2}{\varepsilon}\right)=1, \tag{4·20}$$

so that

$$A^2=-\frac{8\varepsilon}{M(1-\varepsilon^2)}e^{-(N+1)z}. \tag{4·21}$$

Combining Eqs. (4·19) and (4·21) we finally obtain for the dipole moment associated with the localized vibration mode

$$M(z)=\frac{\sqrt{2}\,e}{\sqrt{M}\sqrt{\varepsilon}\sqrt{1-\varepsilon^2}}. \tag{4·22}$$

This expression is identical with the corresponding result for a lattice obeying the cyclic boundary condition, Eq. (2·33).

§ 5. A single defect in a diatomic lattice

We consider an alternating diatomic linear chain composed of N atoms with mass M_1 and N atoms with mass M_2 interacting with nearest neighbor Hooke's law forces. Our discussion will be limited to the case in which a heavier host

atom is replaced by a lighter isotopic impurity atom with mass M'. As stated in the introduction this case exhibits the maximum number of localized vibrational modes and thus can provide an indication of the principal effects to be expected in diatomic lattices.

The time independent equations of motion can accordingly be written as

$$\begin{aligned}&\omega^2 M_2 u_{2n}+\gamma[u_{2n+1}-2u_{2n}+u_{2n-1}]=0\\&\omega^2 M_1 u_{2n+1}+\gamma[u_{2n+2}-2u_{2n+1}+u_{2n}]=\delta_{n,0}\varepsilon M_1\omega^2 u_1,\end{aligned}\tag{5·1}$$

where $M_2<M_1$, $\varepsilon M_1=M_1-M'$, and $0<\varepsilon<1$. By making the transformations

$$\begin{aligned}v_{2n}&=(M_2\omega^2-2\gamma)^{1/2}u_{2n}\\v_{2n+1}&=(M_1\omega^2-2\gamma)^{1/2}u_{2n+1},\end{aligned}\tag{5·2}$$

where positive square roots are to be taken, one can write Eqs. (5·1) in the compact form

$$(M^*\omega^2-2\gamma)v_n+\gamma(v_{n+1}+v_{n-1})=\delta_{n,1}\varepsilon M_1\omega^2\frac{(M_2\omega^2-2\gamma)^{1/2}}{(M_1\omega^2-2\gamma)^{1/2}}v_1 \tag{5·3}$$

where

$$(M^*\omega^2-2\gamma)^{1/2}=(M_1\omega^2-2\gamma)^{1/2}(M_2\omega^2-2\gamma)^{1/2} \tag{5·4}$$

and the atoms have been relabeled.

The Green function solution to Eq. (5·3), assuming cyclic boundary conditions, can be written as

$$v_n=g^*(n-1)\varepsilon M_1\omega^2\frac{(M_2\omega^2-2\gamma)^{1/2}}{(M_1\omega^2-2\gamma)^{1/2}}v_1, \tag{5·5}$$

where the Green's function $g^*(n)$ is given by

$$g^*(n)=\frac{1}{2N}\sum_{s=1}^{2N}\frac{\exp(2\pi isn/2N)}{M^*\omega^2-2\gamma+2\gamma\cos(2\pi s/2N)}. \tag{5·6}$$

The eigenvalue equation, which determines the characteristic values of the frequency ω, is obtained by substituting $n=1$ in Eq. (5·5) yielding

$$1=\varepsilon M_1\omega^2\frac{(M_2\omega^2-2\gamma)^{1/2}}{(M_1\omega^2-2\gamma)^{1/2}}g^*(0). \tag{5·7}$$

Let us consider first the localized mode with frequency above the top of the optical branch. Defining

$$f^2=\frac{M^*\omega^2}{4\gamma}, \tag{5·8}$$

one can evaluate the Green function and obtain

$$g^*(n)=\frac{(-1)^n}{4\gamma}\frac{[f-\sqrt{f^2-1}]^{2|n|}}{f\sqrt{f^2-1}}. \tag{5·9}$$

The characteristic frequency determined by Eqs. (5·7), (5·8) and (5·9) is specified by

$$2\omega^2=\omega_0^2+\frac{2\omega_A^2+\sqrt{(1-\varepsilon^2)^2\omega_0^4+4\varepsilon^2\omega_A^4}}{(1-\varepsilon^2)}, \tag{5·10}$$

where

$$\omega_0^2=2\gamma/M_2 \tag{5·11a}$$

$$\omega_A^2=2\gamma/M_1. \tag{5·11b}$$

The effective dipole moment $M(0)$ can be evaluated using methods similar to those employed for the monatomic lattice. The result is

$$M(0)=-\frac{e\mathcal{E}u_1}{(1-y^4)}\frac{\omega_0\omega}{(\omega^2-\omega_L^2)^{1/2}\omega_A}\left\{\frac{(\omega^2-\omega_0^2)^{1/2}\omega_A}{(\omega^2-\omega_A^2)^{1/2}\omega_0}[1+y^4]+2y^2\right\} \tag{5·12}$$

where

$$y=f-\sqrt{f^2-1} \tag{5·12a}$$

$$\omega_L^2=2\gamma\left(\frac{1}{M_1}+\frac{1}{M_2}\right), \tag{5·12b}$$

and the amplitude u_1 is determined by the normalizing condition

$$M_1\sum_{n\ \mathrm{odd}}u_n^2+M_2\sum_{n\ \mathrm{even}}u_n^2=1+\varepsilon M_1u_1^2. \tag{5·12c}$$

The result obtained from Eq. (5·12c) is

$$M_1u_1^2=\frac{(1-y^8)(\omega^2-\omega_0^2)}{[(\omega^2-\omega_0^2)(1+y^8)+2y^4(\omega^2-\omega_A^2)]-\varepsilon(1-y^8)(\omega^2-\omega_0^2)}. \tag{5·13}$$

The localized mode with frequency in the "forbidden" gap between acoustical and optical branches will be considered next. The Green function for this case can be written as

$$g^*(n)=\frac{(i)^{|n|-1}}{2\gamma}\frac{\omega_0\omega_A x^{|n|}}{\omega\sqrt{\omega_L^2-\omega^2}(\omega_0\omega_A)^{|n|}} \tag{5·14}$$

where

$$x=\omega(\omega_L^2-\omega^2)^{1/2}-(\omega^2-\omega_A^2)^{1/2}(\omega_0^2-\omega^2)^{1/2}. \tag{5·14a}$$

The value of the frequency in the "forbidden" gap is specified by

$$2\omega^2=\omega_0^2+\frac{2\omega_A^2-\sqrt{(1-\varepsilon^2)^2\omega_0^4+4\varepsilon^2\omega_A^4}}{(1-\varepsilon^2)}. \tag{5·15}$$

The effective dipole moment $M(g)$ may be expressed as

$$M(g)=\frac{e\mathcal{E}\omega u_1}{(\omega^2-\omega_A^2)^{1/2}(\omega_L^2-\omega^2)^{1/2}(\omega_0^2\omega_A^2+x^2)}\{\omega_0^2(\omega^2-\omega_A^2)^{1/2}2x$$
$$-(\omega_0^2-\omega^2)^{1/2}(\omega_0^2\omega_A^2-x^2)\}, \tag{5·16}$$

where the amplitude u_1 is determined by the normalization condition Eq. (5·12c) and is given by

$$M_1 u_1^2 = \frac{(\omega_0^2-\omega^2)(\omega_0^4\omega_A^4-x^4)}{2\omega_0^2\omega_A^2(\omega^2-\omega_A^2)x^2+(\omega_0^2-\omega^2)(\omega_0^4\omega_A^4+x^4)-\varepsilon(\omega_0^2-\omega^2)(\omega_0^4\omega_A^4-x^4)}. \tag{5·17}$$

Turning now to the modes with frequencies lying within the acoustical branch we find that the eigenvalue equation (5·7) can be written as

$$1=\varepsilon\frac{\omega^2(\omega_0^2-\omega^2)^{1/2}}{\omega_0\omega_A(\omega_A^2-\omega^2)^{1/2}}\csc\phi\cot N\phi \tag{5·18}$$

where

$$\sin^2\frac{\phi}{2}=f^2=\frac{M^*\omega^2}{4\gamma}. \tag{5·18a}$$

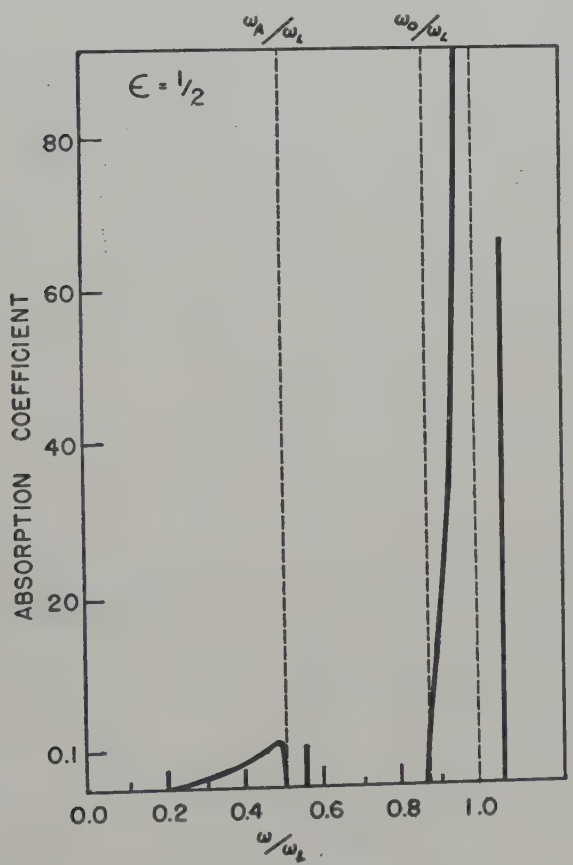

Fig. 6. Absorption in a diatomic linear chain for the case that one of the heavy masses is replaced by a lighter isotope impurity.

In the limit of large N one can evaluate the effective dipole moment $M_a(s)$ for the sth mode in the acoustical branch in terms of the frequency ω for that mode. The result is

$$M_a(s)=\frac{\sqrt{2}\,e\varepsilon}{\sqrt{NM_1}}\frac{\omega^2(\omega_A^2-\omega^2)^{1/2}(\omega_0^2-\omega^2)^{1/2}}{(\omega_L^2-\omega^2)^{1/2}(\omega_L^2-2\omega^2)^{1/2}[\varepsilon^2\omega^2(\omega_0^2-\omega^2)+(\omega_A^2-\omega^2)(\omega_L^2-\omega^2)]^{1/2}}. \tag{5·19}$$

The treatment for the optical branch follows in very similar fashion. The expression for the effective dipole moment $M_o(s)$ for the sth mode in the optical branch is given by

$$M_o(s)=\frac{\sqrt{2}\,e\varepsilon}{\sqrt{NM_1}}\frac{\omega^2(\omega^2-\omega_A^2)^{1/2}(\omega^2-\omega_0^2)^{1/2}}{(\omega_L^2-\omega^2)^{1/2}(2\omega^2-\omega_L^2)^{1/2}[\varepsilon^2\omega^2(\omega^2-\omega_0^2)+(\omega^2-\omega_A^2)(\omega_L^2-\omega^2)]^{1/2}}. \tag{5·20}$$

In order to indicate the qualitative nature of the results for the diatomic case treated above we have made calculations for the specific case in which $M_1=3M_2$ and $\varepsilon=1/2$. The absorption coefficient in arbitrary units is plotted as a function of frequency in Fig. 6. One sees that the absorption in the acoustical branch and at the discrete frequency in the "forbidden" gap is quite small. The absorption coefficient increases very rapidly as one approaches the upper bound ω_L of the optical branch. The absorption coefficient at the discrete frequency above the optical branch is relatively large compared to that associated with the mode in the gap.

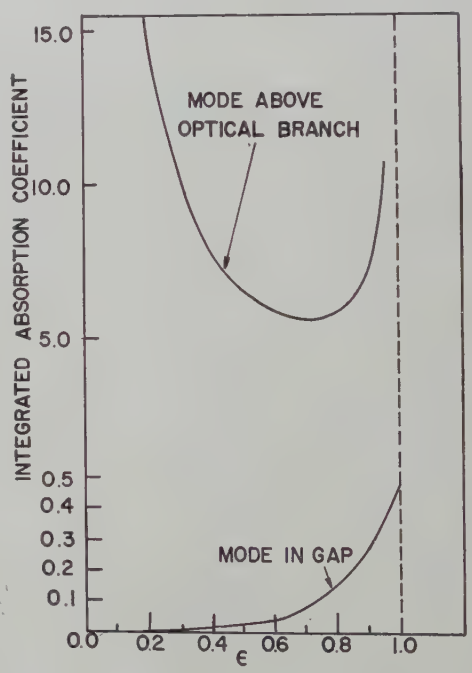

Fig. 7. The integrated absorption coefficient for the two impurity modes in a diatomic linear chain which arise when one of the heavy masses is replaced by a light isotope impurity.

The dependence on ε of the integrated absorption for the localized modes is plotted in Fig. 7 for $M_1=3M_2$. The mode above the optical branch has large absorption for ε near zero and near unity with a minimum at intermediate ε. The mode in the gap has continuously increasing absorption as ε increases but the absorption is always very small.

§ 6. Impurity frequencies in three-dimensional lattices

We conclude with a brief discussion of the frequencies associated with localized

vibration modes due to isotopic impurities in three-dimensional monatomic and diatomic simple cubic lattices having nearest neighbor Hooke's law interactions, both central and non-central.

For a monatomic lattice the eigenvalue equation is

$$1=\varepsilon M\omega^2 g(0,\,0,\,0\,;\,\omega), \tag{6·1}$$

where

$$g(0,\,0,\,0\,;\,\omega)=\frac{2+\alpha}{M\omega_L^2}\left\{\frac{1}{\pi^3}\int_0^{\pi}\!\!\int_0^{\pi}\!\!\int_0^{\pi}\frac{d\phi_1 d\phi_2 d\phi_3}{(2+\alpha)(2f^2-1)-\alpha\cos\phi_1-\cos\phi_2-\cos\phi_3}\right\}, \tag{6·2}$$

$f^2=\omega^2/\omega_L^2$, and α is the ratio γ_1/γ_2 of central to non-central force constants. The integral in curly brackets has been tabulated for a wide range of the parameters α and f.[9] If we denote it by $I(0,\,0,\,0\,;\,\alpha\,;f)$, Eq. (6·1) becomes

$$1=\varepsilon(2+\alpha)f^2 I(0,\,0,\,0\,;\,\alpha\,;f). \tag{6·3}$$

For given values of α and ε this equation must be solved numerically.

Although we know experimentally what ω_L is for the alkali-halides we must still determine the value of α in each case. A simple (though not unique) choice is provided by a calculation of the strain energy of our crystal due to a homogeneous deformation. This leads to the following identifications :[10]

$$c_{11}=\frac{\gamma_1}{a_0},\quad c_{12}=0,\quad c_{44}=\frac{\gamma_2}{2a_0}, \tag{6·4}$$

where a_0 is the lattice parameter. These expressions lead to the result

$$\alpha=\gamma_1/\gamma_2=(c_{11}/2c_{44}). \tag{6·5}$$

For the case of KCl, which is reasonably well approximated by a monatomic crystal, we obtain

$$\alpha=\frac{3.98}{2\times 0.625}=3.18. \tag{6·6}$$

If we consider a U-center in KCl which is believed to be an H^- ion replacing a halide ion, $M=36$, $M'=1$, so that $\varepsilon=35/36$. If we now solve Eq. (6·3) for this choice of parameters with the aid of the results of reference 9) we find

$$f=4.26,\quad \frac{\omega_{\rm loc}}{2\pi}=17.9\times 10^{12}\ \mathrm{sec}^{-1}. \tag{6·7}$$

The experimental result of Pick[2] in this case gives

$$f=4.52,\quad \frac{\omega_{\rm loc}}{2\pi}=19\times 10^{12}\ \mathrm{sec}^{-1}. \tag{6·8}$$

In view of the crudeness of the model, the agreement between theory and experiment is striking.

For a diatomic lattice with masses M_1 and M_2 at alternating lattice points where $M_1 > M_2$, the M^* transformation leads to the following eigenvalue equation

$$1=\varepsilon M_1\omega^2 \begin{matrix}(M_2\omega^2-2\gamma_1-4\gamma_2)^{1/2}\\(M_1\omega^2-2\gamma_1-4\gamma_2)^{1/2}\end{matrix} g^*(0,0,0;\omega) \tag{6·9}$$

in the case that a mass M_1 is replaced by a mass $(1-\varepsilon)M_1$. The function $g^*(0,0,0;\omega)$ is given by

$$g^*(0,0,0;\omega) = \frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi \frac{d\phi_1\,d\phi_2\,d\phi_3}{(M_1\omega^2-2\gamma_1-4\gamma_2)^{1/2}(M_2\omega^2-2\gamma_1-4\gamma_2)^{1/2}+2\gamma_1\cos\phi_1+2\gamma_2\cos\phi_2+2\gamma_2\cos\phi_3}. \tag{6·10}$$

We define the frequencies ω_1^2 and ω_2^2 by

$$\omega_1^2=\frac{2\gamma_1+4\gamma_2}{M_1},\quad \omega_2^2=\frac{2\gamma_1+4\gamma_2}{M_2}. \tag{6·11}$$

In terms of these frequencies $g^*(0,0,0;\omega)$ becomes

$$g^*(0,0,0;\omega) = \frac{2+\alpha}{\pi^3 M_2\omega_2^2}\int_0^\pi\int_0^\pi\int_0^\pi \frac{d\phi_1\,d\phi_2\,d\phi_3}{[(\omega^2/\omega_1^2)-1]^{1/2}[(\omega^2/\omega_2^2)-1]^{1/2}(2+\alpha)-\alpha\cos\phi_1-\cos\phi_2-\cos\phi_3}, \tag{6·12}$$

and the eigenvalue equation takes the form

$$1=\varepsilon\frac{\omega^2}{\omega_1^2}\frac{[(\omega^2/\omega_2^2)-1]^{1/2}}{[(\omega^2/\omega_1^2)-1]^{1/2}}(2+\alpha)\left\{\frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi \frac{d\phi_1\,d\phi_2\,d\phi_3}{[(2+\alpha)/\mu]-\alpha\cos\phi_1-\cos\phi_2-\cos\phi_3}\right\} \tag{6·13}$$

where

$$\mu^{-1}=\left(\frac{\omega^2}{\omega_1^2}-1\right)^{1/2}\left(\frac{\omega^2}{\omega_2^2}-1\right)^{1/2}. \tag{6·14}$$

We are interested in those solutions of Eq. (6·13) for which $\omega^2 > \omega_L^2=\omega_1^2+\omega_2^2$. Inverting Eq. (6·14) we obtain

$$\frac{\omega^2}{\omega_L^2}=f^2=\frac{1}{2}+\frac{1}{2}\left[1+\frac{4\omega_1^2\omega_2^2}{\omega_L^4}\left(\frac{1}{\mu^2}-1\right)\right]^{1/2}. \tag{6·15}$$

In the case of a U-center in NaCl, $M_1=36$, $M_2=23$, $M'=1$, so that $\varepsilon=35/36$ as before. The value of α is still given by Eq. (6·5) and is

$$\alpha=\frac{5.73}{2\times 1.33}=2.154 \tag{6·16}$$

in the present case.

The solution to Eq. (6·13) is obtained with the aid of the material of reference 9) and is found to be

$$f=3.760$$

which should be compared with the experimental value of

$$f=3.542$$

due to Pick.

§ 7. Discussion

The calculations presented in this paper are restricted to the effects of the mass difference of an impurity atom on the infrared absorption due to lattice vibrations. The most striking result is the appearance of a discrete absorption line associated with the localized mode appearing above the maximum frequency of the optical branch when a host atom is replaced by a lighter atom. This effect disappears if the impurity atom is heavier than the host atom. Thus, if the addition of heavy atom impurities leads to experimentally observed discrete absorption lines, their explanation must lie in a change of force constants or some other mechanism rather than in the mass change. This conclusion applies as well to discrete modes which drop down from the optical branches, since the associated absorption is weak and is masked by the continuous in-band absorption.

For not too large concentrations of impurities there are as many discrete impurity modes above the optical branch as there are degrees of freedom associated with the impurity atoms. Not all of these impurity modes are strongly absorbing, however. In the particular cases studied in sections 2 and 3 only the symmetric modes were optically absorbing. For a general number of impurities the impurity modes can no longer be classified as purely symmetric or antisymmetric, but one can still use a qualitative characterization of symmetric or antisymmetric. Only the discrete modes with a large symmetric character will be expected to lead to strong absorption. The width of the optical absorption line may therefore be relatively narrow compared to the width of the impurity band itself. An extreme case would be that in which every second host atom in an initially monatomic chain is replaced by a lighter impurity atom. The impurity band then becomes a significant fraction of the optical branch, but only the largest frequency is optically absorbing. Experimentally the discrete lines associated with hydride ions in alkali halides are observed by Pick to be very narrow.

The rather good agreement between the calculated values of the impurity frequencies given in section 6 and Pick's experimental results is gratifying but may be fortuitous. Our calculations did not take into account the long range Coulomb forces between ions and the coupling between the x-, y- and z-displacement components. It may be, however, that the relation of the impurity frequency to the reststrahl frequency, which is what we actually calculate, is not sensitive to

these details, and is given more accurately by our model than either frequency separately.

Appendix

In this appendix we establish Eqs. (2·14) and (2·15) of the text. We begin by evaluating $g(0)$ which is given by

$$g(0)=\frac{1}{2\gamma N}\sum_{s=1}^{N}\frac{1}{\cos(2\pi s/N)-\cos\phi} \qquad 0<\phi<\pi, \tag{A·1}$$

$$=\frac{1}{2\gamma N}\left\{\frac{1}{2\pi i}\int_{C}\frac{\pi\cot\pi z}{\cos(2\pi z/N)-\cos\phi}dz-\mathrm{Res}\left[z=\frac{N\phi}{2\pi}\right]-\mathrm{Res}\left[z=N-\frac{N\phi}{2\pi}\right]\right\} \tag{A·2}$$

where C is the counterclockwise rectangular contour with corners at $(1/2, -i\alpha)$, $(N+1/2, -i\alpha)$, $(N+1/2, i\alpha)$, and $(1/2, i\alpha)$ which is shown in Fig. A.1. It is easily seen that the contributions to $g(0)$ from BC and DA cancel each other and we are left with

$$g(0)=\frac{\cot[N\phi/2]}{2\gamma\sin\phi}+\frac{\mathrm{Im}}{2\gamma N}\int_{1/2}^{N+1/2}\frac{\cot\pi(x-i\alpha)}{\cos[(2\pi/N)(x-i\alpha)]-\cos\phi}dx \tag{A·3}$$

$$=\frac{\cot[N\phi/2]}{2\gamma\sin\phi} \tag{A·4}$$

in the limit as $\alpha\to\infty$. This is Eq. (2·15).

The recurrence relation satisfied by $g(n)$ can be written as

$$-g(n)\cos\phi+\frac{1}{2}[g(n+1)+g(n-1)]=\frac{\delta_{no}}{2\gamma}. \tag{A·5}$$

For $n\neq 0$ the solution to Eq. (A·5) is given by

$$g(n)=A\cos n\phi+B\sin|n|\phi. \tag{A·6}$$

It is clear however, that $A=g(0)$, while from equations (A·5), and (2·13) we find that

$$g(1)=g(-1)=g(0)\cos\phi+\frac{1}{2\gamma}, \tag{A·7}$$

so that $B=1/2\gamma\sin\phi$. Combining these results we obtain finally that

$$g(n)=g(0)\cos n\phi+\frac{\sin|n|\phi}{2\gamma\sin\phi}, \tag{A·8}$$

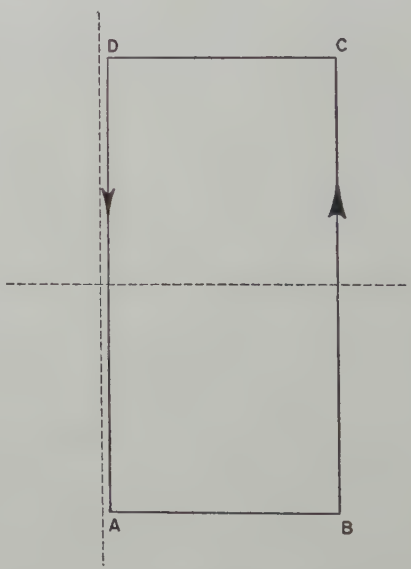

Fig. A.1. The contour for the evaluation of the integral (A·2).

which is Eq. (2·14). It is readily established that the solution (A·8) satisfies the symmetry condition

$$g(n)=g(N-n) \qquad (A\cdot 9)$$

which is implied by Eq. (2·13).

An approximate method for evaluating $g(n)$ has been suggested by Lax.[11]

References

1) N. B. Hannay, *Semiconductors* (Reinhold Publishing Corporation, New York, 1959), p. 474.
2) H. Pick, International Conference on Color Centers, Oregon State College, Corvallis, Oregon, 1959.
3) R. Braunstein, Bull. Am. Phys. Soc. II, **5** (1960), 198.
4) E. W. Montroll and R. B. Potts, Phys. Rev. **100** (1955), 525; **102** (1956), 72.
A. A. Maradudin, P. Mazur, E. W. Montroll and G. H. Weiss, Rev. Mod. Phys. **30** (1958), 175.
5) J. Mahanty, A. A. Maradudin and G. H. Weiss, Prog. Theor. Phys. **20** (1958), 369.
6) P. Mazur, E. W. Montroll and R. B. Potts, Jour. Wash. Acad. Sci. **46** (1956), 2.
7) E. W. Montroll, A. A. Maradudin and G. H. Weiss, *Proceedings of the Stevens Institute Conference on the Many Body Problem* (Interscience Publishers, 1960).
8) H. B. Rosenstock, J. Chem. Phys. **23** (1955), 2415; **27** (1957), 1194.
9) A. A. Maradudin, E. W. Montroll, G. H. Weiss, R. Herman and H. W. Milnes, Memoires of the Royal Belgian Academy of Sciences, in press.
10) C. Kittel, *Introduction to Solid State Physics* (John Wiley and Sons, New York, 1953), p. 53.
11) M. Lax, Phys. Rev. **94** (1954), 1391.

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$M1$-Transition in V^{51} and Configuration Mixing

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The $M1$-transition probability of the 0.321 Mev level of V^{51} has been calculated on the basis of configuration mixing. The calculated values are satisfactory in comparison with experimental value which was recently measured by Delyagin and Preĭsa. In the calculations, the Rosenfeld mixture type of spin exchange forces is used for the two-body interaction between particles in the unfilled shell. For justification of our standpoint, the magnetic moment of V^{51} has been calculated. The result is almost in agreement with the experimental value.

§ 1. Introduction

If outside the filled shell there are only neutrons or only protons characterized by the configuration j^n, the magnetic dipole transition $M1$ alone will be allowed that occurs without a change in seniority quantum number v. V^{51} has three protons outside the closed shells with $Z=20$ and $N=28$. According to the jj-coupling shell model, the configuration of the ground state is $(f_{7/2})^3$ $7/2$ $v=1$, and that of the first excited state (0.321 Mev) is $(f_{7/2})^3$ $5/2$ $v=3$, hence we should expect that the $M1$-transition between these two states vanishes because of the seniority change $\Delta v=2$.

Recently, however, the life time of the 0.321 Mev excited level of V^{51} was measured by Delyagin and Preĭsa[1] using the resonance scattering of γ quanta. They obtained the value $\tau=3.4\times10^{-10}$ sec for the lifetime of the $M1$-transition. This value is large by 300 times in comparison with a single proton transition expected by the shell model (Weisskopf formula[2]). The magnitude of this forbiddeness is the same one as for the l-forbidden transitions.[3] However, we must recognize this as a mere coincidence because in such nucleus the l-forbidden transition cannot occur.

In order to interpret the difficulty, the $M1$-transition probability of V^{51} has been calculated from the standpoint of configuration mixing. For a sort of justification of our standpoint, the magnetic moment of V^{51} has been computed. The pairing energies of the Ca isotopes and the magnetic moment of Ca^{43} have already been calculated on the same basis in the previous work.[4] In Ca^{43} there are three $f_{7/2}$ neutrons in place of protons in V^{51} outside the closed shells, so the two nuclei

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can be considered as equivalent in the shell model. This, in fact, is supported by the extreme resemblance of the level structures[5] experimentally known.

In this calculation, the Rosenfeld mixture type of spin exchange forces is selected for the two-body interaction between nucleons in the unfilled shell, because this kind of spin mixture has been found adequate to interpret the differences of the ground state energies of the Ca isotopes and the magnetic moment of Ca^{43}. The same calculations are applied to the lifetime of the 0.38 Mev level of Ca^{43} which has not yet been measured.

§ 2. Transition probability

It is well confirmed experimentally that the ground and 0.321 Mev level of V^{51} have the spin 7/2 and 5/2. According to the *jj*-coupling shell model, the ground state is assigned as $(f_{7/2}^3)7/2\ v=1$ and 0.321 Mev level, as $(f_{7/2}^3)5/2\ v=3$. As the transition between these states is forbidden by the selection rule on the seniority, non-vanishing effects are obtained by configuration mixing of these states. The matrix elements for the magnetic dipole transition are obtained from the usual magnetic moment operator. The differences from the static moments are that the initial and final states are different and transition probability must be summed on the magnetic quantum numbers.

The transition probability for $M1$-radiation is given by[6]

$$T(M1)=\frac{16\pi}{3}\frac{1}{\hbar}\left(\frac{\omega}{c}\right)^3\frac{3}{4\pi}\left(\frac{e\hbar}{2Mc}\right)^2 B(M1)=0.420\times 10^{13}E^3B(M1)\,\mathrm{sec}^{-1},$$

where photon energy $E=h\omega$ is measured in Mev and the reduced transition probability $B(M1)$ can be written as

$$B(M1)=\frac{1}{2J_i+1}\sum_{M_i}\sum_{M_f}\sum_{q}|\langle J_iM_i|\sum\mu_q|J_fM_f\rangle|^2=\frac{1}{2J_i+1}|\langle J_i\|\sum\mu\|J_f\rangle|^2$$

in terms of the matrix elements of the magnetic moment operator μ_q. Now let us denote the wave functions of the ground and 0.321 Mev level of V^{51} by $\Psi(7/2\ M_f)$ and $\Phi(5/2M_i)$. Then the wave functions $\Psi(7/2\ M_f)$ and $\Phi(5/2\ M_i)$ are written as

$$\Psi=(\psi_0+\sum_{m=1}^{10}\alpha_m\psi_m)/(1+|\alpha_m|^2)^{1/2}$$

$$\Phi=(\varphi_0+\sum_{n=1}^{8}\beta_n\varphi_n)/(1+|\beta_n|^2)^{1/2}$$

where ψ_0 and φ_0 are the shell model wave functions $\psi_0(f_{7/2}{}^3\ 7/2)$ and $\varphi_0(f_{7/2}{}^3\ 5/2)$, and ψ_m and φ_n the wave functions with the excited configurations. In carrying out computations, we adopt as the excited configurations one or two particle jump configurations of $f_{5/2}$ or $p_{3/2}$ particle, i. e. for $J=7/2$ state, 10 excited configurations and for $J=5/2$, 8 configurations. This procedure is the same one with the previous works[4] so that the details will be omitted here. The mixing amplitudes α_m and β_n have been calculated for the two-body interaction with the Rosenfeld spin mixture, which was found adequate to interpret the ground states of the Ca

isotopes. β_n are given for various values of the range parameter λ in Table I. (α_m have already been listed in the previous paper.)

The eigenvalues $E_{7/2}$ and $E_{5/2}$ corresponding to these eigenvectors have been calculated. Experimentally, the spin 7/2 level competes with the 5/2 level in V^{51} and Ca^{43}, but the results of calculations show that the 7/2 level is ground state

Table I

Mixing amplitudes β_n. The wave function Φ of the first excited state is written as $\Phi=(\varphi_0+\sum_{n=1}^{8}\beta_n\varphi_n)/(1+|\beta_n|^2)^{1/2}$, where $\varphi_0=\varphi(f_{7/2}^2: 5/2)$, $\varphi_1=\varphi(f_{7/2}^2(2)\ f_{5/2}: 5/2)$, $\varphi_2=\varphi(f_{7/2}^2(4)\ f_{5/2}: 5/2)$, $\varphi_3=\varphi(f_{7/2}^2(6)\ f_{5/2}: 5/2)$, $\varphi_4=\varphi(f_{7/2}^2(2)\ p_{3/2}: 5/2)$, $\varphi_5=\varphi(f_{7/2}^3(4)p_{3/2}: 5/2)$, $\varphi_6=\varphi(f_{7/2}f_{5/2}^2(2): 5/2)$, $\varphi_7=\varphi(f_{7/2}f_{5/2}^2(4): 5/2)$ and $\varphi_8=\varphi(f_{7/2}p_{3/2}^2(2): 5/2)$.

The wave function Ψ of the ground state has already been listed in Prog. Theor. Phys. **24** (1960), 201.

λ \ β_n	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8
0.2	0.015,4	−0.022,9	−0.035,3	−0.153	−0.173	−0.074,1	−0.009,4	−0.051,1
0.4	0.034,1	−0.021,5	−0.057,4	−0.193	−0.223	−0.107	−0.011,4	−0.052,9
0.6	0.056,7	−0.017,2	−0.086,4	−0.227	−0.277	−0.171	−0.014,0	−0.068,8
0.8	0.061,9	−0.014,8	−0.107	−0.210	−0.267	−0.237	−0.015,8	−0.078,1
10	0.047,4	−0.023,3	−0.117	−0.159	−0.207	−0.282	−0.018,1	−0.055,8

Table II

Non-vanishing reduced matrix elements $\langle n\|\sum\mu\|m\rangle$ of $M1$-transition for the wave functions Ψ_m and Φ_n given in the previous paper and Table I. Values of a defined by $\langle n\|\sum\mu\|m\rangle=a(g_s-g_l)$ are given in the Table, where $g_s=5.585$, $g_l=1$ for protons, and $g_s=-3.826$, $g_l=0$ for neutrons in units of nuclear magneton.

n	m	a	n	m	a
0	3	$\sqrt{33}/7\sqrt{7}$	3	4	$-4\sqrt{55}/7\sqrt{7}$
0	4	$20/7\sqrt{77}$	3	8	$3\sqrt{55}/7\sqrt{14}$
0	5	$-\sqrt{195}/\sqrt{77}$	3	9	$-9/7\sqrt{14}$
1	0	$3\sqrt{2}/\sqrt{7}$	4	6	$16/7\sqrt{21}$
1	1	$-\sqrt{2}/\sqrt{7}$	5	7	$4\sqrt{10}/7\sqrt{7}$
1	8	$-5\sqrt{15}/7\sqrt{7}$	6	3	$-5/7\sqrt{7}$
1	9	$-3\sqrt{33}/7\sqrt{7}$	6	4	$10\sqrt{3}/7\sqrt{7}$
2	0	$-5\sqrt{15}/7\sqrt{7}$	6	5	$-5\sqrt{65}/7\sqrt{7}$
2	1	$-5\sqrt{15}/7\sqrt{7}$	6	8	$-10\sqrt{6}/7\sqrt{7}$
2	3	$-12\sqrt{3}/7\sqrt{7}$	7	3	$2\sqrt{33}/7\sqrt{7}$
2	8	$-15/7\sqrt{14}$	7	4	$-120/7\sqrt{77}$
2	9	$3\sqrt{55}/7\sqrt{14}$	7	5	$-2\sqrt{195}/7\sqrt{77}$
3	0	$-3\sqrt{33}/7\sqrt{7}$	7	9	$-12\sqrt{10}/7\sqrt{7}$
3	1	$-3\sqrt{33}/7\sqrt{7}$	8	10	$20\sqrt{2}/7\sqrt{21}$

and the 5/2 level lies closely above the 7/2 level. Details will be reported together with the level structures of Ca^{42} in the forthcoming paper.

The $M1$ reduced transition probability between states consisting of several independent particles is obtained by the standard method of tensor operators[7]. Only the configurations with one different single particle state give the non-vanishing contributions. Moreover, in the case of $M1$-transition the principal quantum numbers and angular momenta must be equal for both states owing to the selection rule of the reduced matrix element between single particle states $\langle j\|\mu\|j'\rangle$. Non-vanishing elements are given in Table II.

§ 3. Comparison with experimental values and discussions

The calculated values of the reduced transition probability $B(M1)$ and mean lifetime τ are given in Table III for the various values of the range parameter λ where the measured value 0.321 Mev is taken for the energy of γ quanta. Even if the eigenvalues $E_{7/2}$ and $E_{5/2}$ belonging to the wave functions $\Psi(7/2\, M_f)$ and $\Phi(5/2\, M_i)$ are used in computations, the results will not be almost changed. Calculated values τ are satisfactory in comparison with experimental value 3.4×10^{-10} sec which has recently been measured by Delyagin and Preïsa[1] using the method of the resonance scattering. Considering the value 9.7×10^{-13} sec which is expected from a single proton transition (Weisskopf formula[2]), we cannot help but recognize the usefulness of configuration mixing. These situations were shown in calculations on l-forbidden $M1$-transition by Arima et al.[3] even if their calculations were based on the first order perturbation and contained some approximations. Moreover, in order to support our standpoint, the magnetic moment of V^{51} has been calculated. It is found that the calculated values have a good trend toward the experimental value 5.14 from the Schmidt value 5.793. The numerical values are given in Table IV.

Table III

Calculated values of the reduced transition probability $B(M1)$ and lifetime τ of V^{51} and Ca^{43} at photon energies 0.321 and 0.38 Mev. The experimental value of the lifetime of V^{51} for $M1$ radiation is 3.4×10^{-10} sec, and the value of a single proton transition is 9.7×10^{-13} sec.

	V^{51}			Ca^{43}	
λ	$B(M1)(10^{-2})$	$\tau(10^{-10}$ sec)	τ_{exp}	$B(M1)(10^{-2})$	$\tau(10^{-10}$ sec)
0.2	0.663	10.85		0.462	9.398
0.4	1.988	3.620		1.385	3.133
0.6	4.368	1.648	3.4×10^{-10}	3.041	1.427
0.8	5.854	1.230		4.076	1.064
1.0	5.381	1.338		3.747	1.158

Table IV

Calculated values of the magnetic moment of V^{51}, the experimental value is 5.14 and the Schmidt value is 5.793. The unit is the nuclear magneton.

λ \ μ	μ_{cal}	μ_{sch}	μ_{exp}
0.2	5.606		
0.4	5.549		
0.6	5.469	5.793	5.14
0.8	5.362		
1.0	5.257		

Up to date, the radiative transitions which are named " seniority forbidden " have not yet been found except V^{51}. As mentioned in § 1 and § 2, Ca^{43} is the equivalent nucleus with V^{51} in the shell model since Ca^{43} has three $f_{7/2}$ neutrons instead of protons outside the closed shell. Therefore, we can also expect the $M1$-transition, although it is " seniority forbidden ", for first excited state (0.38 Mev) of Ca^{43}. The lifetime for the $M1$-transition of this level has been calculated by the way and is added in Table III.

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References

1) N. N. Delyagin and M. Preĭsa, Soviet Phys. JETP **9** (1959), 1127.
2) J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, New York, 1952), 627.
3) A. Arima, H. Horie and M. Sano, Prog. Theor. Phys. **17** (1957), 567.
4) T. Komoda, Prog. Theor. Phys. **22** (1959), 891; **24** (1960), 201.
5) K. Way et al., *Nuclear Level Schemes*, TID-5300 (1955).
6) A. Bohr and B. R. Mottelson, Kgl Danske Videnskab. Selskab, Mat.-fys. Medd. **27** (1953), No. 16.
S. Moszkowski, *Beta- and Gamma-Ray Spectroscopy*, edited by K. Siegbahn (North-Holland Publishing Company, Amsterdam, 1955), 373.
7) G. Racah, Phys. Rev. **62** (1942), 438; **63** (1943), 367.

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On the Kinetic Equation for a High Temperature Plasma

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Effects of binary and ternary correlations in a high temperature plasma are examined in detail on the basis of the so-called B-B-G-K-Y equation for a system of charged particles. It is shown that the effects of ternary correlation are essential in giving rise to shielding of the interaction between particles and have important influences upon disturbances of the binary correlations. The effects of binary correlation can be divided into the correction terms of the self-consistent field of the Boltzmann-Vlasov equation and the terms representing the collision effects between particles through the shielded interaction. The effects of the collision terms are investigated in detail by deriving a generalized Fokker-Planck equation which can be reduced to the equation derived by Tchen for a special case. The effects of the correction terms of the self-consistent field are examined by deriving an equation of motion of the density fluctuation which describes the longitudinal plasma oscillations. The correction terms give rise to a shift of the k^2-term of the plasma frequency and a damping factor. The frequency of the plasma oscillations is determined to be given by

$$\omega^2=\omega_p^2+(1+\delta)(3\kappa T/m)k^2$$

where the shift due to the binary correlations δ is given as follows,

$$\delta=1.05\{(e^2/\kappa T)^3 n\}^{1/2}.$$

The damping factor γ is determined to be

$$\gamma=0.028\{(e^2/\kappa T)^3 n\}^{1/2}(k/k_d)^2\cdot\omega_p$$

at the limit of $k\to 0$. It is shown that the damping factor due to the long-range binary correlations predominates over the Landau damping in the range of small value of k. On the basis of present investigation, it is concluded that the equation derived by Balescu does not involve any information concerning the plasma oscillations. Since the plasma oscillations are nothing but the appearance of periodical spatial inhomogeneity in the system, it is evident that the assumption of the spatial homogeneity introduced by Temko, Tchen and Balescu in their derivations of the fundamental equations rules out the possible occurrence of the plasma oscillation.

§ 1. Introduction

Although the magnetohydrodynamics[1] is frequently applied to the description of physical phenomena of high temperature plasma, it is necessary to make clear the foundation and the range of applicability of the phenomenological macroscopic theory, basing on the first principle. Investigations of these objects have been developed by many authors.[2)~6)]

Watson and Brueckner,[3),4)] for example, have started from the Boltzmann equation commonly used in the kinetic theory of gases. The distribution func-

tion of charged particles in the phase space is determined by the Boltzmann equation,

$$\frac{\partial}{\partial t}f+(\boldsymbol{v}\cdot\boldsymbol{\nabla})f+\frac{e}{m}\left(\boldsymbol{E}+\frac{1}{c}\boldsymbol{v}\times\boldsymbol{H}\right)\cdot\boldsymbol{\nabla}_{v}f=C[f] \tag{1}$$

where $\boldsymbol{\nabla}_{v}$ is a gradient operator in the velocity space. $\boldsymbol{E}$, $\boldsymbol{H}$ are an external electric and magnetic field, respectively. $C[f]$ is the so-called collision term. When they apply the Boltzmann equation (1) to the investigations of high temperature plasma, however, the collision term presents serious difficulties because of the long range interaction between the charged particles. In order to avoid this obstacle, they have used a prescription proposed by Vlasov[7] and Landau.[8] The Vlasov-Landau prescription suggests to combine the Boltzmann equation with the Maxwell equations in which the electromagnetic field is supposed to be determined self-consistently by the charge-current distribution given as

$$\rho(\boldsymbol{x}, t)=e\int f(\boldsymbol{x}, \boldsymbol{p}, t)\,d\boldsymbol{p} \tag{2a}$$

$$\boldsymbol{j}(\boldsymbol{x}, t)=e\int \boldsymbol{v}f(\boldsymbol{x}, \boldsymbol{p}, t)\,d\boldsymbol{p}. \tag{2b}$$

It should be realized, however, that the electromagnetic fields determined by the charge-current distribution (2a), (2b) do not take account of the effects of many particle correlations. Therefore, we may ask the following question: Does the charge-current fluctuation due to the binary correlations contribute to the internal electromagnetic field, or not? It is one of the objects of the present paper to answer this question.

Now, there are two aspects of the problem, one is dynamical and the other is statistical, concerning the correlation effects of the many particle system. Bohm and Pines[9] have investigated intensively the dynamical aspect of the many particle problem in the electron plasma. According to their investigations, in the electron plasma, there occurs two modes of the dynamical motion, namely the collective oscillations participated by many particles and the random motion of individual particles, and the interaction between electrons is screened off at a distance of the order of Debye length. Yet, it is true that the determination of the eigen mode of the collective oscillations needs a statistical knowledge of the system which determines how every particle participates in the collective oscillations.

Bogoliubov[10] was the first who emphasized the importance of the investigation of the kinetic theory from the dynamical point of view. Following Bogoliubov's method, many Soviet physicists have investigated the problems concerning the theory of plasma. Tolmachev[11] and Temko[12] have derived the Fokker-Planck equation which describes the time development of the momentum distribution of the high temperature plasma. Tchen[13] has investigated the problem much more thoroughly by taking account of the non-Markovian behaviour. Recently, Prigogin

and Balescu[14] have investigated the problem of irreversibility in gases using the method of van Hove[15] and Hugenholtz[16] for the studies of the many particle system, and Balescu[17] has applied their method to the high temperature plasma. Although the Fokker-Planck equation appears to be useful in the investigations of the plasma, the equation cannot account for the occurrence of the plasma oscillations, because the equation is derived under the assumption of spatial homogeneity of the system. It is our second object to examine their approaches to the problem, and to generalize the Fokker-Planck equation allowing the spatial inhomogeneity of the system. The generalized Fokker-Planck equation can be used to derive the equation of motion of the density fluctuations, namely that of the plasma oscillations.

In order to accomplish these objects, we will formulate the problem on the basis of the so-called B-B-G-K-Y equation (§ 2), and seek for the approximate solutions of the fundamental equations (§ 3). In § 4, we will examine contributions of the binary correlations to the internal electric field of the high temperature plasma. The effects of binary collisions are examined in § 5 by reducing the fundamental equation obtained in § 3 to the Fokker-Planck equation. In § 5, it will be concluded also that the complicated set of equations derived by Balescu does not contain any information about the plasma oscillations. The generalized Fokker-Planck equation will be derived in § 6 by taking into account the spatially inhomogeneous effects. In § 7, we will derive the equation of motion of the density fluctuations and will analyze the binary correlation effects on the plasma oscillations. Finally, in § 8 we will summarize the results obtained in the present investigation.

§ 2. Formulation of Problems

It is essential to start with the B-B-G-K-Y equation for the investigations of the problems presented in § 1. Although various authors have derived the B-B-G-K-Y equation by integrating the Liouville equation, we will rederive the equation following the method used by Kadomtsev[18] and Klimontovich.[19]

For the sake of simplicity, let us consider the system of one kind of particles, totalling N in a large volume Ω. No external forces are assumed. For the case of charged particle system, we assume the whole system is immersed in a uniform background of the charge of opposite sign, so that the neutrality of the whole system is insured. We take the following expression of the Hamiltonian,

$$H=\sum_i \frac{p_i^2}{2m}+\frac{1}{2}\frac{\lambda}{\Omega}\sum_{\boldsymbol{k}}\sum_{i\neq j} V(\boldsymbol{k})\, e^{i\boldsymbol{k}(x_i-x_j)} \tag{3}$$

where the term $\boldsymbol{k}=0$ is excluded in the summation over the wave vector $\boldsymbol{k}$, because of the neutrality condition. For the electron plasma immersed in the uniform background of the positive charge, the interaction between the electrons is described by the Coulomb potential, so we have

$$\lambda V(\boldsymbol{k}) = 4\pi e^2/k^2. \tag{4}$$

In discussing the ion system in high temperatur plasma, the electrons can be boiled down as the charged cloud surrounding the every ion, hence the effective interaction between ions can be described by the screened Coulomb potential,[20] so that we have

$$\lambda V(\boldsymbol{k}) = 4\pi e^2/(k^2+k_d^2) \tag{5}$$

where k_d is the abbreviation of the Debye wave number $(4\pi e^2 n/\kappa T)^{1/2}$. Here, κ is the Boltzmann constant and n is the number density of particles N/Ω.

Now, let us introduce the microscopic density of particles in the phase space defined as

$$F(\boldsymbol{x}, \boldsymbol{v}; t) = \sum_i \delta(\boldsymbol{x}-\boldsymbol{x}_i(t))\,\delta(\boldsymbol{v}-\boldsymbol{v}_i(t)). \tag{6}$$

The time variation of this microscopic density is described by the Liouville equation

$$\frac{\partial F}{\partial t} = \{H, F\} \tag{7}$$

where $\{\ ,\ \}$ is the Poisson bracket:

$$\{H, F\} = \sum_i \left\{ \frac{\partial H}{\partial \boldsymbol{x}_i} \cdot \frac{\partial F}{\partial \boldsymbol{p}_i} - \frac{\partial H}{\partial \boldsymbol{p}_i} \frac{\partial F}{\partial \boldsymbol{x}_i} \right\}. \tag{8}$$

Eq. (7) is the partial differential equation which is equivalent to the set of Newton's equations of motion for N particles of the system. Writing down Eq. (7) explicitly, we have

$$\frac{\partial}{\partial t} F + (\boldsymbol{v}\cdot\boldsymbol{\nabla}) F = \frac{\lambda}{\Omega} \sum_k V(\boldsymbol{k})\, i\boldsymbol{k} \cdot \frac{\partial}{\partial \boldsymbol{p}} \iint F(\boldsymbol{x}, \boldsymbol{v}) F(\boldsymbol{x}', \boldsymbol{v}') e^{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x}')}\, d\boldsymbol{x}'\, d\boldsymbol{v}'. \tag{9}$$

Although Eq. (9) is a rigorous equation, it is only sensible to discuss the statistical behavior of N particles by taking the usual statistical average over the initial states of the system. Denoting the average over the initial states of the system by $\langle\ \ \rangle_{Av}$, we define the statistical distribution function $f(\boldsymbol{x}, \boldsymbol{v})$ as

$$f(\boldsymbol{x}, \boldsymbol{v}) = \langle F(\boldsymbol{x}, \boldsymbol{v}) \rangle_{Av}. \tag{10}$$

Then, taking the average of Eq. (9) over the initial states of the system, it is reduced to the following equation,

$$\frac{\partial}{\partial t} f(\boldsymbol{x}, \boldsymbol{v}) + (\boldsymbol{v}\cdot\boldsymbol{\nabla}) f(\boldsymbol{x}, \boldsymbol{v}) = \frac{\lambda}{\Omega} \sum_k V(\boldsymbol{k})\, i\boldsymbol{k} \cdot \frac{\partial}{\partial \boldsymbol{p}} \iint f(\boldsymbol{x}, \boldsymbol{v}; \boldsymbol{x}', \boldsymbol{v}') e^{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x}')}\, d\boldsymbol{x}'\, d\boldsymbol{v}' \tag{11}$$

where the binary distribution function $f(\boldsymbol{x}, \boldsymbol{v}; \boldsymbol{x}', \boldsymbol{v}')$ is defined as to be

$$f(\boldsymbol{x}, \boldsymbol{v}; \boldsymbol{x}', \boldsymbol{v}') = \langle F(\boldsymbol{x}, \boldsymbol{v}) F(\boldsymbol{x}', \boldsymbol{v}') \rangle_{Av}. \tag{12}$$

Let us define a binary correlation function $\varphi(\boldsymbol{x}, \boldsymbol{v}; \boldsymbol{x}', \boldsymbol{v}')$ by

$$f(\boldsymbol{x}, \boldsymbol{v}; \boldsymbol{x}', \boldsymbol{v}') = f(\boldsymbol{x}, \boldsymbol{v}) f(\boldsymbol{x}', \boldsymbol{v}') + \varphi(\boldsymbol{x}, \boldsymbol{v}; \boldsymbol{x}', \boldsymbol{v}'). \tag{13}$$

Then, substituting Eq. (13) into Eq. (11), we get

$$\frac{\partial}{\partial t} f(\boldsymbol{x}, \boldsymbol{v}) + (\boldsymbol{v} \cdot \nabla) f(\boldsymbol{x}, \boldsymbol{v}) + \boldsymbol{F}(\boldsymbol{x}) \cdot \frac{\partial}{\partial \boldsymbol{p}} f(\boldsymbol{x}, \boldsymbol{v}) = \frac{\lambda}{\Omega} \sum_{k} V(\boldsymbol{k})$$

$$\times i\boldsymbol{k} \cdot \frac{\partial}{\partial \boldsymbol{p}} \iint \varphi(\boldsymbol{x}, \boldsymbol{v}; \boldsymbol{x}', \boldsymbol{v}') e^{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x}')} d\boldsymbol{x}' d\boldsymbol{v}', \tag{14}$$

where $\boldsymbol{F}(\boldsymbol{x})$ is defined by

$$\boldsymbol{F}(\boldsymbol{x}) = -\frac{\lambda}{\Omega} \sum_{k} V(\boldsymbol{k}) i\boldsymbol{k} \iint f(\boldsymbol{x}', \boldsymbol{v}') e^{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x}')} d\boldsymbol{x}' d\boldsymbol{v}'. \tag{15}$$

$\boldsymbol{F}(\boldsymbol{x})$ is a resultant force, acting on the particle at the point $(\boldsymbol{x}, \boldsymbol{v})$, due to the other particles of the system. The definition of $\boldsymbol{F}(\boldsymbol{x})$, Eq. (15), shows that it must be determined self-consistently by the kinetic equation of the particle distribution function, Eq. (14). Here, we will remark that the Boltzmann-Vlasov equation does result from Eq. (14) with Eq. (15) by disregarding the effects of the binary correlation which are expressed by the right-hand side of Eq. (14). It is essential, however, to find the binary correlation function $\varphi(\boldsymbol{x}, \boldsymbol{v}; \boldsymbol{x}', \boldsymbol{v}')$ for the purpose of answering the question, how do the binary correlation effects contribute to the behavior of high temperature plasma?

Now, let us derive the equation of $\varphi(\boldsymbol{x}, \boldsymbol{v}; \boldsymbol{x}', \boldsymbol{v}')$. Using Eq. (9), we can construct an equation for the product of $F(\boldsymbol{x}, \boldsymbol{v})$ and $F(\boldsymbol{x}', \boldsymbol{v}')$, which in turn involves the product of three microscopic densities $F(\boldsymbol{x}, \boldsymbol{v})$, $F(\boldsymbol{x}', \boldsymbol{v}')$ and $F(\boldsymbol{x}'', \boldsymbol{v}'')$. Taking the average over the initial states of the system of the latter product, we have

$$\begin{aligned}\langle F(\boldsymbol{x}, \boldsymbol{v}) F(\boldsymbol{x}', \boldsymbol{v}') F(\boldsymbol{x}'', \boldsymbol{v}'') \rangle_{Av} &= \delta(\boldsymbol{x}-\boldsymbol{x}') \delta(\boldsymbol{v}-\boldsymbol{v}') f(\boldsymbol{x}'', \boldsymbol{v}''; \boldsymbol{x}', \boldsymbol{v}') \\ &+ \delta(\boldsymbol{x}'-\boldsymbol{x}'') \delta(\boldsymbol{v}'-\boldsymbol{v}'') f(\boldsymbol{x}'', \boldsymbol{v}''; \boldsymbol{x}, \boldsymbol{v}) + \delta(\boldsymbol{x}-\boldsymbol{x}'') \delta(\boldsymbol{v}-\boldsymbol{v}'') f(\boldsymbol{x}, \boldsymbol{v}; \boldsymbol{x}', \boldsymbol{v}') \\ &+ f(\boldsymbol{x}, \boldsymbol{v}; \boldsymbol{x}', \boldsymbol{v}'; \boldsymbol{x}'', \boldsymbol{v}'').\end{aligned} \tag{16}$$

Then, the equation of $\varphi(\boldsymbol{x}, \boldsymbol{v}; \boldsymbol{x}', \boldsymbol{v}')$ is obtained as follows,

$$\begin{aligned}&\left\{\frac{\partial}{\partial t} + (\boldsymbol{v} \cdot \nabla) + (\boldsymbol{v}' \cdot \nabla')\right\} \varphi(\boldsymbol{x}, \boldsymbol{v}; \boldsymbol{x}', \boldsymbol{v}') = -\left\{\frac{\partial}{\partial t} + (\boldsymbol{v} \cdot \nabla) + (\boldsymbol{v}' \cdot \nabla')\right\} \\ &\times \{f(\boldsymbol{x}, \boldsymbol{v}) f(\boldsymbol{x}', \boldsymbol{v}')\} + \frac{\lambda}{\Omega} \sum_{k} V(\boldsymbol{k}) i\boldsymbol{k} \cdot \left\{\frac{\partial}{\partial \boldsymbol{p}} - \frac{\partial}{\partial \boldsymbol{p}'}\right\} \{f(\boldsymbol{x}, \boldsymbol{v}) f(\boldsymbol{x}', \boldsymbol{v}')\} \\ &\times e^{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x}')} + \frac{\lambda}{\Omega} \sum_{k} V(\boldsymbol{k}) i\boldsymbol{k} \cdot \left\{\frac{\partial}{\partial \boldsymbol{p}} - \frac{\partial}{\partial \boldsymbol{p}'}\right\} \varphi(\boldsymbol{x}, \boldsymbol{v}; \boldsymbol{x}', \boldsymbol{v}') e^{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x}')} \\ &+ \frac{\lambda}{\Omega} \sum_{k} V(\boldsymbol{k}) i\boldsymbol{k} \cdot \left\{\frac{\partial}{\partial \boldsymbol{p}} e^{i\boldsymbol{k}\boldsymbol{x}} + \frac{\partial}{\partial \boldsymbol{p}'} e^{i\boldsymbol{k}\boldsymbol{x}'}\right\} \\ &\times \iint f(\boldsymbol{x}, \boldsymbol{v}; \boldsymbol{x}', \boldsymbol{v}'; \boldsymbol{x}'', \boldsymbol{v}'') e^{-i\boldsymbol{k}\boldsymbol{x}''} d\boldsymbol{x}'' d\boldsymbol{v}''.\end{aligned} \tag{17}$$

Here, the ternary distribution function can be expanded into the following form,

$$f(\boldsymbol{x}, \boldsymbol{v}; \boldsymbol{x}', \boldsymbol{v}'; \boldsymbol{x}'', \boldsymbol{v}'') = f(\boldsymbol{x}, \boldsymbol{v}) f(\boldsymbol{x}', \boldsymbol{v}') f(\boldsymbol{x}'', \boldsymbol{v}'') + f(\boldsymbol{x}, \boldsymbol{v}) \varphi(\boldsymbol{x}', \boldsymbol{v}'; \boldsymbol{x}'', \boldsymbol{v}'') + f(\boldsymbol{x}', \boldsymbol{v}') \varphi(\boldsymbol{x}'', \boldsymbol{v}''; \boldsymbol{x}, \boldsymbol{v}) + f(\boldsymbol{x}'', \boldsymbol{v}'') \varphi(\boldsymbol{x}, \boldsymbol{v}; \boldsymbol{x}', \boldsymbol{v}') + \varphi(\boldsymbol{x}, \boldsymbol{v}; \boldsymbol{x}', \boldsymbol{v}'; \boldsymbol{x}'', \boldsymbol{v}''). \tag{18}$$

Bogoliubov and Tchen have shown that the ternary correlation function $\varphi(\boldsymbol{x}, \boldsymbol{v}; \boldsymbol{x}', \boldsymbol{v}'; \boldsymbol{x}'', \boldsymbol{v}'')$ can be neglected in the case of $(e^2/\kappa T)^3 n \ll 1$. Then, using Eq. (14) and Eq. (18), Eq. (17) is reduced to the following equation for the $\varphi(\boldsymbol{x}, \boldsymbol{v}; \boldsymbol{x}', \boldsymbol{v}')$,

$$\begin{aligned}
&\left\{\frac{\partial}{\partial t} + (\boldsymbol{v}\cdot\nabla) + (\boldsymbol{v}'\cdot\nabla')\right\} \varphi(\boldsymbol{x}, \boldsymbol{v}; \boldsymbol{x}', \boldsymbol{v}') \\
&= -\boldsymbol{F}(\boldsymbol{x}) \cdot \frac{\partial}{\partial \boldsymbol{p}} \varphi(\boldsymbol{x}, \boldsymbol{v}; \boldsymbol{x}', \boldsymbol{v}') - \boldsymbol{F}(\boldsymbol{x}') \cdot \frac{\partial}{\partial \boldsymbol{p}'} \varphi(\boldsymbol{x}, \boldsymbol{v}; \boldsymbol{x}', \boldsymbol{v}') \\
&+ \frac{\lambda}{\Omega} \sum_{\boldsymbol{k}} V(\boldsymbol{k}) i\boldsymbol{k} \cdot \left\{\frac{\partial}{\partial \boldsymbol{p}} - \frac{\partial}{\partial \boldsymbol{p}'}\right\} \{f(\boldsymbol{x}, \boldsymbol{v}) f(\boldsymbol{x}', \boldsymbol{v}')\} e^{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x}')} \\
&+ \frac{\lambda}{\Omega} \sum_{\boldsymbol{k}} V(\boldsymbol{k}) i\boldsymbol{k} \cdot \left\{\frac{\partial}{\partial \boldsymbol{p}} - \frac{\partial}{\partial \boldsymbol{p}'}\right\} \varphi(\boldsymbol{x}, \boldsymbol{v}; \boldsymbol{x}', \boldsymbol{v}') e^{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x}')} \\
&+ \frac{\lambda}{\Omega} \sum_{\boldsymbol{k}} V(\boldsymbol{k}) i\boldsymbol{k} \cdot \frac{\partial}{\partial \boldsymbol{p}} f(\boldsymbol{x}, \boldsymbol{v}) \iint \varphi(\boldsymbol{x}', \boldsymbol{v}'; \boldsymbol{x}'', \boldsymbol{v}'') e^{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x}'')} d\boldsymbol{x}'' d\boldsymbol{v}'' \\
&+ \frac{\lambda}{\Omega} \sum_{\boldsymbol{k}} V(\boldsymbol{k}) i\boldsymbol{k} \cdot \frac{\partial}{\partial \boldsymbol{p}'} f(\boldsymbol{x}', \boldsymbol{v}') \iint \varphi(\boldsymbol{x}, \boldsymbol{v}; \boldsymbol{x}'', \boldsymbol{v}'') e^{i\boldsymbol{k}(\boldsymbol{x}'-\boldsymbol{x}'')} d\boldsymbol{x}'' d\boldsymbol{v}''.
\end{aligned} \tag{19}$$

Now, the set of Eqs. (14), (15) and (19) will completely determine the statistical behavior of the high temperature plasma. These equations are the generalization of Tchen's equations (Eqs. (15) and (16) of reference 13)), which have been derived on the assumption of the spatial homogeneity of the system. The first two terms of the right-hand side of Eq. (19) represent the disturbance of the binary correlation due to the action of the self-consistent field which is determined by the single particle distribution function. The disturbances of the binary correlation induced by the action of the self-consistent field may react back to the self-consistent field through the term of the right-hand side of Eq. (14). Therefore, strictly speaking, it is not rigorous to identify the whole contributions of the binary correlation effects as the collision terms.

Finally, it should be remarked that the assumption of the spatial homogeneity appears to be very restrictive when it is imposed on the intermediate state of the system. The physical state of the system may approach the spatially homogeneous equilibrium state through the intermediate stages in which the spatially inhomogeneous disturbances can take place. In fact these inhomogeneous effects in

the coordinate space manifest themselves as the plasma oscillations through the self-consistent force $\boldsymbol{F}(\boldsymbol{x})$ of Eq. (14). Since the occurrences of the plasma oscillations have been experimentally established by many investigators both in the high temperature plasma[21),22)] and in the electron system of metals,[23),24)] it is very essential to keep these spatial inhomogeneous terms in the fundamental equations of the plasma.

§ 3. Approximate solutions of Eq. (19)

Neglecting the effects of the close distance collisions, Tchen has solved the equation of the binary correlation function by the formal integration along the free particle trajectories. Here, however, because of the terms with the internal self-consistent force $\boldsymbol{F}(\boldsymbol{x})$ and $\boldsymbol{F}(\boldsymbol{x}')$, it is not allowed to assume the free particle trajectories even if we neglect the effects of the close distant collisions. Therefore, we have to seek for an approximate solution of Eq. (19), since we cannot write down the formal integral of Eq. (19). We will introduce a linearization approximation to the problem. Although this might sound as an awkward treatment, it supplies us quite enough information on the behavior of the high temperature plasma. In the following sections, we will derive the generalized Fokker-Planck equation using the approximate solution of the present section, but without using the assumption of slowly varying time dependence of the distribution function, which has been used by Tchen in his derivation of the Fokker-Planck equation. In general, the linearization approximation of the two particle correlation function is a fundamental assumption in the derivation of the Fokker-Planck equation of the many particle system. According to the above discussion, let us introduce the following expansions,

$$f(\boldsymbol{x}, \boldsymbol{v}\,;\, t) = f_0(\boldsymbol{v}) + f_1(\boldsymbol{x}, \boldsymbol{v}\,;\, t) \tag{20}$$

$$\varphi(\boldsymbol{x}, \boldsymbol{v}\,;\, \boldsymbol{x}', \boldsymbol{v}'\,;\, t) = \varphi_0(|\boldsymbol{x}-\boldsymbol{x}'|\,;\, \boldsymbol{v}, \boldsymbol{v}') + \varphi_1(\boldsymbol{x}, \boldsymbol{v}\,;\, \boldsymbol{x}', \boldsymbol{v}'\,;\, t) \tag{21}$$

where it is assumed that the static equilibrium state is spatially homogeneous. Substituting the expansions (20) and (21) into Eq. (19), the equation of $\varphi_0(|\boldsymbol{x}-\boldsymbol{x}'|\,;\, \boldsymbol{v}, \boldsymbol{v}')$ can be derived from Eq. (19) as follows,

$$\begin{aligned}
(\boldsymbol{v}\cdot\nabla + \boldsymbol{v}'\cdot\nabla')\varphi_0(|\boldsymbol{x}-\boldsymbol{x}'|\,;\, \boldsymbol{v}, \boldsymbol{v}') = {} & \frac{\lambda}{\Omega}\sum_{\boldsymbol{k}} V(\boldsymbol{k})\, i\boldsymbol{k}\cdot\left\{\frac{\partial}{\partial \boldsymbol{p}} - \frac{\partial}{\partial \boldsymbol{p}'}\right\} \\
& \times \{f_0(\boldsymbol{v}) f_0(\boldsymbol{v}')\} e^{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x}')} + \frac{\lambda}{\Omega}\sum_{\boldsymbol{k}} V(\boldsymbol{k})\, i\boldsymbol{k}\cdot\left\{\frac{\partial}{\partial \boldsymbol{p}} - \frac{\partial}{\partial \boldsymbol{p}'}\right\} \\
& \times \varphi_0(|\boldsymbol{x}-\boldsymbol{x}'|\,;\, \boldsymbol{v}, \boldsymbol{v}') e^{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x}')} + \frac{\lambda}{\Omega}\sum_{\boldsymbol{k}} V(\boldsymbol{k})\, i\boldsymbol{k}\cdot\frac{\partial}{\partial \boldsymbol{p}} f_0(\boldsymbol{v}) \\
& \times \int \varphi_0(|\boldsymbol{x}'-\boldsymbol{x}''|\,;\, \boldsymbol{v}', \boldsymbol{v}'') e^{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x}'')}\, d\boldsymbol{x}''\, d\boldsymbol{v}'' + \frac{\lambda}{\Omega}\sum_{\boldsymbol{k}} V(\boldsymbol{k})\, i\boldsymbol{k}\cdot\frac{\partial}{\partial \boldsymbol{p}'} f_0(\boldsymbol{v}') \\
& \times \int \varphi_0(|\boldsymbol{x}-\boldsymbol{x}''|\,;\, \boldsymbol{v}\,;\, \boldsymbol{v}'') e^{i\boldsymbol{k}(\boldsymbol{x}'-\boldsymbol{x}'')}\, d\boldsymbol{x}''\, d\boldsymbol{v}''.
\end{aligned} \tag{22}$$

Eq. (22) can be solved by assuming the following form for $\varphi_0(|\boldsymbol{x}-\boldsymbol{x}'|\,;\,\boldsymbol{v},\,\boldsymbol{v}')$,

$$\varphi_0(|\boldsymbol{x}-\boldsymbol{x}'|\,;\,\boldsymbol{v},\,\boldsymbol{v}')=f_0(\boldsymbol{v})f_0(\boldsymbol{v}')\chi(|\boldsymbol{x}-\boldsymbol{x}'|) \tag{23}$$

with

$$\chi(|\boldsymbol{x}-\boldsymbol{x}'|)=\frac{1}{\Omega}\sum_{\boldsymbol{l}} e^{i\boldsymbol{l}(\boldsymbol{x}-\boldsymbol{x}')}\chi(\boldsymbol{l}), \tag{24}$$

$$f_0(\boldsymbol{v})=n(m/2\pi\kappa T)^{3/2}\exp(-mv^2/2\kappa T). \tag{25}$$

Substituting these expressions into Eq. (22), it is reduced to an equation which should be satisfied by $\chi(\boldsymbol{k})$ as follows,

$$\chi(\boldsymbol{k})=-\frac{\lambda}{\kappa T}V(\boldsymbol{k})-n\frac{\lambda}{\kappa T}V(\boldsymbol{k})\chi(\boldsymbol{k})-\frac{1}{\Omega}\sum_{\boldsymbol{l}}\frac{\boldsymbol{k}\cdot(\boldsymbol{k}-\boldsymbol{l})}{k^2}\frac{\lambda}{\kappa T}V(\boldsymbol{k}-\boldsymbol{l})\chi(\boldsymbol{l}). \tag{26}$$

The last term of the right-hand side of Eq. (26) corresponds to the second term of the right-hand side of Eq. (22) which refers to the close distances, while the second term of the right-hand side of Eq. (26) due to the presence of the third particles represents a shielding effect to the first term of the right-hand side of Eq. (26), which represents the collision between the particles through the bare two-body interaction. Therefore, Eq. (26) may be solved by an iteration procedure, since the first two terms of the right-hand side of Eq. (26) represent the main effects of the binary correlations at the far distance.

Firstly, we get

$$\chi(\boldsymbol{k})=\frac{-(\lambda/\kappa T)V(\boldsymbol{k})}{1+(n\lambda/\kappa T)V(\boldsymbol{k})}. \tag{27}$$

For the Coulomb interaction, Eq. (27) is written as

$$\chi(\boldsymbol{k})=\frac{-(k_d^2/n)}{k^2+k_d^2} \tag{28}$$

which gives Debye's expression of the static correlation $\chi(r)\propto\exp(-k_d r)\,r^{-1}$. In order to discuss the contribution of the close distance term of Eq. (26), we will examine in some detail the case of Coulomb interaction. Then, the substitution of Eq. (28) into the last term of the right-hand side of Eq. (26) yields the first iterated solution of Eq. (26) as follows,

$$n\chi^{(1)}(\boldsymbol{k})=-\frac{k_d^2}{k^2+k_d^2}\left\{1-\left(\frac{e^2}{\kappa T}\right)\Phi(\boldsymbol{k})\right\} \tag{29}$$

where $\Phi(\boldsymbol{k})$ is defined by

$$\Phi(\boldsymbol{k})=\frac{1}{2\pi^2}\int d\boldsymbol{l}\frac{1}{l^2+k_d^2}\cdot\frac{\boldsymbol{k}\cdot(\boldsymbol{k}-\boldsymbol{l})}{|\boldsymbol{k}-\boldsymbol{l}|^2}. \tag{30}$$

An approximate evaluation of the function $\Phi(\boldsymbol{k})$ is carried out in the Appendix. Using the result (A·3), Eq. (29) can be written as

$$n\chi^{(1)}(\boldsymbol{k}) = -\frac{k_d^2}{k^2+k_d^2}\left\{1-\sqrt{4\pi n\left(\frac{e^2}{\kappa T}\right)^3}\cdot\frac{k^2}{k_d^2+kk_d}\right\}. \tag{31}$$

Since the dimensionless parameter $(e^2/\kappa T)^3 n$ is extremely small in general, we can conclude that the effects of the close distance collision term can be disregarded for the range of k defined by

$$k<\kappa T/e^2 \tag{32}$$

where the limit $\kappa T/e^2$ is called the Landau cutoff.

Now, let us turn to the time-dependent fluctuation of the distribution function and of the correlation function. The kinetic equation of the distribution function $f_1(\boldsymbol{x}, \boldsymbol{v}; t)$ is reduced from Eqs. (14) and (15) as follows,

$$\left\{\frac{\partial}{\partial t}+(\boldsymbol{v}\cdot\nabla)\right\}f_1(\boldsymbol{x}, \boldsymbol{v}; t) = -\boldsymbol{F}(\boldsymbol{x}, t)\cdot\frac{\partial}{\partial \boldsymbol{p}}f_0(\boldsymbol{v})$$
$$+\frac{\lambda}{\Omega}\sum_{\boldsymbol{k}} V(\boldsymbol{k})i\boldsymbol{k}\cdot\frac{\partial}{\partial \boldsymbol{p}}\iint\varphi_1(\boldsymbol{x}, \boldsymbol{v}; \boldsymbol{x}', \boldsymbol{v}'; t)e^{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x}')}d\boldsymbol{x}'\,d\boldsymbol{v}' \tag{33}$$

and

$$\boldsymbol{F}(\boldsymbol{x}, t) = -\frac{\lambda}{\Omega}\sum_{\boldsymbol{k}} V(\boldsymbol{k})i\boldsymbol{k}\cdot\iint f_1(\boldsymbol{x}', \boldsymbol{v}'; t)e^{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x}')}d\boldsymbol{x}'\,d\boldsymbol{v}'. \tag{34}$$

The equation of $\varphi_1(\boldsymbol{x}, \boldsymbol{v}; \boldsymbol{x}', \boldsymbol{v}'; t)$ is derived from Eq. (19) as follows,

$$\left\{\frac{\partial}{\partial t}+(\boldsymbol{v}\cdot\nabla)+(\boldsymbol{v}'\cdot\nabla')\right\}\varphi_1(\boldsymbol{x}, \boldsymbol{v}; \boldsymbol{x}', \boldsymbol{v}'; t)$$
$$= -\left\{\boldsymbol{F}(\boldsymbol{x}, t)\cdot\frac{\partial}{\partial \boldsymbol{p}}+\boldsymbol{F}(\boldsymbol{x}', t)\cdot\frac{\partial}{\partial \boldsymbol{p}'}\right\}\varphi_0(|\boldsymbol{x}-\boldsymbol{x}'|; \boldsymbol{v}, \boldsymbol{v}')$$
$$+\frac{\lambda}{\Omega}\sum_{\boldsymbol{k}} V(\boldsymbol{k})i\boldsymbol{k}\cdot\left\{\frac{\partial}{\partial \boldsymbol{p}}-\frac{\partial}{\partial \boldsymbol{p}'}\right\}\{f_0(\boldsymbol{v})f_1(\boldsymbol{x}', \boldsymbol{v}'; t)+f_0(\boldsymbol{v}')f_1(\boldsymbol{x}, \boldsymbol{v}; t)\}e^{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x}')}$$
$$+\frac{\lambda}{\Omega}\sum_{\boldsymbol{k}} V(\boldsymbol{k})i\boldsymbol{k}\cdot\frac{\partial}{\partial \boldsymbol{p}}f_1(\boldsymbol{x}, \boldsymbol{v}; t)\cdot\int\varphi_0(|\boldsymbol{x}'-\boldsymbol{x}''|; \boldsymbol{v}', \boldsymbol{v}'')e^{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x}')}d\boldsymbol{x}''\,d\boldsymbol{v}''$$
$$+\frac{\lambda}{\Omega}\sum_{\boldsymbol{k}} V(\boldsymbol{k})i\boldsymbol{k}\cdot\frac{\partial}{\partial \boldsymbol{p}'}f_1(\boldsymbol{x}', \boldsymbol{v}'; t)\int\varphi_0(|\boldsymbol{x}-\boldsymbol{x}''|; \boldsymbol{v}, \boldsymbol{v}'')e^{i\boldsymbol{k}(\boldsymbol{x}'-\boldsymbol{x}'')}d\boldsymbol{x}''\,d\boldsymbol{v}''$$
$$+\frac{\lambda}{\Omega}\sum_{\boldsymbol{k}} V(\boldsymbol{k})i\boldsymbol{k}\cdot\frac{\partial}{\partial \boldsymbol{p}}f_0(\boldsymbol{v})\int\varphi_1(\boldsymbol{x}', \boldsymbol{v}'; \boldsymbol{x}'', \boldsymbol{v}''; t)e^{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x}'')}d\boldsymbol{x}''\,d\boldsymbol{v}''$$
$$+\frac{\lambda}{\Omega}\sum_{\boldsymbol{k}} V(\boldsymbol{k})i\boldsymbol{k}\cdot\frac{\partial}{\partial \boldsymbol{p}'}f_0(\boldsymbol{v}')\int\varphi_1(\boldsymbol{x}, \boldsymbol{v}; \boldsymbol{x}'', \boldsymbol{v}''; t)e^{i\boldsymbol{k}(\boldsymbol{x}'-\boldsymbol{x}'')}d\boldsymbol{x}''\,d\boldsymbol{v}''$$
$$+\frac{\lambda}{\Omega}\sum_{\boldsymbol{k}} V(\boldsymbol{k})i\boldsymbol{k}\cdot\left\{\frac{\partial}{\partial \boldsymbol{p}}-\frac{\partial}{\partial \boldsymbol{p}'}\right\}\varphi_1(\boldsymbol{x}, \boldsymbol{v}; \boldsymbol{x}', \boldsymbol{v}'; t)e^{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x}')}. \tag{35}$$

Eq. (35) tells us that disturbances in the binary correlation result from several different origins. The first term of the right-hand side of Eq. (35) describes the dynamic disturbances caused in the static correlation by change of the state of motion under the action of the internal force $\boldsymbol{F}(\boldsymbol{x}, t)$ and $\boldsymbol{F}(\boldsymbol{x}', t)$. This term represents the part of ternary correlations. Although this term is missing in those theories of Temko and Tchen, it is the essential term in the investigation of the effects of the binary correlation for the internal field of the plasma. The second term is the so-called convection term in velocity space which contributes to the diffusion term and to the friction term. The third and the fourth term represent the static shielding effects. The fifth and the sixth term represent the dynamical shielding effects due to the cross interaction of the pair of particles with a third one. The last term represents a secondary convection effect which refers to the close distance. As it has been explicitly shown that the close distance collision term can be disregarded for the static correlation, here we may also disregard the last term by introducing the Landau cutoff.

The set of equations, (33), (34) and (35), is the fundamental equations of the present investigation of the plasma. Neglecting the effects of the binary correlation, Landau and Berz[25] have discussed the set of Eqs. (33) and (34) as the initial value problem by using the Laplace transformation method. In the present case, however, it is suitable to follow the method of adiabatic process which is familier in the quantum theory of the scattering processes,[26] since the initial condition is given at the time $t=-\infty$. The whole problem can be solved completely for the out-going solutions under the specified initial condition at the time $t=-\infty$. This method has been discussed by van Kampen[27] in connection with the problems of the dispersion relation of the plasma oscillations, and actually it has been applied to the discussions of the excitations of plasma oscillations by Bohr[28] and also by the present author.[29]

In order to solve the set of Eqs. (33), (34) and (35), we introduce the Fourier representation of the functions $f_1(\boldsymbol{x}, \boldsymbol{v}\,;\, t)$ and for $\varphi_1(\boldsymbol{x}, \boldsymbol{v}\,;\, \boldsymbol{x}', \boldsymbol{v}'\,;\, t)$ as follows,

$$f_1(\boldsymbol{x}, \boldsymbol{v}\,;\, t)=\frac{1}{2\pi}\int d\omega\, e^{-i\omega t}\frac{1}{\Omega}\sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\boldsymbol{x}} f_1(\boldsymbol{k}\,;\, \omega\,;\, \boldsymbol{v}) \tag{36}$$

and

$$\varphi_1(\boldsymbol{x}, \boldsymbol{v}\,;\, \boldsymbol{x}', \boldsymbol{v}'\,;\, t)=\frac{1}{2\pi}\int d\omega\, e^{-i\omega t}\frac{1}{\Omega^2}\sum_{\boldsymbol{k}}\sum_{\boldsymbol{l}} e^{i\boldsymbol{k}\boldsymbol{x}} e^{i\boldsymbol{l}\boldsymbol{x}'}\varphi_1(\boldsymbol{k}, \boldsymbol{l}\,;\, \omega\,;\, \boldsymbol{v}, \boldsymbol{v}'). \tag{37}$$

Substituting these expressions in Eq. (35), in which we disregard the last term of the right-hand side, we can write down the formal solution for $\varphi_1(\boldsymbol{x}, \boldsymbol{v}\,;\, \boldsymbol{x}', \boldsymbol{v}'\,;\, t)$ as follows,

$$\varphi_1(\boldsymbol{x}, \boldsymbol{v}\,;\, \boldsymbol{x}', \boldsymbol{v}'\,;\, t)=\frac{1}{2\pi}\int d\omega\, e^{-i\omega t}\frac{1}{\Omega^2}\sum_{\boldsymbol{k}}\sum_{\boldsymbol{l}} e^{i(\boldsymbol{k}+\boldsymbol{l})\boldsymbol{x}} e^{-i\boldsymbol{l}\boldsymbol{x}'}\cdot\varphi_1(\boldsymbol{k}+\boldsymbol{l}, -\boldsymbol{l}\,;\, \omega\,;\, \boldsymbol{v}, \boldsymbol{v}') \tag{38}$$

where $\varphi_1(\boldsymbol{k}+\boldsymbol{l}, -\boldsymbol{l}; \omega; \boldsymbol{v}, \boldsymbol{v}')$ is a solution of the following integral equation,

$$\varphi_1(\boldsymbol{k}+\boldsymbol{l}, -\boldsymbol{l}; \omega; \boldsymbol{v}, \boldsymbol{v}') = +i\Psi(\boldsymbol{k}+\boldsymbol{l}, -\boldsymbol{l}; \omega; \boldsymbol{v}, \boldsymbol{v}')$$
$$+i\lambda V(\boldsymbol{k}+\boldsymbol{l})\delta_+(\omega-(\boldsymbol{k}+\boldsymbol{l})\cdot\boldsymbol{v}+\boldsymbol{l}\cdot\boldsymbol{v}')(\boldsymbol{k}+\boldsymbol{l})\cdot\frac{\partial}{\partial\boldsymbol{p}}f_0(\boldsymbol{v})\int\varphi_1(-\boldsymbol{l}, \boldsymbol{k}+\boldsymbol{l}; \omega; \boldsymbol{v}', \boldsymbol{v}'')d\boldsymbol{v}''$$
$$-i\lambda V(\boldsymbol{l})\delta_+(\omega-(\boldsymbol{k}+\boldsymbol{l})\cdot\boldsymbol{v}+\boldsymbol{l}\cdot\boldsymbol{v}')\boldsymbol{l}\cdot\frac{\partial}{\partial\boldsymbol{p}'}f_0(\boldsymbol{v}')\int\varphi_1(\boldsymbol{k}+\boldsymbol{l}, -\boldsymbol{l}; \omega; \boldsymbol{v}, \boldsymbol{v}'')d\boldsymbol{v}'' \quad (39)$$

with the abbreviation of

$$\Psi(\boldsymbol{k}+\boldsymbol{l}, -\boldsymbol{l}; \omega; \boldsymbol{v}, \boldsymbol{v}')$$
$$=\delta_+(\omega-(\boldsymbol{k}+\boldsymbol{l})\cdot\boldsymbol{v}+\boldsymbol{l}\cdot\boldsymbol{v}')\Big[\lambda V(\boldsymbol{k})\int f_1(\boldsymbol{k}; \omega; \boldsymbol{v}'')d\boldsymbol{v}''\Big\{\boldsymbol{k}\cdot\frac{\partial f_0(\boldsymbol{v})}{\partial\boldsymbol{p}}f_0(\boldsymbol{v}')\chi(\boldsymbol{l})$$
$$+\boldsymbol{k}\cdot\frac{\partial f_0(\boldsymbol{v}')}{\partial\boldsymbol{p}'}f_0(\boldsymbol{v})\chi(\boldsymbol{k}+\boldsymbol{l})\Big\}+\lambda(1+n\chi(\boldsymbol{l}))V(\boldsymbol{l})\boldsymbol{l}\cdot\frac{\partial}{\partial\boldsymbol{p}}f_1(\boldsymbol{k}; \omega; \boldsymbol{v})f_0(\boldsymbol{v}')$$
$$-\lambda(1+n\chi(\boldsymbol{k}+\boldsymbol{l}))V(\boldsymbol{k}+\boldsymbol{l})(\boldsymbol{k}+\boldsymbol{l})\cdot\frac{\partial}{\partial\boldsymbol{p}'}f_1(\boldsymbol{k}; \omega; \boldsymbol{v}')f_0(\boldsymbol{v})$$
$$+\lambda V(\boldsymbol{k}+\boldsymbol{l})(\boldsymbol{k}+\boldsymbol{l})\cdot\frac{\partial}{\partial\boldsymbol{p}}f_0(\boldsymbol{v})f_1(\boldsymbol{k}; \omega; \boldsymbol{v}')-\lambda V(\boldsymbol{l})\boldsymbol{l}\cdot\frac{\partial}{\partial\boldsymbol{p}'}f_0(\boldsymbol{v}')f_1(\boldsymbol{k}; \omega; \boldsymbol{v})\Big]. \quad (40)$$

The factor $\delta_+(\omega-(\boldsymbol{k}+\boldsymbol{l})\cdot\boldsymbol{v}+\boldsymbol{l}\cdot\boldsymbol{v}')$, which is defined as

$$\delta_+(X)=\pi\delta(X)+iP\frac{1}{X} \quad (41)$$

corresponds to the boundary condition for the out-going solutions with the initial condition that

$$\varphi_1(\boldsymbol{x}, \boldsymbol{v}; \boldsymbol{x}', \boldsymbol{v}'; t=-\infty)=0. \quad (42)$$

It should be remarked here that in the kinetic equation of the distribution function $f_1(\boldsymbol{x}, \boldsymbol{v}; t)$, Eq. (33), there appears the function $\varphi_1(\boldsymbol{x}, \boldsymbol{v}; \boldsymbol{x}', \boldsymbol{v}'; t)$ only in the integrated form of $\int\varphi_1(\boldsymbol{x}, \boldsymbol{v}; \boldsymbol{x}', \boldsymbol{v}'; t)d\boldsymbol{v}'$. Therefore, it is sufficient to solve Eq. (39) for the integrated function $\overline{\varphi_1(\boldsymbol{k}+\boldsymbol{l}, -\boldsymbol{l}; \omega; \boldsymbol{v})}$, which is defined by

$$\overline{\varphi_1(\boldsymbol{k}+\boldsymbol{l}, -\boldsymbol{l}; \omega; \boldsymbol{v})}=\int\varphi_1(\boldsymbol{k}+\boldsymbol{l}, -\boldsymbol{l}; \omega; \boldsymbol{v}; \boldsymbol{v}')d\boldsymbol{v}'. \quad (43)$$

By integrating Eq. (39) over the velocity $\boldsymbol{v}'$ and rearranging the terms, we can derive the following integral equation of the function $\overline{\varphi_1(\boldsymbol{k}+\boldsymbol{l}, -\boldsymbol{l}; \omega; \boldsymbol{v})}$

$$\overline{\varphi_1(\boldsymbol{k}+\boldsymbol{l}, -\boldsymbol{l}; \omega; \boldsymbol{v})}=+i\Delta(\boldsymbol{k}+\boldsymbol{l}, -\boldsymbol{l}; \omega; \boldsymbol{v})^{-1}\overline{\Psi(\boldsymbol{k}+\boldsymbol{l}, -\boldsymbol{l}; \omega; \boldsymbol{v})}$$
$$+i\lambda\Delta(\boldsymbol{k}+\boldsymbol{l}, -\boldsymbol{l}; \omega; \boldsymbol{v})^{-1}V(\boldsymbol{k}+\boldsymbol{l})(\boldsymbol{k}+\boldsymbol{l})\cdot\frac{\partial}{\partial\boldsymbol{p}}f_0(\boldsymbol{v})$$
$$\times\int\delta_+(\omega-(\boldsymbol{k}+\boldsymbol{l})\cdot\boldsymbol{v}+\boldsymbol{l}\cdot\boldsymbol{v}')i\Delta(-\boldsymbol{l}, \boldsymbol{k}+\boldsymbol{l}; \omega; \boldsymbol{v}')^{-1}\overline{\Psi(-\boldsymbol{l}, \boldsymbol{k}+\boldsymbol{l}; \omega; \boldsymbol{v}')}d\boldsymbol{v}'$$

$$+i\lambda\Delta(\boldsymbol{k}+\boldsymbol{l},\ -\boldsymbol{l}\,;\,\omega\,;\,\boldsymbol{v})^{-1}V(\boldsymbol{l})\boldsymbol{l}\cdot\frac{\partial}{\partial\boldsymbol{p}}f_0(\boldsymbol{v})$$

$$\times\int\lambda\Delta(-\boldsymbol{l},\ \boldsymbol{k}+\boldsymbol{l}\,;\,\omega\,;\,\boldsymbol{v}')^{-1}V(\boldsymbol{k}+\boldsymbol{l})\,(\boldsymbol{k}+\boldsymbol{l})\cdot\frac{\partial}{\partial\boldsymbol{p}'}f_0(\boldsymbol{v}')$$

$$\times\delta_+(\omega-(\boldsymbol{k}+\boldsymbol{l})\cdot\boldsymbol{v}+\boldsymbol{l}\cdot\boldsymbol{v}')\int(-i)\delta_+(\omega+\boldsymbol{l}\cdot\boldsymbol{v}'-(\boldsymbol{k}+\boldsymbol{l})\cdot\boldsymbol{v}'')$$

$$\times\overline{\varphi_1(\boldsymbol{k}+\boldsymbol{l},\ -\boldsymbol{l}\,;\,\omega\,;\,\boldsymbol{v}'')}\,d\boldsymbol{v}''\,d\boldsymbol{v}' \tag{44}$$

where

$$\Delta(\boldsymbol{k}+\boldsymbol{l},\ -\boldsymbol{l}\,;\,\omega\,;\,\boldsymbol{v})=1+i\lambda V(\boldsymbol{l})\cdot\int\delta_+(\omega-(\boldsymbol{k}+\boldsymbol{l})\cdot\boldsymbol{v}+\boldsymbol{l}\cdot\boldsymbol{v}')\boldsymbol{l}\cdot\frac{\partial}{\partial\boldsymbol{p}'}f_0(\boldsymbol{v}')\,d\boldsymbol{v}' \tag{45}$$

and

$$\overline{\Psi(\boldsymbol{k}+\boldsymbol{l},\ -\boldsymbol{l}\,;\,\omega\,;\,\boldsymbol{v})}=\int\Psi(\boldsymbol{k}+\boldsymbol{l},\ -\boldsymbol{l}\,;\,\omega\,;\,\boldsymbol{v},\,\boldsymbol{v}')\,d\boldsymbol{v}'. \tag{46}$$

The factor Δ represents the dynamical shielding effects due to the cross interaction of pair of particles with the third one. Since the factor Δ is a complex function, let us examine its structure in some detail. Using the definition of δ_+-function, Eq. (41), we obtain the following expression of the factor Δ,

$$\Delta(\boldsymbol{k}+\boldsymbol{l},\ -\boldsymbol{l}\,;\,\omega\,;\,\boldsymbol{v})=1+\frac{\omega_p^2}{nl^2}P\int\frac{1}{\omega-(\boldsymbol{k}+\boldsymbol{l})\cdot\boldsymbol{v}+\boldsymbol{l}\cdot\boldsymbol{v}'}\boldsymbol{l}\cdot\frac{\partial}{\partial\boldsymbol{p}'}f_0(\boldsymbol{v}')\,d\boldsymbol{v}'$$

$$+i(\pi/2)^{1/2}[\omega-(\boldsymbol{k}+\boldsymbol{l})\cdot\boldsymbol{v}/\omega_p]\left(\frac{k_d}{l}\right)^3\exp\left[-\frac{m}{2\kappa T}\{\omega-(\boldsymbol{k}+\boldsymbol{l})\cdot\boldsymbol{v}/l\}^2\right]. \tag{47}$$

It is interesting to notice that the imaginary part has a quite similar structure to the Landau damping factor. The imaginary part of Δ expresses dissipative effects occurring in the plasma, but it can be disregarded for smaller and larger values of $\boldsymbol{l}$.

In the above derived complicated expressions of the function $\overline{\varphi_1(\boldsymbol{k}+\boldsymbol{l},\ -\boldsymbol{l}\,;\,\omega\,;\,\boldsymbol{v})}$, the effects of the presence of the third particles manifest themselves in three places, namely the coefficient Δ^{-1}, the second and the last term of the right-hand side of Eq. (44). The coefficient Δ^{-1} represents the dynamical shielding effects. Although Eq. (44) is still an integral equation of the function $\varphi_1(\boldsymbol{k}+\boldsymbol{l},\ -\boldsymbol{l}\,;\,\omega\,;\,\boldsymbol{v})$, we may disregard the last term of Eq. (44) which represents the secondary shielding effects for the binary correlation function, because the main shielding effects have been taken into account through the factor Δ^{-1} and the second term of the right-hand side of Eq. (44). Therefore, we can explicitly write down the function $\varphi_1(\boldsymbol{k}+\boldsymbol{l},\ -\boldsymbol{l}\,;\,\omega\,;\,\boldsymbol{v})$ as follows,

$$\overline{\varphi_1(\boldsymbol{k}+\boldsymbol{l},\ -\boldsymbol{l}\,;\,\omega\,;\,\boldsymbol{v})}=\overline{\varphi_f}+\overline{\varphi_c^{(1)}}+\overline{\varphi_c^{(2)}}, \tag{48}$$

$$\overline{\varphi_f(\boldsymbol{k}+\boldsymbol{l},\ -\boldsymbol{l}\,;\,\omega\,;\,\boldsymbol{v})}=-\left(\frac{\lambda}{\kappa T}\right)\frac{V(\boldsymbol{k})}{\Delta(\boldsymbol{k}+\boldsymbol{l},\ -\boldsymbol{l}\,;\,\omega\,;\,\boldsymbol{v})}\cdot\overline{f_1(\boldsymbol{k}\,;\,\omega)}$$

$$\times G(\boldsymbol{k}+\boldsymbol{l},\ -\boldsymbol{l}\,;\,\omega\,;\,\boldsymbol{v})f_0(\boldsymbol{v}) \tag{49}$$

with

$$G(\boldsymbol{k}+\boldsymbol{l},\,-\boldsymbol{l}\,;\,\omega\,;\,\boldsymbol{v}) = i\int d\boldsymbol{v}' f_0(\boldsymbol{v}')\,\delta_+(\omega-(\boldsymbol{k}+\boldsymbol{l})\cdot\boldsymbol{v}+\boldsymbol{l}\cdot\boldsymbol{v}')$$
$$\times\Big[\boldsymbol{k}\cdot\boldsymbol{v}\chi(\boldsymbol{l})+\boldsymbol{k}\cdot\boldsymbol{v}'\chi(\boldsymbol{k}+\boldsymbol{l})+\Big(\frac{\lambda}{\kappa T}\Big)\frac{V(\boldsymbol{k}+\boldsymbol{l})}{\Delta(-\boldsymbol{l},\,\boldsymbol{k}+\boldsymbol{l}\,;\,\omega\,;\,\boldsymbol{v}')}(\boldsymbol{k}+\boldsymbol{l}\cdot\boldsymbol{v})$$
$$\times\int d\boldsymbol{v}'' f_0(\boldsymbol{v}'')\,\delta_+(\omega-(\boldsymbol{k}+\boldsymbol{l})\cdot\boldsymbol{v}''+\boldsymbol{l}\cdot\boldsymbol{v}')\,\{(\boldsymbol{k}\cdot\boldsymbol{v}')\chi(\boldsymbol{k}+\boldsymbol{l})+(\boldsymbol{k}\cdot\boldsymbol{v}'')\chi(\boldsymbol{l})\}\Big], \tag{50}$$

$$\overline{f_1(\boldsymbol{k}\,;\,\omega)} = \int f_1(\boldsymbol{k}\,;\,\omega\,;\,\boldsymbol{v})\,d\boldsymbol{v}, \tag{51}$$

and

$$\overline{\varphi_c^{(1)}(\boldsymbol{k}+\boldsymbol{l},\,-\boldsymbol{l}\,;\,\omega\,;\,\boldsymbol{v})} = i\int d\boldsymbol{v}'\,\delta_+(\omega-(\boldsymbol{k}+\boldsymbol{l})\cdot\boldsymbol{v}+\boldsymbol{l}\cdot\boldsymbol{v}')\,\Delta(\boldsymbol{k}+\boldsymbol{l},\,-\boldsymbol{l}\,;\,\omega\,;\,\boldsymbol{v})^{-1}$$
$$\times\Big\{\lambda(1+n\chi(\boldsymbol{l}))\Big[V(\boldsymbol{l})\boldsymbol{l}\cdot\frac{\partial}{\partial\boldsymbol{p}}f_1(\boldsymbol{k}\,;\,\omega\,;\,\boldsymbol{v})f_0(\boldsymbol{v}')$$
$$-V(\boldsymbol{k}+\boldsymbol{l})\,(\boldsymbol{k}+\boldsymbol{l})\cdot\frac{\partial}{\partial\boldsymbol{p}'}f_1(\boldsymbol{k}\,;\,\omega\,;\,\boldsymbol{v}')f_0(\boldsymbol{v})\Big]$$
$$+\lambda\Big[V(\boldsymbol{k}+\boldsymbol{l})\,(\boldsymbol{k}+\boldsymbol{l})\cdot\frac{\partial}{\partial\boldsymbol{p}}f_0(\boldsymbol{v})f_1(\boldsymbol{k}\,;\,\omega\,;\,\boldsymbol{v}')$$
$$-V(\boldsymbol{l})\boldsymbol{l}\cdot\frac{\partial}{\partial\boldsymbol{p}'}f_0(\boldsymbol{v}')f_1(\boldsymbol{k}\,;\,\omega\,;\,\boldsymbol{v})\Big]\Big\}, \tag{52}$$

and

$$\overline{\varphi_c^{(2)}(\boldsymbol{k}+\boldsymbol{l},\,-\boldsymbol{l}\,;\,\omega\,;\,\boldsymbol{v})} = i\lambda\frac{V(\boldsymbol{k}+\boldsymbol{l})}{\Delta(\boldsymbol{k}+\boldsymbol{l},\,-\boldsymbol{l}\,;\,\omega\,;\,\boldsymbol{v})}(\boldsymbol{k}+\boldsymbol{l})\cdot\frac{\partial}{\partial\boldsymbol{p}}f_0(\boldsymbol{v})$$
$$\times\int\delta_+(\omega-(\boldsymbol{k}+\boldsymbol{l})\cdot\boldsymbol{v}+\boldsymbol{l}\cdot\boldsymbol{v}')\,\overline{\varphi_c^{(1)}(-\boldsymbol{l},\,\boldsymbol{k}+\boldsymbol{l}\,;\,\omega\,;\,\boldsymbol{v}')}\,d\boldsymbol{v}'. \tag{53}$$

In the following sections, substituting these expressions of the binary correlation function $\overline{\varphi}_1$ in the linearlized kinetic equation,

$$\Big\{\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\boldsymbol{\nabla}\Big\}f_1(\boldsymbol{x},\,\boldsymbol{v}\,;\,t) = \frac{1}{2\pi}\int d\omega\,e^{-i\omega t}\frac{1}{\Omega}\sum_{\boldsymbol{k}}e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$
$$\times\Big\{i\lambda V(\boldsymbol{k})\boldsymbol{k}\cdot\frac{\partial}{\partial\boldsymbol{p}}f_0(\boldsymbol{v})\overline{f_1(\boldsymbol{k}\,;\,\omega)}+\frac{\lambda}{\Omega}\sum_{\boldsymbol{l}}V(\boldsymbol{l})i\boldsymbol{l}\cdot\frac{\partial}{\partial\boldsymbol{p}}\overline{\varphi_1(\boldsymbol{k}-\boldsymbol{l},\,\boldsymbol{l}\,;\,\omega\,;\,\boldsymbol{v})}\Big\}, \tag{54}$$

we will discuss several aspects of the contributions of the binary correlation effects.

§ 4. Effective field in plasma

Following a similar approach to the present investigation, Kadomtsev[18] investigated the effects of binary correlation on the internal field in the electron-ion

plasma. Assuming that the time dependence of the field may be described as $\boldsymbol{E}\propto e^{i\omega t}$, and, however, that the spatial dependence of the field may be disregarded, Kadomtsev obtained the following results,

$$\begin{aligned}\boldsymbol{E}_{eff}&=\boldsymbol{E}\{1-(12\pi nD^3)^{-1}(T_i/T_i+T_e)^2\} \quad \omega\ll\omega_p\\ &=\boldsymbol{E} \qquad\qquad\qquad\qquad\qquad\qquad\quad \omega\gg\omega_p\end{aligned} \tag{55}$$

where D is a screening radius defined by

$$D^2=\kappa T_e T_i\{4\pi e^2 n(T_e+T_i)\}^{-1}. \tag{56}$$

Now, the electron plasma in the uniform background of the positive ion distribution is known as a useful model of the fully ionized gases, when the ionic temperature is considerably lower than the electronic temperature. In such case, it is well known that the screening radius is determined to be given by $D=\{\kappa T_e/4\pi e^2 n\}^{1/2}$. On the other hand, according to Kadomtsev's investigation, the screening radius is reduced to $D=\{\kappa T_i/4\pi e^2 n\}^{1/2}$ for the limit of $T_e\gg T_i$. This is apparently incorrect, hence Kadomtsev's expressions of the effective field, Eq. (55), also seems to be incorrect. In the present section, we will investigate the effective field in the electron plasma on the basis of the theory developed in the preceeding sections.

In the linearlization approximation, the effective field can be defined, by the rearrangement of the terms of Eq. (54), in terms of the following equation,

$$\begin{aligned}&\left\{\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\nabla\right\}f_1(\boldsymbol{x},\boldsymbol{v};t)+\frac{\partial}{\partial\boldsymbol{p}}\cdot\{\boldsymbol{F}_{eff}(\boldsymbol{x},\boldsymbol{v};t)f_0(\boldsymbol{v})\}\\ &\quad=\frac{1}{2\pi}\int d\omega\, e^{-i\omega t}\frac{1}{\Omega}\sum_{\boldsymbol{k}}e^{i\boldsymbol{k}\boldsymbol{x}}\frac{\lambda}{\Omega}\sum_{\boldsymbol{l}}V(\boldsymbol{l})i\boldsymbol{l}\cdot\frac{\partial}{\partial\boldsymbol{p}}\overline{\varphi_e^{(1)}(\boldsymbol{k}-\boldsymbol{l},\boldsymbol{l};\omega;\boldsymbol{v})}.\end{aligned} \tag{57}$$

The explicit form of $\boldsymbol{F}_{eff}(\boldsymbol{x},\boldsymbol{v};t)$ is written down as follows,

$$\boldsymbol{F}_{eff}(\boldsymbol{x},\boldsymbol{v};t)=\boldsymbol{F}_{eff}^{(1)}(\boldsymbol{x},\boldsymbol{v};t)+\boldsymbol{F}_{eff}^{(2)}(\boldsymbol{x},\boldsymbol{v};t) \tag{58}$$

with

$$\begin{aligned}\boldsymbol{F}_{eff}^{(1)}(\boldsymbol{x},\boldsymbol{v};t)&=-\frac{i}{2\pi}\int d\omega\, e^{-i\omega t}\frac{1}{\Omega}\sum_{\boldsymbol{k}}e^{i\boldsymbol{k}\boldsymbol{x}}\lambda V(\boldsymbol{k})\overline{f_1(\boldsymbol{k};\omega)}\\ &\quad\times\left\{\boldsymbol{k}-\frac{\lambda}{\kappa T}\frac{1}{\Omega}\sum_{\boldsymbol{l}}\frac{V(\boldsymbol{l})\boldsymbol{l}}{\Delta(\boldsymbol{k}-\boldsymbol{l},\boldsymbol{l};\omega;\boldsymbol{v})}G(\boldsymbol{k}-\boldsymbol{l},\boldsymbol{l};\omega;\boldsymbol{v})\right\},\end{aligned} \tag{59}$$

$$\begin{aligned}\boldsymbol{F}_{eff}^{(2)}(\boldsymbol{x},\boldsymbol{v};t)&=-\frac{1}{2\pi}\int d\omega\, e^{-i\omega t}\frac{1}{\Omega}\sum_{\boldsymbol{k}}e^{i\boldsymbol{k}\boldsymbol{x}}\cdot\frac{\lambda^2}{\kappa T}\cdot\frac{1}{\Omega}\sum_{\boldsymbol{l}}\frac{V(\boldsymbol{l})V(\boldsymbol{k}-\boldsymbol{l})}{\Delta(\boldsymbol{k}-\boldsymbol{l},\boldsymbol{l};\omega;\boldsymbol{v})}\\ &\quad\times\boldsymbol{l}\cdot(\boldsymbol{k}-\boldsymbol{l}\cdot\boldsymbol{v})\int\delta_+(\omega-(\boldsymbol{k}-\boldsymbol{l})\cdot\boldsymbol{v}-\boldsymbol{l}\cdot\boldsymbol{v}')\overline{\varphi_e^{(1)}(\boldsymbol{l},\boldsymbol{k}-\boldsymbol{l};\omega;\boldsymbol{v}')}d\boldsymbol{v}'.\end{aligned} \tag{60}$$

In the above definition of the effective field, the term $\overline{\varphi}_f$ and the term $\overline{\varphi}_e^{(2)}$ are separated from the expressions of $\overline{\varphi}_e^{(1)}$, Eq. (48), since they act as an external force for the particle at the point of $(\boldsymbol{x},\boldsymbol{v})$ and induce deviations from the equi-

librium state $f_0(\boldsymbol{v})$. The term $\overline{\varphi}_c^{(1)}$ is separated from other terms as the collision term, because this term represents a transport of correlation by the interaction between the particle at the point $(\boldsymbol{x}, \boldsymbol{v})$ and other particle at the point $(\boldsymbol{x}', \boldsymbol{v}')$ in the phase space. In the expression of the effective field $\boldsymbol{F}_{eff}^{(1)}(\boldsymbol{x}, \boldsymbol{v}; t)$, the term containing the function G corresponds to Eq. (19) of reference 18). The effective field $\boldsymbol{F}_{eff}^{(2)}(\boldsymbol{x}, \boldsymbol{v}; t)$ represents the contributions of the term $\overline{\varphi}_c^{(2)}$, which is missing in Kadomtsev's result.

As we can see from Eq. (57), the corrections of the internal field due to the binary correlation modify the dispersion relation which can be obtained from Eq. (33) by disregarding the correlation term. Let us come back to the problems of the correlation effects on the plasma oscillations in § 7. The problems on the internal field of the plasma are also interesting subjects in connection with the statistical broadening of spectral lines in the plasma. Concerning these problems, many investigations have been developed,[30),31),32)] but none of them did take account of the dynamical correlation effects. An explicit analysis of the above derived expression seems to be very interesting, so it will be reserved for a future occasion.

§ 5. Fokker-Planck equation for high temperature plasma

Now, when the particle density is so high that the particles remain in the force range of the interaction of their neighbours all the time, the concept of collisions does not valid for this situation. Then, instead of the Boltzmann equation, the Fokker-Planck equation must be used to describe the time variation of the statistical state of the system.

In the plasma, the condition is just such one that enforces to employ the Fokker-Planck equation, since the interaction has the infinite force range so that every particle remains always under influences of other particles in the plasma. Although we have mentioned in § 1 about one of the methods for treating the long-range nature of the correlation, which was proposed by Vlasov and Landau, it also seems to be useful to apply the Fokker-Planck equation for the description of the plasma. In fact, following Bogoliubov's method, Tolmachev[11)] and Temko[12)] have derived the Fokker-Planck equation for the plasma, and recently Tchen[13)] has thoroughly investigated the Fokker-Planck equation of the plasma, emphasizing the non-Markovian behaviour of the plasma. However, it has not yet been clarified that what relationship does exist between the descriptions of the plasma on the basis of the Fokker-Planck equation and those of the plasma on the basis of the Boltzmann-Vlasov equation. As it is well known, the latter equation has been commonly used in the discussions of the plasma oscillations, while the theories based on the former equation do not mention about the plasma oscillations at all. In the foregoing section, we have mentioned about the effects of the binary correlation on the internal field of the plasma and on the plasma oscillations. Here,

we will examine the derivation of the Fokker-Planck equation, which describes the change of the momentum distribution of the plasma particles.

Let us integrate Eq. (54) over coordinate space. We obtain the following equation,

$$\frac{\partial}{\partial t}f_1(\boldsymbol{v};t)=\frac{1}{2\pi}\int d\omega\, e^{-i\omega t}\frac{\lambda}{\Omega}\sum_{\boldsymbol{l}}V(\boldsymbol{l})i\boldsymbol{l}\cdot\frac{\partial}{\partial \boldsymbol{p}}\overline{\varphi_c(-\boldsymbol{l},\boldsymbol{l};\omega;\boldsymbol{v})} \tag{61}$$

where

$$\overline{\varphi}_c=\overline{\varphi}_c^{(1)}+\overline{\varphi}_c^{(2)}. \tag{62}$$

The term $\overline{\varphi}_f$ vanishes for $\boldsymbol{k}\to 0$, because of the neutrality condition. Now, let us follow the non-stationary motion of the specified particle of velocity $\boldsymbol{v}$, while other particles of the plasma are assumed to remain in the equilibrium state, which is the same situation as the Brownian motion. For this condition, we can drop the terms with $f_1(\omega;\boldsymbol{v}')$ in the $\overline{\varphi}_c^{(1)}$ term and drop the $\overline{\varphi}_c^{(2)}$ term as a whole. Then, Eq. (61) is reduced to the following equation,

$$\begin{aligned}\frac{\partial}{\partial t}f_1(\boldsymbol{v};t)=&\frac{1}{2\pi}\int d\omega\, e^{-i\omega t}\sum_{\alpha=1}^{3}\frac{\partial}{\partial p_\alpha}\{A^\alpha(\boldsymbol{v};\omega)f_1(\boldsymbol{v};\omega)\}\\&+\frac{1}{2\pi}\int d\omega\, e^{-i\omega t}\sum_{\alpha=1}^{3}\sum_{\beta=1}^{3}\frac{\partial}{\partial p_\alpha}\left\{B^{\alpha\beta}(\boldsymbol{v};\omega)\frac{\partial}{\partial p_\beta}f_1(\boldsymbol{v};\omega)\right\}\end{aligned} \tag{63}$$

where $A^\alpha(\boldsymbol{v};\omega)$, $B^{\alpha\beta}(\boldsymbol{v};\omega)$ are given as follows,

$$A^\alpha(\boldsymbol{v};\omega)=-\frac{\lambda^2}{\Omega}\sum_{\boldsymbol{k}}\frac{V(\boldsymbol{k})^2}{\Delta(-\boldsymbol{k},\boldsymbol{k};\omega;\boldsymbol{v})}k^\alpha\cdot\int\delta_+(\omega+\boldsymbol{k}\cdot(\boldsymbol{v}-\boldsymbol{v}'))\boldsymbol{k}\cdot\frac{\partial}{\partial \boldsymbol{p}'}f_0(\boldsymbol{v}')\,d\boldsymbol{v}', \tag{64}$$

$$\begin{aligned}B^{\alpha\beta}(\boldsymbol{v};\omega)=&+\frac{\lambda^2}{\Omega}\sum_{\boldsymbol{k}}\frac{V(\boldsymbol{k})^2}{\Delta(-\boldsymbol{k},\boldsymbol{k};\omega;\boldsymbol{v})}(1+n\chi(\boldsymbol{k}))k^\alpha k^\beta\\&\times\int\delta_+(\omega+\boldsymbol{k}\cdot(\boldsymbol{v}-\boldsymbol{v}'))f_0(\boldsymbol{v}')\,d\boldsymbol{v}'.\end{aligned} \tag{65}$$

The Fokker-Planck equation (63) is derived under the linearlized approximation but without the assumption that the distribution function varies slowly with time. Although Tchen has derived the Fokker-Planck equation, usidg the assumption of the slowly varying time dependence of the distribution function, it must be kept in our mind that in the plasma there occur oscillations with the high frequency of the order of ω_p. To discuss the problems more consistently, it appears to be necessary to extend the Fokker-Planck equation to include the occurrence of spatially inhomogeneous disturbance of the system. We will come back to this problem in the next section.

Recently, Balescu[17] has derived a complicated set of equations which describes evolution of the momentum distribution of the plasma. By considering the same situation as what we have considered in the reduction of Eq. (63) from Eq. (61), the equations derived by Balescu can be reduced to the following equation,

$$\frac{\partial}{\partial t} f(\boldsymbol{p}\,;\,t) = \sum_{\alpha} \frac{\partial}{\partial p_\alpha} \{A^\alpha(\boldsymbol{p}) f(\boldsymbol{p}\,;\,t)\} + \sum_{\alpha}\sum_{\beta} \frac{\partial}{\partial p_\alpha} \left\{ B^{\alpha\beta}(\boldsymbol{p}) \frac{\partial}{\partial p_\beta} f(\boldsymbol{p}\,;\,t) \right\} \tag{66}$$

with the following abbreviations of $A^\alpha(\boldsymbol{p})$ and $B^{\alpha\beta}(\boldsymbol{p})$,

$$A^\alpha(\boldsymbol{p}) = -\frac{\lambda^2}{\Omega} \sum_{k} \frac{V(\boldsymbol{k})^2}{\Delta_R(\boldsymbol{k}\,;\,\boldsymbol{v})} k^\alpha \int \delta_-(\boldsymbol{k}\cdot(\boldsymbol{v}-\boldsymbol{v}'))\, \boldsymbol{k}\cdot\frac{\partial}{\partial \boldsymbol{p}'} f_0(\boldsymbol{p}')\, d\boldsymbol{p}', \tag{67}$$

$$B^{\alpha\beta}(\boldsymbol{p}) = +\frac{\lambda^2}{\Omega} \sum_{k} \frac{V(\boldsymbol{k})^2}{\Delta_R(\boldsymbol{k}\,;\,\boldsymbol{v})} k^\alpha k^\beta \int \delta_-(\boldsymbol{k}\cdot(\boldsymbol{v}-\boldsymbol{v}')) \left\{ f_0(\boldsymbol{p}') - \frac{n}{8\pi^3} F(\boldsymbol{p}', \boldsymbol{k}) \right\} d\boldsymbol{p}' \tag{68}$$

where $F(\boldsymbol{p}', \boldsymbol{k})$ is a solution of the integral equation, Eq. (3) with Eq. (4) of reference 17). The screening constant Δ_R is defined by

$$\Delta_R(\boldsymbol{k}\,;\,\boldsymbol{v}) = 1 + i\frac{4\pi e^2}{k^2} \int d\boldsymbol{p}'\, \delta_+(\boldsymbol{k}\cdot(\boldsymbol{v}-\boldsymbol{v}'))\, \boldsymbol{k}\cdot\frac{\partial}{\partial \boldsymbol{p}'} f_0(\boldsymbol{p}'). \tag{69}$$

Comparing our equation, Eq. (63), with the above equation, Eq. (66), we can see that Balescu's formalism does not take account of the non-Markovian behavior of the plasma. Furthermore, it should be noticed that information about the plasma oscillations is not involved in Balescu's formalism, since Balescu has developed his formalism on the assumption of spatial homogeneity of the system. As it has been discussed in the above, it is essential to extend the Fokker-Planck equation to take account of the spatial inhomogeneity of the system for the discussions of the plasma oscillations.

§ 6. Generalized Fokker-Planck equation

In the preceding section, we have shown that the Fokker-Planck equation derived by Tchen as well as the equations obtained by Balescu has a very restrictive range of the applicability, because of the assumption of the spatial homogeneity. Although Watson[33] has discussed a generalized Fokker-Planck equation with the term which accounts for the effects of spatial inhomogeneity, yet his derivation of the equation is a kind of semi-phenomenological approach and depends on the special model of the determination of the fluctuating force.

Here, we will derive the generalized Fokker-Planck equation from Eqs. (54) and (44). Since the main effects of the ternary correlation are taken into account by the first term of the right-hand side of Eq. (44), we will neglect the second and the third term in the right-hand side of Eq. (44). Then, instead of Eq. (44) we have a simplified function,

$$\begin{aligned}\overline{\varphi_1(\boldsymbol{k}-\boldsymbol{l}, \boldsymbol{l}\,;\,\omega\,;\,\boldsymbol{v})} = {} & +i\frac{\lambda}{\kappa T} \frac{V(\boldsymbol{k})}{\Delta(\boldsymbol{k}-\boldsymbol{l}, \boldsymbol{l}\,;\,\omega\,;\,\boldsymbol{v})} \overline{f_1(\boldsymbol{k}\,;\,\omega)} \\ & \times \int d\boldsymbol{v}' f_0(\boldsymbol{v}')\, \delta_+(\omega - (\boldsymbol{k}-\boldsymbol{l})\cdot\boldsymbol{v} - \boldsymbol{l}\cdot\boldsymbol{v}') [(\boldsymbol{k}\cdot\boldsymbol{v})\chi(\boldsymbol{l}) + (\boldsymbol{k}\cdot\boldsymbol{v}')\chi(\boldsymbol{k}-\boldsymbol{l})] \\ & + \overline{\varphi_0^{(1)}(\boldsymbol{k}-\boldsymbol{l}, \boldsymbol{l}\,;\,\omega\,;\,\boldsymbol{v})}\end{aligned} \tag{70}$$

where $\bar{\varphi}_c^{(1)}$ is given by Eq. (52). Substituting Eq. (70) in Eq. (54), we can obtain the following equation,

$$\left\{\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\boldsymbol{\nabla}\right\}f_1(\boldsymbol{x},\boldsymbol{v};t)+\frac{\partial}{\partial \boldsymbol{p}}\{\boldsymbol{F}_{eff}^{*}(\boldsymbol{x},\boldsymbol{v};t)f_0(\boldsymbol{v})\}$$

$$=\frac{1}{2\pi}\int d\omega\, e^{-i\omega t}\frac{1}{\Omega}\sum_{\boldsymbol{k}}e^{i\boldsymbol{k}\boldsymbol{x}}\Bigg[\sum_{\alpha=1}^{3}\frac{\partial}{\partial p_\alpha}\{A^\alpha(\boldsymbol{k};\boldsymbol{v};\omega)f_1(\boldsymbol{k};\boldsymbol{v};\omega)\}$$

$$+\sum_{\alpha=1}^{3}\sum_{\beta=1}^{3}\frac{\partial}{\partial p_\alpha}\left\{B^{\alpha\beta}(\boldsymbol{k};\boldsymbol{v};\omega)\frac{\partial}{\partial p_\beta}f_1(\boldsymbol{k};\boldsymbol{v};\omega)\right\}\Bigg]$$

$$+\frac{1}{2\pi}\int d\omega\, e^{-i\omega t}\frac{1}{\Omega}\sum_{\boldsymbol{k}}e^{i\boldsymbol{k}\boldsymbol{x}}\sum_{\alpha=1}^{3}\frac{\partial}{\partial p_\alpha}\left\{\frac{\lambda^2}{\Omega}\sum_{\boldsymbol{l}}\frac{(1+n\chi(\boldsymbol{l}))\,V(\boldsymbol{l})\,V(\boldsymbol{k}-\boldsymbol{l})}{\Delta(\boldsymbol{k}-\boldsymbol{l},\boldsymbol{l};\omega;\boldsymbol{v})}\right.$$

$$\times l^\alpha f_0(\boldsymbol{v})\cdot\int\delta_+(\omega-(\boldsymbol{k}-\boldsymbol{l})\cdot\boldsymbol{v}-\boldsymbol{l}\cdot\boldsymbol{v}')\,(\boldsymbol{k}-\boldsymbol{l})\cdot\frac{\partial}{\partial \boldsymbol{p}'}f_1(\boldsymbol{k};\boldsymbol{v}';\omega)\,d\boldsymbol{v}'$$

$$-\frac{\lambda^2}{\Omega}\sum_{\boldsymbol{l}}\frac{V(\boldsymbol{l})\,V(\boldsymbol{k}-\boldsymbol{l})}{\Delta(\boldsymbol{k}-\boldsymbol{l},\boldsymbol{l};\omega;\boldsymbol{v})}l^\alpha(\boldsymbol{k}-\boldsymbol{l})\cdot\frac{\partial}{\partial \boldsymbol{p}}f_0(\boldsymbol{v})$$

$$\left.\times\int\delta_+(\omega-(\boldsymbol{k}-\boldsymbol{l})\cdot\boldsymbol{v}-\boldsymbol{l}\cdot\boldsymbol{v}')f_1(\boldsymbol{k};\boldsymbol{v}';\omega)\,d\boldsymbol{v}'\right\} \tag{71}$$

where $\boldsymbol{F}_{eff}^{*}$ is given by Eq. (59) by reducing the function G to the simplified function $G^{(0)}$, which is defined as

$$G^{(0)}(\boldsymbol{k}-\boldsymbol{l};\boldsymbol{l};\omega;\boldsymbol{v})=i\int d\boldsymbol{v}'f_0(\boldsymbol{v}')\,\delta_+(\omega-(\boldsymbol{k}-\boldsymbol{l})\cdot\boldsymbol{v}-\boldsymbol{l}\cdot\boldsymbol{v}')$$

$$\times[(\boldsymbol{k}\cdot\boldsymbol{v})\chi(\boldsymbol{l})+(\boldsymbol{k}\cdot\boldsymbol{v}')\chi(\boldsymbol{k}-\boldsymbol{l})]. \tag{72}$$

A^α, $B^{\alpha\beta}$ are defined as follows,

$$A^\alpha(\boldsymbol{k};\boldsymbol{v};\omega)=-\frac{\lambda^2}{\Omega}\sum_{\boldsymbol{l}}\frac{V(\boldsymbol{l})^2}{\Delta(\boldsymbol{k}-\boldsymbol{l},\boldsymbol{l};\omega;\boldsymbol{v})}l^\alpha$$

$$\times\int\delta_+(\omega-(\boldsymbol{k}-\boldsymbol{l})\cdot\boldsymbol{v}-\boldsymbol{l}\cdot\boldsymbol{v}')\boldsymbol{l}\cdot\frac{\partial}{\partial \boldsymbol{p}'}f_0(\boldsymbol{v}')\,d\boldsymbol{v}', \tag{73}$$

$$B^{\alpha\beta}(\boldsymbol{k};\boldsymbol{v};\omega)=+\frac{\lambda^2}{\Omega}\sum_{\boldsymbol{l}}\frac{(1+n\chi(\boldsymbol{l}))\,V(\boldsymbol{l})^2}{\Delta(\boldsymbol{k}-\boldsymbol{l},\boldsymbol{l};\omega;\boldsymbol{v})}l^\alpha l^\beta$$

$$\times\int\delta_+(\omega-(\boldsymbol{k}-\boldsymbol{l})\cdot\boldsymbol{v}-\boldsymbol{l}\cdot\boldsymbol{v}')f_0(\boldsymbol{v}')\,d\boldsymbol{v}'. \tag{74}$$

These coefficients, $A^\alpha(\boldsymbol{k};\boldsymbol{v};\omega)$ and $B^{\alpha\beta}(\boldsymbol{k};\boldsymbol{v};\omega)$, are the generalized friction and diffusion coefficients which depend on the wave vector $\boldsymbol{k}$ and on the frequency ω. These coefficients are composed of the principal value term and the δ-function term, the latter describes proper dissipative phenomena, while the former may act as effective fields which influence the motion of the distribution function $f_1(\boldsymbol{x},\boldsymbol{v};t)$. The ω-dependence of these coefficients represents the non-Markovian

behavior of the plasma, which has been discussed by Tchen. Essential difference, however, lies between Tchen's derivation and our derivation of these coefficients. Since Tchen has assumed the slowly varying time dependence of the distribution function, his results can be reduced from our results simply by assuming that ω is much smaller than such quantities as $(\boldsymbol{k}+\boldsymbol{l})\cdot\boldsymbol{v}$ or $\boldsymbol{l}\cdot\boldsymbol{v}'$. The $\boldsymbol{k}$-dependence of these coefficients takes into account the inhomogeneous spatial correlation effects between different coordinate points. Although Watson[33] has derived the Fokker-Planck equation with the term which accounts for the effects of spatial inhomogeneity, no explicit accounts have been made for the inhomogeneous spatial correlation effects in his derivation of these coefficients. It appears to be very interesting subjects to examine the inhomogeneous spatial correlation effects as well as the non-Markovian effects on these coefficients.

The last term of the right-hand side of Eq. (71) represents the coupling term between the particle at the point $(\boldsymbol{x}, \boldsymbol{v})$ and the particle at the point $(\boldsymbol{x}', \boldsymbol{v}')$. In the discussions of the Brownian motion of a test particle with the velocity v, while all the other particles remain in the equilibrium state, these two terms can be disregarded.

§ 7. Quasi-stationary plasma oscillations

In § 4, we have shown that a part of the binary correlation effects contributes to the internal field of the plasma. Since the dynamical fluctuation of the self-consistent internal field determines the plasma oscillations of the system, any correction of the internal field may modify the characteristic features of the plasma oscillations. In the present section, let us examine the correlation effects on the plasma oscillations.

Now, there are two ways of dealing with the plasma oscillations. A macroscopic, hydrodynamic treatment using Euler's equation of motion together with the Maxwell equations is essentially a phenomenological approach and is confronted with the questions concerning the range of validity of this treatment. Since the fundamental equations of hydrodynamics can be reduced from the Boltzmann equation, the microscopic gas-kinetic treatment using the Boltzmann-Vlasov equation seems to be a fundamental approach to the problem. However, as it is well known, there are controversial ambiguities in the determination of the eigen-mode of oscillations using the gas-kinetic methods. Some of the ambiguities may be attributed purely to inadequate formulations of the problem. It would be worth to notice that the observable plasma oscillations are not the fluctuation of distribution function in the phase space, but the fluctuation of density in the coordinate space. Therefore, it seems to be sufficient to reduce the equation of motion of the density fluctuation from the kinetic equation which contains too much information.

Firstly, by integrating Eq. (71) over the velocity space, we obtain the continuity equation of the density fluctuation,

$$\frac{\partial}{\partial t} n(\boldsymbol{x}, t) + n \operatorname{div} \boldsymbol{V}(\boldsymbol{x}, t) = 0 \tag{75}$$

where the density fluctuation $n(\boldsymbol{x}, t)$ is defined by

$$n(\boldsymbol{x}, t) = \int f_1(\boldsymbol{x}, \boldsymbol{v}; t)\, d\boldsymbol{v}, \tag{76}$$

so that the Fourier component $n(\boldsymbol{k}; \omega)$ of $n(\boldsymbol{x}; t)$ is equal to $\overline{f_1(\boldsymbol{k}; \omega)}$. The macroscopic flow velocity $\boldsymbol{V}(\boldsymbol{x}, t)$ is defined by

$$\boldsymbol{V}(\boldsymbol{x}, t) = \frac{1}{n} \int \boldsymbol{v} f_1(\boldsymbol{x}, \boldsymbol{v}; t)\, d\boldsymbol{v}. \tag{77}$$

Secondly, by multiplying $\boldsymbol{v}/n$ and then integrating over the velocity space, the equation of motion is obtained from Eq. (71) as follows,

$$\frac{\partial}{\partial t} \boldsymbol{V}(\boldsymbol{x}, t) = \frac{1}{m} \boldsymbol{F}_{eff}(\boldsymbol{x}, t) - \frac{1}{n} \int \boldsymbol{v} (\boldsymbol{v} \cdot \nabla) f_1(\boldsymbol{x}, \boldsymbol{v}; t)\, d\boldsymbol{v} - C[\boldsymbol{V}(\boldsymbol{x}, t)], \tag{78}$$

where $C[\boldsymbol{V}(\boldsymbol{x}, t)]$ is a collision term of very complicated structure. $\boldsymbol{F}_{eff}(\boldsymbol{x}, t)$ is given by

$$\boldsymbol{F}_{eff}(\boldsymbol{x}; t) = \frac{1}{n} \int \boldsymbol{F}^*_{eff}(\boldsymbol{x}, \boldsymbol{v}; t) f_0(\boldsymbol{v})\, d\boldsymbol{v}. \tag{79}$$

To simplify the following discussions of the plasma oscillations, let us omit the collision term $C[\boldsymbol{V}]$ from Eq. (78). This is equivalent to taking the following equation instead of Eq. (71),

$$\left\{ \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla \right\} f_1(\boldsymbol{x}, \boldsymbol{v}; t) + \frac{\partial}{\partial \boldsymbol{p}} \{ \boldsymbol{F}^*_{eff}(\boldsymbol{x}, \boldsymbol{v}; t) f_0(\boldsymbol{v}) \} = 0. \tag{80}$$

Now, from Eqs. (75) and (78) with the neglect of the collision term, we can derive the equation of motion of the density fluctuation as follows,

$$\frac{\partial^2}{\partial t^2} n(\boldsymbol{x}, t) + \frac{1}{m} \int \nabla \cdot \boldsymbol{F}^*_{eff}(\boldsymbol{x}, \boldsymbol{v}; t) f_0(\boldsymbol{v})\, d\boldsymbol{v} - \int (\boldsymbol{v} \cdot \nabla)^2 f_1(\boldsymbol{x}, \boldsymbol{v}; t)\, d\boldsymbol{v} = 0. \tag{81}$$

This equation does not close by itself because of the last term. In the hydrodynamic approach, besides the equation of continuity and the equation of motion, the equation of energy conservation is invoked to obtain the closed set of equations. This is equivalent to terminating the moment-equation obtained from the kinetic equation at the finite order. Here we will not follow this procedure, but proceed up to the higher order of the moment equation. Let us construct the moment equation of operator $(\boldsymbol{v} \cdot \nabla)^s$ from Eq. (80), then we get

$$\frac{\partial}{\partial t} \int (\boldsymbol{v} \cdot \nabla)^s f_1(\boldsymbol{x}, \boldsymbol{v}; t)\, d\boldsymbol{v} = \frac{s}{m} \int (\boldsymbol{v} \cdot \nabla)^{s-1} \nabla \cdot \boldsymbol{F}_{eff}(\boldsymbol{x}, \boldsymbol{v}; t) f_0(\boldsymbol{v})\, d\boldsymbol{v} - \int (\boldsymbol{v} \cdot \nabla)^{s+1} f_1(\boldsymbol{x}, \boldsymbol{v}; t)\, d\boldsymbol{v}. \tag{82}$$

By using the Fourier transformation of Eq. (82), we obtain the following relation,

$$\overline{(\boldsymbol{v}\cdot\boldsymbol{k})^s f_1(\boldsymbol{k};\omega;\boldsymbol{v})} = s\frac{\lambda V(\boldsymbol{k})}{m} f_1(\boldsymbol{k};\omega)\frac{1}{\omega}\overline{(\boldsymbol{v}\cdot\boldsymbol{k})^{s-1}\boldsymbol{k}\cdot\boldsymbol{J}(\boldsymbol{v})f_0(\boldsymbol{v})} + \frac{1}{\omega}\overline{\overline{(\boldsymbol{v}\cdot\boldsymbol{k})^{s+1}f_1(\boldsymbol{k};\omega;\boldsymbol{v})}} \tag{83}$$

with the abbreviation of $\boldsymbol{J}(\boldsymbol{v})$,

$$\boldsymbol{J}(\boldsymbol{v}) = \boldsymbol{k} - \frac{\lambda}{\kappa T}\frac{1}{\Omega}\sum_{\tau} V(\boldsymbol{l})\boldsymbol{l}\Delta(\boldsymbol{k}-\boldsymbol{l},\boldsymbol{l};\omega;\boldsymbol{v})^{-1}G^{(0)}(\boldsymbol{k}-\boldsymbol{l},\boldsymbol{l};\omega;\boldsymbol{v}) \tag{84}$$

and the double bar indicates the integration over the velocity $\boldsymbol{v}$. By the successive use of Eq. (83), we get

$$\overline{(\boldsymbol{v}\cdot\boldsymbol{k})^2 f_1(\boldsymbol{k};\omega;\boldsymbol{v})} = \frac{\lambda V(\boldsymbol{k})}{m} f_1(\boldsymbol{k};\omega)\sum_{s=1}^{n}(s+1)\cdot\frac{\overline{(\boldsymbol{v}\cdot\boldsymbol{k})^s\boldsymbol{k}\cdot\boldsymbol{J}(\boldsymbol{v})f_0(\boldsymbol{v})}}{\omega^s} + \frac{\overline{\overline{(\boldsymbol{v}\cdot\boldsymbol{k})^{n+2}f_1(\boldsymbol{k};\omega;\boldsymbol{v})}}}{\omega^n}. \tag{85}$$

We assume that when $\boldsymbol{k}$ is small enough, Eq. (85) can be expressed by the asymptotic expansion series,

$$\overline{(\boldsymbol{v}\cdot\boldsymbol{k})^2 f_1(\boldsymbol{k};\omega;\boldsymbol{v})} = \frac{\lambda V(\boldsymbol{k})}{m} f_1(\boldsymbol{k};\omega)\sum_{s=1}^{\infty}(s+1)\frac{\overline{(\boldsymbol{v}\cdot\boldsymbol{k})^s\boldsymbol{k}\cdot\boldsymbol{J}(\boldsymbol{v})f_0(\boldsymbol{v})}}{\omega^s}. \tag{86}$$

In other words, we can expand the function $\int(\boldsymbol{v}\cdot\boldsymbol{k})^2 f_1(\boldsymbol{k};\omega;\boldsymbol{v})\,d\boldsymbol{v}$ into the asymptotic expansion series Eq. (86) when the function $f_1(\boldsymbol{k};\omega;\boldsymbol{v})$ behaves as $k^{-2+\epsilon}$, $\epsilon>0$, at the limit of $\boldsymbol{k}\to 0$. The series can be expressed symbolically as

$$\sum_{s=1}^{\infty}(s+1)\frac{\overline{(\boldsymbol{v}\cdot\boldsymbol{k})^s\boldsymbol{k}\cdot\boldsymbol{J}(\boldsymbol{v})f_0(\boldsymbol{v})}}{\omega^s} = S_v\int\frac{\boldsymbol{k}\cdot\boldsymbol{J}(\boldsymbol{v})f_0(\boldsymbol{v})}{(1-\boldsymbol{v}\cdot\boldsymbol{k}/\omega)^2}\,d\boldsymbol{v} - \int\boldsymbol{k}\cdot\boldsymbol{J}(\boldsymbol{v})f_0(\boldsymbol{v})\,d\boldsymbol{v} \tag{87}$$

where the symbol S_v defines that the integration should be carried out over each term of the series expansion of the denominators. Here, Eq. (87) present a different implication of the controversial integral,

$$\int\{\omega(\boldsymbol{k})-\boldsymbol{k}\cdot\boldsymbol{v}\}^{-2}f_0(\boldsymbol{v})\,d\boldsymbol{v}$$

which appears in the well-known dispersion relation of the plasma oscillations. Bohm has claimed that it is necessary to have sufficiently rapid decrease of the function $f_0(\boldsymbol{v})$ at large values of $\boldsymbol{v}$ so that the above integral has a definite meaning. This claim is equivalent to requiring the convergence of the infinite series of Eq. (86). We have seen in the above that, although the series does not converge for the Maxwellian distribution function $f_0(\boldsymbol{v})$, it has a meaning of the asymptotic expansion of the function $\int(\boldsymbol{v}\cdot\boldsymbol{k})^2 f_1(\boldsymbol{k};\omega;\boldsymbol{v})\,d\boldsymbol{v}$. Therefore, the above integral is a symbolical representation of the asymptotic expansion of the function

$\int (\boldsymbol{v}\cdot\boldsymbol{k})^2 f_1(\boldsymbol{k};\omega;\boldsymbol{v})\,d\boldsymbol{v}$, and thus it is completely free from the controversial ambiguities concerned with the singularity of the integrand.

One may ask what will happen for $\omega=\boldsymbol{k}\cdot\boldsymbol{v}$. This question is closely connected with the controversial ambiguities of the determination of the eigen-mode of plasma oscillations, and van Kampen[27] has discussed some aspects of the controversial problem. Here we can show that for the special value of $\boldsymbol{k}$ at which the relation $\omega=\boldsymbol{k}\cdot\boldsymbol{v}$ does hold it is essential to take the collision term into consideration. Namely, when $\omega=\boldsymbol{k}\cdot\boldsymbol{v}$, we have

$$\int (\boldsymbol{v}\cdot\nabla)^2 f_1(\boldsymbol{x},\boldsymbol{v};t)\,d\boldsymbol{v}=-\frac{1}{\Omega}\omega(\boldsymbol{k})^2 n(\boldsymbol{k};t)e^{i\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{88}$$

Substituting Eq. (88) in Eq. (81), it is reduced to the following relation,

$$\int \boldsymbol{k}\cdot\boldsymbol{F}^*_{eff}(\boldsymbol{k};\boldsymbol{v};t)f_0(\boldsymbol{v})\,d\boldsymbol{v}=0, \tag{89}$$

which cannot be valid by any means. Therefore, for the special value of $\boldsymbol{k}$, it is necessary to take the collision term into account. Invoking the collision term, Eq. (89) will take the following form,

$$\int \boldsymbol{k}\cdot\boldsymbol{F}^*_{eff}(\boldsymbol{k};\boldsymbol{v};t)f_0(\boldsymbol{v})\,d\boldsymbol{v}=C(\boldsymbol{k};t),$$

which may be regarded as a relation determining the amplitude of density fluctuation. Starting from the Boltzmann-Vlasov equation without the collision term, Landau has discussed the problem of the plasma oscillations as an initial value problem. Using the Laplace transformation, he has found that the frequency of the plasma oscillations has an imaginary part γ_L called Landau damping factor. In the above discussions, we have tried to point out the importance of the collision term in the examination of the controversial case of $\omega=\boldsymbol{k}\cdot\boldsymbol{v}$. Detailed examination of the effects of the collision term will be reserved for some other occasion. Concerning the contributions of the Landau damping, we will discuss in the last paragraph of present section.

Now, let us return to the equation of motion of the density fluctuation, Eq. (81). Using Eqs. (86) and (87), we can rewrite Eq. (81) as

$$\frac{\partial^2}{\partial t^2}\cdot\frac{1}{2\pi}\int d\omega\, e^{-i\omega t}\frac{1}{\Omega}\sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} n(\boldsymbol{k};\omega)+\frac{1}{2\pi}\int d\omega\, e^{-i\omega t}$$
$$\times\frac{1}{\Omega}\sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\frac{\lambda V(\boldsymbol{k})}{m}S_v\int -\frac{\boldsymbol{k}\cdot\boldsymbol{J}(\boldsymbol{v})}{\{1-(\boldsymbol{k}\cdot\boldsymbol{v}/\omega)\}^2}f_0(\boldsymbol{v})\,d\boldsymbol{v}\,n(\boldsymbol{k};\omega)=0. \tag{90}$$

Since $\boldsymbol{J}(\boldsymbol{v})$ defined by Eq. (84) is a complicated complex function, let us simplify it by disregarding the effect of dynamical shielding. Then, by rearranging Eq. (90) in the standard form of the equation of motion of the damping oscillation,

$$\frac{\partial^2}{\partial t^2}n(\boldsymbol{k};t)+2\gamma_C(\boldsymbol{k})\frac{\partial}{\partial t}n(\boldsymbol{k};t)+\omega(\boldsymbol{k})^2 n(\boldsymbol{k};t)=0, \tag{91}$$

we can obtain the following set of equations,

$$\omega(\boldsymbol{k})^2=\omega_p{}^2 n^{-1} S_v \int \frac{\omega(\boldsymbol{k})^2}{(\omega(\boldsymbol{k})-\boldsymbol{k}\cdot\boldsymbol{v})^2} f_0(\boldsymbol{v})\,d\boldsymbol{v}+\omega_p{}^2\left(\frac{\lambda}{\kappa T}\right)\frac{1}{\Omega}$$
$$\times \sum_{l} V(\boldsymbol{l})\,(\boldsymbol{l}\cdot\boldsymbol{e})\,n^{-1} S_v \int \frac{\omega(\boldsymbol{k})}{(\omega(\boldsymbol{k})-\boldsymbol{k}\cdot\boldsymbol{v})^2} P_{v'} \int \frac{1}{\omega(\boldsymbol{l})-(\boldsymbol{k}-\boldsymbol{l})\cdot\boldsymbol{v}-\boldsymbol{l}\cdot\boldsymbol{v}'}$$
$$\times \{(\boldsymbol{e}\cdot\boldsymbol{v})\chi(\boldsymbol{l})+(\boldsymbol{e}\cdot\boldsymbol{v}')\chi(\boldsymbol{k}-\boldsymbol{l})\} f_0(\boldsymbol{v}')\,d\boldsymbol{v}' f_0(\boldsymbol{v})\,d\boldsymbol{v} \tag{92}$$

with the abbreviation of $\boldsymbol{e}=\boldsymbol{k}/|\boldsymbol{k}|$, and

$$\gamma_C(\boldsymbol{k})=\frac{\pi}{2}\frac{\omega_p{}^2}{\omega(\boldsymbol{k})}\left(\frac{\lambda}{\kappa T}\right)\cdot\frac{1}{\Omega}\sum_{l} V(\boldsymbol{l})\,(\boldsymbol{l}\cdot\boldsymbol{e})\,n^{-1}$$
$$\times S_v \frac{\omega(\boldsymbol{k})^2}{(\omega(\boldsymbol{k})-\boldsymbol{k}\cdot\boldsymbol{v})^2}\int \delta(\omega-(\boldsymbol{k}-\boldsymbol{l})\cdot\boldsymbol{v}-\boldsymbol{l}\cdot\boldsymbol{v}')$$
$$\times \{(\boldsymbol{e}\cdot\boldsymbol{v})\chi(\boldsymbol{l})+(\boldsymbol{e}\cdot\boldsymbol{v}')\chi(\boldsymbol{k}-\boldsymbol{l})\} f_0(\boldsymbol{v}')\,d\boldsymbol{v}' f_0(\boldsymbol{v})\,d\boldsymbol{v} \tag{93}$$

where ω_p^2 is defined by

$$\omega_p{}^2=\lambda V(\boldsymbol{k})\,k^2 n/m. \tag{94}$$

For the case of Coulomb potential, let us investigate the long wavelength limit of Eqs. (92) and (93), keeping the terms up to the order of $\boldsymbol{k}^2$. Then, we obtain the frequency of plasma oscillation as

$$\omega(\boldsymbol{k})^2=\omega_p{}^2+3\frac{\kappa T}{m}k^2\left\{1+1.05\sqrt{\left(\frac{e^2}{\kappa T}\right)^3 n}\right\} \tag{95}$$

where the binary correlation effects appears as a small correction factor in the dispersive term of the plasma frequency. Eq. (93) gives the following result for the damping factor,

$$\gamma_C(k)=0.028\sqrt{\left(\frac{e^2}{\kappa T}\right)^3 n}\cdot\left(\frac{k}{k_d}\right)^2\cdot\omega_p \tag{96}$$

which is due to the long-range binary correlation effects. These are the remarkable results. Firstly, Eq. (95) tells us that the binary correlation effects do not modify the Langmuir frequency but modify the dispersive term of the plasma frequency. The recent investigations by Nozierès and Pines,[34] and Kanazawa and his collaborators[35] have shown that the exchange correlation effects modify the dispersive term of the plasma frequency for the electron system in metals. Thus, we may conclude that any kinds of the correlation effects, either quantum or classical, never modifies the Langmuir frequency term of the plasma oscillations. Secondly, Eq. (96) shows that the damping factor due to the long-range binary correlation effects varies as k^2. This is in very contrast with the Landau damping factor γ_L,

$$\gamma_L=\left(\frac{\pi}{8}\right)^{1/2}\left(\frac{k_d}{k}\right)^3\exp\left\{-\frac{1}{2}\left(\frac{k_d}{k}\right)^2\right\}\cdot\omega_p \tag{97}$$

which vanishes at the limit of $k \to 0$ and increases very steeply from 0 to its maximum value $\gamma_L = 0.72 \cdot \omega_p$ at $k = k_d/\sqrt{3}$, then decreases slowly to $\gamma_L = 0.38 \cdot \omega_p$ at $k = k_d$. It takes the values of $\gamma_L = 0.25 \times 10^{-18} \cdot \omega_p$ at $k = 0.1 k_d$ and $\gamma_L = 0.51 \times 10^{-3} \cdot \omega_p$ at $k = k_d$. On the other hand, for the parameters of $n = 10^{13} cm^{-3}$ and $T = 10^4 K$ the correlation damping factor γ_C takes the values of $\gamma_C = 0.61 \times 10^{-7} \cdot \omega_p$ at $k = 0.1 k_d$ and $\gamma_C = 0.24 \times 10^{-6} \cdot \omega_p$ at $k = 0.2 k_d$. Therefore, we may conclude that the long-range correlation damping predominates over the Landau damping in the range of small values of k for the typical plasma. For the larger values of k, such as $k > k_d$, the two-body collision damping factor

$$\gamma_{collision} \approx \log[(\kappa T/e^2)^3 n^{-1}] \cdot \sqrt{\left(\frac{e^2}{\kappa T}\right)^3 n} \cdot \omega_p \tag{98}$$

is expected to be essential. Comparing Eq. (98) with Eqs. (96) and (97), we find that the long-range correlation damping factor has the same order of magnitude of the two-body collision damping factor, while the Landau damping factor becomes abnormally large in the range of k of $(0.5 \sim 1) k_d$. As it has been discussed in the preceding paragraph, it seems to be essential to take account of the effects of the collision term in discussing the Landau damping. According to the present formulation of the theory of the plasma oscillations, in which the collision term is neglected, the Landau damping has no place to come into the theory. Thorough investigation is necessary to clarify the various mechanism of damping phenomena.

§ 8. Summaries

In the present paper, we have obtained the following results:

1) It has been explicitly shown that the effects of binary correlations are composed of the terms giving rise to corrections of the self-consistent field and the collision terms. Therefore, it is not rigorous to treat the correlation effects as the collision terms as a whole.

2) Kadomtsev's calculation of the internal field of the plasma committed errors. For instance, it does not give a correct result for the limiting case of the electron plasma in the uniform background of the positive charge distribution. The correct expression of the internal field for the electron plasma has been given in § 4.

3) The collision terms are examined in detail by reducing the kinetic equation to the Fokker-Planck equation. The friction and diffusion coefficients take account of the non-Markov effects which have been discussed by Tchen. Furthermore, comparing our results with the results derived form Balescu's equation, it is concluded that Balescu's equations neither involve any information of the plasma oscillations, nor take account of the effects of non-Markovian behavior of the plasma.

4) The generalized Fokker-Planck equation (71) is derived by taking account of the effects of the spatial inhomogeneity. The friction and diffusion coefficients, Eqs. (73) and (74), take account of the inhomogeneous spatial correlation effects.

5) The equation of motion of the plasma oscillations has been derived from the generalized kinetic equation. It has been found that the correction term to the self-consistent field gives rise to the shift in the k^2-term of the plasma frequency and to the damping factor of the plasma oscillations. It appears to be quite general conclusions that any kinds of the correlation effects do not modify the Langmiur frequency ω_p at the limit of $k \to 0$.

6) The contribution of the correlation damping factor has been examined in comparison with that of the Landau damping factor γ_L, although the latter type of damping has not been derived within the framework of the present formulation of the problem. It is argued that it is very necessary to take into account the collision term in order to clarify various mechanisms of the damping phenomena.

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Appendix

Here we will evaluate the function $\Phi(\boldsymbol{k})$,

$$\Phi(\boldsymbol{k}) = \frac{1}{2\pi^2} \int d\boldsymbol{l} \frac{1}{l^2 + k_d^2} \frac{\boldsymbol{k} \cdot (\boldsymbol{k} - \boldsymbol{l})}{|\boldsymbol{k} - \boldsymbol{l}|^2} . \tag{30}$$

By integrating over the angular variables, we get

$$\Phi(k) = \frac{2}{\pi} \int_0^\infty dl \frac{l^2}{l^2 + k_d^2} \left\{ 1 + \frac{k^2 - l^2}{4kl} \log \left| \frac{k^2 + l^2 + 2kl}{k^2 + l^2 - 2kl} \right| \right\} . \tag{A·1}$$

It is easy to observe that this converges at the upper limit of l. To evaluate Eq. (A·1) we may use the following approximate expression,

$$\log \left| \frac{k^2 + l^2 + 2kl}{k^2 + l^2 - 2kl} \right| \doteqdot \frac{4kl}{k^2 + l^2} \tag{A·2}$$

which is valid for both extremely large and small values of k. Then, $\Phi(k)$ is simply evaluated to be

$$\Phi(k) = \frac{k^2}{k + k_d} . \tag{A·3}$$

References

1) H. Alfvén, *Cosmical Electrodynamics* (Clarendon Press, Oxford, 1950).
2) G. F. Chew, M. L. Goldberger and F. E. Low, Proc. Roy. Soc. **A 236** (1956), 112.
3) K. M. Watson, Phys. Rev. **102** (1956), 12.
4) K. A. Brueckner and K. M. Watson, Phys. Rev. **102** (1956), 19.
5) S. T. Belyaev, *Plasma Physics and Problem of Controlled Thermonuclear Reactions* (Pergamon Press, 1959), Vol. **3**, 56.

6) L. I. Rudakov and R. Z. Sagdeev, *Plasma Physics and Problem of Controlled Thermonuclear Reactions* (Pergamon Press, 1959), Vol. **3**, 321.
7) A. Vlasov, J. Phys. U. S. S. R. **9** (1945), 25.
8) L. Landau, J. Phys. U. S. S. R. **10** (1946), 25.
9) D. Bohm and E. P. Gross, Phys. Rev. **75** (1949), 1851.
D. Bohm and D. Pines, Phys. Rev. **85** (1952), 338.
10) N. N. Bogoliubov, *Problems of Dynamical Theory in Statistical Physics* (Gostekhizdat, Moscow 1946). (Translation by E. K. Gora, Providence College, 1959).
11) V. V. Tolmachev, Doklady. Akad. Nauk S. S. S. R. **113** (1957), 301 (translation: Soviet Phys. Doklady **2** (1957), 85).
12) S. V. Temko, J. Exp. Theoret. Phys. U. S. S. R. **31** (1956), 1021 (translation: Soviet Phys. JETP **4** (1957), 898).
13) C. M. Tchen, Phys. Rev. **114** (1959), 394.
14) I. Prigogine and R. Balescu, Physica **25** (1959), 281, 302.
15) L. van Hove, Physica **21** (1955), 517.
16) N. M. Hugenholtz, Physica **23** (1957), 481.
17) R. Balescu, Physica **25** (1959), 324.
18) B. B. Kadomtsev, J. Exper. Theoret. Phys. U. S. S. R. **33** (1957), 151 (translation: Soviet Phys. JETP **6** (1958), 117).
19) Ju. L. Klimontovic, J. Exper. Theoret. Phys. U. S. S. R. **34** (1958), 173 (translation: Soviet Phys. JETP. **7** (1958), 11).
20) P. S. Zyrianov, J. Exper. Theoret. Phys. U. S. S. R. **29** (1955), 193 (translation: Soviet Phys. JETP **2** (1956), 247).
21) H. J. Merril and H. W. Webb, Phys. Rev. **55** (1939), 1191.
22) S. Kojima, K. Kato and S. Hagiwara, J. Phys. Soc. Japan **12** (1957), 1277.
23) D. Pines, Rev. Modern Phys. **28** (1956), 184.
24) H. Watanabe, J. Phys. Soc. Japan **11** (1956), 112.
25) F. Berz, Proc. Phys. Soc. **69B** (1956), 939.
26) P. A. M. Dirac, *The Principles of Quantum Mechanics* (Clarendon Press, Oxford, 1947), 3rd ed., 195.
27) N. G. van Kampen, Physica **21** (1955), 949.
28) A. Bohr, Dan. Mat. Fys. Medd. **24** (1948), No. 19.
29) Y. H. Ichikawa, Prog. Theor. Phys. **23** (1960), 512.
30) M. Baranger and B. Mozer, Phys. Rev. **115** (1959), 521.
31) O. Theimer and R. Gentry, Phys. Rev. **116** (1959), 787.
32) T. Morita, Prog. Theor. Phys. **23** (1960), 1211.
33) K. M. Watson, *Series of Lectures on Physics of Ionized Gases* (Los Alamos Scientific Laboratory of the University of California, 1956) T-701.
34) P. Nozieres and D. Pines, Phys. Rev. **111** (1958), 442.
35) H. Kanazawa, S. Misawa and E. Fujita, Prog. Theor. Phys. **23** (1960), 426.

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On Radiative Corrections due to Soft Photons

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It is shown that we can take the contribution from soft photons exactly into account in the calculation of the S-matrix in Q. E. D. The conditions are examined under which the contribution from soft photons to cross sections is factorized.

Since the discussion by Bloch and Nordsieck,[1] it has become a well-known fact that the number of soft photons has no clear meaning and is not an observable and that we can observe soft photons only collectively. Many authors have considered this problem, and especially Jauch and Rohrlich[2] have given a compact form of the contribution from soft photons. However, since they approximated the currents by classical ones from the beginning, they could not examine the validity of this approximation. Also they did not refer to a method for taking in the contribution from hard photons.*

In this note we shall show that we can take the contribution from soft photons exactly into account in the calculation of the S-matrix in Q. E. D., and shall examine the conditions under which the contribution from soft photons to the cross section of an arbitrary scattering problem is factorized.

Preliminaries

For an arbitrary division of interaction Hamiltonian

$$H(t)=H_h(t)+H_s(t),$$

S-matrix is given by

$$S=T\exp\left[-i\int_{-\infty}^{\infty}dt\,H(t)\right]$$

$$=\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int dt_1\cdots dt_n\,T[U(\infty,-\infty)H_h(t_1))\cdots H_h(t_n)],\quad (1\cdot1)^{**}$$

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* These problems have been discussed by Lomon[3] from a point of view somewhat different from ours.

** This expression of S-matrix represents transitions by H_h between eigenstates of H_0+H_s. The eigenstates are given by $U(0,-\infty)\Phi_n$,[4] where Φ_n's are eigenstates of H_0, free Hamiltonian. In our case, since the transformation operator $U(0,-\infty)$ is not unitary but contains infrared divergences (see (2·11)), Hilbert space expanded by Φ_n is unitarily inequivalent to the one expanded by $U(0,-\infty)\Phi_n$ in which the number of soft photons has no meaning.

where

$$U(t,\ t_0)=T\exp\Big[-i\int_{t_0}^{t}d\tau\,H_s(\tau)\Big]. \tag{1·2}$$

In general, the following relation holds,[5]

$$T\exp\Big[-i\int_{t_0}^{t}d\tau\,H(\tau)\Big]=\prod_{n=0}^{\infty}\exp\Big[-i\int_{t_0}^{t}d\tau\,H^{(n)}(\tau)\Big], \tag{1·3}$$

where

$$H^{(0)}(t)=H(t),$$

$$H^{(n)}(t)=\sum_{m=1}^{\infty}(-i)^m\frac{m}{(m+1)!}$$

$$\times\Big[\cdots\Big[H^{(n-1)}(t),\ \int_{t_0}^{t}d\tau\,H^{(n-1)}(\tau)\Big],\ \int_{t_0}^{t}d\tau\,H^{(n-1)}(\tau)\Big],\ \cdots,\ \int_{t_0}^{t}d\tau\,H^{(n-1)}(\tau)\Big].$$

This relation can be proved as follows:

We shall define $H^{(1)}(t)$ by

$$T\exp\Big[-i\int_{t_0}^{t}d\tau\,H(\tau)\Big]=\exp\Big[-i\int_{t_0}^{t}d\tau\,H(\tau)\Big]T\exp\Big[-i\int_{t_0}^{t}d\tau\,H^{(1)}(\tau)\Big]. \tag{1·4}$$

In order to determine the unknown operator $H^{(1)}(t)$, we differentiate Eq. (1·4) by t, multiply it by $\exp[i\int_{t_0}^{t}d\tau H(\tau)]$ from the left, and get

$$\exp\Big[i\int_{t_0}^{t}d\tau\,H(\tau)\Big](-i)H(t)\,T\exp\Big[-i\int_{t}^{t_0}d\tau\,H(\tau)\Big]$$

$$=\Big\{\exp\Big[i\int_{t_0}^{t}d\tau\,H(\tau)\Big]\frac{d}{dt}\exp\Big[-i\int_{t_0}^{t}d\tau\,H(\tau)\Big]+(-i)H^{(1)}(t)\Big\}$$

$$\times T\exp\Big[-i\int_{t_0}^{t}d\tau\,H^{(1)}(\tau)\Big].$$

Rewriting the left-hand side of this equation by using (1·4), we get

$$H^{(1)}(t)=\exp\Big[i\int_{t_0}^{t}d\tau\,H(\tau)\Big]\Big(H(t)-i\frac{d}{dt}\Big)\exp\Big[-i\int_{t_0}^{t}d\tau\,H(\tau)\Big].$$

Finally, using the relations

$$e^{-A}Be^{A}=\sum_{m=0}^{\infty}\frac{1}{m!}[\cdots[B,\overbrace{A],A],\cdots,A}^{m}],$$

$$e^{-A}\frac{d}{dt}e^{A}=\sum_{m=0}^{\infty}\frac{1}{(m+1)!}[\cdots[\dot{A},\overbrace{A],A],\cdots,A}^{m}],$$

we get

$$H^{(1)}(t)=\sum_{m=1}^{\infty}(-i)^{m}\frac{m}{(m+1)!}\left[\cdots\left[H(t),\overbrace{\int_{t_0}^{t}d\tau\,H(\tau)\Big],\int_{t_0}^{t}d\tau\,H(\tau)\Big],\cdots,\int_{t_0}^{t}d\tau\,H(\tau)\Big]}^{m}\right]. \tag{1·5}$$

By repeated use of (1·4) and (1·5), the relation (1·3) is proved.

S-matrix in Q. E. D.

The interaction Hamiltonian of the system of photons and electrons is given by

$$H(t)=ie\int d\boldsymbol{x}\,d\boldsymbol{p}\,d\boldsymbol{q}\,d\boldsymbol{k}:\bar{\psi}(q,x)\gamma_{\mu}\psi(p,x):A^{\mu}(k,x), \tag{2·1}$$

where

$$\psi(p,x)=\frac{1}{(2\pi)^{3/2}}\sqrt{\frac{m}{E_p}}\sum_{r=1,2}(u_r(p)b_r(p)e^{ipx}+v_r(p)d_r^{*}(p)e^{-ipx}),$$

$$A^{\mu}(k,x)=A^{\mu+}(k,x)+A^{\mu-}(k,x),$$

$$A^{\mu+}(k,x)=\frac{1}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\omega}}a^{\mu}(k)e^{ikx}=(A^{\mu-}(k,x))^{*},$$

and $a^{\mu}(k)$, $b_r(p)$ and $d_r(p)$ are annihilation operators of photon, electron and positron, respectively. We shall devide the interaction Hamiltonian as follows:

$$H_s(t)=ie\int d\boldsymbol{x}\,d\boldsymbol{p}\,d\boldsymbol{q}\,d\boldsymbol{k}\,\theta(\Lambda-\omega):\bar{\psi}(q,x)\gamma_{\mu}\psi(p,x):A^{\mu}(k,x), \tag{2·2}$$

$$H_h(t)=H(t)-H_s(t),$$

where Λ is a constant satisfying

$$\Lambda\ll m. \tag{2·3}$$

Using the relation (1·3), we shall calculate $U(t,t_0)$ under the condition (2·3). Creation and annihilation operators must be treated carefully, because their commutation relations are singular. First, we shall integrate $H_s(t)$ by t. Since $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{k}$ contained in $H_s(t)$ satisfy the relations $\boldsymbol{q}=\boldsymbol{p}\pm\boldsymbol{k}$ and $|\boldsymbol{k}|<\Lambda$, we can use the following approximations,

$$E_q = E_{p\pm k} = E_p \pm \frac{\boldsymbol{pk}}{E_p},$$

$$E_q - E_p \mp \omega = \pm \frac{(pk)}{E_p},$$

and

$$E_q + E_p \mp \omega = 2E_p.$$

The time dependence of a part of $H_s(t)$ which causes an electron-pair creation or annihilation is $\sim e^{\pm 2iE_p t}$, and that of the other part of $H_s(t)$ is $\sim e^{\pm i(pk/E_p)}$. Therefore the magnitude of the former is smaller than the latter multiplied by Λ/E after the time integration is performed, hence the pair interaction can be neglected. Further, using the relations

$$\bar{u}_s(q)\gamma_\mu u_r(p) = \bar{v}_s(q)\gamma_\mu v_r(p) = -i\frac{p_\mu}{m}\delta_{rs} + O\left(\frac{\Lambda}{m}\right),$$

we get

$$(-i)\int_{t_0}^{t} d\tau\, H_s(\tau) = F - F^*, \tag{2·4}$$

where

$$F = -e\int d\boldsymbol{p}\, d\boldsymbol{k}\,\theta(\Lambda-\omega)\,\frac{p_\mu}{pk} \times \sum_r \left(b_r^*(p+k)\,b_r(p) - d_r^*(p+k)\,d_r(p)\right) A^{\mu+}(k)\left(e^{i\cdot pk/E_p\cdot t} - e^{i\cdot pk/E_p\cdot t_0}\right). \tag{2·5}$$

In a similar manner, we get

$$\begin{aligned}
&(-i)\int_{t_0}^{t} d\tau\left[H_s(\tau),\ \frac{(-i)}{2}\int_{t_0}^{\tau} d\tau'\, H_s(\tau')\right] \\
&= i\frac{e^2}{(2\pi)^3}\int d\boldsymbol{p}\, d\boldsymbol{k}\,\frac{\theta(\Lambda-\omega)}{2\omega}\,\frac{m^2}{E_p(pk)}\,N(p)\left\{(t-t_0) - \frac{E_p}{pk}\sin\left(\frac{pk}{E_p}(t-t_0)\right)\right\} \\
&+ i\frac{e^2}{(2\pi)^3}\int d\boldsymbol{p}\, d\boldsymbol{p}'\, d\boldsymbol{k}\,\frac{\theta(\Lambda-\omega)}{2\omega}\,\frac{(pp')}{(pk)\,(p'k)} \\
&\times \sum_{r,r'} : \left(b_r^*(p+k)\,b_r(p) - d_r^*(p+k)\,d_r(p)\right)\left(b_{r'}^*(p')\,b_{r'}(p'+k)\right. \\
&\left. - d_{r'}^*(p')\,d_{r'}(p'+k)\right) : \\
&\left[\sin\frac{pk}{E_p}(t-t_0) - \frac{pk}{E_p}\,\frac{1}{(p/E_p - p'/E_{p'},\, k)}\left\{\sin\left(\frac{p}{E_p} - \frac{p'}{E_{p'}},\, k\right)t\right.\right. \\
&\left.\left. - \sin\left(\frac{p}{E_p} - \frac{p'}{E_{p'}},\, k\right)t_0\right\}\right],
\end{aligned} \tag{2·6}$$

where $N(p)=\sum_r (b_r^*(p)b_r(p)+d_r^*(p)d_r(p))=N^+(p)+N^-(p)$. The first term is a part of self energy of an electron and the second is that of Coulomb energy. Since the order of the magnitudes of both terms is the same as that of Λ, we can ignore the contribution from (2·6) in comparison with that from (2·4). In the same way, all the other factors in (1·3) except the first can be approximated by 1. Therefore

$$U(t, t_0)=\exp\left[-i\int_{t_0}^{t} d\tau\, H_s(\tau)\right]=\exp[F-F^*]. \tag{2·7}$$

Since the same approximation leads us to the following commutation relations,

$$[F, F^*]=\frac{e^2}{(2\pi)^3}\int d\boldsymbol{p}\, d\boldsymbol{p}'\, d\boldsymbol{k}\frac{\theta(\Lambda-\omega)}{2\omega}\frac{(pp')}{(pk)(p'k)}$$
$$\times\sum_{r,r'}(b_r^*(p+k)b_r(p)-d_r^*(p+k)d_r(p))(b_{r'}^*(p')b_{r'}(p'+k)-d_{r'}^*(p')d_{r'}(p'+k))$$
$$\times(e^{i\cdot pk/E_p\cdot t}-e^{i\cdot pk/E_p\cdot t_0})(e^{-i\cdot p'k/E_{p'}\cdot t}-e^{-i\cdot p'k/E_{p'}\cdot t_0}), \tag{2·8}$$

and

$$[F,[F, F^*]]=[[F, F^*], F^*]=0,$$

$U(t, t_0)$ can be expressed by the normal product with respect to $A^\mu(k)$ as follows:

$$U(t, t_0)=\exp[F-F^*]=\exp[-(1/2)[F, F^*]]\exp[-F^*]\exp[F]. \tag{2·9}$$

It will be obvious that we can use the approximation

$$\sum_r (b_r^*(p+k)b_r(p)-d_r^*(p+k)d_r(p))$$
$$=\sum_r (b_r^*(p)b_r(p)-d_r^*(p)d_r(p))=N^+(p)-N^-(p) \tag{2·10}*$$

in (2·9), as far as the order of $U(t, t_0)$ and the operator of electron field is not exchanged. In special cases in which both or either of t, t_0 become $\pm\infty$, omitting oscillating terms, we get

$$U(\infty, -\infty)=1,$$
$$U(0, -\infty)=\exp\left[-\frac{1}{2}\frac{e^2}{(2\pi)^3}\int d\boldsymbol{p}\, d\boldsymbol{p}'\, d\boldsymbol{k}\frac{\theta(\Lambda-\omega)}{2\omega}\frac{(pp')}{(pk)(p'k)}\right.$$
$$\left.\times(N^+(p)-N^-(p))(N^+(p')-N^-(p'))\right]$$

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* The result expressed by (2·9) and (2·10) contains the fact that the overlapping of the infrared divergences caused by the overlapping of a real photon and another real or virtual photon is disentangled by symmetrizing over the order of their emission and absorption, which was proved by Suura-Yennie.[6] The most general proof of the cancellation of infrared divergences in perturbation theory was given by Nakanishi.[7]

$$\times\exp\left[e\int d\boldsymbol{p}\,d\boldsymbol{k}\,\theta(\Lambda-\omega)\frac{p_\mu}{pk}(N^+(p)-N^-(p))A^{\mu-}(k)\right]$$

$$\times\exp\left[-e\int d\boldsymbol{p}\,d\boldsymbol{k}\,\theta(\Lambda-\omega)\frac{p_\mu}{pk}(N^+(p)-N^-(p))A^{\mu+}(k)\right],$$

$$U(\infty,0)=U^*(0,-\infty), \tag{2·11}$$

where

$$A^\mu(k)=A^\mu(k,0).$$

From relations (1·1), (2·9) and (2·10), we can calculate the S-matrix, taking into account the contribution from soft photons exactly.

General Jauch-Rohrlich factor

The nth order term of S-matrix in perturbation series with respect to $H_h(t)$ is given by

$$S^{(n)}=(-i)^n\int dt_1\cdots dt_n\,\theta(t_1-t_2)\,\theta(t_2-t_3)\cdots\theta(t_{n-1}-t_n)$$

$$\times U(\infty,t_1)\,H_h(t_1)\,U(t_1,t_2)\,H_h(t_2)\cdots U(t_{n-1},t_n)\,H_h(t_n)\,U(t_n,-\infty). \tag{3·1}$$

We shall examine the t_i dependence of the integrand in (3·1). Inserting complete orthonormal sets into before and after $H_h(t_i)$, we see that the t_i dependence of $H_h(t_i)$ is decided by the difference ΔE between energies of states at before and after $H_h(t_i)$ due to emission or absorption of a hard photon:

$$H_h(t_i)\sim e^{i\Delta E t_i}.$$

t_i dependences of $U(t_{i-1},t_i)$ and $U(t_i,t_{i+1})$ are decided by soft photons contained in those states (see (2·9), (2·5) and (2·8)):

$$U(t_{i-1},t_i),\ U(t_i,t_{i+1})\sim\sum e^{i\Sigma(pk/E)t_i}.$$

If the inequality

$$|\Delta E_i|\geqslant|\textstyle\sum pk/E|\sim nO(\Lambda),\quad n:\text{ integer} \tag{3·2}$$

holds, we can use the following approximation in the integrand in (3·1):

$$U(t_{i-1},t_i)\,H_h(t_i)\,U(t_i,t_{i+1})=U(t_{i-1},0)\,H_h(t_i)\,U(0,t_{i+1}). \tag{3·3}$$

When all the momenta of electrons and hard photons which transit the vertex $H_h(t_i)$ are determined by the momenta of electrons and hard photons in initial and final states, the condition (3·2) is satisfied for small enough Λ, as far as we are concerned with a finite angle scattering. Also when those contain integral variables, (3·2) is satisfied in the greater part of the integration region (the integration does not any more cause infrared divergence). Therefore the condition (3·2) is satisfied and the approximation (3·3) can be used for all t_i's. Since $U(0,0)=1$, (3·1) can be approximated by

$$S^{(n)}=U(\infty, 0)\frac{(-i)^n}{n!}\int dt_1\cdots dt_n T[H_h(t_1)\cdots H_h(t_n)]U(0, -\infty). \quad (3\cdot4)$$

This result means that we can neglect the processes of electron internal lines emitting and absorbing soft photons. Since (3·4) holds for all n, we get

$$S=U(\infty, 0)S_h U(0, -\infty), \quad (3\cdot5)$$

where

$$S_h=T\exp\left[-i\int_{-\infty}^{\infty} dt\, H_h(t)\right].$$

Using (3·5), we shall seek for physically meaningful cross sections. From (3·5) and (2·11), we get

$$\begin{aligned}&\langle k_1', \cdots k_n', k_{n+1}', \cdots k_{l'}'; p_1', \cdots p_{n_1'}'; q_1', \cdots q_{n_2'}'|S|k_1, \cdots k_l; p_1, \cdots p_{n_1}; q_1, \cdots q_{n_2}\rangle\\ &=\langle k_1', \cdots k_n'|U(\infty, 0; p'; q')U(0, -\infty; p; q)|0\rangle\\ &\times\langle k_{n+1}', \cdots k_{l'}'; p_1', \cdots p_{n_1'}'; q_1', \cdots q_{n_2'}'|S_h|k_1, \cdots k_l; p_1, \cdots p_{n_1}; q_1, \cdots q_{n_2}\rangle, \quad (3\cdot6)\end{aligned}$$

where

$$\begin{aligned}&|k_1, \cdots k_l; p_1, \cdots p_{n_1}; q_1, \cdots q_{n_2}\rangle\\ &\quad=a^*(k_1)\cdots a^*(k_l)b^*(p_1)\cdots b^*(p_{n_1})d^*(q_1)\cdots d^*(q_{n_2})|0\rangle,\\ &|\boldsymbol{k}_j|>\Lambda,\\ &|\boldsymbol{k}_j'|\begin{cases}<\Lambda & j=1, \cdots n,\\ >\Lambda & j=n+1, \cdots l',\end{cases}\\ &U(0, -\infty)|p_1, \cdots p_{n_1}; q_1, \cdots q_{n_2}\rangle=U(0, -\infty; p; q)|p_1\cdots p_{n_1}; q_1, \cdots q_{n_2}\rangle,\end{aligned}$$

and $U(0, -\infty; p, q)$ does not contain other operators than those of soft photons. Therefore, $\Delta E(\leq\Lambda)$ being energy resolution of the final state of the scattering, the cross section is generally given by

$$d\sigma=b(\Delta E)d\sigma_0, \quad (3\cdot7)$$

where $d\sigma_0$ is the cross section determined by S_h and may contain radiative corrections due to virtual hard photons up to any order. The general Jauch-Rohrlich factor $b(\Delta E)$ is given by

$$\begin{aligned}b(\Delta E)=&\sum_{n=0}^{\infty}\frac{1}{n!}d\boldsymbol{k}_1\cdots d\boldsymbol{k}_n\theta(\Delta E-\sum\omega_i)\\ &\times\sum_{\text{polarization}}|\langle k_1, \cdots k_n|U(\infty, 0; p'; q')U(0, -\infty; p; q)|0\rangle|^2. \quad (3\cdot8)\end{aligned}$$

Substituting in (3·8) the following relation derived from (2·11):

$$\langle k_1,\cdots k_n|U(\infty,0;p';q')U(0,-\infty;p;q)|0\rangle$$
$$=\prod_{l=1}^{n}\frac{e}{(2\pi)^{3/2}}\frac{\theta(\Lambda-\omega_l)}{\sqrt{2\omega_l}}(j(k)\epsilon_l)\cdot\exp\left[-\frac{1}{2}\frac{e^2}{(2\pi)^3}\int dk\frac{\theta(\Lambda-\omega)}{2\omega}j_\mu(k)j^\mu(k)\right],$$

where

$$j^\mu(k)=\sum\left(\frac{p^\mu}{pk}-\frac{q^\mu}{qk}-\frac{p'^\mu}{p'k}+\frac{q'^\mu}{q'k}\right),$$

and ϵ_l is the polarization vector of photon k_l, we get[2),8)]

$$b(\Delta E)=\frac{1}{\pi}\int_{-\infty}^{\infty}dx\frac{\sin x}{x}\exp\left[\alpha C\int_0^{(\Lambda/\Delta E)x}\frac{dy}{y}(e^{iy}-1)\right]$$
$$=\frac{2}{\pi}\left(\frac{\Delta E}{\Lambda\gamma}\right)^{\alpha C}\int_0^{\infty}dx\ \frac{\sin x}{x^{1+\alpha C}}\cos\left[\alpha C\,Si\left(\frac{\Lambda}{\Delta E}x\right)\right]\exp\left[\alpha C\,Ci\left(\frac{\Lambda}{\Delta E}x\right)\right]$$
$$=1-\alpha C\log\frac{\Lambda}{\Delta E}+\cdots. \qquad (3\cdot9)$$

Here, $\log\gamma$ is Euler's constant, $Ci(x)=-\int_x^{\infty}dt\cos t/t$, $Si(x)=\int_0^x dt\ \sin t/t$, α is the fine structure constant and C is defined by the following equation:

$$\frac{1}{(2\pi)^2}\int dk\frac{\theta(\Lambda-\omega)}{2\omega}j^2(k)=C\int d\omega\frac{\theta(\Lambda-\omega)}{\omega}.$$

C can be easily calculated by using the following relations:

$$\frac{1}{(2\pi)^2}\int\frac{dk}{\omega}\frac{p^2}{(pk)^2}=-\frac{1}{\pi}\int\frac{d\omega}{\omega},$$
$$\frac{1}{(2\pi)^2}\int\frac{dk}{\omega}\frac{(pq)}{(pk)(qk)}=-\frac{1}{\pi}2u\coth 2u\int\frac{d\omega}{\omega},$$

where

$$(p-q)^2=4m^2\sinh^2 u.$$

For some processes of actual interest, values of C are given as follows:

1) Electron (positron) scattering in an external field, Compton scattering $(p\to p')$ and electron-positron annihilation $(p+p'\to\gamma'\mathrm{s})$:

$$C=-\frac{2}{\pi}(1-2u\coth 2u),$$

where

$$4m^2\sinh^2 u=(p-p')^2.$$

2) Electron-electron (positron-positron) scattering $(p_1+p_2\to p_1'+p_2')$;

$$C=-(4/\pi)(1+2u\coth 2u-2v\coth 2v-2w\coth 2w),$$

where

$$4m^2 \sinh^2 u = (p_1 - p_2)^2 = (p_1' - p_2')^2,$$
$$4m^2 \sinh^2 v = (p_1 - p_1')^2 = (p_2 - p_2')^2$$
$$4m^2 \sinh^2 w = (p_1 - p_2')^2 = (p_2 - p_1')^2.$$

Radiative correction due to soft photons is obtained in compact form in (3·7) and (3·9). Unfortunately, $b(\Delta E)$ given by (3·9) contains unphysical quantity Λ, hence cannot be compared with experimental results. At present, we can only point out that this Λ dependence is to be cancelled by that of $d\sigma_0$ which is determined by radiative corrections due to virtual hard photons.

References

1) F. Bloch and A. Nordsieck, Phys. Rev. **52** (1937), 54.
2) J. M. Jauch and F. Rohrlich, Helv. Phys. Acta **27** (1954), 613; *Theory of Photons and Electrons* (Addison-Wesley Press, Cambridge, 1955), pp. 390-405.
3) E. L. Lomon, Nuclear Phys. **1** (1956), 101.
4) M. Gell-Mann and F. Low, Phys. Rev. **84** (1951), 350.
5) As to operator calculus the following papers are to be referred: R. P. Feynman, Phys. Rev. **84** (1951), 108, I. Fujiwara, Prog. Theor. Phys. **7** (1952), 433, and K. Yamazaki, Prog. Theor. Phys. **7** (1952), 449.
6) D. R. Yennie and H. Suura, Phys. Rev. **105** (1957), 1378.
7) N. Nakanishi, Prog. Theor. Phys. **19** (1958), 159.
8) E. L. Lomon, Phys. Rev. **113** (1959), 726.

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Applications of Mellin Transforms to Some Problems of Statistical Mechanics

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Use of Mellin transforms is made to get an asymptotic form of an integral containing a power series, where the term by term integration cannot be allowed. Examples are given concerning problems of statistical mechanics, for example, the partition function of an electron gas,[1] the correlation energy of a free electron gas,[2] the partition function of a hard sphere gas at zero-temperature limit.[4]

§ 1. The partition function of an electron gas

To get the partition function of an electron gas, Abe[1] has calculated

$$S_2=2\pi\rho^2\int_0^\infty\left[e^{-\beta q(r)}-1+\beta q(r)-\frac{1}{2}\beta^2q^2(r)\right]r^2dr,$$

$$q(r)=\epsilon^2e^{-\kappa r}/r$$

and got

$$S_2/\rho=\lambda^2\left(\frac{\log 3}{12}+\frac{\gamma}{6}-\frac{11}{72}\right)+\frac{\lambda^2\log\lambda}{12}+O(\lambda^3),$$

$$\lambda=\kappa/\beta\epsilon^2.$$

His calculation is not so easy.

We modify the formula of Mellin transform

$$e^{-x}=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma(s)x^{-s}ds,$$

$$\sigma=\text{real part of } s>0$$

to get

$$e^{-x}-1+x-\frac{x^2}{2}=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma(s)x^{-s}ds,\quad -3<\sigma<-2.$$

Replacing x by $\beta\epsilon^2e^{-\kappa r}/r$, we have

$$S_2/2\pi\rho^2=\int_0^\infty r^2dr\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma(s)[\beta\epsilon^2e^{-\kappa r}/r]^{-s}ds,\quad -3<\sigma<-2,$$

$$= (\beta\epsilon^2)^3 \int_0^\infty t^2\,dt\, \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \Gamma(s)\, e^{\lambda s t} t^s\, ds$$

where $t=r/\beta\epsilon^2$, $\kappa\beta\epsilon^2=\lambda$. We denote the right member deprived of its coefficient $(\beta\epsilon^2)^3$ by J and interchange the order of integration. By virtue of the restriction $-3<\sigma<-2$, the integration with respect to t converges at $t=0$ as well as at $t=\infty$. So we have

$$\int_0^\infty e^{\lambda s t} t^{s+2}\, dt = \Gamma(s+3)\,(-\lambda s)^{-3-s},$$

hence we have

$$J = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \Gamma(s)\,\Gamma(s+3)\,\lambda^{-3-s}(-s)^{-3-s}\, ds, \quad -3<\sigma<-2.$$

The integrand of J has poles of second order at $s=-3, -4, \cdots$. Translating the path of integration to the left, we get by the calculus of residue the asymptotic expansion

$$J = \sum_{n=3}^{\infty} \frac{n^{n-3}}{\Gamma(n+1)\,\Gamma(n-2)} \left[\log n + \frac{n-3}{n} - \frac{\Gamma'(n+1)}{\Gamma(n+1)} - \frac{\Gamma'(n-2)}{\Gamma(n-2)} + \log\lambda\right]\lambda^n$$

of which the first term

$$\frac{1}{6}\left[\log 3 + 2\gamma - \frac{11}{6} + \log\lambda\right]\lambda^3$$

gives the result of Abe.

§ 2. The correlation energy of an electron gas

The required integral[2] is

$$J = \int_{-\infty}^{\infty} du \int_0^1 \frac{dq}{q} \sum_{n=2}^{\infty} \frac{(-1)^n}{n} [R(u)]^n \left(\frac{4\alpha r}{\pi q^2}\right)^{n-2},$$

$$R(u) = 1 - u \arctan u^{-1}.$$

We modify the formula of Mellin transform

$$\log(1+x) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\pi}{s \sin \pi s} x^s\, ds, \quad 0<\sigma<1$$

to get

$$\log(1+x) - x = -\sum_{n=2}^{n} \frac{(-)^n x^n}{n} = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\pi}{s \sin \pi s}\, x^s\, ds, \quad 1<\sigma<2.$$

This formula enables one to write

$$\sum_{n=2}^{\infty}\frac{(-)^n}{n}[R(u)]^n\left(\frac{4\alpha r}{\pi q^2}\right)^{n-2}=\frac{-1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\pi}{s\sin\pi s}[R(u)]^s\left(\frac{4\alpha r}{\pi q^2}\right)^{s-2}ds,\quad 1<\sigma<2$$

and to integrate J with respect to q at first, by virtue of the restriction $\sigma<2$, thereby getting

$$\int_0^1\frac{1}{q}\left(\frac{1}{q^2}\right)^{s-2}dq=\int_0^1 q^{3-2s}dq=\frac{1}{4-2s}.$$

If we put

$$\int_0^{\infty}[R(u)]^s du=\varphi(s),$$

$\varphi(s)$ is regular for $\sigma>\frac{1}{2}$ and has a pole of first order at $s=\frac{1}{2}$, with the residue $1/2\sqrt{3}$. We have then

$$J=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\pi}{\sin\pi s}\frac{\varphi(s)}{s(s-2)}\left(\frac{4\alpha r}{\pi}\right)^{s-2}ds,\quad 1<\sigma<2.$$

The expansion for small r is given by translating the path of integration to the right

$$J=-\frac{1}{2}\left[\varphi(2)\log\frac{4\alpha r}{\pi}-\frac{1}{2}\varphi(2)+\varphi'(2)\right]+\cdots$$

or

$$J=-\frac{\pi}{6}(1-\log 2)\left[\log\frac{4\alpha r}{\pi}-\frac{1}{2}+\frac{\varphi'(2)}{\varphi(2)}\right]+\cdots$$

where $\varphi(2)=\pi(1-\log 2)/3$ is used. This result is that of Gell-Mann and Brueckner.

The asymptotic expansion for large r is given by translating the path of integration to the left,

$$J=\frac{\pi^2}{16\alpha r}-\frac{2\pi}{3\sqrt{3}}\left(\frac{\pi}{4\alpha r}\right)^{3/2}+\cdots,$$

which is the first two terms of Nozières and Pines.[3]

§ 3. The partition function of a hard sphere gas

Lee and Yang[4] have evaluated the integral for large D

$$\Delta=-\pi^{-1/2}\int_0^{\infty}q^{1/2}dq\psi$$

with

$$\psi=\sum_m\left[\log(1+D\lambda_m)-D\lambda_m+\frac{1}{2}D^2\lambda_m^2\right],$$

$$\lambda_m=\int_0^1 dt(\cos 2\pi mt)\exp[-qt(1-t)]$$

to get the pressure of hard sphere gas at zero-temperature.

If we denote the ψ with λ_m replaced by its approximate value Λ_m

$$\Lambda_m=\frac{2q}{q^2+(2\pi m)^2}$$

by ψ', and use the formula of Mellin transform

$$\log(1+x)-x+\frac{1}{2}x^2=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\pi}{s\sin\pi s}x^s\,ds,\quad 2<\sigma<3,$$

we have

$$\psi'=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\pi}{s\sin\pi s}D^s\sum_m\Lambda_m^s\,ds.$$

$\sum\Lambda_m^s$ converges for $\sigma>\frac{1}{2}$. If we integrate at first with respect to q, we see

$$\int_0^\infty q^{1/2}\Lambda_m^s\,dq=\int_0^\infty\frac{2^s q^{s+1/2}}{[q^2+(2\pi m)^2]^s}\,dq=2^{3/2}(\pi m)^{-s+3/2}\Gamma\left(\frac{s}{2}-\frac{3}{4}\right)\Gamma\left(\frac{s}{2}+\frac{3}{4}\right)/\Gamma(s).$$

This integral converges for $\sigma>3/2$. So, summing with respect to m, for $\sigma>5/2$, we have

$$\sum_m\int_0^\infty q^{1/2}\Lambda_m^s\,dq=2^{3/2}\pi^{3/2-s}\zeta\left(s-\frac{3}{2}\right)\Gamma\left(\frac{s}{2}-\frac{3}{4}\right)\Gamma\left(\frac{s}{2}+\frac{3}{4}\right)/\Gamma(s)$$

where $\zeta(s)$ is Riemann's Zeta function. Consequently, we have

$$\int_0^\infty\psi' q^{1/2}\,dq=-\frac{2^{3/2}\pi^{-1/2}}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\pi/2\cdot(s-1/2)}{\sin\pi(s/2-1/4)}\Gamma(-s)\zeta\left(\frac{5}{2}-s\right)\left(\frac{D}{2}\right)^s ds,$$

$$5/2<\sigma<3.$$

If we translate the path of integration to the position $2<\sigma<5/2$, there remains the residue at $s=5/2$, $4(2D)^{5/2}/15$, giving the result of Lee and Yang. The asymptotic expansion thereof becomes

$$\int_0^\infty\psi' q^{1/2}\,dq=\frac{4}{15}(2D)^{5/2}+\frac{3\pi^{1/2}}{8}\zeta\left(\frac{1}{2}\right)D^2+\frac{2\pi}{3}(2D)^{3/2}+\cdots.$$

The above examples may suffice to show the usefulness of Mellin transforms

to obtain an asymptotic expansion. The technique of Mellin transform has been used by Husimi[5] to evaluate the "partitio numerorum" who has emphasized its prime importance.

References

1) R. Abe, Prog. Theor. Phys. **22** (1959), 213.
2) M. Gell-Mann and K. A. Brueckner, Phys. Rev. **106** (1957), 364.
3) P. Nozières and D. Pines, Phys. Rev. **111** (1958), 442, formula (30).
4) T. D. Lee and C. N. Yang, Phys. Rev. **117** (1960), 19.
5) K. Husimi, Proc. Phys. Math. Soc. Japan, **20** (1938), 912.

Letters to the Editor

The opinions expressed in these columns do not necessarily reflect those of the Board of Editors. Communications should be submitted in duplicate and should be held to within 100 lines (pica tipe) on standard size letter paper (approx. 21×30 cm.), so that each letter may be arranged into two pages when printed. Do not forget to count in enough space for formulas, figures or tables.

On High Energy Charge Exchange Collision

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July 21, 1960

The polarization and asymmetry of the high energy neutrons ejected in the high energy proton-nucleus collisions have been measured by Donaldson et al.[1]

The transition matrix for the neutron ejected in high energy proton-nucleus collision or the charge exchange reaction, within the framework of the impulse approximation, is given by

$$T=\langle \Psi(T)\chi^{(-)}_{k_{fn}} | \sum_j t_{0j} | \Phi(T_0)\chi^{(+)}_{k_{ip}} \rangle, \quad (1)$$

where $\Phi(T_0)$ and $\Psi(T)$ are the wave functions of initial and final target nucleus with the isotopic spin T_0 and T, respectively, and t_{0j} is the scattering spin matrix defined in free two-body scattering,[2] and $\chi^{(+)}_{k_{ip}}$ and $\chi^{(-)}_{k_{fn}}$ are the wave functions of the incident proton and the ejected neutron, respectivery.

If one uses the isotopic spin operators τ_p and τ_n for the incident proton and the neutron in the target nucleus, one obtains t_{0j} as follows,

$$t_{0j}=\frac{1}{4}(3+\tau_p\tau_n)t_1+\frac{1}{4}(1-\tau_p\tau_n)t_0$$

$$=\frac{1}{4}(3t_1+t_0)+\frac{1}{4}(t_1-t_0)\tau_p\tau_n, \quad (2)$$

where t_1 and t_0 denote the scattering matrices of the charge triplet and singlet states for the n-p system. The first term of the second line in Eq. (2) relates to the ordinary scattering amplitude and the second relates to the charge exchange scattering amplitude. In $C^{12}(p, n)N^{12*}$ investigated here, the former vanishes by the selection rule for the isotopic spin of the target nucleus.

Since the energy of the incident proton is very high, the proton almost does not remain in the nucleus though the energy of ejected neutron has so high energy as to be equal to that of the incident proton. Then one can express $\Psi(T)$ by the product of the wave functions of the residual nucleus and of the recoiled proton distorted due to the interaction between the proton and the residual nucleus.

If the recoiled proton is not affected by the interaction due to the residual nucleus, one may put unity to the ratio

of the spin flip to the non-spin flip terms of the reduced nuclear matrix elements for the recoiled proton, then the polarization or asymmetry is given by

$$P(\theta) = \frac{2\,\mathrm{Re}(A+B)^{*}C}{|A|^2+|C|^2+|B|^2+|C|^2+|E|^2+|F|^2}, \quad (3)$$

where $A=(1/2)(A_1-A_0)$, $C=(1/2)(C_1-C_0)$, etc.; here A_1, A_0, etc., have the same meanings as in ref. 2). It is important to note that the coefficients A, C, etc., for the free p-n scattering matrix are $A=(1/2)(A_1+A_0)$, $C=(1/2)(C_1+C_0)$, etc., while the coefficients for the charge exchange collision, $C^{12}(p, n)N^{12*}$, are $A=(1/2)(A_1-A_0)$, $C=(1/2)(C_1-C_0)$, etc., though Eq. (3) has the same form with the polarization for the free p-n scattering.

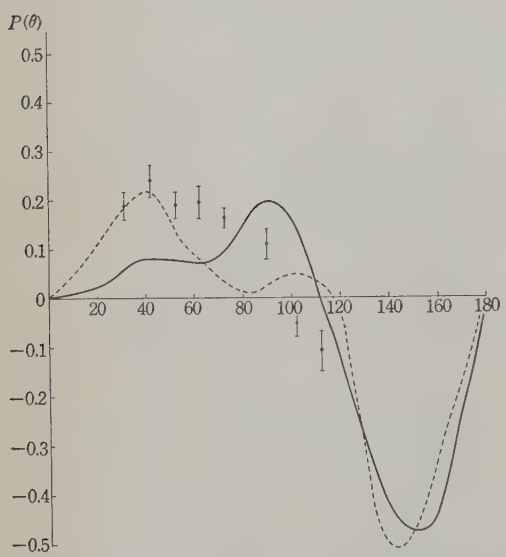

Fig. 1. The experimental polarization of the neutron is measured in coincidence with associated scattered proton. The solid curve shows the result calculated by using the Gammel-Thaler phase shifts for p-n system at 310 Mev and the dashed one shows the result calculated by the V-shaped angular distribution discussed by Gammel and Thaler.[7]

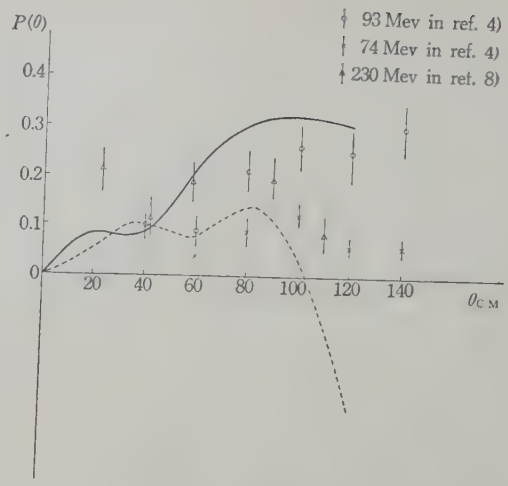

Fig. 2. The experimental polarizations of the neutrons are measured with the classified energy of the ejected neutron 73 Mev and 94 Mev.[4] The parameters stated in the text are set by 1, 0.8, 0.3, 0.1 for scattering angles $0°<\theta_{C.M.}<20°$, $20°<\theta_{C.M.}<50°$, $50°<\theta_{C.M.}<90°$, $90°<\theta_{C.M.}$, respectively in order to reproduce the data of Wilson et al.[4] by using the Gammel-Thaler phase shifts at 156 Mev. Also the parameter is set by 1 for $0°<\theta_{C.M.}<100°$ to reproduce the data of Roberts et al.[8] by using the Gammel-Thaler phase shifts at 230 Mev. The solid and dashed curves show the results for 156 Mev and 230 Mev, respectively.

The results calculated by using Eq. (3) are compared with the experimental data in Fig. 1. To calculate the coefficients of the scattering spin matrix, we have used the two-body scattering phase shifts derived from the Gammel-Thaler potential.[3] As shown in Fig. 1, the polarization of the neutron quasi-elastically scattered from C^{12} agrees with the experimental data measured by Donaldson et al.,[1] and the results are very sensitive to the sets of two-

body scattering phase shifts. The study of the charge exchange of the quasi-elastic scattering supplements the knowledge on the free p-n scattering.

By making use of the quasi-elastic scattering, one cannot explain the experimental data at 45° measured by Wilson et al.[4] of the 74 Mev ejected neutrons, though the energy of the neutron quasi-elastically scattered is about 73 Mev at 45° as bombarding energy is 145 Mev. The reason is that in their results the two nucleons interacting in the nucleus are not measured in coincidence with the associated scattered nucleons, their results would contain the contributions of some other inelastic scattering processes. The neutron ejected from p-C^{12} collision contains the neutron scattered from some levels corresponding to the excitations of the target nucleus. In order to explain the polarization including the nucleon, one may use as a free parameter the ratio of the spin flip to the non-spin flip terms of the reduced nuclear matrix elements for the recoiled proton, adopted by Thaler et al.[5] When the nucleons are scattered from several levels, the parameter depends on the scattering angles. Then, if one may use for the parameters a large value, say, 1, and a small one, say, 0.1, in the small angles and the large ones, respectively, one may explain the polarization or asymmetry measured by Wilson et al.[4] Introducing the parameter, one need not take account of the effect of the multiple scattering in the light nucleus for the ejected nucleons, which were considered by Squires.[6]

The authors would like to express their appreciation of the discussions with Dr. S. Hatano. They also wish to thank Prof. M. Kobayasi for his encouragement.

1) R. E. Donaldson and H. Bradner, Phys. Rev. **99** (1955), 892.
2) L. Wolfenstein, *Annual Review of Nuclear Science* Vol. 6 (1956), p. 43.
3) J. L. Gammel and R. M. Thaler, Phys. Rev. **107** (1957), 291, 1337.
4) S. G. Carpenter and R. Wilson, Phys. Rev. **113** (1959), 650.
5) A K. Kerman, H. McManus and R. M. Thaler, *Annals of Physics* 8 (1959), 551.
6) E. J. Squires, Proc. Phys. Soc. **A 72** (1958), 433.
7) J. L. Gammel and R. M. Thaler, *Progress in Elementary Particle and Cosmic Ray Physics* Vol. V (1960), p. 97.
8) A. Roberts, J. Tinlot and E. M. Hafner, Phys. Rev. **95** (1954), 1099.

A Note on Exchange Reaction

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August 1, 1960

Austern, Butler and McManus[1] (A-B-M) put forward the theory on exchange reactions as (n, p) and (p, n). They considered their theory is applicable to (p, p') as well. According to their theory, the scattering amplitude is expressed as a product of the square

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root of the sticking factor times the dynamical factor. The dynamical factor is represented by the two-body scattering amplitude. The sticking factor is estimated outside the nuclear surface. The dynamical factor is insensitive to the angular variation. Then the angular distribution is determined mainly by the sticking factor expressed as $j_{dl}(\Delta \boldsymbol{k}\boldsymbol{R})$.

The aim of this note is to point out that to the inelastic scattering their theory is applicable but to the exchange reaction their theory may not be applied. At first sight, the antisymmetrization may seem to remove this difficulty. However, even if we took into account the antisymmetrization, the difficulty would remain.

The experimental results on (p, n) is very scanty compared with (p, p') and it is limited to light nuclei.[2] However, the existing data seem to show that the angular distribution of (p, n) is not so well reproduced by A-B-M, whereas in (p, p') fits are good by A-B-M in most of the cases.

Throughout this note, we name the incident particle o, a specific particle in the target nucleus p, the residual particles as a whole q. To aid understanding, first we describe the cases of non-exchange; the elastic scattering and the inelastic scattering. As for the applicability of the impulse aproximation to the elastic scattering, we have to take into account the correlation between target particles.[3] The treatment in this note of the elastic scattering by a simple impulse approximation is written only for the sake of understanding of the exchange reaction.

[I] The elastic scattering

In the impulse approximation, the matrix element involving the particle o and p is given by

$$\mathfrak{H}_{fi}=\sum_{mn}(\Phi_f, \phi_m)(\phi_m V_{op}\psi_n)(\phi_n, \Phi_i), \tag{I·1}$$

where

$$\begin{aligned}
\phi_m &= (1/2\pi)^{3/2\times 3} \\
&\quad \times \exp[i(\boldsymbol{k}_o\boldsymbol{r}_o+\boldsymbol{k}_p\boldsymbol{r}_p+\boldsymbol{k}_q\boldsymbol{r}_q)], \\
\phi_n &= (1/2\pi)^{3/2\times 3} \\
&\quad \times \exp[i(\boldsymbol{k}_o^{0}\boldsymbol{r}_o+\boldsymbol{k}_p^{0}\boldsymbol{r}_p+\boldsymbol{k}_q^{0}\boldsymbol{r}_q)], \qquad (I·2) \\
\psi_n &= (1/2\pi)^{3/2\times 3}\exp[i\boldsymbol{k}_q^{0}\boldsymbol{r}_q] \\
&\quad \times \exp[i(\boldsymbol{k}_o^{0}+\boldsymbol{k}_p^{0})(\boldsymbol{r}_o+\boldsymbol{r}_p)/2] \\
&\quad \times \chi_{(1/2)(k_o^0-k_p^0)}(\boldsymbol{r}_o-\boldsymbol{r}_p), \\
\Phi_{i,f} &= (1/2\pi)^{3/2\times 2} \\
&\quad \times \exp\left[i\boldsymbol{k}_{i,f}\left(\boldsymbol{r}_o-\frac{\boldsymbol{r}_p+(A-1)\boldsymbol{r}_q}{A}\right)\right] \\
&\quad \times \psi_\alpha(\boldsymbol{r}_p-\boldsymbol{r}_q)\Psi(\xi). \qquad (I·3)
\end{aligned}$$

Here $\chi_{(1/2)(k_o^0-k_p^0)}(\boldsymbol{r}_o-\boldsymbol{r}_p)$ is the wave function describing the two-body scattering. $\psi_\alpha(\boldsymbol{r}_p-\boldsymbol{r}_q)\Psi(\xi)$ is the wave function of the target nucleus. ξ represents all the coordinates of the target except the center of mass of the target and the coordinate $\boldsymbol{r}_p-\boldsymbol{r}_q$. According to the Appendix of the previous paper,[3] (I·1) is expressed as

$$\begin{aligned}
\mathfrak{H}_{fi} &= (1/2\pi)^{3/2\times 3}\int d\boldsymbol{k}_q \\
&\times \int \exp\left[-i\left(\frac{(A-1)(-\boldsymbol{k}_i-\boldsymbol{k}_q)-\boldsymbol{k}_q}{A}\right)\boldsymbol{r}\right] \\
&\times \psi_\alpha(\boldsymbol{r})\,d\boldsymbol{r}\int \exp[-i(\boldsymbol{k}_f+\boldsymbol{k}_q/2)\boldsymbol{r}_{op}] \\
&\times V_{op}\chi_{k_i+k_q/2}(\boldsymbol{r}_{op})\,d\boldsymbol{r}_{op} \\
&\times \int \exp\left[i\left(\frac{-(A-1)(\boldsymbol{k}_q+\boldsymbol{k}_f)-\boldsymbol{k}_q}{A}\right)\boldsymbol{r}'\right] \\
&\times \psi_\alpha^{*}(\boldsymbol{r}')\,d\boldsymbol{r}', \qquad (I·4)
\end{aligned}$$

where $\boldsymbol{r}=\boldsymbol{r}_p-\boldsymbol{r}_q$.

The dynamical factor in (I·4) is approximated by

$$\int \exp[-i(\boldsymbol{k}_f+\boldsymbol{k}_\eta/2)\boldsymbol{r}_{op}] \times V_{op}\chi_{(k_i+k_q/2)}(\boldsymbol{r}_{op})\,d\boldsymbol{r}_{op} \simeq \int \exp[-i\boldsymbol{k}_f\boldsymbol{r}_{op}]\,V_{op}\chi_{k_i}(\boldsymbol{r}_{op})\,d\boldsymbol{r}_{op}. \tag{I·5}$$

Then (I·4) becomes

$$\mathfrak{H}_{fi}=(1/2\pi)^{3/2}\int \exp[-i\Delta\boldsymbol{k}\boldsymbol{r}]\,|\phi_a(\boldsymbol{r})|^2\,d\boldsymbol{r} \times \int \exp[-i\boldsymbol{k}_f\boldsymbol{r}_{op}]\,V_{op}\chi_{k_i}(\boldsymbol{r}_{op})\,d\boldsymbol{r}_{op}, \tag{I·6}$$

where $\Delta\boldsymbol{k}=\boldsymbol{k}_f-\boldsymbol{k}_i$. In (I·6), the assumption is $1/A\ll 1$.

[II] The inelastic scattering

In this case, we have

$$\mathfrak{H}_{fi}=(1/2\pi)^{3/2}\int \exp[-i\Delta\boldsymbol{k}\boldsymbol{r}] \times \phi_a(\boldsymbol{r})\,\phi_b^*(\boldsymbol{r})\,d\boldsymbol{r} \times \int \exp[-i\boldsymbol{k}_f\boldsymbol{r}_{op}]\,V_{op}\chi_{k_i}(\boldsymbol{r}_{op})\,d\boldsymbol{r}_{op}. \tag{II·1}$$

This leads to A-B-M.

[III] The exchange reaction

In this case, (I·3) is replaced by

$$\Phi_i=(1/2\pi)^{3/2\times 2} \times \exp\left[i\boldsymbol{k}_i\left(\boldsymbol{r}_o-\frac{\boldsymbol{r}_p+(A-1)\boldsymbol{r}_q}{A}\right)\right] \times \phi_a(\boldsymbol{r}_p-\boldsymbol{r}_q)\,\Psi(\xi),$$

$$\Phi_f=(1/2\pi)^{3/2\times 2} \times \exp\left[i\boldsymbol{k}_f\left(\boldsymbol{r}_p-\frac{\boldsymbol{r}_o+(A-1)\boldsymbol{r}_q}{A}\right)\right] \times \phi_b(\boldsymbol{r}_o-\boldsymbol{r}_q)\,\Psi(\xi).$$

Corresponding to (I·4), we obtain

$$\mathfrak{H}_{fi}=(1/2\pi)^{3/2\times 3}\int d\boldsymbol{k}_i \times \int \exp\left[-i\left(\frac{-(A-1)(\boldsymbol{k}_q+\boldsymbol{k}_i)-\boldsymbol{k}_q}{A}\right)\boldsymbol{r}_{pq}\right] \times \phi_a(\boldsymbol{r}_{pq})\,d\boldsymbol{r}_{pq} \times \int \exp[i(\boldsymbol{k}_f+\boldsymbol{k}_q/2)\boldsymbol{r}_{op}] \times V_{op}\chi_{(k_i+k_q/2)}(\boldsymbol{r}_{op})\,d\boldsymbol{r}_{op} \times \int \exp\left[i\left(\frac{-(A-1)(\boldsymbol{k}_f+\boldsymbol{k}_q)-\boldsymbol{k}_q}{A}\right)\boldsymbol{r}_{oq}\right] \times \phi_b(\boldsymbol{r}_{oq})\,d\boldsymbol{r}_{oq}. \tag{III·1}$$

The comparison of the dynamical factor in (III.1) with that of (I·4) is very interesting. The sign of the exponent is reversed. Physically this meaning is as follows.

The dynamical factor of (III.1) is expressed alternatively as

$$\int \exp[-i(\boldsymbol{k}_f+\boldsymbol{k}_q/2)\boldsymbol{r}_{op}] \times P_x V_{op}\chi_{(k_i+k_q/2)}(\boldsymbol{r}_{op})\,d\boldsymbol{r}_{op},$$

where P_x is the position exchange operator of o and p. This means, in the case [III], after the interaction V_{op}, the position of the particles o and p is exchanged.

To the dynamical factor of (III.1), the approximation as (I·5) is no longer guaranteed. Then it is not surprising that in the exchange scattering, the angular distribution is not of A-B-M type.

1) Austern, Butler and McManus, Phys. Rev. **92** (1953), 350.
2) Y. Ishizaki and T. Saji, private communication on the experiment made at Institute for Nuclear Study, Tokyo University.
3) T. Sasakawa, Prog. Theor. Phys. **24** (1960), 241.

Mass Differences of Charged and Neutral States of Elementary Particles

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August 15, 1960

Although the mass values of more than a dozen elementary particles are now well established, a satisfactory theoretical deduction of these numbers is still lacking. During the past few years a number of attempts have been made to find empirical connections among the available data,[1] but most of these suggestions appear somewhat artificial. Moreover, none has succeeded in yielding mass numbers that are acceptably close to the present values.

In particular, as part of the general problem, there appears the question of the relatively small mass differences between the charged particles and their neutral counterparts, suggesting that they are in reality different charge states of the same particles. With the improved experimental mass values now available it was found possible to discover a strikingly simple numerical correlation which may be of significance.

The accompanying table lists the known pairs of charged and neutral particles. Each pair has been assigned an integer n. The pairs appear here in order of increasing mass, except that the positions of the kaons and nucleons are interchanged.

The second column gives the product of the experimental mass of the neutral member by the difference in mass between the charged and neutral components. In each case, the indicated departure is the probable error.

The next column lists the values of the function

$$R=(2326)2^n \quad (n=0, 1, 2, 3\cdots), \qquad (1)$$

and the final column of numbers represents the difference between each R and the absolute value of $M_0\Delta M$, expressed as a fraction of the probable error. The agreement is surprisingly good.

For some pairs, the neutral particle has a greater mass than the charged one, so that the negative value of ΔM must be used in comparing with Eq. (1). The mass differences between members of a pair are probably attributable to electromagnetic self-energy effects, but the reason for these reversals of sign are not apparent.

It is to be noted that a duality exists

Particle multiplet	n	$M_0\Delta M$, exper., in units of electron rest mass	$R=(2326)2^n$	Departure, fraction of P.E.	Source of data
Pions	0	2370 ± 108	2326	0.41	Ref. 2
Nucleons	1	$-(4652\pm160)$	4652	0	Ref. 2
Kaons	2	$-(9130\pm1200)$	9304	0.14	Ref. 3
Sigmas	3	Σ^-: $20{,}130\pm2780$ Σ^+: $-(10{,}900\pm2590)$	18,610	0.55 (3.0)	Ref. 4 Ref. 2
Xi's	4	40,200 (?)	37,220	—	Ref. 2

for the sigmas. Two different values of $M_0 \Delta M$ are obtained, the first one being much closer to the number given by Eq. (1). In the case of the Xi-hyperon pair, there is some doubt at present as to whether the difference between the experimental mass values is real.[2]

For what it may signify, it may be mentioned that the numerical factor 2326 in Eq. (1) is very close to the mass numbers of the Σ^0 and Σ^+ hyperons (2328 and 2333, respectively).

1) R. Fürth, Nature **162** (1948), 772; **166** (1950), 727.
G. Stephenson, Nature **166** (1950), 655; **165** (1950), 1018.
I. M. Freeman, Nature **165** (1950), 933.
Y. Nambu, Prog. Theor. Phys. **7** (1952), 595.
2) R. E. Marshak, Science **132** (1960), 269.
3) F. S. Crawford, *et al.*, Phys. Rev. Letters **2** (1959), 112.
4) L. Rosenfeld, Bull. Am. Phys. Soc. **3** (1958).

Effect of the Particle-Hole Pair Interactions in Nuclear Many-Boby Problem

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The dynamical aspect of nuclear collective motion has recently been considered.[1] These authors applied Sawada's method in the problem of electron plasma to nuclear many-body system. Sawada's method,[2] or the method of particle-hole pair, however, has much different effect, when applied to the nuclear many-body system. If we apply Sawada's method to infinite nuclear medium as it is, it is well known that we obtain an imaginary value as phonon energy and the collective mode may be dissipative. The above result may be caused by the fact that the nuclear size is infinite or the spectrum of single particle energy in unperturbed system is continuous. This occurs because of attractive nuclear interaction, being different from repulsive Coulomb interaction in electron system. This difficulty may be avoided by taking account of finite size for nuclear medium or discretness of single particle energy levels.[1]

Another difference, considered here, is the screening effect. The collection of bubble type diagram may give the effective interaction introduced by Hubbard.[3] (Fig. 1.) Fourier component

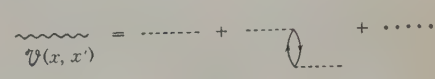

Fig. 1.

of effective interaction $v(x, x')$ is given as follows,

$$v(k, \lambda) = v(k)/[1 - v(k) W(k, \lambda)], \quad (1)$$

where, $v(k)$ is Fourier component of residual nuclear interaction and $W(k, \lambda)$ represents Fourier component of particle-hole propagator. Dispersion relation which determines the phonon energy for transferred momentum k is given as

$$1=(1/\Omega)\sum_{i}^{occ.}\sum_{j}^{unocc.}\frac{\mathcal{P}\cdot 2v(k)\,(E_i-E_j)}{[(E_i-E_j)^2-\omega^2(k)]}\,, \qquad (2)^*$$

where Ω is normalization volume, ω is phonon energy and E_i's are the energies of discrete single particle levels. $\mathcal{P}$ means Cauchy's principal part. If we put Min.$|E_i-E_j|=\Delta$, Eq. (1) may be rewritten in configuration space approximately as

$$\mathcal{V}(\rho)=\frac{a}{\rho}\cdot\int_0^\infty\frac{pv(p)}{\sqrt{1+(\Delta/\Omega)\,v(p)}}\times\sin p\rho\,dp, \qquad (3)$$

in terms of the dispersion relation, where $\rho=k_F r$ and $p=k/k_F$ respectively. (k_F is Fermi momentum.) a represents a positive definite constant. In the case of repulsive Coulomb interaction $v(p)=+\text{const.}/p^2$, so $pv(p)$ has a singularity of $1/p$ at $p=0$. But the denominator of Eq. (3), which represents the effect of particle-hole pair interactions, has also a singularity $1/p$ at $p=0$. So particle-hole pair interaction screens the singularity of Fourier component of Coulomb interaction at $p=0$.[4] Nuclear many-body system, however, is not the case. That is, $v(p)$may not have the singularity at low-momentum side. If the hard-core singularity has already been screened by Brueckner's method,[5] $v(p)$ with screened high-momentum side may have the form illustrated by Fig. 2. Then Eq. (3) shows us that the denominator, or the effect of the particle-hole pair interactions increases the ratio of the low-momentum component of the interaction $\mathcal{V}(p)$ to the high-component

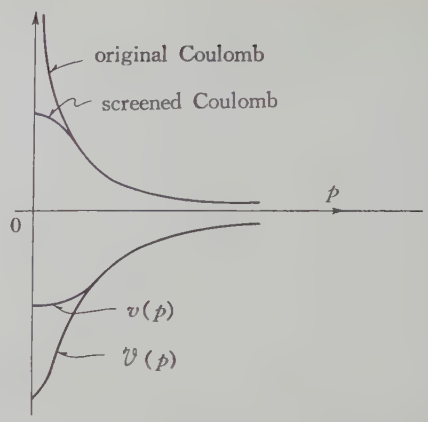

Fig. 2. Upper is the Coulomb case and the lower is the nuclear case. $\mathcal{V}(p)$ is normalized to $v(p)$ at high-momentum side, and the scale is arbitrary.

one in comparison with original interaction $v(p)$. Thus, we may conclude that in nuclear many-body system the effect of particle-hole pair produces the effective interaction in which the probability of low-momentum transfer is more favourable than that of the original one. In other words, the method of particle-hole pair transforms the short-range nuclear interaction to a long-range one in contrast with the screening effect of Coulomb interaction in many-electron case. A full note will be published soon in this issue. The author wishes to express sincere thanks to Professors S. Takagi and H. Tanaka for his valuable discussions.

1) S. Takagi, Prog. Theor. Phys. **21** (1959), 194.
K. Ikeda et al., Prog. Theor. Phys. **22** (1959), 663.
T. Marumori, Prog. Theor. Phys. **24** (1960), 331.
2) K. Sawada, Phys. Rev. **106** (1957), 372.
K. Sawada et al., Phys. Rev. **108** (1957), 507.
3) J. Hubbard, Proc. Roy. Soc. **240** (1957), 539.

* We abopt the natural unit $\hbar=c=1$.

A. Klein and R. Prange, Phys. Rev **112** (1958), 994.
4) H. Iwamoto, private communication.
5) K. A. Brueckner, in *The Many Body Problem*, (Dunod, Paris, 1959).

Possible Methods for Detecting Intergalactic Hydrogen

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From the cosmological point of view it is very important to know how much hydrogen exists in the intergalactic space. According to the steady state cosmology, for example, the intergalactic matter is created so that it will be of the simplest form, hydrogen, and its density should be related to the rates of creation and condensation into galaxies. Without going into such a fundamental theory, the density of hydrogen is known to be in close connection with the evolution of galaxies and other related problems.

Since hydrogen is tenuous, it is supposed to be ionized and is subject to the recombination at a rate of about

$$q \simeq 3\times 10^{-10} T^{-3/4} n_e n_p \text{ sec}^{-1} \text{ cm}^{-3}, \quad (1)$$

where for definiteness we assume the temperature as high as $T \simeq 5\times 10^5 °\text{K}$ and the density $n_e = n_p \simeq 10^{-4} \text{ cm}^{-3}$.[1)]

The recombination gives rise to radiations in a wide spectral range. Among them visible light is hard to observe against the background due to galactic light and Lyman radiations, belonging both to line and continuous spectra, are quickly absorbed by interstellar gases. Only such a spectral band seems to be transparent that is a continuous one due to the transitions between S states, particularly due to the transition from $2S$ to $1S$ states.[2)] This transition gives rise to the emission of two photons. Most part of the spectrum of these photons lies between Lα and the Balmer edge, and the Balmer continuum will not appreciably mask this continuous spectrum.[3),4)] It is almost needless to mention that the collisional deexcitation occurring in the stellar atmosphere is of little importance in a tenuous gas under consideration. Thus we have good reasons for the observability of this spectrum.

Taking into account that appreciable part of the recombination process goes through the $2S$ state, we estimate the emission rate as[3)]

$$j(2S \to 1S + 2h\nu) \simeq 10^{-22} \text{ photons sec}^{-1} \text{ cm}^{-3}. \quad (2)$$

For this radiation the optical depth in the intergalactic space may be as large as 10^7 *l. y.*, the radiation produced at farther distances being masked by galaxies nearby. Then the intensity expected at the top of the earth atmosphere, high enough to be free from the atmospheric absorption, is estimated as

$$I(2S \to 1S + 2h\nu) \simeq 10^3 \Delta\Omega \text{ photons sec}^{-1} \text{ cm}^{-2}, \quad (3)$$

where $\Delta\Omega$ is the solid angle covered by a telescope to be used. This is strong enough to be observable,[5)] provided that there were no noise to disturb this intergalactic radiation.

The most serious noise is presumed to come from the Galactic halo, if it contains an ionized gas of hydrogen of the density greater than $10^{-3}\,\mathrm{cm}^{-3}$. If this noise is stronger, our proposed method will be used for observing hydrogen in the Galactic halo rather than in the intergalactic space; this noise itself is of great astrophysical importance.

There would be some more methods, if there were anti-matter in the intergalactic space. Then we would expect radiations from the annihilation processes of negaton-positon pairs and of proton-antiproton pairs. The former would give a line spectrum at 0.51 Mev, and the latter a continuous spectrum with a center at about 100 Mev, both being subject to the Hubble shift. Since the optical depths for these radiations could be as large as 10^8 to 10^9 *l.y.* on account of their penetrability, their intensities would be higher than that given in (3), if the amount of anti-matter were comparable to that of ordinary matter. Consequently, the 100 Mev radiations would have to be detected by cosmic ray observations. Since there seems to be no evidence for this, the copious existence of anti-matter in the universe may be thought as negative.

1) See, for example, F. D. Kahn and L. Woltjer, Ap. J. **130** (1959), 705.
2) L. Spitzer and J. L. Greenstein, Ap. J. **104** (1951), 407, have suggested the importance of this process in connection with the luminosity of a planetary nebula. The author is grateful to Professor W. Unno who called his attention to the above paper and stimulated him by suggestive comments.
3) This process is regarded as the most important cause of the ultraviolet radiation from Galactic clouds, as was discussed by the author. S. Hayakawa, Publ. Astr. Soc. Japan, **12** (1960), 110, 113.
4) This and related processes are analyzed in more detail and are applied to the ultraviolet radiation from Galactic clouds. The work will soon be published by S. Hayakawa, S. Nishimura and K. Takayanagi.
5) See, for example, J. E. Kupperian, Jr, A. Bogges III and J. E. Milligan, Ap. J. **128** (1958), 453.
6) P. Morrison, Nuovo Cimento, Suppl. **8** (1958), 858.

Electric Polarizability of Nucleon

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As to the electric polarizability of neutron: α_n, several experiments have been carried out,[1] but the smallness of its effect makes it possible only to determine the upper limit of $|\alpha|$——$|\alpha| < 2\times10^{-41}\,\mathrm{cm}^3$. Foldy[2] pointed out that if the electric polarizability of neutron came dominantly from its anomalous magnetic moment, α_n must be negative and very small——$\alpha_n = -6.2\times10^{-44}\mathrm{cm}^3$. He also said that it may occur as in the case of radius of neutron that meson cloud effect is very small. Breit and Rustgi[3] estimated α_n by making use of the cross section of photomeson production and got the result: $1.1\times10^{-43} < \alpha_n < 2.2\times10^{-42}\,\mathrm{cm}^3$. It is interesting derive the Breit-Rustgi relation from more familiar dispersion treatment, and to refine the estimation of the electric

polarization of nucleon (α_p and α_n).

Energy shift of any particle in an external electromagnetic field is

$$\begin{aligned}\Delta\mathcal{E} &= \int d^3y \langle\psi(A)|j_\mu(y,0) \\ &\quad \times A_\mu^{(e)}(y,0)|\psi(A)\rangle \\ &= \int d^3y \langle\psi_0|j_\mu(y,0)A_\mu^{(e)}(y,0)|\psi_0\rangle \\ &\quad + i\int d^3y \int d^4x \\ &\quad \times \langle\psi_0|P(j_\nu(x), j_\mu(y,0))|\psi_0\rangle \\ &\quad \times A_\nu^{(e)}(x)A_\mu^{(e)}(y,0) + \cdots, \qquad (1)\end{aligned}$$

where $|\psi(A)\rangle$ and $|\psi_0\rangle$ are the nucleon states distorted and not distorted by the external field, respectively. As usual, we define the electric polarizability: α by

$$\Delta\mathcal{E} = -\frac{1}{2}\alpha E^2. \qquad (2)$$

From Eqs. (1) and (2) we can obtain

$$\alpha = \left[\frac{\partial^2 F(k)}{\partial k_0^2}\right]_{k=0} \qquad (3)$$

where

$$F(k) = i\int d^4x \langle\psi_0|P(j_3(x), j_3(0))|\psi_0\rangle e^{ikx}. \qquad (4)$$

It should be noted that in Eq. (3) the direction of derivative is not on the mass shell ($k^2 - k_0^2 = 0$), but partial derivative with respect to k_0. It can, however, be shown that

$$\left[\frac{d^2 F_E(\omega)}{d\omega^2}\right]_{\omega=0} = \left[\frac{\partial^2 F(k)}{\partial k_0^2}\right]_{k=0}, \qquad (5)$$

where $F_E(\omega)$ is the amplitude of photon scattering of electric multipoles only. As the imaginary part of $F(\omega)$ in the neighbourhood of $\omega = 0$ is small, and by making use of the dispersion relation

$$\mathrm{Re}\,F_E(\omega) = \frac{1}{2\pi^2} P\int_0^\infty d\omega' \frac{\omega'^2 \sigma_E(\omega')}{\omega'^2 - \omega^2}, \qquad (6)$$

we finally obtain

$$\alpha = \frac{1}{\pi^2} P\int_0^\infty d\omega' \frac{\sigma_E(\omega')}{\omega'^2}, \qquad (7)$$

where $\sigma_E(\omega)$ means the total cross section of electron transition to any state from photon state with energy ω in the laboratory system.

As an integrand of Eq. (7) is $\sigma_E(\omega)/\omega^2$, only the lower energy part of $\sigma_E(\omega)$ contributes to the value of α, so we take only the cross sections of photoproduction and γ-N scattering, but the latter is small and not so important.

Polarizability of proton: α_p

It is well known that the reaction $\gamma + p \to \pi^0 + p$ can be understood that, for E_γ(lab.) $\lesssim$ 450 Mev, photon is absorbed from $M1$ state only and the reaction occurs through the ($I=3/2$, $J=3/2$) resonance. On the other hand, the reaction $\gamma + p \to \pi^+ + n$ cannot be interpreted as S and P waves only, except E_γ(lab.) $\lesssim$ 250 Mev. But, fortunately, for the evaluation of the electric polarizability higher energy cross section is not so important. So we assume that the total cross section is made by S-wave pion (from $E1$ absorption) and P-wave pion (from $M1$ absorption through $I=3/2$, $J=3/2$) only. As a result, $\sigma(\pi^+) - (1/2)\cdot\sigma(\pi^0)$ contains only S-wave pion that comes from the absorption of $E1$ photon.

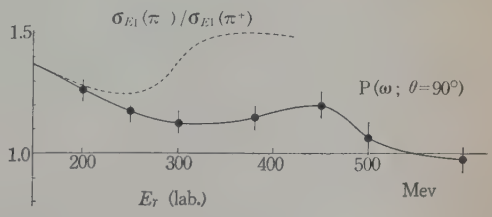

Fig. 1.

From the data of photomeson production cross section, we obtain α_p

$$\alpha_p = (0.75 \pm 0.25) \times 10^{-42}\ \mathrm{cm}^3 \quad (8)$$

with the cutoff E_γ(lab.) $=500$ Mev. The errors come from the cutoff, the assumption $\sigma_E(\omega) \approx \sigma(\pi^+) - (1/2)\cdot\sigma(\pi^0)$ and the neglect of the cross section of γ-N scattering of lower energy photon. In the error estimation we used the data of Compton scattering of 60 Mev photon,[4] and the lower bound of α_p is estimated by the extrapolation of threshold S-wave photo-pion data with the cutoff at 300 Mev.

Polarizability of neutron: α_n

As we have no neutron target, we must extract the cross section of $(\gamma + n \to \pi^- + p)$ reaction: $\sigma_-(\omega)$ from the ratio $\sigma_-(\omega)/\sigma_+(\omega)$ of deuteron[5] and the cross section of $(\gamma + p \to \pi^+ + n)$ reaction $\sigma_+^{\pi}(\omega)$. As photo-π^0 production is known to occur by $M1$ photon-absorption for low energy region and the magnitude of the anomalous magnetic moment of neutron is approximately equal to that of the proton, so it is natural to take $\sigma_-^{M1}(\omega) = \sigma_+^{M1}(\omega)$ for low energy region. So, the fact that the ratio $\sigma_-(\omega)/\sigma_+(\omega)$ is different from unity (Fig. 1) comes from the difference of $E1$ photo-pion production. We obtain

$$\sigma_n^{E1}(\omega) = \mathrm{P}(\omega\,;\,\theta = 90°) \times \left\{\sigma_+(\omega) + \frac{3}{14}\sigma_0(\omega)\right\} - \frac{5}{7}\sigma_0(\omega), \quad (9)$$

where

$$\mathrm{P}(\omega\,;\,\theta = 90°) \equiv \left[\frac{d\sigma_-(\omega)}{d\Omega} \Big/ \frac{d\sigma_+(\omega)}{d\Omega}\right]_{\theta = 90°},$$

σ_0: total cross section for $\gamma + p \to \pi^0 + p$.

With the use of $\sigma_n^{E1}(\omega)$ thus obtained, the electric polarizability of neutron is

$$\alpha_n = (1.0 + 0.4) \times 10^{-42}\ \mathrm{cm}^3. \quad (10)$$

Error comes from the experimental error of the ratio $\mathrm{P}(\omega\,;\,\theta = 90°)$, the assumption $\sigma_-^{M1}(\omega) \approx \sigma_+^{M1}(\omega)$ and the three sources of error that is written above with respect to the error of α_p.

1) Aleksandrov, Soviet Phys. JETP **6** (1958), 228: Langsdorf, Lane and Monahan, Phys. Rev. **107** (1957), 1077.
2) Foldy, Phys. Rev. Letters **3** (1959), 105.
3) Breit and Rustgi, Phys. Rev. **114** (1959), 830.
4) High Energy Physics Conference at CERN (1958).
5) Neugebauer, Wales and Walker, Phys. Rev. Letters **2** (1959), 429.

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Space-Time Description of Collision and Decay Processes

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On the basis of Lorentz invariance and the existence of the vacuum, scattering amplitudes are expressed in terms of Wightman functions. It is done with the aid of observation functions, four-dimensional wave packets vanishing outside the space-time regions of observation of colliding particles. Restrictions must be imposed on each of these regions in order to assure the one-particle character of observed particles, which may be either stable or unstable. Our expression, in a form modified under the additional assumption of microcausality, is compared with that of Lehmann, Symanzik and Zimmermann and the physical meaning of the asymptotic condition is clarified. Extension of our method of description to processes involving composite particles is given, and some relativistic problems concerned with unstable particles are also discussed from the above space-time point of view.

§ 1. Introduction and summary

Although several efforts[1)-3)] to work out formulations of quantum field theory have clarified the axiomatic basis of the current local theory to some extent, further study of the problem seems desirable in view of its importance. In these formulations are always discussed transition amplitudes from initial states in the infinite past to final states in the infinite future, the S-matrix elements. However, it is nothing but an idealization of real physical processes, which is made inevitable by the adoption of the asymptotic condition[1)] as one of the fundamental principles. Moreover, in a real process any state cannot be observed instantaneously. The author believes that these facts should receive more attention.* The present work is an attempt to describe, from a space-time point of view, collision and decay processes in a unified and physically intuitive way.

Throughout this paper we assume the invariance under inhomogeneous proper Lorentz transformations and the existence of the vacuum state, Ψ_0, which is defined as the unique, Lorentz-invariant and lowest-energy state. From these two assumptions it follows that there exist displacement operators P_μ with the properties

$$\frac{\partial \phi(x)}{\partial x_\mu} = -i[P_\mu, \phi(x)], \quad [P_\mu, P_\nu]=0, \tag{1·1}$$

$$P_\mu \Psi_0 = 0. \tag{1·2}$$

The closure of the domain of P_μ is a Hilbert space $\mathfrak{H}$, which forms the object of

* Stueckelberg[4)] once considered this problem in perturbation theory.

our physical interest. We consider a model of a neutral scalar field described by a hermitian field operator in the Heisenberg representation, $\phi(x)$, by means of which $\mathfrak{H}$ can be generated from the vacuum state. We do not require the principle of microcausality until otherwise stated.

In § 2 we shall construct physical one-particle states, making use of four-dimensional wave packets which represent average momenta of colliding particles and their space-time regions of observation. The temporal extension of these regions, revealing the time indeterminacy of the observations, should be restricted by physical requirements (*the particle condition*) necessary to assure the one-particle character of observed particles. In § 3 scattering amplitudes are expressed in terms of Wightman functions,[5] the vacuum expectation values of products of field operators. Wightman functions can be replaced by Green's functions under the additional assumption of microcausality. After a modification our expression is compared with that of Lehmann, Symanzik and Zimmermann[1] (hereafter referred to as LSZ). The former is wider than the latter as it is applicable also to a colliding system involving unstable particles. This is achieved by the replacement of the asymptotic condition with the particle condition.

Next in § 4, our method of description is extended to a system involving composite particles. The limiting process of time averaging in deriving the Bethe-Salpeter equation is shown to be unnecessary in our method. Finally in § 5 the representation of physical states of unstable particles is discussed, and the elongation of the lifetime of a moving unstable particle is explained with the aid of the concept of the volatile fluid. In the Appendix the mass and the lifetime of an unstable spinor particle is defined under the assumption of CTP-invariance.

§ 2. Representation of states with the one-particle character

In order to get expressions for scattering or decay amplitudes, we begin our discussions with studying how to construct a physical one-particle state, which may be either stable or unstable. A representation for a one-particle state was previously given by Nishijima[6] as

$$\Psi=\Big[\int d^3x f(\boldsymbol{x},t)\phi(\boldsymbol{x},t)+\int d^3x f'(\boldsymbol{x},t)\dot{\phi}(\boldsymbol{x},t) + \iint d^3x_1 d^3x_2 f(\boldsymbol{x}_1,\boldsymbol{x}_2,t)\phi(\boldsymbol{x}_1,t)\phi(\boldsymbol{x}_2,t)+\cdots\Big]\Psi_0, \qquad (2\cdot1)$$

the integrations being extended over the three-dimentional space. In this representation one requires an infinite number of functions called contravariant components, which represent virtually dissociated states of the physical one-particle state. With a single term of them we can obtain an approximate eigenstate of the momentum operator $P_i(i=1, 2, 3)$, but *not* of the energy operator P_0. In this point lies the difference of relativistic field theory from the nonrelativistic quantum mechanics. In the latter the energy of a particle is determined by its momentum,

while in the former the energy of a state is never determined by its momentum, because the state can take arbitrarily high mass configurations. A physical one-particle state could not be obtained without giving a state an approximately definite energy-momentum lying on the mass shell. In this section we attempt to get a compact and intuitive representation for physical states, extending the integrations over the four-dimensional space-time.

A particle moving with an average momentum p_μ must have been prepared somehow in a space-time region R. Such a state can be represented in the form

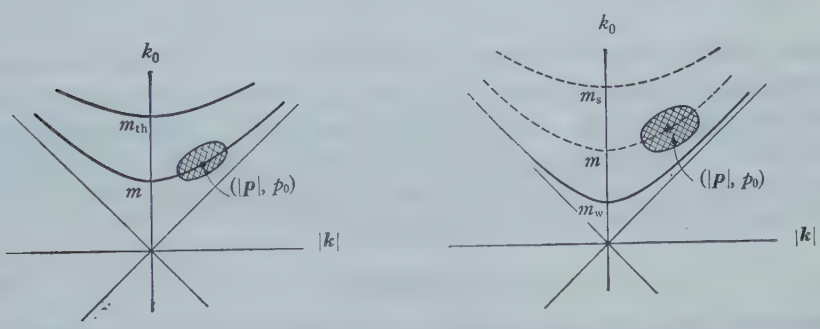
$$\Psi_h=\frac{1}{N^{1/2}}\int d^4x h(x\,;\,p)\phi(x)\Psi_0, \tag{2·2}$$

where $h(x\,;\,p)$ behaves approximately like $(2\pi)^{-2}e^{ip\cdot x}$ inside the preparation region R, and vanishes outside it. The Fourier transform of $h(x\,;\,p)$,

$$\tilde{h}(k\,;\,p)=\frac{1}{(2\pi)^2}\int d^4x e^{-ik\cdot x}h(x\,;\,p) \tag{2·3}$$

is then approximately equal to unity for $k_\mu\sim p_\mu$, and is negligibly small otherwise (Fig. 1). That the state Ψ_h is an approximate eigenstate of P_μ is easily seen from (1·1), (1·2) and the properties of $h(x\,;\,p)$, namely

$$P_\mu\Psi_h=\frac{1}{N^{1/2}}\int d^4x h(x\,;\,p)i\frac{\partial\phi(x)}{\partial x_\mu}\Psi_0\doteqdot\frac{-i}{N^{1/2}}\int d^4x\frac{\partial h(x\,;\,p)}{\partial x_\mu}\phi(x)\Psi_0. \tag{2·4}$$



a) Stable case b) Unstable case

Fig. 1. $\tilde{h}(k\,;\,p)$ is approximately equal to unity in the shaded region, and is negligibly small outside it.

The normalization constant N is given by

$$N=\iint d^4x\,d^4x' h^*(x\,;\,p)\,W(x,\,x')h(x'\,;\,p)=2\pi\int d^4k\theta(k_0)\rho(-k^2)|\tilde{h}(k\,;\,p)|^2. \tag{2·5}$$

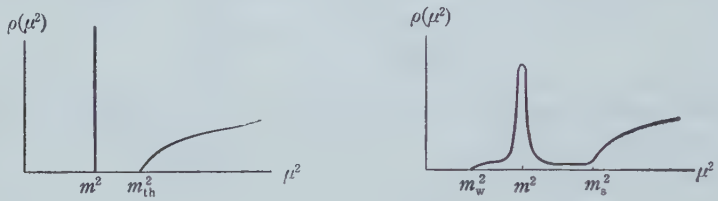
where

$$W(x, x') \equiv \langle \phi(x) \cdot \phi(x') \rangle_0 = i\Delta^{(+)\prime}(x-x'), \tag{2·6}$$

and $\Delta^{(+)\prime}(x)$ may be represented as a superposition of free functions $\Delta^{(+)}(x\,;\,\mu^2)$:[7]

$$\Delta^{(+)\prime}(x) = \int_0^\infty d\mu^2 \rho(\mu^2) \Delta^{(+)}(x\,;\,\mu^2). \tag{2·7}$$

The behaviour of the weight function $\rho(\mu^2)$ is shown in Fig. 2 for a stable and an unstable particle. m_{th} or m_{w} represents the threshold value of continuous mass-spectrum for each case. For an unstable particle it is assumed that it has two kinds of interactions, strong and weak, and that it would become stable if weak interactions were switched off. m_{s} represents the threshold value of continuous mass-spectrum for this fictitious case.



a) Stable case b) Unstable case

Fig. 2. The behaviour of the weight function $\rho(\mu^2)$ for a stable and an unstable case.

Several restrictions, which may be called the particle condition, should be imposed on an observation region R in order to assure the one-particle character of a state we are observing. Let us denote by Δt the time indeterminacy unavoidable to observe a one-particle state, which is due to the uncertainty principle. We shall first consider a particle with the zero average momentum. For the case of a stable particle, Δt must satisfy the inequality

$$(\Delta t)^{-1} \ll m_{\text{th}} - m \tag{2·8}$$

to eliminate continuous energy spectrum. We shall also need the condition that $h(x\,;\,p)$ contains scarcely any negative energy frequency, which is satisfied by the inequality

$$(\Delta t)^{-1} \ll m. \tag{2·9}$$

For the case of an unstable particle, besides an inequality corresponding to (2·8),

$$(\Delta t)^{-1} \ll m_{\text{s}} - m, \tag{2·10}$$

Δt must satisfy another one for the decay Q-value to be sharply defined

$$(\Delta t)^{-1} \ll m - m_{\text{w}}. \tag{2·11}$$

On the other hand, it must be shorter than the lifetime, $\tau_0(=\gamma^{-1})$, of the particle,[8]

$$(\Delta t)^{-1} \gg \gamma. \tag{2·12}$$

It may be noted that the state Ψ_h does not sensitively depend on the time indeterminacy Δt so far as it satisfies the above inequalities. For an unstable particle moving with an average momentum $\boldsymbol{p}$, these inequalities should be replaced by

$$\begin{aligned}(\Delta t)^{-1} &\ll \sqrt{m_s^2+\boldsymbol{p}^2}-\sqrt{m^2+\boldsymbol{p}^2},\\ (\Delta t)^{-1} &\ll \sqrt{m^2+\boldsymbol{p}^2}-\sqrt{m_w^2+\boldsymbol{p}^2}\end{aligned} \tag{2·13}$$

and

$$(\Delta t)^{-1} \gg \frac{m}{\sqrt{m^2+\boldsymbol{p}^2}}\gamma. \tag{2·14}$$

At first sight the replacement of the right-hand side of (2·14) with $m^{-1}\sqrt{m^2+\boldsymbol{p}^2}\gamma$ might be preferred. It is not right, however, as will be explained in § 5 in connection with the elongation of the lifetime of a moving unstable particle.

We now discuss a two-particle state characterized by average momenta p_1, p_2 and observation regions R_1, R_2. This state can be represented by

$$\Psi_{h_1 h_2}=\frac{1}{N^{1/2}}\iint dx_1 dx_2 h_1(x_1)h_2(x_2)\phi(x_2)\phi(x_1)\Psi_0, \tag{2·15}$$

where the normalization constant N is given by

$$N=\int\cdots\int dx_1 dx_2 dx_1' dx_2' h_1^*(x_1)h_2^*(x_2)\,W(x_1, x_2, x_2', x_1')h_1(x_1')h_2(x_2'). \tag{2·16}$$

For the sake of simplicity, we restrict ourselves to the case that the two regions of observation are separated with a macroscopic spacelike distance, so that the observations of the two particles could be done independently.* It can then be shown that the normalization constant is approximately equal to the product of the ones for the two particles,

$$N\approx N_1\cdot N_2. \tag{2·17}$$

Two-particle states may be divided into three classes according to the relative configuration of the particles: incoming, outgoing and collisionless.[9] When the two particles collide after their observation (preparation), the state is called incoming, and when they collide before their observation (detection), the state is called outgoing. When the two particles never collide before and after their observation, the state is called collisionless, which is of no physical interest and will not be discussed any more.

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* This assumption is too restrictive, and may probably be replaced by a weaker one such that the wave propagating from a region of observation should not pass through another one.

§ 3. Expressions for scattering amplitudes

As we have become fairly familiar with the physical meaning of the states introduced in the preceding section, we are now ready to proceed to the description of scattering processes. A scattering experiment consists of the preparation of an incoming state followed by the detection of an outgoing one, as is schematically shown in Fig. 3. We restrict ourselves to two-particle elastic scattering, as generalization to multi-particle cases does not offer any difficulty. A scattering amplitude of an incoming two-particle state $\Psi_{h_1 h_2}$ characterized by normalized observation functions h_1 and h_2 into an outgoing one $\Psi_{h_3 h_4}$ characterized by h_3 and h_4 is given by an inner product of the two states,

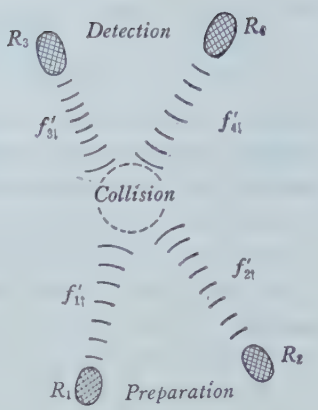

Fig. 3. A schematic picture of a scattering experiment.

$$\langle \Psi_{h_3 h_4} | \Psi_{h_1 h_2} \rangle = \int \cdots \int dx_3 dx_4 dx_1 dx_2 h_3^*(x_3) h_4^*(x_4) \times W(x_3, x_4, x_2, x_1) h_1(x_1) h_2(x_2). \quad (3\cdot1)$$

Thus we have obtained expressions for scattering amplitudes in terms of Wightman functions.

Next we shall rewrite our expressions for scattering amplitudes under the additional assumption of microcausality, which requires

$$[\phi(x), \phi(x')] = 0, \quad \text{for } (x-x')^2 > 0. \quad (3\cdot2)$$

We have assumed that the two regions of preparation and those of detection should be spacelikely separated from each other, respectively. Then we are allowed to replace the Wightman function $W(x_3, x_4, x_2, x_1)$ in (3·1) with $\langle T[\phi(x_3), \phi(x_4)] \cdot T[\phi(x_1), \phi(x_2)] \rangle_0$. Since the regions of detection should be situated future timelikely to those of preparation, we can further replace the latter with a Green's function

$$G(x_3, x_4, x_1, x_2) \equiv \langle T[\phi(x_3), \phi(x_4), \phi(x_1), \phi(x_2)] \rangle_0. \quad (3\cdot3)$$

After all, a Wightman function in the expression (3·1) can be replaced by a corresponding Green's function. It is convenient to introduce another function, $I(x_3, x_4, x_1, x_2)$, defined by

$$G(x_3, x_4, x_1, x_2) = \int \cdots \int dx_3' dx_4' dx_1' dx_2' \Delta_F'(x_3 - x_3') \Delta_F'(x_4 - x_4') \times I(x_3', x_4', x_1', x_2') \Delta_F'(x_1' - x_1) \Delta_F'(x_2' - x_2), \quad (3\cdot4)$$

which is obtained from the Green's function by eliminating its external lines. Then with the use of functions

$$f'_{i1}(x) = i \int dx' \Delta_F'(x - x') h_i(x'), \quad (i = 1, 2) \quad (3\cdot5a)$$

$$f'_{i\downarrow}(x)=-i\int dx'\,\Delta'^{*}_{F}(x-x')h_i(x'),\quad (i=3,\,4) \tag{3·5b}$$

the expression (3·1) can be rewritten, through the change in the order of integration, as

$$\langle\Psi_{h_3h_4}|\Psi_{h_1h_2}\rangle=\int\cdots\int dx_3dx_4dx_1dx_2f'^{*}_{3\downarrow}(x_3)f'^{*}_{4\downarrow}(x_4)$$
$$\times I(x_3,\,x_4,\,x_1,\,x_2)f'_{1\uparrow}(x_1)f'_{2\uparrow}(x_2). \tag{3·6}$$

The functions $f'_{\uparrow}(x)$ and $f'_{\downarrow}(x)$ are related to the matrix element,

$$g(x)\equiv\langle\Psi_0|\phi(x)|\Psi_h\rangle=\int dx'\,W(x-x')h(x'), \tag{3·7}$$

in such a way that $g(x)$ is equal to $f'_{\uparrow}(x)$ $(f'_{\downarrow}(x))$ unless x is situated in the past (future) of the observation region.

We can also give another expression with a more conventional form,

$$\langle\Psi_{h_3h_4}|\Psi_{h_1h_2}\rangle=\int\cdots\int dx_3dx_4dx_1dx_2f^{*}_{3\downarrow}(x_3)f^{*}_{4\downarrow}(x_4)$$
$$\times K_3K_4K_1K_2G(x_3,\,x_4,\,x_1,\,x_2)f_{1\uparrow}(x_1)f_{2\uparrow}(x_2), \tag{3·8}$$
$$K_i\equiv\Box_{x_i}-m^2,$$

where $f_{i\uparrow}(x)$ and $f_{i\downarrow}(x)$ are obtained by replacing the primed propagators in (3·5) with the corresponding unprimed ones. The above expression is for stable particles. For unstable ones we have only to use $K\equiv\Box-\kappa^2$, and

$$\Delta_F(x-x';\,\kappa^2)=\frac{-i}{(2\pi)^4}\int d^4k\,e^{ik\cdot(x-x')}\frac{1}{k^2+\kappa^2} \tag{3·9}$$

as the unprimed propagator in defining the functions, $f_{i\uparrow}(x)$ and $f_{i\downarrow}(x)$. κ represents the complex mass[10] of the unstable particle under consideration (see also the Appendix),

$$\kappa=m-i\frac{\gamma}{2}. \tag{3·10}$$

By the way, it is remarked that we may approximate the functions $-if_{i\uparrow}'(x)$ and $if'_{i\downarrow}(x)$ by $f'_{i\,\mathrm{ret}}(x)$ and $f'_{i\,\mathrm{adv}}(x)$,

$$f'_{i\,\mathrm{ret}}(x)=\int dx'\,\Delta'_{\mathrm{ret}}(x-x')h_i(x'),\quad (i=1,\,2) \tag{3·11a}$$

$$f'_{i\,\mathrm{adv}}(x)=\int dx'\,\Delta'_{\mathrm{adv}}(x-x')h_i(x')\quad (i=3,\,4) \tag{3·11b}$$

respectively, because the observation functions $h_i(x)$ contain scarcely any negative frequency spectrum by the restriction imposed upon them and the equalities

$$\Delta_F'(k)=\Delta'_{\mathrm{ret}}(k),\quad \Delta'^{*}_{F}(k)=\Delta'_{\mathrm{adv}}(k) \tag{3·12}$$

hold for $k_0>0$.

Similar results were obtained by LSZ[1] with the use of the asymptotic condition. The difference is that in our description $f_i(x)$ are solutions of inhomogeneous Klein-Gordon (K-G) equations with the observation functions as source functions,

$$(\Box-m^2)f_{i\uparrow}(x)=-h_i(x), \quad (i=1, 2) \tag{3·13a}$$

$$(\Box-m^2)f_{i\downarrow}(x)=-h_i(x), \quad (i=3, 4) \tag{3·13b}$$

while in the formulation of LSZ the corresponding functions satisfy the homogeneous K-G equation. The appearance of the source terms is due to the fact that a scattering system is not closed by itself but is connected with external systems through the processes of preparation and detection. When the regions of preparation are brought to the infinite past and those of detection to the infinite future, our expression (3·8) reduces exactly to that of LSZ, since $f_{i\uparrow}(x)$ and $f_{i\downarrow}(x)$ now satisfy the free K-G equation in the whole space-time. We may correspond this limiting process to the so-called asymptotic condition.

§ 4. Composite particles and the Bethe-Salpeter equation

We now discuss a system with a stable composite particle of mass M besides an elementary one of mass m. A composite state can be represented as

$$\Psi_{h_c}=\iint dx\,d\xi\,h_c(x, \xi)\,\phi\left(x-\frac{\xi}{2}\right)\phi\left(x+\frac{\xi}{2}\right)\Psi_0, \tag{4·1}$$

where $h_c(x, \xi)$ is normalized by

$$\int\cdots\int dx\,d\xi\,dx'\,d\xi'\,h_c^*(x, \xi)\,W\left(x+\frac{\xi}{2},\ x-\frac{\xi}{2},\ x'-\frac{\xi'}{2},\ x'+\frac{\xi'}{2}\right)h_c(x', \xi')=1. \tag{4·2}$$

$h_c(x, \xi)$ must, as a function of x, satisfy the same requirement as $h(x)$ does for an elementary particle, while it is arbitrary as a function of ξ. We may then take a function which is non-vanishing only for space-like ξ, and replace the ordinary product of field operators in (4·1) by the time-ordered one under the assumption of microcausality.

We can easily generalize our expressions for scattering amplitudes obtained in the previous section to collision systems involving composite particles. As an illustration we consider an elastic scattering of an elementary particle and a composite one. The scattering amplitude from an initial state $\Psi_{h_1 h_c}$ to a final state $\Psi_{h_2 h_{c'}}$ is given by

$$\langle\Psi_{h_2 h_{c'}}|\Psi_{h_1 h_c}\rangle=\int\cdots\int dx_2\,dx'\,d\xi'\,dx_1\,dx\,d\xi\,h_2^*(x_2)h_{c'}^*(x', \xi')$$
$$\times W\left(x_2,\ x'+\frac{\xi'}{2},\ x'-\frac{\xi'}{2},\ x-\frac{\xi}{2},\ x+\frac{\xi}{2},\ x_1\right)h_1(x_1)h_c(x, \xi). \tag{4·3}$$

We can also give another expression,

$$\langle \Psi_{h_2 h_{c'}} | \Psi_{h_1 h_c} \rangle = \int\cdots\int dx_2 dx' d\xi' dx_1 dx d\xi f_2^*{}_{\downarrow}(x_2) f_{c'}^*{}_{\downarrow}(x', \xi')$$
$$\times K_2 K_{c'} K_1 K_c G\left(x_2, x'+\frac{\xi'}{2}, x'-\frac{\xi'}{2}, x-\frac{\xi}{2}, x+\frac{\xi}{2}, x_1\right) f_{1\uparrow}(x_1) f_{c\uparrow}(x, \xi), \tag{4·4}$$

where

$$K_c = \Box_x - M^2, \quad K_{c'} = \Box_{x'} - M^2,$$

and the functions $f_{c\uparrow}(x, \xi)$ and $f_{c'\downarrow}(x, \xi)$ are defined by

$$f_{c\uparrow}(x, \xi) = i\int dx' \Delta_F(x-x'; M^2) h_c(x', \xi), \tag{4·5a}$$

$$f_{c'\downarrow}(x, \xi) = -i\int dx' \Delta_F^*(x-x'; M^2) h_{c'}(x', \xi). \tag{4·5b}$$

With the use of the function defined by

$$g(x, \xi) = \left\langle \Psi_0 | \phi\left(x+\frac{\xi}{2}\right) \phi\left(x-\frac{\xi}{2}\right) | \Psi_{h_c} \right\rangle$$
$$= \iint dx' d\xi' W\left(x+\frac{\xi}{2}, x-\frac{\xi}{2}, x'-\frac{\xi'}{2}, x'+\frac{\xi'}{2}\right) h_c(x', \xi'), \tag{4·6}$$

the normalization condition (4·2) of the function $h_c(x, \xi)$ may be rewritten as

$$\iint dx d\xi h_c^*(x, \xi) g(x, \xi) = \iint dx d\xi g^*(x, \xi) h_c(x, \xi) = 1. \tag{4·7}$$

Our expression (4·4) and the normalization (4·7) are to be compared with the results of Nishijima.[3]

The Bethe-Salpeter amplitude for the state Ψ_{h_c} is given by

$$g_{BS}(x_1, x_2) = \langle \Psi_0 | T[\phi(x_1), \phi(x_2)] | \Psi_{h_c} \rangle$$
$$= \iint dx' d\xi' \left\langle T[\phi(x_1), \phi(x_2)] \cdot T\left[\phi\left(x'-\frac{\xi'}{2}\right), \phi\left(x'+\frac{\xi'}{2}\right)\right] \right\rangle_0 h_c(x', \xi'). \tag{4·8}$$

We here introduce another function, the Feynman amplitude,

$$g_F(x_1, x_2) \equiv \iint dx' d\xi' G\left(x_1, x_2, x'-\frac{\xi'}{2}, x'+\frac{\xi'}{2}\right) h_c(x', \xi'), \tag{4·9}$$

which is equal to $g_{BS}(x_1, x_2)$ except for the case that at least one of the two points is in or on the past light cone of the preparation region of this composite particle.

As is well known, Green's function $G(x_1, x_2, y_1, y_2)$ satisfies an integral equation,

$$G(x_1, x_2, y_1, y_2) = \Delta_F'(x_1-y_1)\Delta_F'(x_2-y_2) + \Delta_F'(x_1-y_2)\Delta_F'(x_2-y_1)$$
$$+\int\cdots\int dx_1' dx_2' dy_1' dy_2' \Delta_F'(x_1-x_1')\Delta_F'(x_2-x_2')K(x_1', x_2', y_1', y_2')$$
$$\times G(y_1', y_2', y_1, y_2). \tag{4·10}$$

Multiplying both sides of the above equation by $h_c((y_1+y_2/2), y_1-y_2)$ and integrating them with y_1 and y_2, we get the Bethe-Salpeter equation,

$$g_F(x_1, x_2) = \int\cdots\int dx_1' dx_2' dy_1' dy_2' \Delta_F'(x_1-x_1')\Delta_F'(x_2-x_2')$$
$$\times K(x_1', x_2', y_1', y_2')g_F(y_1', y_2'). \tag{4·11}$$

The contribution from the non-interacting terms of (4·10) nearly vanishes due to the particle condition imposed upon the preparation function $h_c(x, \xi)$.

A similar consideration was previously given by Allcock.[11] His argument differs from ours in that $h_c(x, \xi)$ satisfies the free K-G equation as a function of x, while in our treatment it represents a source term of the function $f_{c\uparrow}(x, \xi)$. In their derivation of the B-S equation Gell-Mann and Low[12] used a limiting process, which is a kind of time averaging. In our description an integration with time appears as a result of the time indeterminacy of the preparation process.

It may also be remarked that with our method we can express the matrix element of any dynamical variable between two composite states in terms of Wightman functions.

§ 5. Physical states of unstable particles

In the case of unstable particles the use of the asymptotic condition is certainly impossible and the space-time description in which physical aspects receive more emphasis seems to be inevitable. The above way of description of collision processes can immediately be applied to decay processes, too. Not to repeat similar discussions we concentrate our attention to the representation of physical states of unstable particles, which has been one of the most obscure points in the theory of unstable particles. The state vector corresponding to a physical unstable particle was represented in § 2 as

$$\Psi_h = \int dx h(x; p)\phi(x)\Psi_0, \tag{5·1}$$

where p_μ denotes an average four-momentum of the particle lying on the mass shell with the mass $m(=Re\,\kappa)$. It is a relativistic generalization of the representation given by Nakanishi[13] in the Lee model. It may formally be rewritten in a form

$$\Psi_h = \int dx f_\uparrow(x: p)j(x)\Psi_0, \tag{5·2}$$

where

$$(\Box-\kappa^2)\phi(x)=-j(x), \tag{5·3}$$

$$f_{\uparrow}(x\,;\,p)=i\int dx'\,\Delta_F(x-x'\,;\,\kappa^2)h(x'). \tag{5·4}$$

For simplicity of the discussion, we here make use of an exact eigenstate of P_μ with an eigenvalue p_μ', $|p'\rangle$, which is not a normalizable vector of the Hilbert space $\mathfrak{H}$. If we take a scalar product of Ψ_h and $|p'\rangle$, we get

$$\langle p'|\Psi_h\rangle=\int dx\langle p'|j(x)|\Psi_0\rangle f_{\uparrow}(x\,;\,p)=\langle p'|j(0)|\Psi_0\rangle\tilde{f}_{\uparrow}(p'\,;\,p)$$

$$=\langle p'|j(0)|\Psi_0\rangle\frac{1}{p'^2+\kappa^2}\tilde{h}(p'\,;\,p); \tag{5·5}$$

where $\tilde{f}_{\uparrow}(p')$ and $\tilde{h}(p';\,p)$ represent the Fourier transforms of $f_{\uparrow}(x\,;\,p)$ and $h(x\,;\,p)$, respectively. As a function of $-p'^2$ the vertex function $\langle p'|j(0)|\Psi_0\rangle$ has no singularity at κ^2 in the first unphysical sheet of the complex $(-p'^2)$-plane.

The mass-spectral function of the state Ψ_h defined by

$$\rho_h(\mu^2)=\sum_{-p^2=\mu^2}|\langle p'|\Psi_h\rangle|^2 \tag{5·6}$$

has a peak at $\mu^2\approx m$ with the width approximately equal to γ. It shows no such bump due to strong interactions as $\rho(\mu^2)$ does. The cutoff width of the spectrum is determined by the time-indeterminacy unavoidable for the preparation of this particle.*

Finally we discuss the propagation of a wave representing a physical unstable particle. The function $g(x\,;\,p)$,

$$g(x\,;\,p)\equiv\langle\Psi_0|\phi(x)|\Psi_h\rangle=\int dx'\,W(x,\,x')h(x'\,;\,p) \tag{5·7}$$

is equal to $f_{\uparrow}'(x\,;\,p)$,

$$f_{\uparrow}'(x\,;\,p)=\int dx'\,\Delta_F'(x-x')h(x \tag{5·8}$$

if we restrict x in the future of the preparation region. The latter may be approximated by $f_{\uparrow}(x\,;\,p)$ because of the properties of $\tilde{h}(p';\,p)$ (see Fig. 1). $f_{\uparrow}(x\,;\,p)$ behaves approximately like $e^{i\tilde{p}\cdot x}$ inside the propagation path of the wave, while it is negligibly small outside it. (Though the boundary of the path cannot be sharply defined, the concept of the path is helpful to clarify the physical image.) Here $\tilde{p}_\mu$ is a complex four-momentum given by

$$\tilde{p}_\mu\equiv p_\mu-i\frac{\gamma}{2m}p_\mu=\frac{\kappa}{m}p_\mu, \tag{5·9}$$

---

* Matthews and Salam[14] defined the mass and lifetime of an unstable particle by the first and second moments of the spectral function $\rho(\mu^2)$. It is not physical, apart from a possible divergence, to use $\rho(\mu^2)$ with a bump due to strong interactions as a mass distribution function of an unstable particle.

$$p_\mu=\left(\frac{m\boldsymbol{v}}{\sqrt{1-v^2}},\ i\frac{m}{\sqrt{1-v^2}}\right).$$

It is often convenient to attach damping waves with complex four-momenta to unstable particles as to attach plane waves to stable ones. Let us consider a damping wave,

$$e^{i\tilde{p}\cdot x}=e^{ip\cdot x}e^{(\gamma/2m)p\cdot x}, \tag{5·10}$$

corresponding to an unstable particle moving with a velocity $\boldsymbol{v}$. At any spatially fixed point it damps exponentially with time in the form of $\exp((-\gamma p_0/2m)x_0)$. However, its characteristic time, $\sqrt{1-v^2}\,\tau_0$, is measured by Euler's method of description for this 'volatile fluid' with proper volatilization time τ_0, and it does not correspond to the lifetime of the particle. The point of observation should be moved with time in accordance with the movement of this fluid. In other words, the characteristic time should be measured by Lagrange's method of description for the fluid. Then the wave damps like $\exp(-(\gamma m/2p_0)x_0)$ as time goes on, because

$$-\frac{\gamma}{2m}p\cdot x=\frac{\gamma}{2m}(p_0x_0-\boldsymbol{p}\cdot\boldsymbol{x})\to\frac{\gamma}{2m}(p_0x_0-\boldsymbol{p}\cdot\boldsymbol{v}x_0)=\frac{\gamma m}{2p_0}x_0. \tag{5·11}$$

The lifetime of an unstable particle moving with a velocity v is thus elongated from τ_0 to τ, where

$$\tau=\frac{\tau_0}{\sqrt{1-v^2}}. \tag{5·12}*$$

The damping wave should also be cutoff in the direction of its movement, since the time duration necessary for a wave packet to pass through a spatially fixed point with a phase velocity v^{-1} must be much shorter than the lifetime measured *à la* Euler. $\sqrt{1-v^2}\tau_0$. Then we get the inequality,

$$l\ll\frac{\sqrt{1-v^2}}{v}\tau_0, \tag{5·13}$$

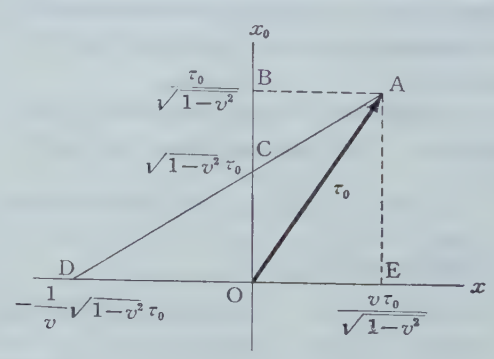

Fig. 4. Propagation of 'volatile fluid', $e^{i\tilde{p}x}$. $\overline{OB}$ and $\overline{OC}$ are the characteristic times measured *à la* Lagrange and Euler, respectively. $\overline{DCA}$ represents a constant amplitude line for this damping wave.

where l represents the spatial extension of the packet in the direction of its movement.

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* Møller[15] once derived this relation neglecting γ^2 compared to m^2

Appendix

The mass and the lifetime of an unstable spinor particle

We have restricted ourselves only to Bose fields. Some complication arises for unstable spinor particles due to the appearance of Dirac matrices. We summarise here the results of various invariances for the Wightman functions,

$$S^{(+)\prime}_{\alpha\beta}(x)=i\left\langle\psi_\alpha\left(\frac{x}{2}\right)\bar{\psi}_\beta\left(-\frac{x}{2}\right)\right\rangle_0, \tag{A·1}$$

$$S^{(-)\prime}_{\alpha\beta}(x)=i\left\langle\bar{\psi}_\beta\left(\frac{x}{2}\right)\psi_\alpha\left(-\frac{x}{2}\right)\right\rangle_0. \tag{A·2}$$

CTP-invariance:

$$S^{(-)\prime}(x)=-\gamma_5 S^{(+)\prime}(-x)\gamma_5, \tag{A·3}$$

C-invariance:

$$S^{(-)\prime}(x)=-C\{S^{(+)\prime}(-x)\}^T C^{-1}, \tag{A·4}$$

T-invariance:

$$S^{(+)\prime}(x)=\gamma_4\gamma_5 C\{S^{(+)\prime}(-\boldsymbol{x}, x_0)\}^T C^{-1}\gamma_5\gamma_4. \tag{A·5}$$

P-invariance

$$S^{(+)\prime}(x)=\gamma_4 S^{(+)\prime}(-\boldsymbol{x}, x_0)\gamma_4. \tag{A·6}$$

Under the assumption of CTP-invariance the propagation function,

$$S'_{F\alpha\beta}(x)=\left\langle T\left[\psi_\alpha\left(\frac{x}{2}\right), \bar{\psi}_\beta\left(-\frac{x}{2}\right)\right]\right\rangle_0, \tag{A·7}$$

is expressed as

$$S_F'(x)=\int_0^\infty d\mu^2\{\rho_1(\mu^2)S_F(x;\mu^2)+\rho_2(\mu^2)\Delta_F(x;\mu^2)$$
$$+\rho_3(\mu^2)\gamma_5 S_F(x;\mu^2)+\rho_4(\mu^2)\gamma_5\Delta_F(x;\mu^2). \tag{A·8}$$

If T- or P-invariance is assumed in addition to the CTP-invariance, it follows that

$$\rho_3(\mu^2)\equiv\rho_4(\mu^2)\equiv 0. \tag{A·9}$$

In order to get a definition of the mass m and the lifetime γ^{-1} of an unstable spinor particle it seems convenient to use complex four-vectors, $\tilde{p}_\mu(=(\kappa/m)p_\mu)$, introduced in § 5. A spinor $u(\tilde{p})$ is defined by the Dirac equation with the complex mass κ,

$$(i\gamma\cdot\tilde{p}+\kappa)u(\tilde{p})=0. \tag{A·10}$$

As is easily seen it is equal to a constant multiple of the free spinor, $u(p)$. The complex mass κ may then be determined by

$$[\bar{u}(\tilde{p})S_F'(\tilde{p})u(\tilde{p})]^{-1}=0, \tag{A·11}$$

where the Fourier transform of the propagation function should be analytically continued in an 'appropriate' way explained below. Because of the equalities,

$$\bar{u}(\tilde{p})\gamma_5 u(\tilde{p})=\bar{u}(\tilde{p})\gamma_5\gamma_\mu u(\tilde{p})=0, \tag{A·12}$$

the parity-nonconserving terms in the expression (A·8) are not relevant for the definition of κ. Then we introduce a function defined by

$$F(z)=-i\int_0^\infty d\mu^2\frac{(-\sqrt{z}+\mu)\rho_1(\mu^2)+\rho_2(\mu^2)}{\mu^2-z}, \tag{A·13}$$

and we consider an analytic continuation of this function in the complex z-plane with the cut shown in Fig. 5. The definition of the complex mass determined by a pole appearing under the real positive axis gives a clear meaning to the intuitive definition of (A·11).

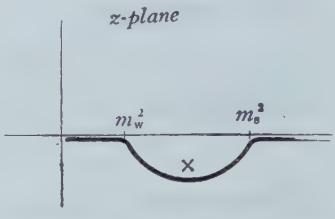

Fig. 5. The branch cut in the complex z-plane for the function $F(z)$. The cross represents the pole of $F(z)$ at $z=\kappa^2$.

We may normalize the field operator $\psi(x)$ by

$$\lim_{z\to\kappa^2}(z-\kappa^2)F(z)=e^{i\alpha} \tag{A·14}$$

with a real constant α. Although the way of normalization is not a unique one, it coincides with that of Lehmann[7] for a stable particle if decay interactions were switched off hypothetically. The renormalization constant is then given by

$$Z_2^{-1}=\int_0^\infty d\mu^2\rho_1(\mu^2). \tag{A·15}$$

References

1) H. Lehmann, K. Symanzik and W. Zimmermann, Nuovo Cimento **1** (1955), 205
2) H. Lehmann, K. Symanzik and W. Zimmermann, Nuovo Cimento **6** (1957), 319
V. Glaser, H. Lehmann and W. Zimmermann, Nuovo Cimento **6** (1957), 1122.
3) K. Nishijima, Prog. Theor. Phys. **17** (1957), 765; Phys. Rev. **111** (1958), 995.
4) E. C. G. Stueckelberg, Phys. Rev. **81** (1951), 130.
5) A. Wightman, Phys. Rev. **101** (1956), 806.
6) K. Nishijima, Prog. Theor. Phys. **10** (1953), 549.
7) H. Lehmann, Nuovo Cimento **11** (1954), 342.
8) H. Araki, Y. Munakata, M. Kawaguchi and T. Gotô, Prog. Theor. Phys. **17** (1957), 419.
9) K. Yamamoto, Prog. Theor. Phys. **20** (1958), 857.
10) R. E. Peierls, *Proc. of the* 1954 *Glasgow Conf.* (London, 1955), p. 296.
M. Lévy, Nuovo Cimento **13** (1959), 115.
11) G. R. Allcock, Phys. Rev. **104** (1956), 1799.
12) M. Gell-Mann and F. Low, Phys. Rev. **84** (1951), 350.
13) N. Nakanishi, Prog. Theor. Phys. **20** (1958), 822.
14) P. T. Matthews and A. Salam, Phys. Rev. **112** (1958), 283.
15) C. Møller, Mat. Fys. Skr. Dan. Vid. Selsk. **22**, No. 19 (1945–46).

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On Some Peculiar Role of the Cosmological Constant in the General Theory of Relativity and in the Schrödinger Theory of Non-Symmetric Field

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In this paper, some peculiar role of the so-called cosmological constant Λ is investigated from the two alternative standpoints of the general theory of relativity and the Schrödinger theory of non-symmetric field. It is revealed then that, in spite of a rather drastically different character of the latter theory from the former, there exists a peculiar solution whose existence is assured by the role of Λ in question and whose physical meaning is equivalent to that of Bertotti's solution in general relativity.

§ 1. Introduction

As is well-known, the so-called cosmological (or Λ-) term is originally inserted into Einstein's gravitational equations in 1916 with the intention of weaving Mach's principle into them. However, not only is such an idea broken down by de Sitter through his empty model-universe, but we have nowadays no positive reason to retain this term.[1] There is of course an opposite opinion that the cosmological constant Λ should be retained as an adjustable parameter included in various theoretical formulae (in the general relativistic cosmology) to be compared with the observational data. But this is barely the case at the cost of spoiling one important function of a theory, i.e. the prediction of phenomena,* because the field equations themselves are not definite from this standpoint.

In addition to these situations, it is found that Einstein's field equations with Λ-term permit us to obtain some peculiar cosmological solutions[2),3)] in which the transition $\Lambda \to 0$ is intrinsically prohibited in contrast with the situation in the de Sitter model-universe. The peculiarities of Gödel's model,[2)] however, lie in the existence of "spin" throughout the universe as a whole rather than in the role of the cosmological constant in his model, so we shall not go further into this model. On the other hand, the main peculiarity of Nariai's model,[3)] in which Λ should be positive, lies in the decomposition of space-time into two kinds of 2-spaces with constant curvatures. Recently, Bertotti[4)] has found another peculiar solution

* It must be remarked that there is no adjustable parameter in the Schwarzschild solution in the 1916 theory, because the only integration constant "m" in it is determined at the stage of Newtonian approximation.

which is a generalization of Nariai's such that a uniform (constant) electromagnetic field is attached to the latter.

Now, in contrast with the general theory of relativity, the Λ-term appears as an inevitable request in the Schrödinger[5] theory of non-symmetric field, i. e. in the Schrödinger theory Λ should be intrinsically non-vanishing, though his field equations are formally reduced to those of Einstein's[6] generalized theory of gravitation if we put $\Lambda=0$. In this respect, the cosmological constant Λ in Schrödinger's theory should be regarded as a fundamental constant* (though we have not got as yet any information concerning its sign and numerical value), but not as a mere adjustable parameter. Then it will be interesting to inquire whether or not such a peculiar role of Λ as found in general relativity is also played in the Schrödinger theory. The reason is that there would be a slight possibility that a peculiar solution of Nariai-Bertotti's type is automatically prohibited in his theory, if we take it into consideration that it is not a mere unified field theory of gravitation and electromagnetism, but a kind of the unitary theory of field.

In Part I of this paper, we shall compare Nariai's** space-time with de Sitter's empty model-universe and then shall investigate several properties of spherical electromagnetic waves in Bertotti's space-time, in order to illustrate the peculiarities of a curved space-time of Nariai-Bertotti's type. In Part II, we shall verify that, in the Schrödinger theory as well, there exists such a peculiar solution that the symmetric part $g_{\mu\nu}$ of the non-symmetric field $g_{\mu\nu}(\neq g_{\nu\mu})$ is of the same form as the metric tensor of Nariai's space-time (cf. (2·8)), and that several components of the anti-symmetric part $g_{\mu\nu}$ are non-vanishing.

If we introduce the metric tensor into Schrödinger's theory in a suitable manner and stipulate various physical quantities describing the electromagnetic field in conformity with the metric tensor thus introduced, we may say that our solution has the same physical meaning as Bertotti's solution of Einstein-Maxwell's equations in general relativity.

Part I

§ 2. Comparison between de Sitter's empty model-universe and Nariai's peculiar one

Before entering into the main subject, we shall compare de Sitter's empty space-time with Nariai's peculiar one, both of them being derived from Einstein's field equations for an empty region, i. e.

$$\mathring{R}_{\mu\nu}=\Lambda\mathring{g}_{\mu\nu},\ (\mu,\nu=0,1,2,3). \qquad (2\cdot1)^{***}$$

* Here it must be remarked that the phrase "cosmological" attached to Λ is due to its historical origin.

** We shall hereafter use this terminology for the peculiar space-time mentioned before.

*** In order to discriminate from the non-symmetric field, the symbol "°" is attached to $g_{\mu\nu}$ and $R_{\mu\nu}$ on the top of them.

It is well known that the metric for a spherically symmetric space-time can in general be written as

$$ds^2=Cdt^2-Adr^2-r^2(d\theta^2+\sin^2\theta d\varphi^2), \tag{2·2}$$

or

$$ds^2=e^{\nu}d\bar{t}^2-e^{\mu}[d\bar{r}^2+\bar{r}^2(d\theta^2+\sin^2\theta d\varphi^2)], \tag{2·3}$$

where C and A are positive functions of (r, t), μ and ν are functions of $(\bar{r}, \bar{t})$; (2·2) is always reduced to (2·3) by a suitable transformation $(r, t)\rightarrow(\bar{r}, \bar{t})$ and vice versa, except in the case $e^{\mu}=\text{const}\cdot(\bar{r})^{-2}$.

According to Birkhoff's theorem, the only solution of (2·1) with (2·2) is

$$C=1/A=1-2m/r-\Lambda r^2/3, \tag{2·4}$$

where m is an integration constant. This solution is reduced to the Schwarzschild solution by the transition $\Lambda\rightarrow 0$. On the other hand, putting $m=0$, we obtain de Sitter's empty space-time. Then we have

$$ds^2=(1-r^2/\Re^2)\,dt^2-\frac{dr^2}{(1-r^2/\Re^2)}-r^2(d\theta^2+\sin^2\theta d\varphi^2), \tag{2·5}$$

where $\Lambda\Re^2=3$. If we put

$$\left.\begin{aligned}\rho&=r\exp(-t/\Re)/(1-r^2/\Re^2)^{1/2},\\ \tau&=t+(1/2)\Re\ln(1-r^2/\Re^2),\end{aligned}\right\} \tag{2·6}$$

(2·5) can be rewritten as

$$ds^2=d\tau^2-\exp(2\tau/\Re)[d\rho^2+\rho^2(d\theta^2+\sin^2\theta d\varphi^2)], \tag{2·7}$$

which belongs to the class specified by (2·3).

A slight revision is however necessary to the above theorem, because, as was shown by one of the present authors (H.N.),[3] there exists such a solution of (2·1) with (2·3) as $e^{\mu}\cdot(\bar{r})^2=\text{constant}$.* This solution can also be represented by the following metric:

$$ds^2=(1-r^2/a^2)\,dt^2-\frac{dr^2}{(1-r^2/a^2)}-a^2(d\theta^2+\sin^2\theta d\varphi^2), \tag{2·8}$$

in which the coordinates (t, r, θ, φ) are analogous to those in (2·5), where $\Lambda a^2=1$. If we perform the transformation of coordinates (2·6) with a in place of $\Re$, (2·8) is reduced to

$$ds^2=d\tau^2-\exp(2\tau/a)\,d\rho^2-a^2(d\theta^2+\sin^2\theta d\varphi^2). \tag{2·9}$$

As is easily seen from the above two expressions, Λ should be positive in order that the signature of the metric tensor $\overset{\circ}{g}_{\mu\nu}$ is $(+---)$, in addition to the fact that it is impossible in the above metrics to take the limit $\Lambda\rightarrow 0$, in contrast

* Such a situation is intimately connected with the fact that, as was shown by Takeno,[7] it is insufficient to take (2·2) as the most general expression of the spherically symmetric line-element.

with the situation in de Sitter's metric specified by (2·5) or (2·7).

So far, comparison has been made in referring to special kinds of coordinate systems. The essential difference between de Sitter's and Nariai's space-times lies however in the following points: The former represents a 4-space with constant curvature, because the metric tensor $\mathring{g}_{\mu\nu}$ satisfies not only (2·1), but also

$$\mathring{R}_{\lambda\mu\nu\sigma}=(\Lambda/3)(\mathring{g}_{\lambda\sigma}\mathring{g}_{\mu\nu}-\mathring{g}_{\lambda\nu}\mathring{g}_{\mu\sigma}). \tag{2·10}$$

On the other hand, the latter has such a decomposed structure that both of the 2-spaces, i. e. (t, r)· and (θ, φ)-spaces, are of constant curvature $1/a^2$. As a result, the latter is represented as a section of two 5-dimensional hyper-spheres embedded in the Euclidean 6-space.[3]

§ 3. Spherical electromagnetic waves in Bertotti's space-time

In order to illustrate more clearly the peculiarity of a curved space-time with metric of the type (2·8), we shall look into some properties of the spherical electromagnetic wave in Bertotti's space-time whose metric is, say, of the form

$$ds^2=d\tau^2-\exp(2\tau/b)\,d\rho^2-a^2(d\theta^2+\sin^2\theta d\varphi^2), \tag{3·1}$$

where b is a constant not necessarily equal to a (cf. (2·9)). The coordinates $(\tau, \rho, \theta, \varphi)$ in (3·1) can be regarded as corresponding to those in the metric (2·7) for de Sitter's space-time, the latter being usually called the co-moving coordinates in the sense that both an observer and light sources are at rest relative to them.

On the other hand, Maxwell's equations (without 4-current) for the electromagnetic field $F_{\lambda\mu}$ in a curved space-time with metric $ds^2=\mathring{g}_{\mu\nu}dx^\mu dx^\nu$ can be written as

$$\left.\begin{aligned} &F_{\lambda\mu,\nu}+F_{\mu\nu,\lambda}+F_{\nu\lambda,\mu}=0,\\ &\{(-\mathring{g})^{1/2}F^{\mu\nu}\}_{,\nu}=0,\end{aligned}\right\} \tag{3·2}$$

where $\mathring{g}=\det(\mathring{g}_{\mu\nu})$. Inserting $\mathring{g}_{\mu\nu}$ in (3·1) into (3·2), we obtain

$$\left.\begin{aligned} &(a\cdot\sin\theta H_\rho)_{,\rho}+(e^{\tau/b}\sin\theta H_\theta)_{,\theta}+(e^{\tau/b}H_\varphi)_{,\varphi}=0,\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ &(a\cdot e^{\tau/b}H_\varphi)_{,\tau}+(aE_\theta)_{,\rho}-(e^{\tau/b}E_\rho)_{,\theta}=0,\\ &(a\cdot\sin\theta E_\rho)_{,\rho}+(e^{\tau/b}\sin\theta E_\theta)_{,\theta}+(e^{\tau/b}E_\varphi)_{,\varphi}=0,\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ &-(a\cdot e^{\tau/b}E_\varphi)_{,\tau}+(aH_\theta)_{,\rho}-(e^{\tau/b}H_\rho)_{,\theta}=0,\end{aligned}\right\} \tag{3·3}$$

where $(E_\rho, E_\theta, E_\varphi, H_\rho, H_\theta, H_\varphi)$ stand for the physical components[8] (scalars) of $(F_{10}, F_{20}, F_{30}, F_{23}, F_{31}, F_{12})$ respectively.

The metric (3·1) is form-invariant under the transformation of coordinate, $\rho\to\rho+$const., so that, though the same symbol is used, ρ in (3·1) has a somewhat

different meaning from the radial coordinate in (2·7). In spite of such a situation, we shall hereafter call a progressive wave in the direction of ρ the "spherical" wave. Then (3·3) can be solved as follows:

i) TM-wave ($H_\rho=0$ and $E_\rho=u/a^2$):

$$\left.\begin{aligned} E_\theta&=\frac{\exp(-\tau/b)}{l(l+1)a}\frac{\partial^2 u}{\partial\theta\partial\rho}, & H_\theta&=\frac{1}{l(l+1)a\cdot\sin\theta}\frac{\partial^2 u}{\partial\varphi\partial\tau},\\ E_\varphi&=\frac{\exp(-\tau/b)}{l(l+1)a\cdot\sin\theta}\frac{\partial^2 u}{\partial\varphi\partial\rho}, & H_\varphi&=-\frac{1}{l(l+1)a}\frac{\partial^2 u}{\partial\theta\partial\tau}, \end{aligned}\right\} \tag{3·4}$$

ii) TE-wave ($E_\rho=0$ and $H_\rho=v/a^2$):

$$\left.\begin{aligned} H_\theta&=\frac{\exp(-\tau/b)}{l(l+1)a}\frac{\partial^2 v}{\partial\theta\partial\rho}, & E_\theta&=-\frac{1}{l(l+1)a\cdot\sin\theta}\frac{\partial^2 v}{\partial\varphi\partial\tau},\\ H_\varphi&=\frac{\exp(-\tau/b)}{l(l+1)a\cdot\sin\theta}\frac{\partial^2 v}{\partial\varphi\partial\rho}, & E_\varphi&=\frac{1}{l(l+1)a}\frac{\partial^2 v}{\partial\theta\partial\tau}, \end{aligned}\right\} \tag{3·5}$$

in which both u and v are functions of the type

$$u(\text{or } v)=\Xi(\xi)\,e^{-ik\rho}P_l^m(\cos\theta)\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}m\varphi, \tag{3·6}$$

with

$$\frac{d^2\Xi}{d\xi^2}+\left\{k^2+\left(\frac{b}{a}\right)^2\cdot\frac{l(l+1)}{\xi^2}\right\}\Xi=0, \tag{3·7}$$

where

$$\xi\equiv-b\cdot\exp(-\tau/b), \tag{3·8}$$

and others with the usual notation.

Now we shall consider only the TM-wave, because almost all of the following discussions are also applied to the TE-wave. As is recognized from (3·6)–(3·8), the separation constant l concerning the angle θ contributes to the temporal (but not spatial) factor $\Xi(\xi)$ of u, and the amplitude of u does not depend on the radial coordinate ρ. Accordingly, we cannot identify, say, the case $l=1$ and $m=0$ with the dipole field in the true sense of words. What properties has this dipole-like field in our space-time, then? From (3·7) and (3·8), we obtain

$$\Xi\simeq\text{const.}\times e^{ik\xi}, \tag{3·9}$$

provided that

$$k\cdot a\gg 2^{1/2}\cdot\exp(\tau/b)\quad\text{for } l=1. \tag{3·10}$$

Inserting (3·9) into (3·6) with $l=1$ and $m=0$, we obtain

$$u\simeq u_0\cdot\sin\{k(\xi-\rho)\}\cos\theta, \tag{3·11}$$

where u_0 is an arbitrary constant, and we have replaced the factor $e^{ik(\xi-\rho)}$ by $\sin\{k(\xi-\rho)\}$ without loss of generality. From (3·4) and (3·11), we obtain

$$\left.\begin{aligned} E_\rho &\simeq (u_0/a^2)\sin\{k(\xi-\rho)\}\cos\theta, \\ E_\theta &\simeq H_\varphi \simeq (k\cdot a/2e^{\tau/b})\,(u_0/a^2)\cos\{k(\xi-\rho)\}\sin\theta, \end{aligned}\right\} \tag{3·12}$$

and other components $=0$.

Now let W and $\mathfrak{S}=(S_\rho, S_\theta, S_\varphi)$ be the physical components of the energy density and Poynting's vector respectively for the radiation field in question.[8] Then, for (3·12) we have

$$W \simeq S_\rho \simeq (k\cdot u_0/2ae^{\tau/b})^2\cos^2\{k(\xi-\rho)\}\sin^2\theta, \tag{3·13}$$

with $|S_\theta/S_\rho|\simeq 0$ (cf. (3·10)) and $S_\varphi=0$. Since the temporal variation of the amplitude of W per one cycle of oscillation is negligible (cf. (3·10) and (3·8)), the time mean of (3·13) is

$$\overline{W} \simeq \overline{S}_\rho \simeq (3L/8\pi a^2)\sin^2\theta. \tag{3·14}$$

where L is the total radiation defined by

$$L \equiv \int_0^{2\pi} d\varphi \int_0^{\pi} \overline{S}_\rho \cdot a^2 \sin\theta\, d\theta, \quad \text{(cf. (3·1))}. \tag{3·15}$$

The average values of $\overline{W}$ and $\overline{S}$ per unit solid angle are, then,

$$\langle \overline{W} \rangle \simeq \langle \overline{S}_\rho \rangle \simeq L/4\pi a^2, \tag{3·16}$$

with

$$L = L_0 \cdot \exp(-2\tau/b), \tag{3·17}$$

where $L_0 \equiv (\pi/3)\,(ku_0)^2$.

Since $\langle \overline{S}_\rho \rangle$ physically means the mean flux of radiation per unit time and per unit solid angle, the expression (3·16) shows us that, in contrast with the situation in de Sitter's space-time, the apparent luminosity of light source at the origin is independent of the radial coordinate ρ specifying the position of an observer. Thus, in Bertotti's space-time, the luminosity distance[9] d_L between a source and an observer can be defined by

$$d_L = a, \tag{3·18}$$

irrespective of the value of ρ. On the other hand, if we define the distance in question from the standpoint of appealing to the geodesic deviation of null-geodesics,[10] we obtain

$$d_G = \rho \cdot \exp(\tau/b), \tag{3·19}$$

as shown in the Appendix. Such a discrepancy of two kinds of distance, both being defined in a covariant manner, is of course due to the peculiarity of Bertotti's space-time, while, say, in de Sitter's space-time we have[10]

$$d_L = d_G = \rho \cdot \exp(\tau/\mathfrak{R}). \tag{3·20}$$

Part II

§ 4. Schrödinger's equations for the non-symmetric field and their peculiar solution

According to Schrödinger,[5] the field equations for the non-symmetric field $g_{\mu\nu}$ can be written as

$$\left.\begin{aligned}
&(\boldsymbol{E}_1): \quad g_{\underset{+-}{\mu\nu};\sigma} \equiv g_{\mu\nu,\sigma} - \Gamma^{\lambda}_{\mu\sigma} g_{\lambda\nu} - \Gamma^{\lambda}_{\sigma\nu} g_{\mu\lambda} = 0, \\
&(\boldsymbol{E}_2): \quad \Gamma^{\sigma}_{\underset{\vee}{\mu\sigma}} = 0, \\
&(\boldsymbol{E}_3): \quad R_{\underline{\mu\nu}} = \Lambda g_{\underline{\mu\nu}}, \\
&(\boldsymbol{E}_4): \quad R_{\underset{\vee}{[\mu\nu,\sigma]}} = \Lambda g_{\underset{\vee}{[\mu\nu,\sigma]}},
\end{aligned}\right\} \tag{4·1}$$

with the usual notation. If we put $k_{\mu\nu} \equiv g_{\mu\nu} = 0$, $(\boldsymbol{E}_2)$ and $(\boldsymbol{E}_4)$ are satisfied auto matically and $(\boldsymbol{E}_3)$ with $(\boldsymbol{E}_1)$ is reduced to (2·1) with $\overset{\circ}{g}_{\mu\nu} = h_{\mu\nu} (\equiv g_{\mu\nu})$. Accordingly, we have only to consider such a case that at least some of k's do not vanish, in order to examine whether or not (4·1) automatically prohibits a peculiar solution as elucidated in Part I.

Following (2·8), we assume that the non-symmetric field $g_{\mu\nu}$ is spherically symmetric and static. Then we have, without loss of generality,[11]*

$$g_{\mu\nu} = \begin{pmatrix} C & -\alpha & 0 & 0 \\ \alpha & -A & 0 & 0 \\ 0 & 0 & -B & \beta\sin\theta \\ 0 & 0 & -\beta\sin\theta & -B\sin^2\theta \end{pmatrix}, \tag{4·2}$$

where $A(>0)$, $B(>0)$, $C(>0)$, α and β are functions of r** only. It is easily seen that $B(r)$ in the above $g_{\mu\nu}$ corresponds to the constant a^2 in (2·8), so that we assume as follows

$$B = \text{const.} \quad (>0). \tag{4·3}$$

Now it is generally impossible to solve $(\boldsymbol{E}_1)$ uniquely with respect to Γ's when $g_{\mu\nu}$ is given by (4·2), as shown in [0], i.e. a necessary and sufficient condition that $(\boldsymbol{E}_1)$ is solved uniquely with respect to Γ's is that we have

$$B^2 \neq \beta^2 \quad \text{or} \quad \alpha \neq 0. \tag{4·4}$$

(In the following, we shall omit an exceptional case $B^2 = \beta^2$ and $\alpha = 0$ for brevity's sake.) It is difficult to find out a general condition such that the assumption (4·3) contradicts with the field equations (4·1) for $g_{\mu\nu}$ specified by (4·2) and (4·4), though we can prove that it is the case when $\alpha = 0$ and $\beta \neq$ const. Accordingly,

---

* We shall hereafter refer to this paper as [0].

** Here the spherical polar coordinates are used, such that $x^0 = t$, $x^1 = r$, $x^2 = \theta$, $x^3 = \varphi$.

we shall adopt another method of approach later on.

Generalizing (4·3), we assume that

$$B=\text{const.}\ (>0)\ \text{and}\ \beta=\text{const.} \tag{4·5}$$

Then, from (4·4), we must have

$$\text{i)}\ \alpha\neq 0,\quad \text{or}\quad \text{ii)}\ B^2\neq\beta^2\ \text{ if }\ \alpha=0. \tag{4·6}$$

Solving $(\boldsymbol{E}_1)$ with (4·2), (4·5) and (4·6), we obtain (cf. § 2, [0])

$$\left.\begin{aligned}
&\Gamma^1_{\underline{11}}=A'/A,\ \Gamma^3_{\underline{23}}=\cot\theta,\ \Gamma^2_{\underline{33}}=-\sin\theta\cos\theta,\\
&\Gamma^0_{\underline{10}}=(1/L)\,\{\alpha(\alpha'-\alpha A'/2A)-AC'/2\},\\
&\Gamma^1_{\underline{00}}=(1/AL)\,\{2C\alpha(\alpha'-\alpha A'/2A)-C'(\alpha^2+AC)/2\},\\
&\Gamma^1_{\underset{\vee}{10}}=(1/L)\,\{C(\alpha'-\alpha A'/2A)-\alpha C'/2\}\\
&\qquad\text{(the other }\Gamma'\text{s}=0),
\end{aligned}\right\} \tag{4·7}$$

where the dash denotes differentiation with respect to r, and $L\equiv\alpha^2-AC$. Similarly we have

$$\left.\begin{aligned}
&R_{00}=-\Gamma^1_{\underline{00},1}-\Gamma^1_{\underline{00}}(\Gamma^1_{\underline{11}}-\Gamma^0_{\underline{10}})-(\Gamma^1_{\underset{\vee}{01}})^2,\\
&R_{11}=\Gamma^0_{\underline{10},1}+\Gamma^0_{\underline{10}}(\Gamma^0_{\underline{10}}-\Gamma^1_{\underline{11}}),\\
&R_{22}=R_{33}/\sin^2\theta=-1,\\
&R_{10}=-R_{01}=\Gamma^1_{\underset{\vee}{01},1},\ R_{23}=-R_{32}=0,\\
&\text{(the other }R\text{'s}=0).
\end{aligned}\right\} \tag{4·8}$$

Taking account of (4·7), we obtain from $(\boldsymbol{E}_2)$

$$\left.\begin{aligned}
&\text{i)}\ \text{The case }\alpha\neq 0:\ AC=(1+\gamma)\alpha^2,\ (\gamma=\text{const.}>0),\\
&\text{ii)}\ \text{The case }\alpha=0\text{ and }B^2\neq\beta^2:\ (\boldsymbol{E}_2)\text{ is automatically satisfied.}
\end{aligned}\right\} \tag{4·9}$$

Moreover, in each of the above alternative cases, $(\boldsymbol{E}_4)$ is satisfied automatically, since $R_{\underset{\sim}{\mu\nu,\sigma}}=g_{\underset{\sim}{\mu\nu,\sigma}}=0$. Substitution of (4·7) and (4·9) in (4·8) permits us to obtain

$$\left.\begin{aligned}
&R_{00}=-(C'/2A)'-(C'/2A)\,(A'/2A-C'/2C),\\
&R_{11}=(C'/2C)'+(C'/2C)\,(C'/2C-A'/2A),\\
&R_{22}=R_{33}/\sin^2\theta=-1.
\end{aligned}\right\} \tag{4·10}$$

Inserting (4·2) and (4·10) into $(\boldsymbol{E}_3)$, we get

$$C''-C'(C'/2C+A'/2A)+2\Lambda CA=0, \tag{4·11}$$

$$\Lambda B=1. \tag{4·12}$$

Since (4·12) is of the same form as $\Lambda a^2=1$ in (2·8), Λ should be positive in order that the signature of $h_{\mu\nu}$ is $(+---)$.

Now since (4·11) is the only equation for two unknowns A and C, we can apparently obtain no unique solution. But it is not the case, because such an arbitrariness is eliminated by suitable transformations of coordinates:

i) *The case* $\alpha \neq 0$. Taking account of (4·9), we shall put

$$r^* = (1+\gamma)^{1/2} \int |\alpha(r)| dr. \tag{4·13}$$

Then, in the coordinate system specified by $(t, r^*, \theta, \varphi)$ we have

$$C^* = 1/A^* = C,\ B^* = B,\ \alpha^* = \pm (1+\gamma)^{-1/2},\ \beta^* = \beta, \tag{4·14}$$

since $h_{\mu\nu}$ and $k_{\mu\nu}$ are a covariant tensor and W-tensor of the second rank respectively;[12] the last three expressions show us that the assumptions we have so far built up are preserved under the transformation of coordinate (4·13). From (4·11) and the first expression in (4·14), we obtain

$$\frac{d^2C^*}{dr^{*2}} + 2\Lambda = 0,$$

hence

$$C^* = c_1 + c_2 r^* - \Lambda r^{*2}, \tag{4·15}$$

where c_1 and c_2 are integration constants. If we further perform the following transformation of coordinates:

$$r^{**} = (r^* - c_2/2\Lambda)/(c_1 + c_2^2/4\Lambda)^{1/2},\ t^{**} = (c_1 + c_2^2/4\Lambda)^{1/2} \cdot t, \tag{4·16}$$

we obtain

$$\left.\begin{aligned} &C^{**} = 1/A^{**} = 1 - \Lambda r^{**2},\ B^{**} = B^* = B, \\ &\alpha^{**} = \alpha^* = \pm (1+\gamma)^{-1/2},\quad \beta^{**} = \beta^* = \beta. \end{aligned}\right\} \tag{4·17}$$

ii) *The case* $\alpha = 0$ *and* $B^2 \neq \beta^2$. Since A and C are positive functions of r only, we can put without loss of generality

$$AC = g^2(r), \text{ say.} \tag{4·18}$$

Then, through the transformation of coordinate such that

$$r^* = \int g(r)\, dr, \tag{4·19}$$

(A, B, C, α, β) are transformed as follows:

$$C^* = 1/A^* = C,\ B^* = B,\ \alpha^* = 0,\ \beta^* = \beta, \tag{4·20}$$

which is of the same form as (4·14), except for the third expression. Henceforth the situation is the same as that in the case i).

To sum up the above, it may be said that Schrödinger's field equations (4·1) are satisfied by the non-symmetric field of the form (4·2) with

$$C = 1/A = 1 - \Lambda r^2,\ AB = 1, \tag{4·21}$$

$$\alpha = \text{const.},\quad \beta = \text{const.} \tag{4·22}$$

If we interpret the symmetric part of $g_{\mu\nu}$, $h_{\mu\nu}$, as the metric tensor, then (4·21) is equivalent to the metric (2·8) for Nariai's space-time, in spite of the existence of non-vanishing $k_{\mu\nu}=g_{\underset{\sim}{\mu\nu}}$ specified by (4·22). If we take it into consideration that (1) (4·21) satisfies (4·1), even if $\alpha=\beta=0$, and (2) $k_{\mu\nu}$ is an antisymmetric W-tensor in terms of which the electromagnetic field could be described, it seems apparent that, so far as the above solution is concerned at least, the existence of an electromagnetic field does not influence upon the metric tensor, in contrast with the situation in Bertotti's solution of Einstein-Maxwell's equations in general relativity. But it is not the case, as shown in the following sections.

§ 5. Introduction of the Riemannian metric

In contrast with the general theory of relativity in its orthodox form, the metric tensor in the Schrödinger theory would be an *a posteriori* concept to be defined in a suitable manner, rather than a major premise of the theory. As far as we know, the symmetric part of $g_{\mu\nu}$, $h_{\mu\nu}$, is usally identified with the metric tensor, on account of the fact that his field equations (4·1) are automatically reduced to (2·1) in general relativity when $k_{\mu\nu}=0$. Such an identification is, however, not compulsory, because it seems to be more plausible to impose the following conditions as qualifications for the metric tensor $\mathring{h}_{\mu\nu}$ (say) in question:

(1) $\mathring{h}_{\mu\nu}$ should be a covariant symmetric tensor of the second rank which is automatically reduced to $h_{\mu\nu}$ in the limit when $k_{\mu\nu}\to 0$.

(2) $\mathring{h}_{\mu\nu}$ should be a tensor concomitant of $g_{\mu\nu}$ itself, because the field equations (4·1) are after all to be solved with respect to $g_{\mu\nu}$.

As is easily seen, the following expression satisfies these two conditions:

$$\mathring{h}_{\mu\nu}=h_{\mu\nu}+\zeta k_{\mu\alpha}k_{\nu\beta}h^{\alpha\beta}, \tag{5·1}$$

where $h^{\alpha\beta}h_{\alpha\gamma}=\delta_{\gamma}^{\ \beta}$, and ζ is a parameter whose sign and numerical value are not yet determined. Moreover, (5·1) is the simplest of the possible expressions for $\mathring{h}_{\mu\nu}$.[12] Accordingly, we shall stipulate the metric for a Riemannian space-time in the Schrödinger theory in the following way:

$$ds^2=\mathring{h}_{\mu\nu}dx^{\mu}dx^{\nu}. \tag{5·2}$$

Then, to what space-time does our peculiar solution obtained in § 4 correspond? From (4·2) and (5·1), we have

$$\mathring{h}_{\mu\nu}=\begin{bmatrix} (1-\zeta\alpha^2)C & 0 & 0 & 0 \\ 0 & -(1-\zeta\alpha^2)A & 0 & 0 \\ 0 & 0 & -a^2 & 0 \\ 0 & 0 & 0 & -a^2\sin^2\theta \end{bmatrix}, \tag{5·3}$$

where

$$a^2\equiv B(1+\zeta\beta^2/B^2). \tag{5·4}$$

In order that the signature of $h_{\mu\nu}$ is $(+\ -\ -\ -)$, we must have

$$\alpha^{-2}>\zeta>-(B/\beta)^2. \tag{5·5}$$

Inserting the second expression of (4·21) in (5·4), we get

$$\Lambda^{-2}-a^2\Lambda^{-1}+\zeta\beta^2=0,$$

or, since Λ should be positive irrespective of the sign and numerical value of ζ (cf. (4·21)),

$$\Lambda^{-1}=(a^2/2)\cdot[1+(1-4\zeta\beta^2/a^4)^{1/2}], \tag{5·6}$$

where ζ must satisfy the following inequality relation:

$$\zeta\leq(a^2/2\beta)^2. \tag{5·7}$$

Similarly, we have from (5·6)

$$a^2\leq 2/\Lambda. \tag{5·8}$$

If we insert (5·3) into (5·2) with $x^\mu=(t, r, \theta, \varphi)$, we obtain

$$ds^2=(1-\zeta\alpha^2)(C\cdot dt^2-dr^2/C)-a^2(d\theta^2+\sin^2\theta d\varphi^2), \tag{5·9}$$

where the first expression of (4·21) is used. Now let us perform such a transformation of coordinates as

$$\bar{t}=(1-\zeta\alpha^2)^{1/2}\cdot t, \qquad \bar{r}=(1-\zeta\alpha^2)^{1/2}\cdot r. \tag{5·10}$$

Then (5·9) is rewritten as

$$ds^2=e^\nu\cdot d\bar{t}^2-e^{-\nu}\cdot d\bar{r}^2-a^2(d\theta^2+\sin^2\theta d\varphi^2), \tag{5·11}$$

with

$$e^\nu=1-\bar{r}^2/b^2, \tag{5·12}$$

where (cf. (5·6))

$$b=a\cdot\left[\frac{1}{2}(1-\zeta\alpha^2)\{1+(1-4\zeta\beta^2/a^4)^{1/2}\}\right]. \tag{5·13}$$

The expression (5·13) is also rewritten as

$$1-(b/a)^2=\zeta\cdot[\alpha^2+(\beta/ab)^2(1-\zeta\alpha^2)^2],$$

so that

$$b/a\gtreqless 1 \text{ according as } \zeta\gtreqless 0. \tag{5·14}$$

Since b is not equal to a in general, the metric (5·11) with (5·12) shows us that our peculiar solution specified by (4·21) and (4·22) corresponds to Bertotti's space-time mentioned before.[4] It must be remarked however that such a correspondence between our solution and Bertotti's is due to the fact that the metric tensor in the Schrödinger theory is defined by (5·1).

From the above reasoning, it is suggested that our solution has also an intimate connection with Bertotti's one in general relativity with respect to the

physical interpretation of the electromagnetic field specified by $k_{\mu\nu}$ and $\mathring{h}_{\mu\nu}$. As a preparation for approaching this problem, we shall apply the transformation of coordinates (5·10) to $\mathring{h}_{\mu\nu}$ and $g_{\mu\nu}$ given by (5·3) and (4·2) with (4·21) and (4·22) respectively. Then we have

$$\mathring{\bar{h}}_{\mu\nu}=\begin{bmatrix} e^{\nu} & 0 & 0 & 0 \\ 0 & -e^{-\nu} & 0 & 0 \\ 0 & 0 & -a^2 & 0 \\ 0 & 0 & 0 & -a^2\sin^2\theta \end{bmatrix}, \tag{5·15}$$

and

$$\bar{g}_{\mu\nu}=\begin{bmatrix} \dfrac{e^{\nu}}{(1-\zeta\alpha^2)} & -\dfrac{\alpha}{(1-\zeta\alpha^2)} & 0 & 0 \\ \dfrac{\alpha}{(1-\zeta\alpha^2)} & -\dfrac{e^{-\nu}}{(1-\zeta\alpha^2)} & 0 & 0 \\ 0 & 0 & -B & \beta\sin\theta \\ 0 & 0 & -\beta\sin\theta & -B\sin^2\theta \end{bmatrix}. \tag{5·16}$$

§ 6. The electromagnetic field predicted by our solution

At the present stage of the theories of non-symmetric field, there are no definite definitions for various physical quantities, such as the field strength $F_{\lambda\mu}$, the current density $\mathfrak{J}^{\alpha}$ and the energy-momentum tensor $E_{\mu\nu}$, etc., describing the electromagnetic field. Such a situation is mainly due to the fact that these theories are unitary theories of field, so that, for instance, $\mathfrak{J}^{\alpha}$ should be defined without recourse to the concept of matter field. Another reason is that there exists a variety of possibilities in the lowering and raising of tensorial indices for any tensor, this arbitrariness being due to the reducibility of the non-symmetric field $g_{\mu\nu}$ itself. In order to eliminate the second ambiguity, however, we shall exclusively use the metric tensor $\mathring{h}_{\mu\nu}$ or its reciprocal $\mathring{h}^{\mu\nu}$ ($\mathring{h}^{\mu\nu}\mathring{h}_{\mu\lambda}=\delta_{\lambda}^{\nu}$) in the lowering or raising of tensorial indices.

Though the first ambiguity cannot be eliminated, we shall henceforth adopt the following definitions for $F_{\lambda\mu}$, $\mathfrak{J}^{\alpha}$ and $E_{\mu\nu}$:

(i) *The field strength.* According to Ikeda[13]* and Hlavatý,[14] we have a strong reason to stipulate that

$$F_{\lambda\mu}=\epsilon_{\lambda\mu\alpha\beta}g^{\alpha\beta}(-g)^{1/2}/2!, \tag{6·1}$$

where $\epsilon_{\lambda\mu\alpha\beta}$ is Levi-Civita's symbol, and $g^{\alpha\beta}g_{\alpha\gamma}=\delta_{\gamma}^{\beta}$, $g=\det(g_{\mu\nu})$. In order to establish a comparison between this and Maxwell's theory, we shall further define the electric field $\mathfrak{E}$ and magnetic field $\mathfrak{H}$ as follows:

* His two papers will be cited as [I] and [II] hereafter.

$$E_i = F_{io},\quad H_i = -\mathring{h}_{ij}\epsilon^{jsto}F_{st}/2!(-h)^{1/2},\quad (i=1,2,3), \tag{6·2}$$

in which the latter is different from the corresponding one in [II] only through the appearance of $\mathring{h}_{ij}$ and $\mathring{h}$ in place of h_{ij} and g (not h!) respectively, where $\mathring{h}=\det(\mathring{h}_{\mu\nu})$.

(ii) *The current density.* The most important property of the current density $\mathfrak{J}^\alpha$ is its conservation character specified by $\mathfrak{J}^\alpha_{,\alpha}=0$. Such a character is automatically assured by the existence of a super-potential $\mathfrak{U}^{[\alpha\beta]}$ such that

$$\mathfrak{J}^\alpha = -\mathfrak{U}^{[\alpha\beta]}{}_{,\beta}. \tag{6·3}$$

In view of the monistic character of the Schrödinger theory, we shall postulate that $\mathfrak{U}^{[\alpha\beta]}$ should be a contravariant anti-symmetric tensor density of the second rank constructed from $g_{\mu\nu}$. Then we have, say,*

$$\mathfrak{U}^{[\alpha\beta]} = \left\{\begin{matrix} \epsilon^{\alpha\beta\lambda\mu}k_{\lambda\mu}/2! \\ \text{or} \\ \mathring{F}^{\alpha\beta}\cdot(-\mathring{h})^{1/2} \end{matrix}\right\}, \tag{6·4}$$

where $\mathring{F}^{\alpha\beta}=h^{\alpha\lambda}h^{\beta\mu}F_{\lambda\mu}$; the former corresponds to Einstein's[6] expression for $\mathfrak{J}^\alpha$, the latter being a generalization of Hlavaty's[14] definition.

(iii) *The energy-momentum tensor.* As far as we know, there has been made no proposal as to the expression for $E_{\mu\nu}$, so that as for the energy-momentum tensor we shall stipulate that

$$E_{\mu\nu} = -F_{\mu\alpha}F_{\nu\beta}\mathring{h}^{\alpha\beta} + \frac{1}{4}\mathring{h}_{\mu\nu}F_{\alpha\beta}\mathring{F}^{\alpha\beta}, \tag{6·5}$$

in a manner analogous to Maxwell's theory.

According to the above definitions, we shall now look into what electromagnetic field is predicted from our solution obtained in § 4. If we make use of (5·16),** (6·1) enables us to obtain

$$F_{10} = \left(\frac{1-\alpha^2}{B^2+\beta^2}\right)^{1/2}\cdot\frac{\beta}{(1-\zeta\alpha^2)},\quad F_{23} = -\left(\frac{B^2+\beta^2}{1-\alpha^2}\right)^{1/2}\cdot\alpha\sin\theta, \tag{6·6}$$

and other components=0. Inserting (5·15) and (5·16) into (6·2), we have

$$E_1 = F_{10},\quad H_1 = -\left(\frac{B^2+\beta^2}{1-\alpha^2}\right)^{1/2}\cdot(\alpha/a^2)\,e^{-\nu}, \tag{6·7}$$

whose physical components are given by

$$\left.\begin{aligned} E_r &\equiv (-\mathring{h}^{00}\mathring{h}^{11})^{1/2}\cdot E_1 = \left(\frac{1-\alpha^2}{B^2+\beta^2}\right)^{1/2}\cdot\frac{\beta}{(1-\zeta\alpha^2)}, \\ H_r &\equiv (-\mathring{h}_{00}\mathring{h}^{11})^{1/2}\cdot H_1 = -\left(\frac{B^2+\beta^2}{1-\alpha^2}\right)^{1/2}\cdot(\alpha/a^2). \end{aligned}\right\} \tag{6·8}$$

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* Arbitrariness in the definition of the required super-potential is similar to that in the definition of the energy-momentum complex of the gravitational field in general relativity.[15]

** From now on we shall omit the bar-symbol on the top of t, r, and g's.

These expressions show us that both $\mathfrak{E}=(E_r, 0, 0)$ and $\mathfrak{H}=(H_r, 0, 0)$ are uniform (const.) and parallel to the direction of r-axis; this is the same as the situation in Bertotti's solution.

In the next, from (6·4), (6·6), (5·15) and (5·16), we obtain

$$\mathfrak{U}^{[\alpha\beta]}=\left[\begin{array}{cc|cc} 0 & \beta\sin\theta & & \\ -\beta\sin\theta & 0 & & 0 \\ \hline & & 0 & -\dfrac{\alpha}{(1-\zeta\alpha^2)} \\ & 0 & \dfrac{\alpha}{(1-\zeta\alpha^2)} & 0 \end{array}\right] \quad \text{or}$$

$$\left[\begin{array}{cc|cc} 0 & -E_r a^2\sin\theta & & \\ E_r a^2\sin\theta & 0 & & 0 \\ \hline & & 0 & H_r \\ & 0 & -H_r & 0 \end{array}\right]. \tag{6·9}$$

according to the alternative definitions in (6·4). Inserting (6·9) into (6·3), we have

$$\mathfrak{I}^{\alpha}=0, \quad \text{for both } \mathfrak{U}^{[\alpha\beta]}\text{'s}, \tag{6·10}$$

which is also equivalent to the corresponding equation in Bertotti's solution.

Moreover, from (6·6), (5·15) and (6·8), we have

$$F_{\alpha\beta}\mathring{F}^{\alpha\beta}=2\cdot[(F_{10})^2\mathring{h}^{00}\mathring{h}^{11}+(F_{23})^2\mathring{h}^{22}\mathring{h}^{33}]=2(\mathfrak{H}^2-\mathfrak{E}^2). \tag{6·11}$$

Inserting (5·15), (6·6) and (6·11) into (6·5), we obtain the following physical components of E's:

$$W\equiv E_{00}/(\mathring{h}_{00})=\sigma, \tag{6·12}$$

$$\left.\begin{aligned} &E_{rr}\equiv E_{11}/(-\mathring{h}_{11})=-\sigma, \quad (\text{other } E\text{'s}=0), \\ &E_{\theta\theta}\equiv E_{22}/(-\mathring{h}_{22})=\sigma, \\ &E_{\varphi\varphi}\equiv E_{33}/(-\mathring{h}_{33})=\sigma, \end{aligned}\right\} \tag{6·13}$$

where

$$\sigma\equiv(\mathfrak{E}^2+\mathfrak{H}^2)/2. \tag{6·14}$$

It is remarkable that (6·11) and (6·12) with (6·14) are of the same form as the corresponding expressions in the usual Maxwell theory. However, such a similarity is broken down, if we use the expression due to Ikeda (cf. [II]) as the definition of H_i. From (6·12) and (6·14), we see that the positive-definite

character of the energy density is automatically assured. The relation (6·13) is characteristic of our present solution, and this together with (6·12) leads us to the following result: The eigenvalues of the matrix $(E_{\mu\nu})$ are $(\sigma, \sigma, -\sigma, -\sigma)$, as in Bertotti's solution.

Summarizing the above, we may say that our peculiar solution of Schrödinger's field equations (4·1) is of the same physical meaning as Bertotti's in general relativity.

Remark. These peculiar roles of the cosmological constant in both theories seem to suggest us that, as soon as the constant is introduced, we have to be more careful in any discussion of the interrelationship between space-time and various physical fields in which gravitation plays a leading role. This is particularly the case in the cosmological problem, because we have a possibility to reconcile the peculiar situation in § 3 with the observational practice in the extragalactic astronomy, though we do not enter into this problem here.

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Appendix

(A) *Geodesic deviation of null-geodesics in Bertotti's space-time*

Now we shall adopt (3·1) as the metric for Bertotti's space-time, and let $x^\mu=(\tau, \rho, \theta, \varphi)$ be the co-moving coordinates such that the light source S is at the origin and the tangential vector λ^α to the world line of an observer O is written as

$$\lambda^\alpha=\delta_0{}^\alpha, \quad (\alpha=0, 1, 2, 3). \tag{A·1}$$

Next, we shall consider a null-geodesic, whose tangential vector is specified by $p^\alpha=dx^\alpha/d\sigma$ (σ is a non-vanishing parameter), passing through S and O. As is easily verified, it is stipulated by

$$p^0=b/\sigma, \quad p^1=k/\sigma^2, \quad p^2=p^3=0. \tag{A·2}$$

with

$$b\cdot\sigma=k\cdot\exp(\tau/b), \tag{A·3}$$

where k is an integration constant. Eliminating σ from (A·2) and (A·3), we have

$$\rho-\rho_0=-b\cdot\exp(-\tau/b), \tag{A·4}$$

where ρ_0 is an integration constant. From (A·3) and (A·4), we obtain

$$\sigma=k/(\rho_0-\rho). \tag{A·5}$$

On the other hand, the geodesic deviation specified by η^λ for null-geodesics should satisfy[10]

$$\frac{\partial^2\eta^\lambda}{\partial\sigma^2}+\mathring{R}^\lambda_{\alpha\nu\beta}p^\alpha\eta^\nu p^\beta=0, \tag{A·6}$$

with

$$\mathring{g}_{\alpha\beta}p^\alpha\eta^\beta=\mathring{g}_{\alpha\beta}\lambda^\alpha\eta^\beta=0. \tag{A·7}$$

Solving (A·6) and (A·7) in conjunction with (A·1)—(A·3), we obtain

$$\eta^0=\eta^1=0,\ \eta^2=A\sigma+B,\ \eta^3=C\sigma+D, \tag{A·8}$$

where A, B, C and D are integration constants. From (3·1) and (A·8), we get

$$\eta\equiv(\mathring{g}_{\alpha\beta}\eta^\alpha\eta^\beta)^{1/2}=a\cdot[(A\sigma+B)^2+(C\sigma+D)^2\sin^2\theta]^{1/2}. \tag{A·9}$$

Substitution of (A·5) into the above expression permits us to obtain

$$\eta=a\cdot\left[\left\{\frac{kA}{(\rho_0-\rho)}+B\right\}^2+\left\{\frac{kC}{(\rho_0-\rho)}+D\right\}^2\sin^2\theta\right]^{1/2}. \tag{A·10}$$

However, we have $\eta=0$ at $\rho=0$, since the source S from which the null-geodesic is issued is at $\rho=0$. Accordingly, we must have $B=-kA/\rho_0$ and $D=-kC/\rho_0$, $(\rho_0>0)$. Inserting these expressions into (A·10), we get

$$\eta=a\cdot|k|\cdot(A^2+C^2\sin^2\theta)^{1/2}/\rho_0(\rho_0-\rho), \tag{A·11}$$

where

$$\rho_0>\rho\geq 0, \tag{A·12}$$

because $\sigma\to\infty$ for $\rho\to\rho_0$.

(B) *Definition of spatial distance in terms of the geodesic deviation of null-geodesics*

There have been made many proposals[9] on the invariant definition of spatial distance between two time-like world lines in a curved space-time. The most recent of them is the one due to Newman and Goldberg.[10] According to them, the distance d_G between S and O is represented by

$$d_G=\Theta^{-1}\cdot\eta, \tag{B·1}$$

with

$$\Theta=d\eta/dl\Big|_{\mu=0};\ dl^2=(v_\alpha p^\alpha)^2d\sigma, \tag{B·2}$$

where η is a scalar given by (A·11) in the present case, Θ stands for the angle between the projected curves of two null-geodesics issued from S into the 3-space with metric dl^2 and v^α is the parallel transfer of λ^α (cf. (A·1)) along one of the two null-geodesics in question.

From the definition of v^α, we have $v_\alpha p^\alpha=p^0=b/\sigma$ (cf. (A·1) and (A·2)), therefore the second expression of (B·2) enables us to obtain

$$dl=b\cdot d(\ln\sigma). \tag{B·3}$$

From $(A\cdot 5)$, $(A\cdot 11)$ and $(B\cdot 3)$, we have

$$d\eta/dl=(a/b)\,|k|\cdot(A^2+C^2\sin^2\theta)^{1/2}/(\rho_0-\rho). \qquad (B\cdot 4)$$

Inserting this into the first expression of $(B\cdot 2)$, we obtain

$$\theta=(a/b)\,|k|\cdot(A^2+C^2\sin^2\theta)^{1/2}/\rho_0, \qquad (B\cdot 5)$$

Inserting $(A\cdot 11)$ and $(B\cdot 5)$ into $(B\cdot 1)$, we get

$$d_G=b\cdot\rho/(\rho_0-\rho),$$

or, if we make use of $(A\cdot 4)$,

$$d_G=\rho\cdot\exp(\tau/b). \qquad (B\cdot 6)$$

References

1) A. Einstein, *The Meaning of Relativity*, Princeton, 1953.
2) K. Gödel. Rev. Mod. Phys. **21** (1949), 447; Proc. Inter. Cong. Math, **1** (1950), 175.
3) H. Nariai, Sci. Rep. Tohoku Univ. **34** (1950), 160; **35** (1951), 62.
4) B. Bertotti, Phys. Rev., **116** (1959), 1331.
5) E. Schrödinger, *Space-Time Structure*, Cambridge, 1952
6) A. Einstein, loc. cit., Appendix II.
7) H. Takeno, J. Math. Soc. Japan **3** (1951), 317.
8) Y. Ueno, Prog. Theor. Phys. **12** (1954), 461.
9) See, for example, G. Temple, Proc. Roy. Soc. **A168** (1938), 122; H. Bondi, *Cosmology*, Cambridge, 1952.
10) E. Newman and J. N. Goldberg, Phys. Rev. **114** (1959), 1391.
11) H. Takeno, M. Ikeda and S. Abe, Prog. Theor. Phys. **6** (1951), 837.
12) M. Ikeda and S. Abe, Tensor **7** (1957), 59.
M. Ikeda, Tensor **7** (1957), 117.
13) M. Ikeda, Prog. Theor. Phys. **12** (1954), 17; **13** (1955), 265.
14) V. Hlavatý, Nieuw. Arch. Wisk. **(3)II** (1954), 103.
15) See, for example, J. N. Goldberg, Phys. Rev. **111** (1958), 315; P. G. Bergmann, Phys. Rev. **112** (1958), 287.

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On the Semi-Classical Treatments of the Meson-Nucleon Scattering

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The quantum mechanical treatment of the static meson-nucleon scattering which has close connection with the semi-classical one is presented. On this footing the semi-classical treatments of the p-wave pion-nucleon scattering, recently developed independently by Jackson and by Weisskopf, are examined. As a result it is shown that their treatments are not complete and that, if treated correctly, their effective range formula should be modified largely. This would result in the deviation from the experimental behavior. The applications to the charged and neutral scalar theories are also given and compared with the Castillejo-Dalitz-Dyson theory.

§ 1. Introduction and summary

As is well-known,[1] Chew and Low succeeded in explaining the large p-wave phase shift of the pion-nucleon scattering. They derived the integral equations for the T-matrix elements in the so-called one-meson approximation. By the excellent use of mathematical technics they solved these integral equations and obtained the effective range formula*

$$k^3 \cot \delta_\alpha/\omega = 1/\lambda_\alpha \cdot (1-\omega r_\alpha),$$

$$\lambda_\alpha = \frac{2}{3} f^2 \begin{bmatrix} -4 \\ -1 \\ 2 \end{bmatrix} \text{ for } \begin{cases} I=J=1/2 \\ I=1/2,\ J=3/2 \text{ and } I=3/2,\ J=1/2 \\ I=J=3/2, \end{cases} \quad (1)$$

where k and ω are pion momentum and pion energy respectively; r_α are the effective ranges; δ_α are phase shifts. The expected linear dependence of the quantity $k^3 \cot \delta_{33}/\omega$ on ω had been found experimentally[2] and the intercept at zero energy led to a value for the renormalized coupling constant, $f^2=0.08$. Taking the cutoff energy $\omega_{max} \simeq 6$ they obtained the right resonance energy $\omega_{res} \simeq 2$. Their method is characteristic of the second quantization method and its relation to the classical theory does not seem so clear. On the other hand, Jackson[3] and Weisskopf[4] independently developed the semi-classical theories from the different viewpoints,** in which the quantum mechanical variables are only σ and τ, and they directly treated the Klein-Gordon equation for the meson fields, and got the effective range formula,

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* We shall put $c=\hbar=\mu$ (μ: pion mass)$=1$.

** As to the difference between the theories by Jackson and by Weisskopf, see § 4.

$$k^3 \cot \delta_\alpha/\omega - 1/\omega = 1/\lambda_\alpha \cdot (1-\omega r_\alpha), \qquad (2)$$

which is essentially equivalent to that of Chew and Low except for the addition of the so-called Serber term, $-1/\omega$. Their treatments are very interesting and intuitive, but they seem to be forced to use physically ambiguous assumptions inevitable to the semi-classical theory.

In this paper we shall try to make clear the theoretical background of the semi-classical theories and their connections with the usual quantum field theory. In § 2 general considerations on our method are given. It is shown that the phase shifts defined as in usual quantum mechanics from the matrix elements of the meson field operators between the physical nucleon states and the physical nucleon plus one-meson states with the energy below the threshold for meson production are identified with the phase shifts of the S-matrix elements. In § 3 this method is applied to the p-wave pion-nucleon scattering and with a certain approximation we get the same equation as what Jackson has derived from the semi-classical viewpoint. Then we exactly* solve this equation which has a non-local but separable form. Our result does not satisfy the crossing symmetry (see the footnote in p. 1172). Aside from this the difference between our result and the result by Chew and Low consists in replacing two powers of virtual energy, ω_p^2, by that of real energy, ω_k^2, in the expression for h (see Eq. (32), Eq. (33) and Eq. (34)). The difference seems delicate as mentioned above but this difference is decisive in deriving the effective range formula. In § 4 the relations of our method to the semi-classical ones are discussed. It is pointed out that the effective range formula of the above mentioned authors should be modified largely, i.e. the Serber term must be replaced as follows:

$$-\frac{1}{\omega} \to \frac{A}{\omega}, \quad A \gg 1.$$

This would result in the deviation from the observed linear dependence of the quantity $k^3 \cot \delta_{33}/\omega$ on ω.

In the Appendix our method is applied to the charged and neutral scalar theories. In comparison with the result of Castillejo, Dalitz and Dyson[5] we find the same difference (see Eq. (A10) and Eq. (A11)) as in the case of § 3.

§ 2. General considerations

As mentioned in § 1, we directly treat the Klein-Gordon equation for the meson field operators with static source

$$(\Box - 1)\phi(x) = j(x). \qquad (3)$$

Taking the matrix elements of this equation between the physical nucleon states

* Jackson treated Eq. (24) with the improper assumption (see § 4, i)), while we solve exactly Eq. (26) which is the momentum representation of Eq. (24).

and the physical nucleon plus one-meson incoming states, we can replace

$$\frac{\partial\phi}{\partial t}=i[H,\phi]\rightarrow -i\omega_k\phi. \tag{4}$$

Here ω_k and $\boldsymbol{k}$ are the energy and momentum of incident meson respectively. Then Eq. (3) takes the form

$$(\nabla^2+\boldsymbol{k}^2)\langle N|\phi(x)|N,\omega_k\rangle=\langle N|j(x)|N,\omega_k\rangle. \tag{5}$$

By the usual technics of quantum mechanics we solve Eq. (5) and ask the phase shifts of the matrix elements $\langle N|\phi(x)|N,\omega_k\rangle$. We will show that these phase shifts are identified with the phase shifts of the S-matrix elements. But the illustration will be given with the neglect of the nucleon spin for simplicity. The extension to the inclusion of the nucleon spin is straightforward.

We shall start from the equations[6];

$$\phi_{out}(x)=S^+\phi_{in}(x)S=\exp[-i\eta]\phi_{in}(x)\exp[i\eta], \tag{6}$$

$$\phi_{out}(x)=\phi_{in}(x)+\int\Delta(x-x')j(x')\,d^4x', \tag{7}$$

$$\phi(x)=\phi_{in}(x)-\int\Delta_R(x-x')j(x')\,d^4x', \tag{8}$$

where

$$S\equiv\exp[i\eta].$$

From Eq. (6) we can easily show for the energy below the threshold for meson production

$$\langle N|\phi_{out}(x)|N,\omega_k\rangle=\sum_{|\boldsymbol{p}|=|\boldsymbol{k}|}\langle N|\phi_{in}(x)|N,\omega_p\rangle\langle N,\omega_p|\exp[i\eta]|N,\omega_k\rangle,$$

and for $|\boldsymbol{x}|\equiv r\rightarrow\infty$

$$\approx e^{-i\omega t}\sum_l(2l+1)i^lP_l(\cos\theta)\frac{1}{2kr}\{e^{i[kr-(1/2)(l+1)\pi]}+e^{-i[kr-(1/2)(l+1)\pi]}\}\exp[i\eta_l], \tag{9}$$

where $\theta=\widehat{\boldsymbol{x},\boldsymbol{k}}$, and the fact $\langle N|\phi_{in}(x)|N,\omega_k\rangle=e^{-ik\cdot x-i\omega t}$ is used. On the other hand, from Eq. (7)

$$\langle N|\phi_{out}(x)|N,\omega_k\rangle=\langle N|\phi_{in}(x)|N,\omega_k\rangle+\int\Delta(x-x')\langle N|j(x')|N,\omega_k\rangle\,d^4x'. \tag{10}$$

After some calculations using the integral representation of the function $\Delta(x)$, the second term of Eq. (10) can be rewritten as follows:

$$\int\Delta(x-x')\langle N|j(x')|N,\omega_k\rangle\,d^4x'$$

$$=e^{-i\omega t}\frac{1}{4\pi}\int d^3\boldsymbol{x}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\{e^{ik|\boldsymbol{x}-\boldsymbol{x}'|}-e^{-ik|\boldsymbol{x}-\boldsymbol{x}'|}\}\langle N|j(\boldsymbol{x}',0)|N,\omega_k\rangle, \tag{11}$$

and for $r\rightarrow\infty$

$$\approx e^{-i\omega t}\frac{1}{r}\left\{e^{ikr}\frac{1}{4\pi}\int d^3\boldsymbol{x}'\,e^{-i\boldsymbol{p}\cdot\boldsymbol{x}'}\langle N|j(\boldsymbol{x}',0)|N,\omega_k\rangle\right.$$

$$\left.-e^{-ikr}\frac{1}{4\pi}\int d^3\boldsymbol{x}'\,e^{i\boldsymbol{p}\cdot\boldsymbol{x}'}\langle N|j(\boldsymbol{x}',0)|N,\omega_k\rangle\right. \tag{12}$$

Here we used $e^{ik|\boldsymbol{x}-\boldsymbol{x}'|}\approx e^{ik(r-\boldsymbol{x}\cdot\boldsymbol{x}'/r)}=e^{ikr-i\boldsymbol{p}\cdot\boldsymbol{x}'}$, $\boldsymbol{p}\equiv k\boldsymbol{x}/r$. Putting Eq. (12) into Eq. (10) and comparing the coefficient of e^{ikr}/r with that of Eq. (9) we obtain

$$f(\theta)\equiv\sum_l\frac{1}{2ik}(2l+1)\,(\exp[i\eta_l]-1)\,P_l(\cos\theta)$$

$$=\frac{-1}{4\pi}\int d^3\boldsymbol{x}'\,e^{-i\boldsymbol{p}\cdot\boldsymbol{x}'}\langle N|j(\boldsymbol{x}',0)|N,\omega_k\rangle. \tag{13}$$

Here $f(\theta)$ is the scattering amplitude defined as in usual quantum mechanics from the phase shifts of the S-matrix elements. As for the equation which should be obtained from the comparison of the coefficient of e^{-ikr}/r, we can easily show it is identical with Eq. (13). By the analogous method we can derive from Eq. (8)

$$\langle N|\phi(x)|N,\omega_k\rangle=\langle N|\phi_{in}(x)|N,\omega_k\rangle$$

$$-e^{-i\omega t}\frac{1}{4\pi}\int d^3\boldsymbol{x}'\frac{e^{ik|\boldsymbol{x}-\boldsymbol{x}'|}}{|\boldsymbol{x}-\boldsymbol{x}'|}\langle N|j(\boldsymbol{x}',0)|N,\omega_k\rangle, \tag{14}$$

and for $r\to\infty$

$$\approx\langle N|\phi_{in}(\boldsymbol{x})|N,\omega_k\rangle-e^{-i\omega t}\frac{e^{ikr}}{r}\frac{1}{4\pi}\int d^3\boldsymbol{x}'\,e^{-i\boldsymbol{p}\cdot\boldsymbol{x}'}\langle N|j(\boldsymbol{x}',0)|N,\omega_k\rangle$$

$$=e^{-i\omega t}\left\{e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}+\frac{e^{ikr}}{r}f(\theta)\right\}, \tag{15}$$

where we used Eq. (13). Thus $\langle N|\phi(x)|N,\omega_k\rangle$ in the region $r\to\infty$ are represented as in usual quantum mechanics from the phase shifts which are identified with the phase shifts of the S-matrix elements.

§ 3. *p*-wave pion-nucleon scattering

In this section we shall apply the method developed in § 2 to the p-wave pion-nucleon scattering. As usual we take the interaction Hamiltonian of the form

$$H_{int}=F_0\int\rho(\boldsymbol{x}')\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}'\tau\phi(x')\,d^3\boldsymbol{x}'. \tag{16}$$

Then the basic equations to be solved are as follows:

$$(\Box-1)\phi(x)=-F_0\tau\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\rho(\boldsymbol{x}), \tag{17}$$

$$\dot{\tau}=\frac{1}{i}[\tau,H]=-2F_0\int\tau\times(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}'\phi(x'))\rho(\boldsymbol{x}')\,d^3\boldsymbol{x}', \tag{18}$$

$$\dot{\boldsymbol{\sigma}}=\frac{1}{i}[\boldsymbol{\sigma},H]=-2F_0\int\boldsymbol{\sigma}\times\boldsymbol{\nabla}'(\tau\phi(x'))\,\rho(\boldsymbol{x}')\,d^3\boldsymbol{x}'. \tag{19}$$

Following the thought of line in § 2 we obtain from Eq. (17)

$$(\boldsymbol{\nabla}^2+\boldsymbol{k}^2)\langle N|\phi(x)|N,\omega_k\rangle=-F_0\boldsymbol{\nabla}\rho(\boldsymbol{x})\cdot\langle N|\tau\boldsymbol{\sigma}|N,\omega_k\rangle. \tag{20}$$

By using Eqs. (18) and (19) we can represent the right-hand side of Eq. (20) as

$$\begin{aligned}\langle N|\tau\boldsymbol{\sigma}|N,\omega_k\rangle&=-\frac{1}{i\omega_k}\langle N|[\tau\boldsymbol{\sigma}]^{\cdot}|N,\omega_k\rangle\\&=-\frac{1}{i\omega_k}\langle N|\dot{\tau}\boldsymbol{\sigma}+\tau\dot{\boldsymbol{\sigma}}|N,\omega_k\rangle\\&=\frac{2F_0}{i\omega_k}\langle N|\int[\boldsymbol{\sigma}\times\boldsymbol{\nabla}'\phi(x')+\tau\times\boldsymbol{\nabla}'\phi(x')]\rho(\boldsymbol{x}')\,d^3\boldsymbol{x}'|N,\omega_k\rangle.\end{aligned} \tag{21}$$

Putting Eq. (21) into Eq. (20) we obtain the equation

$$\begin{aligned}(\boldsymbol{\nabla}^2+\boldsymbol{k})\langle N|\phi(x)|N,\omega_k\rangle=&-\frac{2F_0^2}{i\omega_k}\boldsymbol{\nabla}\rho(\boldsymbol{x})\\&\cdot\int\langle N|\boldsymbol{\nabla}'\tau\times\phi(x')+\boldsymbol{\sigma}\times\boldsymbol{\nabla}'\phi(x')|N,\omega_k\rangle\rho(\boldsymbol{x}')\,d^3\boldsymbol{x}'.\end{aligned} \tag{22}$$

Until now no approximation has been used, but here we shall approximate the term $\langle N|\tau\times\phi|N,\omega_k\rangle$ and $\langle N|\boldsymbol{\sigma}\times\boldsymbol{\nabla}'\phi|N,\omega_k\rangle$ in the right-hand side of Eq. (22) as follows:

$$\begin{aligned}&\langle N|\tau\times\phi|N,\omega_k\rangle=\langle N|\tau|N\rangle\times\langle N|\phi|N,\omega_k\rangle,^*\\&\langle N|\boldsymbol{\sigma}\times\boldsymbol{\nabla}'\phi|N,\omega_k\rangle=\langle N|\boldsymbol{\sigma}|N\rangle\times\langle N|\boldsymbol{\nabla}'\phi|N,\omega_k\rangle,\end{aligned} \tag{23}$$

and neglect all such terms as

$$\langle N,\omega_p|\phi|N,\omega_k\rangle,\ \langle N,\omega_p,\omega_q|\phi|N,\omega_k\rangle,\cdots\cdots,$$

which correspond to multi-meson amplitudes. We add here a remark about our approximation. In Eq. (22) we take the order $\tau\times\phi$, etc., and not the order $-\phi\times\tau$, etc. If we took the latter, our equation in the above approximation should become free equation by the pseudo-scalarity of the meson fields, $\langle N|\phi(x)|N\rangle=0$.

Putting Eq. (23) into Eq. (22), we get

$$(\boldsymbol{\nabla}^2+\boldsymbol{k}^2)\phi_k(\boldsymbol{x})=-\frac{2F^2}{i\omega_k}\boldsymbol{\nabla}\rho(\boldsymbol{x})\cdot\int[\boldsymbol{\nabla}'\tau\times\phi_k(\boldsymbol{x}')+\boldsymbol{\sigma}\times\boldsymbol{\nabla}'\phi(\boldsymbol{x}')]\rho(\boldsymbol{x}')\,d^3\boldsymbol{x}', \tag{24}$$

where

$$\phi_k(\boldsymbol{x})\equiv\langle N|\phi(\boldsymbol{x},0)|N,\omega_k\rangle.$$

Here we defined new renormalized coupling constant F^2 as follows:

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* We call our approximation the "zero-meson approximation".

$$F_0^2\langle N_\alpha|\sigma|N_\beta\rangle \equiv F^2\sigma_{\alpha\beta},$$
$$F_0^2\langle N_\alpha|\tau|N_\beta\rangle \equiv F^2\tau_{\alpha\beta}. \tag{25}$$

This definition of renormalization differs from the usual one and this renormalized coupling constant is not necessarily positive definite so far as we consider only Eq. (25) (see also § 4, ii) Eq. (50)). Eq. (24) can be thought of as a wave equation for $\phi_k(\boldsymbol{x})$ with the right-hand side involving a non-local potential operators and this equation coincides entirely with Jackson's equation (3·5), which he derived from the semi-classical viewpoint.

Now we can solve Eq. (24) exactly. Transforming Eq. (24) into momentum representation, we get

$$(\boldsymbol{p}^2-\boldsymbol{k}^2)\,\phi_k(\boldsymbol{p}) = (2\pi)^{-3}\frac{F^2}{\omega_k}2i\boldsymbol{p}v(\boldsymbol{p})\cdot\int v(\boldsymbol{p}')[\boldsymbol{p}'\tau\times\phi_k(\boldsymbol{p}')+\sigma\times\boldsymbol{p}'\phi_k(\boldsymbol{p}')]d^3\boldsymbol{p}', \tag{26}$$

where

$$\phi_k(\boldsymbol{p}) = (2\pi)^{-3}\int d^3\boldsymbol{x}\,e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\phi_k(\boldsymbol{x}),$$
$$v(\boldsymbol{p}) = \int d^3\boldsymbol{x}\,e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\rho(\boldsymbol{x}).$$

Then we decompose Eq. (26) into the eigenstates of total isotopic spin and total angular momentum

$$(\boldsymbol{p}^2-\boldsymbol{k}^2)\phi_k{}^\alpha(\boldsymbol{p}) = (2\pi)^{-3}\frac{2F^2}{\omega_k}\xi_\alpha\boldsymbol{p}v(\boldsymbol{p})\cdot\int v(\boldsymbol{p}')\,\boldsymbol{p}'\,\phi_k(\boldsymbol{p}')\,d^3\boldsymbol{p}', \tag{27}$$

where

$$\xi_\alpha = \begin{bmatrix}-4\\-1\\2\end{bmatrix} \text{ for } \begin{cases} I=J=1/2 \\ I=1/2,\ J=3/2 \text{ and } I=3/2,\ J=1/2 \\ I=J=3/2. \end{cases}$$

Considering the incoming wave boundary condition of the one-meson states $|N,\omega_k\rangle$ we get the solution of Eq. (27) as follows:

$$\phi_k{}^\alpha(\boldsymbol{p}) = \delta(\boldsymbol{p}-\boldsymbol{k}) + \frac{(2\pi)^{-3}(2F^2/\omega_k)\xi_\alpha\boldsymbol{p}v(\boldsymbol{p})\cdot\boldsymbol{C}_k}{\boldsymbol{p}^2-\boldsymbol{k}^2-i\epsilon}, \tag{28}$$

where $\boldsymbol{C}_k = \int v(\boldsymbol{p}')\,\boldsymbol{p}'\phi_k(\boldsymbol{p}')\,d^3\boldsymbol{p}'$. Using the definition of $\boldsymbol{C}_k$ we obtain the equation to determine $\boldsymbol{C}_k$,

$$\boldsymbol{C}_k = v(\boldsymbol{k})\boldsymbol{k} + (2\pi)^{-3}(2F^2/\omega_k)\xi_\alpha\int\frac{(\boldsymbol{C}_k\cdot\boldsymbol{p})\,\boldsymbol{p}v^2(\boldsymbol{p})}{\boldsymbol{p}^2-\boldsymbol{k}^2-i\epsilon}d^3\boldsymbol{p}$$
$$= v(k)\boldsymbol{k} + (2\pi)^{-3}(2F^2/3\omega_k)\xi_\alpha\boldsymbol{C}_k\int\frac{\boldsymbol{p}^2v^2(p)}{\boldsymbol{p}^2-\boldsymbol{k}^2-i\epsilon}d^3\boldsymbol{p}. \tag{29}*$$

Then

* We assumed here that $v(\boldsymbol{k})$ depends only on $k=|\boldsymbol{k}|$.

$$C_k=\frac{v(k)\boldsymbol{k}}{1-\frac{2}{\pi^2}\lambda_\alpha\frac{1}{\omega_k}\int\frac{\boldsymbol{p}^2v^2(p)}{\boldsymbol{p}^2-\boldsymbol{k}^2-i\epsilon}d^3\boldsymbol{p}}\,, \tag{30}$$

where $$\lambda_\alpha=\frac{2}{3}f^2\tilde{\xi}_\alpha,\ f^2\equiv F^2/4\pi.$$

And using Eq. (13) in § 2 we obtain for the scattering amplitude

$$f(\theta)=\frac{3}{2ik}(e^{i\eta}-1)\cos\theta=\frac{3\lambda_\alpha(1/\omega_k)k^2v^2(k)\cos\theta}{1-\frac{2}{\pi^2}\lambda_\alpha\frac{1}{\omega_k}\int\frac{\boldsymbol{p}^2v^2(p)}{\boldsymbol{p}^2-\boldsymbol{k}^2-i\epsilon}d^3\boldsymbol{p}}\,, \tag{31}$$

and for $h_\alpha\equiv e^{i\delta_\alpha}\sin\delta_\alpha/k^3v^2(k)$

$$\begin{aligned}h_\alpha(\omega_k)&=\frac{\lambda_\alpha(1/\omega_k)}{1-\frac{\lambda_\alpha}{\omega_k}\frac{2}{\pi}\int_0^\infty\frac{v^2(p)p^4}{\boldsymbol{p}^2-\boldsymbol{k}^2-i\epsilon}dp}=\frac{\lambda_\alpha(1/\omega_k)}{1-\frac{\lambda_\alpha}{\omega_k}\frac{2}{\pi}\int_1^\infty\frac{v^2(p)p^3\omega_p}{\omega_p^2-\omega_k^2-i\epsilon}d\omega_p}\\&=\frac{\lambda_\alpha(1/\omega_k)}{1-\frac{\lambda_\alpha}{\omega_k}\left(\frac{1}{\pi}\int\frac{v^2(p)p^3}{\omega_p-\omega_k-i\epsilon}d\omega_p+\frac{1}{\pi}\int\frac{v^2(p)p^3}{\omega_p+\omega_k}d\omega_p\right)}\,.\end{aligned} \tag{32}$$

It is interesting to compare Eq. (32) with Eq. (47) of Chew and Low

$$h_\alpha^{\text{C.L.}}(\omega_k)=\frac{\lambda_\alpha(1/\omega_k)}{1-\lambda_\alpha\omega_k\frac{1}{\pi}\int\frac{v^2(p)}{\omega_p^2}\frac{p^3}{\omega_p-\omega_k-i\epsilon}d\omega_p-\frac{1}{\pi}\int\frac{v^2(p)}{\omega_p^2}\frac{H_\alpha(\omega_p)}{\omega_p+\omega_k}d\omega_p}\,, \tag{33}$$

where $H_\alpha(\omega_p)$ are the functions to be determined from the crossing relation. It must be noted that Eq. (32) does not satisfy the crossing symmetry.* But aside from this the difference between Eq. (32) and Eq. (33) consists in the second term in the denominator

$$\frac{\lambda_\alpha}{\omega_k}\frac{1}{\pi}\int\frac{v^2(p)p^3}{\omega_p-\omega_k-i\epsilon}d\omega_p\leftrightarrow\lambda_\alpha\omega_k\frac{1}{\pi}\int\frac{v^2(p)}{\omega_p^2}\cdot\frac{p^3}{\omega_p-\omega_k-i\epsilon}d\omega_p\,, \tag{34}$$

that is, the change $\omega_p^2\leftrightarrow\omega_k^2$. This change seems very subtle but it causes the decisive change for the effective range formula.

We can derive from Eq. (32) the equation

$$\begin{aligned}\frac{v^2(k)k^3\cot\delta_\alpha}{\omega_k}&=\frac{1}{\lambda_\alpha}-\frac{2}{\pi\omega_k}P\int\frac{v^2(p)p^3}{\omega_p^2-\omega_k^2}\omega_p d\omega_p\\&=\frac{1}{\lambda_\alpha}-\frac{1}{\omega_k}r'(\omega_k),\end{aligned} \tag{35}$$

* This is clear from the fact that we consider only "the repetition of the Born approximation" (see Eq. (27)).

where $$r'(\omega_k)=\frac{2}{\pi}P\int\frac{v^2(p)p^3}{\omega_p^2-\omega_k^2}\omega_p\,d\omega_p.$$

It is seen that Eq. (35) takes contrasting behavior with the effective range formula (1) by Chew and Low: Eq. (35) approaches the Born term $1/\lambda_\alpha$ for $\omega_k\to\infty$, while Eq. (1) does for $\omega_k\to 0$.

For $\omega_k\ll\omega_{max}$ it would be reasonable to neglect the dependence of $r'(\omega_k)$ on ω_k, and there is no linear dependence of the quantities $v^2(k)k^3\cot\delta_\alpha/\omega_k$ on ω_k in the right-hand side of Eq. (35). Its behavior on energy ω_k for $(1/\lambda_{33})>0$ would be as shown in Fig. 1 (a) which differs very much from that of Fig 1 (b) by Chew and Low, and so from the experimental one.

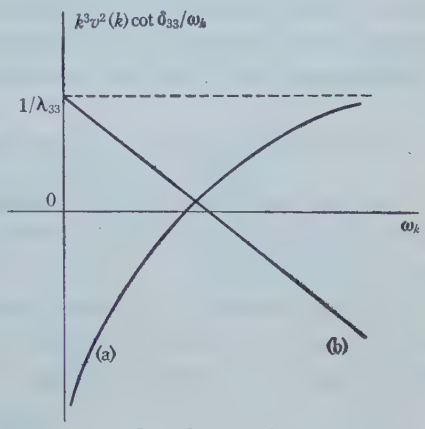

Fig. 1. $k^3v^2(k)\cot\delta_{33}/\omega_k$ versus ω_k.
(a) : according to Eq. (35)
(b) : according to Eq. (1) by Chew and Low.

In order to see this more clearly, we deform Eq. (35) as follows:

$$k_b^3v^2(k_b)\cot\delta_b-k_a^3v^2(k_a)\cot\delta_a$$
$$=\frac{1}{\lambda}(\omega_b-\omega_a)-(\omega_b^2-\omega_a^2)\frac{2}{\pi}P\int\frac{v^2(p)p^3\omega_p d\omega_p}{(\omega_p^2-\omega_a^2)(\omega_p^2-\omega_b^2)},\qquad(36)$$

and choosing $\omega_a=0$, $k_a=i$,

$$\frac{k^3v^2(k)\cot\delta}{\omega_k}+\frac{A}{\omega_k}=\frac{1}{\lambda}-\omega_k r(\omega_k),\qquad(37)$$

where

$$r(\omega_k)=\frac{2}{\pi}\int\frac{v^2(p)p^3}{\omega_p^2-\omega_k^2}\frac{1}{\omega_p}d\omega_p\approx\frac{2}{\pi}\omega_{max},$$
$$A=\frac{2}{\pi}\int\frac{v^2(p)p^3}{\omega_p}d\omega_p\approx\frac{2}{3\pi}\omega_{max}^3.\qquad(38)$$

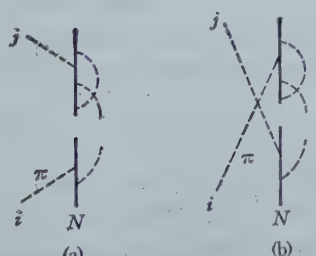

Fig. 2. Graphs for (3-3) state.

For $\omega_k\ll\omega_{max}$ it would also be reasonable to neglect the dependence $r(\omega_k)$ on ω_k and it corresponds to the usual effective range. Then it is seen that the quantities $k^3v^2(k)\cot\delta_\alpha/\omega_k+A/\omega_k$ (in contrast with the quantities $k^3v^2(k)\cot\delta_\alpha/\omega_k$ in usual case) would be linearly dependent on ω_k. As it would be seen from the comparison of r and A in Eq. (38), this change would be decisive and Eq. (37) could not explain the experimental behavior of

the (3–3)-state phase shift.

We noted above that our solution does not satisfy crossing symmetry. It may be said that graphically our solution for the (3–3) state contains only such graph as Fig. 2 (a) but the solution by Chew and Low contains many crossed graphs, for example, Fig. 2 (b) as well as Fig. 2 (a).

§ 4. Discussions: On the methods of Jackson and of Weisskopf

In this section we shall discuss about the semi-classical theories of Jackson and of Weisskopf in the light developed in the previous sections.

i) *Jackson's theory*

For convenience we briefly summarize his theory. His basic equations are Eqs. (16), (17), (18) and (19), in which $\phi(x)$ are considered as c-number funcions, not as operators. Then he assumes in the right-hand side of Eq. (17)

$$\sigma(t)=\sigma+\sigma_1 e^{-i\omega t},$$
$$\tau(t)=\tau+\tau_1 e^{-i\omega t},$$

where σ_1 and τ_1 are assumed to be small compared with σ and τ, which are time independent Pauli matrices. He considers only linear terms in σ_1 and τ_1, and he gets Eq. (24) by use of Eqs. (18) and (19). Then he considers the effective range approach to the solution of Eq. (24) in complete analogy with the derivation used in nucleon-nucleon scattering, and gets

$$k_b^3 \cot\delta_b - k_a^3 \cot\delta_a = \frac{1}{\lambda_\alpha}(\omega_b-\omega_a)+(\omega_b^2-\omega_a^2)\int_0^\infty\left(W_a^\alpha W_b^\alpha - \frac{1}{r^2} - U_a^\alpha U_b^\alpha\right)dr, \tag{39}$$

where $W^\alpha(r)=k^2 r[\cot\delta_\alpha j_1(kr)-n_1(kr)]$ and $U^\alpha(r)$ is the solution divided by r of the equation which is obtained by decomposing Eq. (24) into the eigen-states of total isotopic spin and total angular momentum.

Choosing $\omega_a=0$, $k_a=i$ and assuming, in his words, "to prevent an exponentially growing solution at infinity",*

$$\cot\delta=i \text{ for } \omega_a=0,\ k_a=i, \tag{40}$$

he gets the final result of the form

$$\frac{k^3\cot\delta_\alpha}{\omega_k}-\frac{1}{\omega_k}=\frac{1}{\lambda_\alpha}(1-r_\alpha\omega_k) \tag{2}$$

Here such an approximation as

$$\int_0^\infty\left(W_a^\alpha W_b^\alpha-\frac{1}{r^2}-U_a^\alpha U_b^\alpha\right)dr\approx\int_0^\infty\left((W_0^\alpha)^2-\frac{1}{r^2}-(U_0^\alpha)^2\right)\equiv-\frac{1}{\lambda_\alpha}r_\alpha \tag{41}$$

* The assumption (40) for the (1, 3), (3, 1) and (3, 3) states has no physical backgrounds, because the ordinary spin and/or iso-spin of these states differ from that of one-nucleon state.

is made. Eq. (2) is the effective range formula with the Serber term and it is usually said that the Serber term has no essential role in the considered energy region, so Eq. (2) can explain the experimental results as well as Eq. (1) does. This is apparent contradiction with the result of § 3 in which we dealt the same equation (24). This contradiction comes from his improper assumption (40).

In order to see this we derive

$$k_b^3 \cot \delta_b - k_a^3 \cot \delta_a = \frac{1}{\lambda}\left(\frac{\omega_b}{v^2(k_b)} - \frac{\omega_a}{v^2(k_a)}\right) + \frac{v^2(k_a)\omega_a^2 - v^2(k_b)\omega_b^2}{v^2(k_a)v^2(k_b)}$$

$$\times \frac{2}{\pi} P\int \frac{v^2(p)p^3\omega_p d\omega_p}{(\omega_p^2-\omega_b^2)(\omega_p^2-\omega_a^2)} + \frac{v^2(k_b)-v^2(k_a)}{v^2(k_a)v^2(k_b)} \frac{2}{\pi} P\int \frac{v^2(p)p^3\omega_p^3 d\omega_p}{(\omega_p^2-\omega_a^2)(\omega_p^2-\omega_b^2)} \tag{42}$$

corresponding to Eq. (39) by use of Eq. (35). For

$$v^2(k_b) \approx v^2(k_a) \text{ and } \omega_a = 0, \tag{43}$$

the third term in the right-hand side of Eq. (42) disappears and Eq. (42) resembles the form of Eq. (2). But the assumption (40) can only be satisfied by taking

$$v^2(k) < 0 \text{ for } \omega_k = 0,\ k = i, \tag{44}$$

for on account of Eq. (35)

$$v^2(i) = -\frac{2}{\pi}\int \frac{v^2(p)p^3}{\omega_p} d\omega_p \approx -\frac{2}{3\pi}\omega_{max}^3 < 0.$$

The condition (44) obviously contradicts the condition (43) for the energy region

$$1 \leq \omega_k \ll \omega_{max}$$

in which $v^2(k) \approx 1$ is required.

ii) *Weisskopf's theory*

He considers the Klein-Gordon equation

$$(\Box - 1)\phi(x) = 0 \text{ for } r > r_0 \tag{45}$$

and he represents $z^3 \cot \delta (z = r_0 k)$ in terms of logarithmic derivatives f_α of the complete meson wave functions $\phi_\alpha(x)$ at $r = r_0$. Then he developes $z^3 \cot \delta_\alpha$ to second order in ω by developing f_α in ω, and determines these developing coefficients in comparison with the solution of the meson scattering at zero energy. After some detailed considerations on the coefficients he gets the equation

$$\frac{k^3 \cot \delta_\alpha}{\omega_k} - \frac{1}{\omega_k} = \frac{1}{\lambda_\alpha} - \frac{\omega_k}{r_0}. \tag{46}$$

We will show next that his prescriptions on the meson scattering at zero energy are essentially equivalent to ours, but only in the Born approximation in his case.

From Eqs. (16), (17), (18) and (19) we obtain analogously to Eq. (22),

$$(\nabla^2+k^2)\langle N|\phi_i(x)|N,\omega_k\rangle$$
$$=\frac{F_0^2}{\omega_k}\int\langle N|[\tau_i\sigma\cdot\Delta\rho(x),\tau_j\sigma\cdot\nabla'\rho(x')]|N\rangle\langle N|\phi_j(x')|N,\omega_k\rangle d^3x'. \quad (47)$$

Transforming Eq. (47) into momentum representation, we get*

$$(p^2-k^2)\phi_k^i(p)=-(2\pi)^{-3}\frac{F_0^2}{\omega_k}v(p)\int\langle N|[\tau_i\sigma\cdot p,\tau_j\sigma\cdot p']|N\rangle\phi_k^j(p')v(p')d^3p'. \quad (48)$$

Here we make "zero-meson approximation" once more,

$$\langle N|[\tau_i\sigma\cdot p,\tau_j\sigma\cdot p']|N\rangle=[\langle N|\tau_i\sigma\cdot p|N\rangle,\langle N|\tau_j\sigma\cdot p'|N\rangle].$$

Then Eq. (48) takes the form

$$(p^2-k^2)\phi_k^i(p)=(2\pi)^{-3}\frac{F^2}{\omega_k}v(p)\int[\tau_i\sigma\cdot p,\tau_j\sigma\cdot p']\phi_k^j(p')v(p')d^3p', \quad (49)$$

where we defined the new renormalized coupling constant as

$$F_0\langle N|\tau\sigma|N\rangle\equiv F\tau\sigma \quad (50)$$

which coincides with the definition by Chew and Low.

Decomposing Eq. (49) into the eigenstates, we again get the same equation as Eq. (27) except the difference of the definition of F (i.e. now by Eq. (50), previously by Eq. (25)).

Now it would be clear that Weisskopf's treatment corresponds to taking $\langle N|\phi(x')|N,\omega_k\rangle=e^{ik\cdot x'}$ (i. e. $\phi_k^i(p)=\delta(p-k)$) in the right-hand side of Eq. (49), i.e. to Born approximation. So if we solved Eq. (49) exactly, we should get the results as obtained in § 3.

In conclusion the author would like to express his sincere gratitude to Professor T. Inoue, Professor Y. Munakata, Dr. S. Tanaka, Dr. M. Ida and Dr. M. Uehara for their continual encouragement and invaluable discussions, and especially to Dr. M. Ida, whose advices and comments are very much appreciated. Thanks are also due to other members of Yukawa Laboratory.

Appendix

The charged and neutral scalar theories

We proceed analogously to the method of § 3. The notation used is the same as that in the paper by Castillejo, Dalitz and Dyson. Corresponding to Eqs. (16), (17) and (18),

* As it would be easily seen, Eq. (26) is obtained from Eq. (48) by the actual calculation of the commutator in the right-hand side of Eq. (48).

$$H_{Ch}=G_0\int U(x)[Q_0\phi(x)+Q_1\phi^*(x)]\,d^3\boldsymbol{x}, \tag{A1}$$

$$(\Box-1)\phi(x)=G_0U(x)Q_1, \tag{A2}$$

$$\dot{Q}_1=\frac{1}{i}[Q_1,\ H]=\frac{1}{i}G_0(-\tau_3)\int U(x)\phi(x)\,d^3\boldsymbol{x}, \tag{A3}$$

where τ_3 is defined by $\tau_3=Q_0Q_1-Q_1Q_0$. Corresponding to Eq. (22) and Eq. (24)

$$(\boldsymbol{\nabla}^2+\boldsymbol{k}^2)\langle N|\phi(x)|N,\ \omega_k{}^\alpha\rangle=-G_0{}^2\frac{1}{\omega_k}U(x)\int\langle N|\tau_3\phi(x')|N,\ \omega_k{}^\alpha\rangle U(x')\,d^3\boldsymbol{x}', \tag{A4}$$

$$(\boldsymbol{\nabla}^2+\boldsymbol{k}^2)\phi_k{}^\alpha(x)=-G^2(-1)^\alpha\frac{1}{\omega_k}U(x)\int U(x')\phi_k{}^\alpha(x')\,d^3\boldsymbol{x}'. \tag{A5}$$

Corresponding to Eq. (26), Eq. (28) and Eq. (30),

$$(\boldsymbol{p}^2-\boldsymbol{k}^2)\phi_k{}^\alpha(\boldsymbol{p})=G^2(-1)^\alpha\frac{1}{\omega_k}v(\boldsymbol{p})C_k{}^\alpha, \tag{A6}$$

where

$$C_k{}^\alpha=\int v(\boldsymbol{p}')\phi_k{}^\alpha(\boldsymbol{p}')\,d^3\boldsymbol{p}',$$

$$\phi_k{}^\alpha(\boldsymbol{p})=\delta(\boldsymbol{p}-\boldsymbol{k})+\frac{G^2(-1)^\alpha\dfrac{1}{\omega^2}v(\boldsymbol{p})C_k{}^\alpha}{\boldsymbol{p}^2-\boldsymbol{k}^2-i\epsilon}, \tag{A7}$$

$$C_k{}^\alpha=\frac{v(\boldsymbol{k})}{1-G^2(-1)^\alpha\dfrac{1}{\omega_k}\displaystyle\int\frac{v^2(\boldsymbol{p})}{\boldsymbol{p}^2-\boldsymbol{k}^2-i\epsilon}d^3\boldsymbol{p}}. \tag{A8}$$

For $f=e^{i\delta}\sin\delta/k$

$$f=\frac{2\pi^2G^2(-1)^\alpha\dfrac{1}{\omega_k}v^2(\boldsymbol{k})}{1-G^2(-1)^\alpha\dfrac{1}{\omega_k}\displaystyle\int\frac{v^2(\boldsymbol{p})}{p^2-k^2-i\epsilon}d^3\boldsymbol{p}}. \tag{A9}$$

For

$$H_\alpha(\omega_k)\equiv-1/h_\alpha(\omega_k)=-k|v^2(\boldsymbol{k})|/\omega_kf_\alpha(k),$$

$$H_\alpha(\omega_k)=-\frac{(-1)^\alpha}{\Lambda}+\frac{1}{\omega_k}\frac{2}{\pi}\int\frac{\omega_x x|v(x)|^2}{\omega_x{}^2-\omega_k{}^2-i\epsilon}d\omega_x. \tag{A10}$$

If we compare this with the simplest solution by C. D. D.[5],

$$H_\alpha^{C.D.D.}(\omega_k)=-\frac{(-1)^\alpha}{\Lambda}+\omega_k\frac{2}{\pi}\int\frac{1}{\omega_x}\frac{x|v(x)|^2}{\omega_x{}^2-\omega_k{}^2-i\epsilon}d\omega_x, \tag{A11}$$

the same kind of difference as in § 3 can be seen.

For neutral scalar case we would obtain the free equation

$$(\boldsymbol{\nabla}^2+\boldsymbol{k}^2)\langle N|\phi(x)|N,\ \omega_k\rangle=G_0U(x)\langle N|N,\ \omega_k\rangle=0, \tag{A12}$$

because of orthogonality of the states $|N\rangle$ and $|N, \omega_k\rangle$. And scattering can not take place.

References

1) G. F. Chew and F. E. Low, Phys Rev. **101** (1956), 1570.
2) S. J. Lindenbaum and L. C. L. Yuan, Phys. Rev. **100** (1955), 306.
   G. Puppi and A. Stanghellini, Nuovo Cimento **5** (1957), 1305.
3) J. D. Jackson, *The physics of Elementary Particles* (Princeton University Press, **1958**), Part I, Chap. 3.
4) V. F. Weisskopf, Phys. Rev. **116** (1959), 1615.
5) L. Castillejo, R. H. Dalitz and F. J. Dyson, Phys. Rev. **101** (1956), 453.
6) C. N. Yang and D. Feldman, Phys. Rev. **79** (1950), 972.
   H. Lehman, K. Symanzik and W. Zimmerman, Nuovo Cimento **6** (1957), 319.

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Theory of Finite Nuclei in Statistical Method with Correlation Correction. I

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A statistical method for atomic nuclei is developed, in which the effective potential is the density-dependent non-local reaction matrix in the nuclear matter. This method is applied to the calculation of the surface thickness, t, and the surface energy, E_s, of heavy nuclei, and the following results are obtained:*

$$t = (2.8 \sim 3.0) \times 10^{-13}\,\text{cm}$$

$$E_s = (23 \sim 28)\, A^{2/3}\,\text{Mev}.$$

§ 1. Introduction and summary

The Thomas-Fermi-Dirac statistical method has been applied to atomic nuclei by many authors.[1] Some of them succeeded in reproducing the binding energy as a function of mass number.** However, the potentials used by them have the unrealistic exchange character which leads only to exchange integrals and not to direct integrals. This was necessary in order to guarantee the nuclear saturation, but is incompatible with nucleon-nucleon scattering experiments. At present, both nucleon-nucleon scattering and saturation are explained by the existence of the hard core in the two-body potential. It is apparent, however, that we cannot apply the statistical method without modifying it when the hard core is present.

In this work, this difficulty is solved by making use of the results derived in the study of nuclear matter.[2] The recent development in the theory of nuclear matter shows us that owing to Pauli's principle the correlation of nucleons is not very complicated. According to this theory it is a good approximation to neglect all correlations except two-body correlation, and this two-body correlation and the singular potential are reduced to a density dependent non-local effective potential (reaction matrix) which is attractive in low and moderate density and repulsive in high density. This density dependence of the potential guarantees the saturation.

We apply the statistical method to atomic nuclei using this effective po tential. We can solve the Hartree-Fock equation with this potential as was proposed by Brueckner, Gammel and Weitzner.[2] Therefore, our method is to Brueckner et al.'s what the Thomas-Fermi-Dirac method is to the Hartree-Fock method in

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* t is the 90% to 10% distance.

** For example, Gombás et al. could reproduce the binding energy as a function of mass number but the central density was one order of magnitude greater than observed.

the atomic physics. If we want the wave functions, we must solve the Hartree-Fock equation. However, if we want only density distributions and binding energies, we need not solve the Hartree-Fock equation, because many works on atomic physics[3] teach us that there is little difference between the answers of the two methods. The difference will be smaller in the nuclear problem than in the atomic problem as the shell structure is less significant in nuclei. Needless to say, the calculation in the statistical method is far simpler than that in the Hartree-Fock method.

In the following we consult the effective potential calculated by Brueckner et al., but besides we will use several other effective potentials, as the nuclear surface thickness is very sensitive to the range of the potential, and as the real two-body potential has not completely been determined from experiments as yet. As to the non-locality of the effective potential we assume that it is the same as that of the effective potential in the cluster development method by Iwamoto and Yamada.[4]

Given an effective potential, we can calculate the density distributions and the total energies of nuclei with various mass-numbers by the variation principle. As the first example we calculate the surface thickness and the surface energy of heavy nuclei using the effective potentials which reproduce the properties of the nuclear matter. We get the following results:*

$$t=(2.8\sim3.0)\times10^{-13}\ \text{cm},$$

$$E_s=(23\sim28)A^{2/3}\ \text{Mev},$$

for the effective potential proposed by Brueckner et al., which are to be compared with the experimental data,

$$t=(2.2\pm0.3)\times10^{-13}\ \text{cm}^{5)},$$

$$E_s=(18\sim26)A^{2/3}\ \text{Mev}.^{6)}$$

In the above calculation we have neglected the Coulomb interaction. The inclusion of it, the application of this method to lighter nuclei, and the elaboration of the method will be found in the subsequent paper.

In § 2 we formulate the method, in § 3 we calculate the surface thickness and the surface energy, and in § 4 we give discussions.

§ 2. The statistical method with correlation correction

It is not easy to solve a finite nucleus starting from the Hamiltonian of the system. In order to solve the problem we must introduce some approximations. In this work our main assumption is that the energy of the system is a functional only of the one-body density distribution function (the statistical method), so our task is to look for the best expression for the energy within the above mentioned restriction. The statistical method was originally applied to the atomic physics

* In this work we have not evaluated the theoretical errors due to approximations.

(the Thomas-Fermi-Dirac method), but we cannot apply the Thomas-Fermi-Dirac method to atomic nuclei without modification since the local Fermi gas model is a very bad approximation. Owing to the singular two-body interaction there are much correlations in nuclei and besides we cannot get a satisfactory solution of the problem.*

We remove these difficulties by taking into account the two-body correlation. In the atomic physics the Thomas-Fermi-Dirac model was derived from the Hartree-Fock approximation, and here we also start from the Hartree-Fock approximation with correlation correction. As is well known, Brueckner et al. have shown[2] that we may treat a finite nucleus in the Hartree-Fock approximation in which the effective two-body potential is a density-dependent non-local reaction matrix in the nuclear matter. Then, the energy of a finite nucleus is given by

$$E=-\frac{\hbar^2}{2m}\sum_i\int\varphi_i^*(\boldsymbol{r})\,\Delta\varphi_i(\boldsymbol{r})\,dv+\sum_{i>j}\int(\varphi_i^*(\boldsymbol{r}_1)\varphi_j^*(\boldsymbol{r}_2)-\varphi_j^*(\boldsymbol{r}_1)\varphi_i^*(\boldsymbol{r}_2))\times(\boldsymbol{r}_{12}|K_\rho|\boldsymbol{r}_{12}')\varphi_i(\boldsymbol{r}_1')\varphi_j(\boldsymbol{r}_2')\,dv_1dv_2dv_1'dv_2', \tag{2·1}$$

where $(\boldsymbol{r}_{12}|K_\rho|\boldsymbol{r}_{12}')$ is the reaction matrix which is density-dependent and non-local. The reaction matrix is known to be divided into two parts, the core part, K_{core}, and the attractive part, K_a. The core part, K_{core}, is of short range, repulsive and density dependent, and the attractive part, K_a, is of longer range, attractive and almost density-independent.

We can understand these properties of the reaction matrix (the effective potential) most intuitively by consulting cluster development method.[4] The zero-th order term of the development, $E^{(0)}$, is given by

$$\begin{aligned}E^{(0)}=&-\frac{\hbar^2}{2m}\sum_i\int\varphi_i^*(\boldsymbol{r})\,\Delta\varphi_i(\boldsymbol{r})\,dv+\sum_{i>j}\int(\varphi_i^*(\boldsymbol{r})\varphi_j^*(\boldsymbol{r}')-\varphi_j^*(\boldsymbol{r})\varphi_i^*(\boldsymbol{r}'))\\&\times\left[f(|\boldsymbol{r}-\boldsymbol{r}'|)\,V(|\boldsymbol{r}-\boldsymbol{r}'|)f(|\boldsymbol{r}-\boldsymbol{r}'|)-\frac{\hbar^2}{m}f(|\boldsymbol{r}-\boldsymbol{r}'|)\,\Delta f(|\boldsymbol{r}-\boldsymbol{r}'|)\right]\\&\times\varphi_i(\boldsymbol{r})\varphi_j(\boldsymbol{r}')\,dv\,dv'\\&-\frac{2\hbar^2}{m}\sum_{i>j}\int(\varphi_i^*(\boldsymbol{r})\varphi_j^*(\boldsymbol{r}')-\varphi_j^*(\boldsymbol{r})\varphi_i^*(\boldsymbol{r}'))f(|\boldsymbol{r}-\boldsymbol{r}'|)\,\nabla f(|\boldsymbol{r}-\boldsymbol{r}'|)\\&\times\nabla\varphi_i(\boldsymbol{r})\varphi_j(\boldsymbol{r}')\,dv\,dv',\end{aligned} \tag{2·2}$$

where $f(r)$ is the correlation function. We can easily see that this expression is just the energy in the Hartree-Fock approximation with a non-local potential. The first term in the right-hand side of (2·2) is the kinetic energy due to the single particle motion. In the second term we find $fVf-\hbar^2/m\cdot f\,\Delta f$ instead of the real potential, V. The potential V appears in a form fVf; that is to say, its sin-

* If the two-body potential is of Serber type as nucleon-nucleon scattering experiments indicate, there is no saturation.

gularity is reduced by the correlation functions. The term which contains $-\hbar^2/m \cdot f \Delta f$ means a kinetic energy due to the correlation between nucleons. The last term is the cross term in the kinetic energy between correlations and single particle motions, and this corresponds to the non-locality of the reaction matrix, $(\boldsymbol{r}|K_\rho|\boldsymbol{r}')$. Thus we can see that the effective potential is much weaker than the real two-body potential, and that we can divide the region of the potential into two parts, the outer part where correlation functions are almost density independent, and the inner part where correlation functions vary with the density. The effective potential in the outer region corresponds to K_a, and that in the inner region to K_{core}. If the effect of many-body correlation is included in the effective potential, it will change a little but its properties mentioned above will not change.

As nucleon-nucleon scattering experiments show that the nuclear forces in the odd states are fairly weak, we assume that K_a is of Serber type and K_{core} is of Wigner type. For the sake of convenience we rewrite the reaction matrix as

$$(r|K|\boldsymbol{r}') = \delta(\boldsymbol{r}-\boldsymbol{r}')K_a(r)\frac{1+P_x}{2} + \delta(\boldsymbol{r}-\boldsymbol{r}')K_{core}(r) + K_{non\text{-}local}, \quad (2\cdot3)$$

where

$$K_a(r) = \int d\boldsymbol{r}'(\boldsymbol{r}|K_a|\boldsymbol{r}'),$$

$$K_{core}(r) = \int d\boldsymbol{r}'(\boldsymbol{r}|K_{core}|\boldsymbol{r}').$$

Substituting the expression (2·3) in Eq. (2·1),

$$\begin{aligned}
E = &-\frac{\hbar^2}{2m}\sum_i \int \varphi_i^*(\boldsymbol{r})\Delta\varphi_i(\boldsymbol{r})\,dv \\
&+\frac{3}{16}\sum_{i,j}\int \varphi_i^*(\boldsymbol{r})\varphi_j^*(\boldsymbol{r}')K_a(|\boldsymbol{r}-\boldsymbol{r}'|)\varphi_i(\boldsymbol{r})\varphi_j(\boldsymbol{r}')\,dv\,dv' \\
&+\frac{3}{16}\sum_{i,j}\int \varphi_i^*(\boldsymbol{r})\varphi_j^*(\boldsymbol{r}')K_a(|\boldsymbol{r}-\boldsymbol{r}'|)\varphi_i(\boldsymbol{r}')\varphi_j(\boldsymbol{r})\,dv\,dv' \\
&+\frac{1}{2}\sum_{i,j}\int \varphi_i^*(\boldsymbol{r})\varphi_j^*(\boldsymbol{r}')K_{core}(|\boldsymbol{r}-\boldsymbol{r}'|)\varphi_i(\boldsymbol{r})\varphi_j(\boldsymbol{r}')\,dv\,dv' \\
&-\frac{1}{8}\sum_{i,j}\int \varphi_i^*(\boldsymbol{r})\varphi_j^*(\boldsymbol{r}')K_{core}(|\boldsymbol{r}-\boldsymbol{r}'|)\varphi_i(\boldsymbol{r}')\varphi_j(\boldsymbol{r})\,dv\,dv' \\
&+\frac{1}{2}\sum_{i,j}\int [\varphi_i^*(\boldsymbol{r}_1)\varphi_j^*(\boldsymbol{r}_2)-\varphi_j^*(\boldsymbol{r}_1)\varphi_i^*(\boldsymbol{r}_2)] \\
&\times(\boldsymbol{r}_{12}|K_{non\text{-}local}|\boldsymbol{r}_{12}')\varphi_i(\boldsymbol{r}_1')\varphi_j(\boldsymbol{r}_2')\,dv_1\,dv_2\,dv_1'\,dv_2' \\
&=-\frac{\hbar^2}{2m}\int \Delta\rho(\boldsymbol{r}',\boldsymbol{r})|_{\boldsymbol{r}'=\boldsymbol{r}}\,dv
\end{aligned}$$

$$+\frac{3}{16}\int\rho(\boldsymbol{r})K_a(|\boldsymbol{r}-\boldsymbol{r}'|)\rho(\boldsymbol{r}')\,dv\,dv'$$

$$+\frac{3}{16}\int\rho^2(\boldsymbol{r},\boldsymbol{r}')K_a(|\boldsymbol{r}-\boldsymbol{r}'|)\,dv\,dv'$$

$$+\frac{1}{2}\int\rho(\boldsymbol{r})K_{core}(|\boldsymbol{r}-\boldsymbol{r}'|)\rho(\boldsymbol{r}')\,dv\,dv'$$

$$-\frac{1}{8}\int\rho^2(\boldsymbol{r},\boldsymbol{r}')K_{core}(|\boldsymbol{r}-\boldsymbol{r}'|)\,dv\,dv'$$

$$+\frac{1}{2}\int[\rho(\boldsymbol{r}_1,\boldsymbol{r}_1')\rho(\boldsymbol{r}_2,\boldsymbol{r}_2')-\rho(\boldsymbol{r}_1,\boldsymbol{r}_2')\rho(\boldsymbol{r}_2,\boldsymbol{r}_1')]$$

$$\times(\boldsymbol{r}_{12}|K_{non\text{-}local}|\boldsymbol{r}_{12}')\,dv_1\,dv_2\,dv_1'\,dv_2', \qquad (2\cdot4)$$

where

$$\rho(\boldsymbol{r},\boldsymbol{r}')=\sum_{i:\,neutron\ and\ proton}\varphi_i^*(\boldsymbol{r})\varphi_i(\boldsymbol{r}').$$

$$\rho(\boldsymbol{r})=\sum_i|\varphi_i(\boldsymbol{r})|^2=\rho(\boldsymbol{r},\boldsymbol{r}). \qquad (2\cdot5)^{*,**}$$

Now, let us simplify this Hartree-Fock expression. In the statistical method it is assumed that the energy can be expressed by the one-body density distribution function only; that is, the phase relation between single particle wave functions plays minor role. Thus we must look for the best expression for the energy, E, within the above restriction. In the usual statistical method (Thomas-Fermi-Dirac model), from the assumption of the local Fermi-gas we use

$$\rho(\boldsymbol{r},\boldsymbol{r}')=\frac{4}{(2\pi)^3}\int_0^{K_F}e^{i\boldsymbol{K}\cdot(\boldsymbol{r}'-\boldsymbol{r})}\,d\boldsymbol{K}$$

$$\left.\begin{aligned}\rho^2(\boldsymbol{r},\boldsymbol{r}')&=\rho(\boldsymbol{r})\rho(\boldsymbol{r}')\times\frac{9(\sin K_F|\boldsymbol{r}-\boldsymbol{r}'|-K_F|\boldsymbol{r}-\boldsymbol{r}'|\cos K_F|\boldsymbol{r}-\boldsymbol{r}'|)^2}{K_F^6|\boldsymbol{r}-\boldsymbol{r}'|^6}\\&\equiv\rho(r)\rho(r')g(K_F|\boldsymbol{r}-\boldsymbol{r}'|),\end{aligned}\right\} \qquad (2\cdot6)$$

and

$$-\frac{\hbar^2}{2m}\int\Delta\rho(\boldsymbol{r},\boldsymbol{r}')|_{r'=r}\,dv=\frac{3}{40}(12\pi^2)^{2/3}\frac{\hbar^2}{m}\int\rho^{5/3}(r)\,dv. \qquad (2\cdot7)$$

No one has ever investigated the correction to the expression (2·6), but corrections to the expression (2·7) for the kinetic energy have been investigated

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* In this work, we neglect the Coulomb interaction, so the density distribution of protons is equal to that of neutrons. $\rho(\boldsymbol{r})=\rho_{proton}(\boldsymbol{r})+\rho_{neutron}(\boldsymbol{r})=2\rho_{proton}(\boldsymbol{r})$.

** In the following we write $\rho(\boldsymbol{r})$ as $\rho(r)$, since the surface energy is minimum for the spherical nucleus.

by many authors.[7] This correction, however, turned out to be small for our case in which the gradient of the density distribution is small,[7] so we neglect this correction in the following.*

As to the non-locality of the effective potential, we approximate it by the non-locality of the effective potential in the cluster development method:

$$\begin{aligned} E_{non\text{-}local} &= -\frac{2\hbar^2}{m}\sum_{i>j}\int[\varphi_i^*(\boldsymbol{r})\varphi_j^*(\boldsymbol{r}')-\varphi_j^*(\boldsymbol{r})\varphi_i^*(\boldsymbol{r}')] \\ &\times f(|\boldsymbol{r}-\boldsymbol{r}'|)\nabla f(|\boldsymbol{r}-\boldsymbol{r}'|)\nabla\varphi_i(\boldsymbol{r})\varphi_j(\boldsymbol{r}')\,dv\,dv' \\ &= -\frac{\hbar^2}{4m}\int\nabla\rho(r)\nabla f^2(|\boldsymbol{r}-\boldsymbol{r}'|)\rho(r')\,dv\,dv' \\ &+\frac{\hbar^2}{16m}\int\nabla f^2(|\boldsymbol{r}-\boldsymbol{r}'|)\nabla\rho^2(\boldsymbol{r},\boldsymbol{r}')\,dv\,dv', \end{aligned} \tag{2·8}$$

Thus, the expression for energy becomes

$$\begin{aligned} E &= C_k\int\rho^{5/3}(r)\,dv \\ &+\frac{3}{16}\int\rho(r)K_a(|\boldsymbol{r}-\boldsymbol{r}'|)\rho(r')\,dv\,dv' \\ &+\frac{3}{16}\int\rho(r)K_a(|\boldsymbol{r}-\boldsymbol{r}'|)g\left[K_F\left(\frac{\boldsymbol{r}+\boldsymbol{r}'}{2}\right)\times|\boldsymbol{r}-\boldsymbol{r}'|\right]\rho(r')\,dv\,dv' \\ &+\frac{1}{2}\int\rho(r)K_{core}(|\boldsymbol{r}-\boldsymbol{r}'|)\rho(r')\,dv\,dv' \\ &-\frac{1}{8}\int\rho(r)K_{core}(|\boldsymbol{r}-\boldsymbol{r}'|)g\left[K_F\left(\frac{\boldsymbol{r}+\boldsymbol{r}'}{2}\right)\times|\boldsymbol{r}-\boldsymbol{r}'|\right]\rho(r')\,dv\,dv' \\ &-\frac{\hbar^2}{4m}\int\nabla\rho(r)\nabla f^2(|\boldsymbol{r}-\boldsymbol{r}'|)\rho(r')\,dv\,dv' \\ &+\frac{\hbar^2}{16m}\int\nabla f^2(|\boldsymbol{r}-\boldsymbol{r}'|)\nabla\rho^2(\boldsymbol{r},\boldsymbol{r}')\,dv\,dv', \end{aligned} \tag{2·9}$$

where $C_k=3/40(12\pi^2)^{2/3}\cdot\hbar^2/m$, and $K_F(r)=(3\pi^2/2\cdot\rho(r))^{1/3}$. In the expression (2·9) there appears the correlation function, $f(r)$. It is also density dependent and should be found as the solution of the Bethe-Goldstone equation, but as the magnitude of the last two terms is not so large, we do not need the detailed behavior of $f(r)$.

Besides (2·9), we have a subsidiary condition,

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* We can include this correction. Then the nuclear surface thickness increases a little. If this correction is 1/4 of the Weizsäcker term, $(1/4)\times(\hbar^2/8m)\int(d\rho)^2/\rho\,dv$, the 90% to 10% distance, t, increases by 0.2×10^{-13}cm, and if the correction is 1/8, t increases by 0.1×10^{-13}cm.

$$A=\int\rho(r)\,dv, \tag{2·10}$$

where A is the mass number. (2·9) and (2·10) are the basic formulas of our theory. By making use of the variation principle we can find the density distribution, $\rho(r)$, and the energy, E, from (2·9) and (2·10).

§ 3. The surface thickness and the surface energy

As the first example of our theory, we will calculate the surface thickness and the surface energy of heavy nuclei in this section. We choose as our trial function.

$$\rho(r)=\begin{cases}\rho & 0\leq r\leq c-z\\ \rho(c+z-r)/2z & c-z\leq r\leq c+z\\ 0 & c+z\leq r\end{cases} \tag{3·1}*$$

which contains two adjustable parameters—one of the three parameters, c, z and ρ, is a function of the other two parameters through the subsidiary condition (2·10). We can try with trial functions which contain more parameters, but it is needless because at present we can determine from experiments only two parameters on the density distribution which correspond to c and z.

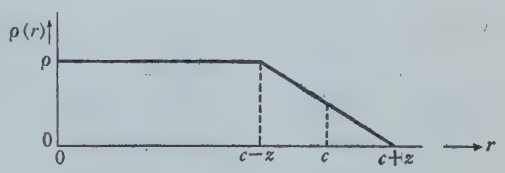

Fig. 1

If we let $c\to\infty$, we are considering the nuclear matter. Then,

$$A\simeq\frac{4\pi}{3}c^3\rho,$$

$$E\simeq\frac{4\pi}{3}c^3\Big(C_k\rho^{5/3}+\frac{3}{16}\rho^2\int K_a(r)\,dv+\frac{3}{16}\rho^2\int K_a(r)g(K_Fr)\,dv$$
$$+\frac{1}{2}\rho^2\int K_{core}(r)\,dv-\frac{1}{8}\rho^2\int K_{core}(r)g(K_Fr)\,dv\Big), \tag{3·2}$$

and the energy per nucleon ε is

$$\varepsilon=\frac{E}{A}=C_k\rho^{2/3}+\frac{3}{16}\rho\int K_a(r)\,dv+\frac{3}{16}\rho\int K_a(r)g(K_Fr)\,dv$$
$$+\frac{1}{2}\rho\int K_{core}(r)\,dv-\frac{1}{8}\rho\int K_{core}(r)g(K_Fr)\,dv. \tag{3·3}$$

Given $K_a(r)$ and $K_{core}(r)$, we can calculate the density, the binding energy,

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* c is the distance to the point where ρ drops to a half of its central value, and z is $\frac{1}{1.6}t$ (t is the 90% to 10% distance).

and the compressibility of the nuclear matter from (3·3). However, we will not concern ourselves with this problem, and we will use potentials which reproduce the properties of the nuclear matter, as we are interested in the surface energy and the surface thickness. As to the ρ dependence of K_{core}, we assume that $K_{core}(r)=\rho^{1/3}V_{core}(r)$ for the sake of simplicity. In this paper we try five potentials with various ranges all for two cases, (i) $r_0=1.0\times10^{-13}$ cm* and $\varepsilon=-15$ Mev/*particle*, and (ii) $r_0=1.1\times10^{-13}$ cm and $\varepsilon=-15$ Mev/*particle*. The potentials are**

(A)

$$\frac{1+P_x}{2}\begin{cases}0 & 1.1<r\\ -230 & 0.4<r<1.1\\ 0 & r<0.4\end{cases}+\frac{912\,\delta(r-0.4)\rho^{1/3}(r)}{4\pi\times(0.4)^2}$$

(A)′

$$\frac{1+P_x}{2}\begin{cases}0 & 1.1<r\\ -282 & 0.4<r<1.1\\ 0 & r<0.4\end{cases}+\frac{1310\,\delta(r-0.4)\rho^{1/3}(r)}{4\pi\times(0.4)^2}$$

(B)

$$\frac{1+P_x}{2}\begin{cases}0 & 1.6<r\\ -36.3 & 0.8<r<1.6\\ -363 & 0.4<r<0.8\\ 0 & r<0.4\end{cases}+\frac{875\,\delta(r-0.4)\rho^{1/3}(r)}{4\pi\times(0.4)^2}$$

(B)′

$$\frac{1+P_x}{2}\begin{cases}0 & 1.6<r\\ -44.4 & 0.8<r<1.6\\ -444 & 0.4<r<0.8\\ 0 & r<0.4\end{cases}+\frac{1260\,\delta(r-0.4)\rho^{1/3}(r)}{4\pi\times(0.4)^2}$$

(C)

$$\frac{1+P_x}{2}\begin{cases}0 & 2.0<r\\ -340\,e^{-1.5r}/r & 0.6<r<2.0\\ -230 & 0.4<r<0.6\\ 0 & r<0.4\end{cases}+\frac{780\,\delta(r-0.4)\rho^{1/3}(r)}{4\pi\times(0.4)^2}$$

(C)′

$$\frac{1+P_x}{2}\begin{cases}0 & 2.0<r\\ -415\,e^{-1.5r}/r & 0.6<r<2.0\\ -280 & 0.4<r<0.6\\ 0 & r<0.4\end{cases}+\frac{1150\,\delta(r-0.4)\rho^{1/3}(r)}{4\pi\times(0.4)^2}$$

(D)

$$\frac{1+P_x}{2}\begin{cases}0 & 2.0<r\\ -353\,e^{-1.5r}/r & 0.7<r<2.0\\ -177 & 0.4<r<0.7\\ 0 & r<0.4\end{cases}+\frac{770\,\delta(r-0.4)\rho^{1/3}(r)}{4\pi\times(0.4)^2}$$

(D′)

$$\frac{1+P_x}{2}\begin{cases}0 & 2.0<r\\ -431\,e^{-1.5r}/r & 0.7<r<2.0\\ -216 & 0.4<r<0.7\\ 0 & r<0.4\end{cases}+\frac{1140\,\delta(r-0.4)\rho^{1/3}(r)}{4\pi\times(0.4)^2}$$

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* $c\equiv r_0A^{1/3}$.

** A, B, C, D and E correspond to case (i) and A′, B′, C′, D′ and E′ correspond to case (ii). The unit of length is 10^{-13} cm, and the unit of energy is Mev.

(E)

$$\frac{1+P_x}{2}\begin{cases} 0 & 2.0<r \\ -620\,e^{-2r}/r & 0.7<r<2.0 \\ -220 & 0.4<r<0.7 \\ 0 & r<0.4 \end{cases}$$

$$+\frac{809\,\delta(r-0.4)\,\rho^{1/3}(r)}{4\pi\times(0.4)^2}$$

(E)′

$$\frac{1+P_x}{2}\begin{cases} 0 & 2.0<r \\ -756\,e^{-2r}/r & 0.7<r<2.0 \\ -268 & 0.4<r<0.7 \\ 0 & r<0.4 \end{cases}$$

$$+\frac{1180\,\delta(r-0.4)\,\rho^{1/3}(r)}{4\pi\times(0.4)^2}\,.$$

Now, let us calculate the surface energy and the surface thickness. Substituting (3·1) in (2·9), we find

$$C_k\int\rho^{5/3}(r)\,dv=\frac{4\pi}{3}C_k\rho^{5/3}\left[c^3-\frac{3}{4}c^2z\right], \tag{3·4}$$

$$\frac{3}{16}\int\rho(r)\,K_a(|\boldsymbol{r}-\boldsymbol{r}'|)\,\rho(r')\,dv\,dv'=-\frac{1}{3}\pi^2\rho^2c^3\int\frac{dK_a(r)}{dr}r^3\,dr$$

$$+\frac{1}{3}\pi^2\rho^2c^2\int\frac{dK_a(r)}{dr}\left(zr^3+\frac{3}{20}\frac{r^5}{z}-\frac{1}{64}\frac{r^6}{z^2}\right)dr+\mathrm{O}(c) \tag{3·5}*$$

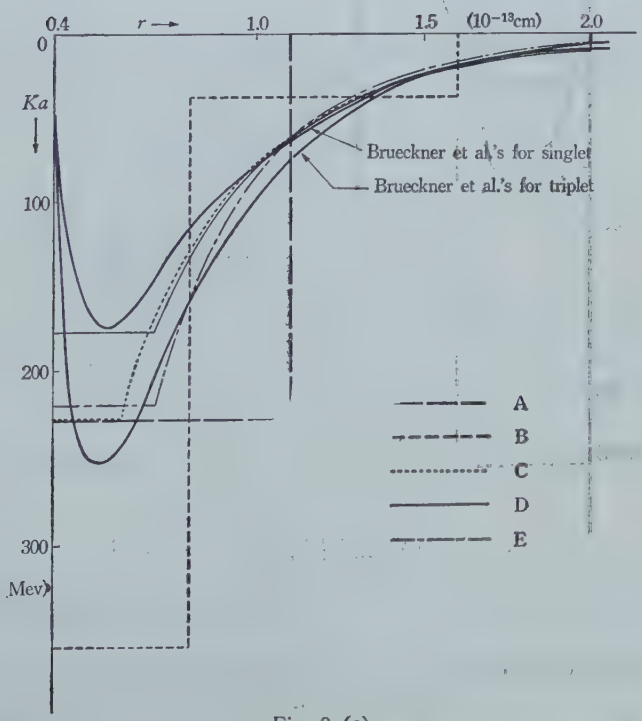

Fig. 2 (a).

in the Taylor expansion in z/c. The term containing c^3 is the volume energy and the term containing c^2 is the surface energy. The third integral

$$\frac{3}{16}\int\rho(r)g\left[K_F\left(\frac{\boldsymbol{r}+\boldsymbol{r}'}{2}\right)\times|\boldsymbol{r}-\boldsymbol{r}'|\right]K_a(|\boldsymbol{r}-\boldsymbol{r}'|)\rho(r')\,dv\,dv'$$

difficult to calculate analytically, and we approximate it by

$$\frac{3}{16}\int\rho(r)g[p\times|\boldsymbol{r}-\boldsymbol{r}'|]K_a(|\boldsymbol{r}-\boldsymbol{r}'|)\rho(r')\,dv\,dv'.$$

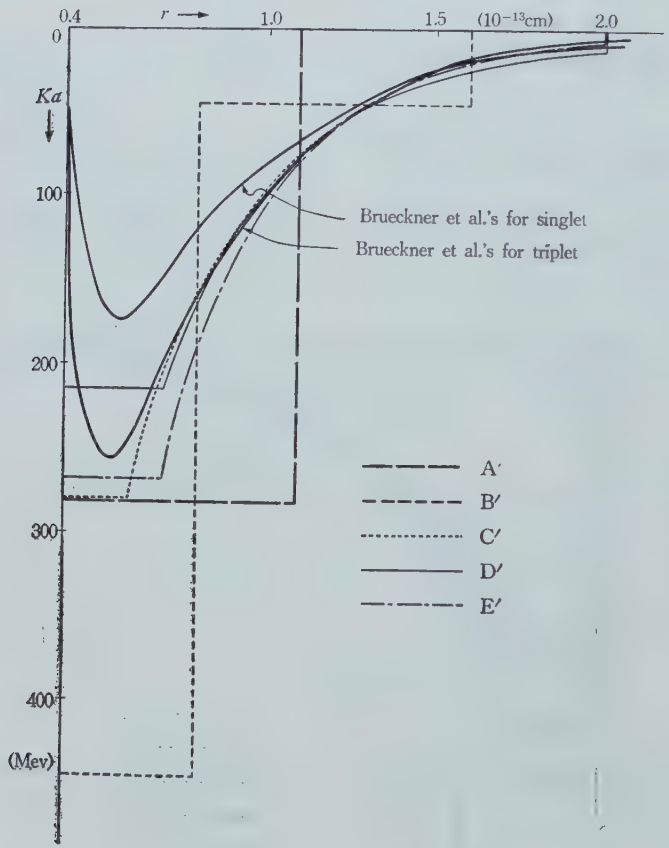

Fig. 2 (b).

* This formula is derived in Appendix A, and is valid to the potentials which satisfy the condition

$$K_a(r)=0, \quad \text{if } r>2z. \tag{3·6}$$

All potentials used by us satisfy (3·6).

and

$$p=\frac{6}{7}K_{F_0},$$

where K_{F_0} stands for the Fermi momentum at the center ($r=0$), and $6/7\cdot K_{F_0}$ is the expectation value of the Fermi momentum in the surface region. This integral becomes as (3·5),

$$-\frac{1}{3}\pi^2\rho^2c^3\int\frac{d(K_a(r)g(pr))}{dr}r^3dr$$
$$+\frac{1}{3}\pi^2\rho^2c^2\int\frac{d(K_a(r)g(pr))}{dr}\left(zr^3+\frac{3}{20}\frac{r^5}{z}-\frac{1}{64}\frac{r^6}{z^2}\right)dr+O(c). \tag{3·7}$$

In the same way, the fourth integral,

$$\frac{3}{8}\int\rho^{4/3}(r)\,V_{core}(|\boldsymbol{r}-\boldsymbol{r}'|)\rho(r')\,dv\,dv'$$
$$\simeq-\frac{2}{3}\pi^2\rho^{7/3}c^3\int\frac{dV_{core}(r)}{dr}r^3dr$$
$$+\frac{2}{3}\pi^2\rho^{7/3}c^2\int\frac{dV_{core}(r)}{dr}\left(\frac{6}{5}zr^3+\frac{3}{20}\frac{r^5}{z}\right)dr, \tag{3·8}$$

where we have neglected a term which is proportional to r^6/z^2 because V_{core} is of short range. Finally, the non-local term is

$$-\frac{\hbar^2}{4m}\int\nabla\rho(r)\nabla f^2(|\boldsymbol{r}-\boldsymbol{r}'|)\rho(r')\,dv\,dv'$$
$$+\frac{\hbar^2}{16m}\int\nabla f^2(|\boldsymbol{r}-\boldsymbol{r}'|)\nabla\rho^2(\boldsymbol{r},\boldsymbol{r}')\,dv\,dv'$$
$$\simeq-\frac{\hbar^2}{4m}\int\nabla\rho(r)\nabla f^2(|\boldsymbol{r}-\boldsymbol{r}'|)\rho(r')\,dv\,dv'$$
$$=-\frac{\pi^2\hbar^2}{m}\frac{\rho^2}{2z}c^2\left\{\frac{4}{3}a^3-\frac{1}{4}\frac{a^4}{z}\right\}, \tag{3·9}*$$

where we have assumed that

$$f^2(r)=\begin{cases}0 & |r|<a\\ 1 & |r|\geq a.\end{cases} \tag{3·10}**$$

Collecting (3·4), (3·5), (3·7), (3·8) and (3·9), the surface energy, E_s, becomes

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* See Appendix B.

** We choose 0.5×10^{-13} cm as the value of a.

$$E_s=(a_1 z+a_2/z+a_3/z^2)c^2, \tag{3·11}$$

where

$$a_1=-\pi C_k\rho^{5/3}+\frac{1}{3}\pi^2\rho^2\int\frac{dK_a(r)}{dr}r^3dr$$
$$+\frac{1}{3}\pi^2\rho^2\int\frac{d(K_a(r)g(pr))}{dr}r^3dr$$
$$+\frac{4}{5}\pi^2\rho^{7/3}\int\frac{dV_{core}(r)}{dr}r^3dr$$
$$a_2=\frac{1}{20}\pi^2\rho^2\int\frac{dK_a(r)}{dr}r^5dr+\frac{1}{20}\pi^2\rho^2\int\frac{d(K_a(r)g(pr))}{dr}r^5dr$$
$$+\frac{1}{10}\pi^2\rho^{7/3}\int\frac{dV_{core}(r)}{dr}r^5dr-\frac{2\pi^2\hbar^2}{3m}\rho^2a^3,$$

and

$$a_3=-\frac{\pi^2\rho^2}{192}\int\frac{dK_a(r)}{dr}r^6dr-\frac{\pi^2\rho^2}{192}\int\frac{d(K_a(r)g(pr))}{dr}r^6dr$$
$$+\frac{\pi^2\hbar^2\rho^2}{8m}a^4.$$

We will neglect a_3c^2/z^2 as this term is smaller than the other two terms by a factor of 1/20. Thus

$$E_s=c^2(a_1z+a_2/z). \tag{3·12}$$

By the variation principle,* we find

$$z=\sqrt{a_2/a_1}, \tag{3·13}$$

and

$$E_s=2\sqrt{a_1a_2}c^2=2\sqrt{a_1a_2}\left(\frac{3}{4\pi\rho}\right)^{2/3}A^{2/3}. \tag{3·14}$$

Numerical results are given in the Table.

As is seen from Fig. 2, our effective potentials C, D, E, C′, D′ and E′ are very similar to the effective potential by Brueckner et al., so in our approximation Brueckner et al.'s[2] potential gives

$$t=(2.8\sim3.0)\times10^{-13}\text{ cm},$$
and
$$E_s=(23\sim28)A^{2/3}\text{ Mev}. \tag{3·15}$$

The potentials A, B, A′ and B′ have no corresponding real two-body potentials, they are used to show how the surface thickness and the surface energy are sensitive to the range of the effective potential.

* c and ρ are not arbitrary parameters, z is the only one arbitrary parameter. c and ρ are determined from nuclear matter (3·3). The change in z does not affect c and ρ as $A=(4\pi/3)\rho c(c^2+z^2)$.

Table

potential	a_1/ρ^2 Mev× 10^{-39} cm^3	a_2/ρ^2 Mev× 10^{-39} cm^3	the 1st term of a_2/ρ^2	the 2nd term of a_2/ρ^2	the 3rd & 4th terms of a_2/ρ^2	$t=(1.6z)$ 10^{-13} cm	E_s $A^{2/3}$ Mev
A	130	225	180	135	−90	2.1	20
B	130	300	240	150	−90	2.4	23
C	130	450	345	190	−85	3.0	28
D	130	460	355	190	−85	3.0	28
E	130	395	310	170	−85	2.8	26
A′	168	305	225	180	−100	2.2	18
B′	168	385	295	190	−100	2.4	20
C′	168	580	425	250	−95	3.0	24
D′	168	600	435	260	−95	3.0	25
E′	168	520	380	235	−95	2.8	23

§ 4. Discussions

In this work we have developed a statistical method for treating a finite nucleus. This is a simple but powerful method. This method will be justified whenever the corresponding Hartree-Fock method is justified. By making use of the effective potential we have taken into account the effects of the two-body correlation. The validity of our method essentially lies on the correctness of our assumption that the probability of many-body correlation is negligible. At present, there is no experimental indication whether the probability is small or not. There may exist some experiments for which many-body correlations play an important role, but it is very probable that they are negligible for the density and the energy.

As a first example, we have applied the method for the calculation of the surface thickness and the surface energy of heavy nuclei. In the calculation we have made two approximations, (i) the simple density dependence of K_{core}, $\rho^{1/3}V_{core}(r)$ and (ii) the approximation made in the calculation of the exchange integral of $K_a(r)$. In spite of these approximations the results are nearly satisfactory. As was expected, the results are sensitive to the form of the potential, especially to its range. For the effective potential proposed by Brueckner et al., we have obtained a little bigger value for t. In the subsequent paper we shall get rid of the above two avoidable approximations and examine if the difference decreases or not, and also we shall apply the method to lighter nuclei there. In this paper the Coulomb interaction was neglected, but it is easily taken into account. However, notice that the Coulomb interaction modified by correlation functions, $f\cdot 1/|\boldsymbol{r}-\boldsymbol{r}'|\cdot f$ should be used instead of the real Coulomb interaction, $1/|\boldsymbol{r}-\boldsymbol{r}'|$.

The author would like to express his gratitude to Prof. H. Miyazawa and Dr. Y. Wada for discussions and hearty encouragement.

Appendix A

$$I \equiv \int \rho(r)\, W(|\boldsymbol{r}-\boldsymbol{r}'|)\, \rho(r')\, dv\, dv'$$

$$= 8\pi^2 \int r^2\, dr\, r'^2\, dr' \int_{-1}^{1} d\omega\, \rho(r)\, W(\sqrt{r^2+r'^2-2rr'\omega})\, \rho(r').$$

At first we treat a square well potential,

$$W(r) = \begin{cases} W & r<a, \\ 0 & r>a, \end{cases}$$

then

$$I = 8\pi^2 \int r^2\, dr\, r'^2\, dr' \int_{\omega}^{1} \rho(r)\, W(\sqrt{r^2+r'^2-2rr'\omega})\, \rho(r')\, d\omega,$$

where

$$\omega = \begin{cases} 1 & |r-r'|>a, \\ \dfrac{r^2+r'^2-a^2}{2rr'} & r+r'>a>|r-r'|, \\ -1 & a>r+r'. \end{cases}$$

Then

$$I = \int \rho(r)\, W(r, r')\, \rho(r')\, r^2\, dr r'^2\, dr',$$

where

$$W(r, r') = \begin{cases} 0 & |r-r'|>a, \\ 8\pi^2 W \dfrac{a^2-(r-r')^2}{2rr'} & r+r'>a>|r-r'|, \\ 16\pi^2 W & a>r+r'. \end{cases}$$

For the following density distribution,

$$\rho(r) = \begin{cases} \rho & r<c-z, \\ \rho\dfrac{c+z-r}{2z} & c-z<r<c+z, \\ 0 & c+z<r, \end{cases}$$

we get the following result after straightforward but tedious calculations.

$$I = \frac{16}{9}\pi^2\rho^2 W \left\{ c^3a^3 - c^2za^3 - \frac{3}{20}c^2a^5/z + \frac{1}{64}c^2a^6/z^2 \right\},$$

if $a<2z$.

This result is easily extended to the potential $V(r)$, provided $V(r)$ vanishes for $r>2z$:

$$I=-\frac{16}{9}\pi^2\rho^2c^3\int\frac{dV(r)}{dr}r^3dr$$
$$+\frac{16}{9}\pi^2\rho^2c^2\int\frac{dV(r)}{dr}\left\{zr^3+\frac{3}{20}\frac{r^5}{z}-\frac{1}{64}\frac{r^6}{z^2}\right\}.$$

Appendix B

$$I\equiv-\frac{\hbar^2}{4m}\int\nabla\rho(r)\nabla f^2(|\boldsymbol{r}-\boldsymbol{r}'|)\rho(r')\,dv\,dv'$$
$$=-\frac{\hbar^2}{4m}\int\frac{d\rho(r)}{dr}\frac{d}{dr}f^2(|\boldsymbol{r}-\boldsymbol{r}'|)\rho(r')\,dv\,dv'. \qquad (B\cdot1)$$

If we assume

$$f(r)=\begin{cases}0 & r<a,\\ 1 & r>a,\end{cases}$$

$$\frac{d}{dr}f^2(|\boldsymbol{r}-\boldsymbol{r}'|)=\frac{d|\boldsymbol{r}-\boldsymbol{r}'|}{dr}\frac{df^2(|\boldsymbol{r}-\boldsymbol{r}'|)}{d|\boldsymbol{r}-\boldsymbol{r}'|}=\frac{r-r'\cos\theta}{|\boldsymbol{r}-\boldsymbol{r}'|}\delta(|\boldsymbol{r}-\boldsymbol{r}'|-a). \qquad (B\cdot2)$$

Substituting (B·2) in (B·1),

$$I=-\frac{\pi^2\hbar^2}{m}\int\frac{d\rho(r)}{dr}(r^2-r'^2+a^2)\rho(r')r'\,dr\,dr'.$$

This becomes after straightforward but tedious calculations,

$$I=-\frac{\pi^2\hbar^2}{m}\rho^2c^2\left\{\frac{2}{3}a^3/z-\frac{1}{8}a^4/z^2\right\},$$

for the density distribution (3·1).

References

1) E. Majorana, Z. Phys. **82** (1933), 137.
C. F. Weizsäcker, Z. Phys. **96** (1935), 431.
S. Flügge, Z. Phys. **96** (1935), 459.
K. Nakabayashi, Z. Phys. **97** (1935), 211.
H. A. Bethe and R. F. Bacher, Rev. Mod. Phys. **8** (1936), 33.
F. S. Wang, Z. Phys. **100** (1936), 734.
H. Volz, Z. Phys. **105** (1937), 537.
H. Euler, Z. Phys. **105** (1937), 553.
L. M. Yang, Proc. Phys. Soc. **A64** (1951), 632.
J. M. Luttinger and J. H. D. Jensen, Phys. Rev. **86** (1952), 907.
P. Gombás, Acta Phys. Hung. **1** (1952), 329; **2** (1952), 223; Zeitschrift Fortschritte Phy· **5** (1957), 159; and papers cited there.
2) K. A. Brueckner and J. L. Gammel, Phys. Rev. **109** (1958), 1023.
K. A. Brueckner, J. L. Gammel and H. Weitzner, Phys. Rev. **110** (1958), 431.

K. A. Brueckner, A. M. Lockett and M. Rotenberg, Bull. Amer. Phys. Soc. Ser. II. **4** (1959), 156, and papers cited in these papers.

3) P. Gombás, *Die statistische Theorie des Atoms und ihre Anwendungen* (Springer, Wien, 1949).
4) F. Iwamoto and M. Yamada, Prog. Theor. Phys. **17** (1957), 543; **18** (1957), 345.
5) D. G. Ravenhall, Rev. Mod. Phys. **30** (1958), 430.
6) A. G. W. Cameron, Can. J. Phys. **35** (1951), 1021.
A. E. S. Green, Rev. Mod. Phys. **30** (1958), 569; Phys. Rev. **95** (1954), 1006.
7) C. F. Weizsäcker, Z. Phys. **96** (1935), 431.
W. J. Swiatecki, Proc. Phys. Soc. **A64** (1951), 226; **A68** (1955), 285.
R. Berg and L. Wilets, Proc. Phys. Soc. **A68** (1955), 229.
M. A. Naqvi, Nucl. Phys. **10** (1959), 256

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Cluster Expansion Technique in the Many-Fermion Systems. I*

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A new cluster expansion technique is developed for the calculation of the energy expectation value of the many-fermion systems with singular potentials. The wave function, which is studied in this paper, is of the form

$$\Psi=\prod_{(i,j)} f(r_{ij}) \sum_{P} (-1)^{p} P\{\chi(1, 2)\chi(3, 4)\cdots\cdots\chi(2N-1,2N)\},$$

which, in a special case, coincides with Jastrow wave function. The formula for the energy expectation value is written in terms of the complex cluster integrals. The method introduced in the derivation of this formula has an analogy to the classical cluster expansion method in the theory of an imperfect classical gas of multicomponents.

§ 1. Introduction

In the calculation of the energy expectation value of the many-fermion systems with short-range repulsions, Jastrow[1] first introduced the cluster expansion technique. The more systematic treatments of this cluster method were given later by several people.[2)-4)] The wave function which was studied in their works was of the form (Jastrow wave function)

$$\Psi_J=\prod_{(i,j)} f(r_{ij})\ \ Det\ \ (plane\ wave). \tag{1}$$

As well-known, the correlation factor $\prod f(r_{ij})$ in front of the Slater determinant is due to short-range repulsions, and the introduction of the correlation factor of this type yields a finite result for the energy expectation value.

On the other hand, another example of the cluster expansion technique was given by us in the discussion of the Bardeen-Cooper-Schrieffer theory of superconductivity in the configuration space. The wave function, which was studied in our previous paper[5] hereafter referred to as A, was of the form (BCS wave function)

$$\Psi_0=\sum_{P} (-1)^{p} P\{\chi(1, 2)\chi(3, 4)\cdots\chi(2N-1, 2N)\}, \tag{2}$$

where P is a permutation operator working on the indices of $2N$ coordinates and p its parity. The two-fermion wave function $\chi(i, j)$ is

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$$\chi(i,\ j)=\psi(r_{ij})\frac{\alpha(i)\beta(j)-\alpha(j)\beta(i)}{\sqrt{2}}, \tag{3}$$

where $\psi(r)$ is a function giving the spatial correlation between two fermions, and α and β the two spin functions. As shown in A, the energy expectation value of the BCS ground state is written in terms of the quantum-mechanical ring-cluster integrals b_l defined by

$$b_l=\frac{(-1)^{l-1}}{\Omega l}\int\psi(r_{12})\psi(r_{23})\cdots\psi(r_{2l,1})\,d\boldsymbol{r}^{2l}, \tag{4}$$

where Ω is the volume of the system. The factor (-1) originates from the Fermi-Dirac statistics of particles.

The purpose of the present paper is to develop a new cluster expansion technique for the calculation of the energy expectation value of the many-fermion systems with singular potentials. The wave function, which is studied in this paper, is of the form*

$$\Psi=\prod_{(i,j)}f(r_{ij})\,\Psi_0, \tag{5}$$

where Ψ_0 is the pair-correlated wave function defined by (2). The correlation factor $\prod f(r_{ij})$ in front of BCS wave function is due to the singularity of potentials. It is easy to see that (5) is equivalent to Jastrow wave function (1), when $\psi(r)=\sum_{k\leq k_f}\exp(i\boldsymbol{k}\boldsymbol{r})$, where k_f is the radius of the Fermi sphere.

The content of this paper is as follows. § 2 is concerned with the normalization integral of the wave function (5). In this section, we shall introduce the complex cluster integrals and show that the normalization integral expressed in terms of the complex cluster integrals has the same form as that of the partition function in Mayer's theory[6] of an imperfect classical gas. In § 3, the formula for the energy expectation value will be written in terms of the complex cluster integrals. The method, which we adopt in the derivation of this formula, has an analogy to the classical cluster expansion method in the theory[7] of an imperfect classical gas of multicomponents.

§ 2. Normalization integral of the wave function

The problem at hand is the calculation of the normalization integral:

$$\langle\Psi|\Psi\rangle=(2N)!\int\prod_{(i,j)}f^2(r_{ij})\,W_N(\tau^{2N})\,d\tau^{2N}, \tag{6}$$

where

$$W_N(\tau^{2N})=\chi(1,\ 2)\chi(3,\ 4)\cdots\chi(2N-1,\ 2N)\sum_P(-1)^p \times P\{\chi(1,\ 2)\chi(3,\ 4)\cdots\chi(2N-1,\ \ 2N)\}. \tag{7}$$

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* The wave function of this type may be obtained by the extension of the BCS wave function to the system with singular potentials.

τ^{2N} stands for the set of $2N$ coordinates (space coordinates $\boldsymbol{r}_1 \cdots \boldsymbol{r}_{2N}$ and spin coordinates $s_1, \cdots s_{2N}$).

As shown in Appendix 1, $W_N(\tau^{2N})$ can be written in the sum of the products of the ring-cluster functions $B_\nu(\tau^{2\nu})$ as follows,

$$W_N(\tau^{2N}) = \sum \Pi B_\nu(\tau^{2\nu}), \tag{8}$$

where

$$B_\nu(\tau^{2\nu}) = (-1)\chi(j_1, j_{2\nu})\chi(j_2, j_3)\cdots\chi(j_{2\nu}, j_1). \tag{9}$$

Equation (8) is analogous to the development introduced by Beth and Uhlenbeck[8] and Kahn[9] in the calculation of the Slater sum in the quantum statistical mechanics.* The minus sign in (9) originates from the Fermi-Dirac statistics of particles. The group of 2ν fermions represented by the function $B_\nu(\tau^{2\nu})$ will be called a ring-cluster of species ν. In (9), $\sum\Pi$ denotes a sum of products of B_ν-functions over all possible divisions of $2N$ fermions into separate groups (or ring-clusters) containing even number of fermions. The number of the products which represent m_1 clusters of two fermions, m_2 clusters of four fermions, ……, m_ν clusters of 2ν fermions,……, is

$$2^N(N)!\prod_\nu \frac{(2^{\nu-1}/\nu)^{m_\nu}}{m_\nu!}, \tag{10}$$

according to Appendix A of A.

Inserting (8) into (6) and using (10) in consideration of the fact that $\Pi f^2(r_{ij})$ is a symmetrical function in the $2N$ space coordinates, we have

$$\langle \Psi | \Psi \rangle = 2^N(N!)^2(2N)! \sum_{\sum_\nu m_\nu = m} \prod_\nu \frac{(2^{\nu-1}/\nu)^{m_\nu}}{m_\nu!} W_N(\boldsymbol{m}), \tag{11}$$

where

$$W_N(\boldsymbol{m}) = \int \prod_{(i,j)} f^2(r_{ij}) \prod_\nu [B_\nu(\tau^{2\nu})]^{m_\nu} d\tau^{2N}. \tag{12}$$

Throughout this paper, any bold-face symbol will be employed to indicate a set of numbers, one for each species. Thus the bold-face $\boldsymbol{m}$ indicates a set of numbers of ring-clusters of different species, m_ν of species ν: $\boldsymbol{m} = m_1, m_2, \cdots, m_\nu, \cdots$. Also, the expression $[B_\nu(\tau^{2\nu})]^{m_\nu}$ denotes the product of m_ν ring-cluster functions $B_\nu(\tau^{2\nu})$ with the different set of 2ν coordinates:

$$[B_\nu(\tau^{2\nu})]^{m_\nu} \equiv B_\nu(\tau_1^{2\nu}) B_\nu(\tau_2^{2\nu}) \cdots B_\nu(\tau_{m_\nu}^{2\nu}). \tag{13}$$

Following Jastrow, we now introduce the function h_{ij} defined by

$$h_{ij} = f^2(r_{ij}) - 1. \tag{14}$$

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* It is recommended that the following reference besides references (8) and (9) should be read on this point: J. de Boer, *Reports on Progress in Physics*, XII (1948-49) p. 336.

Then, $W_N(\boldsymbol{m})$ can be written as

$$W_N(\boldsymbol{m}) = \int \prod_{(i,j)} (1+h_{ij}) \prod_{\nu} [B_\nu(\tau^{2\nu})]^{m_\nu} d\tau^{2N}. \tag{15}$$

Furthermore, we introduce the B_ν^*-functions defined by

$$B_\nu^*(\tau^{2\nu}) = \prod_{1 \leq i < j \leq 2\nu} (1+h_{ij}) B_\nu(\tau^{2\nu}), \tag{16}$$

where h_{ij} plays the role of connecting two fermions among the 2ν fermions composing the ring-cluster of species ν (or the $B_\nu(\tau^{2\nu})$-function). The diagram showing the role of the functions h_{ij} in (16) is shown in Fig. 1(a). The group of 2ν fermions represented by the function $B_\nu^*(\tau^{2\nu})$ will be called a "molecule" of species ν. Hereafter the molecules will be specified by the Greek letters and the fermions by the italic letters.

With this definition of the B_ν^*-functions, (15) can be rewritten as

$$W_N(\boldsymbol{m}) = \int \prod'_{(m,n)} (1+h_{mn}) \prod_{\nu} [B_\nu^*(\tau^{2\nu})]^{m_\nu} d\tau^{2N} \tag{17}$$

where $\prod'_{(m,n)}$ denotes the product of the functions $(1+h_{mn})$, in which the fermion m belongs to the one molecule of species μ and the fermion n to the other molecule of species ν, that is, h_{mn} plays a role of connecting two molecules. The diagram showing the role of the functions h_{mn} in (17) is shown in Fig. 1(b).

Fig. 1. Comparison of the role of the h-functions in (16) with that of the h-functions in (17); a dotted line represents an h-connection.

In the development of the product $\prod'_{(m,n)}(1+h_{mn})$, which occurs in (17), into the sum of the products of the h-functions, we now pay attention to the sum of the products of the h-functions connecting the two molecules, μ and ν. We introduce the $g_{\mu\nu}$-function defined by

$$g_{\mu\nu} = \sum \prod h_{mn}. \tag{18}$$

Sum over all products of the h-functions connecting two molecules, μ and ν.

The diagram showing the role of the $g_{\mu\nu}$-function is shown in Fig. 2. The g-function is analogous to a certain extent to Mayer's f-function in the statistical mechanics of an imperfect classical gas.

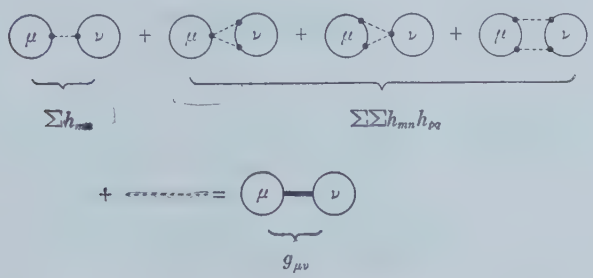

Fig. 2. Diagrammatic representation of $g_{\mu\nu}$, (cf. (18)); a bold-face line represents a g-connection.

With this definition of the $g_{\mu\nu}$-functions, the product $\prod'_{(m,n)}(1+h_{mn})$ can be re written as

$$\prod'_{(m,n)}(1+h_{mn})=\prod_{(\mu,\nu)}(1+g_{\mu\nu}), \tag{19}$$

where the sets (μ, ν) are taken over all pairs of molecules (B_ν*-functions) contributing to (17). Inserting (19) into (17), we then obtain

$$W_N(\boldsymbol{m})=\int\prod_{(\mu\nu)}(1+g_{\mu\nu})\prod_{\nu}[B_\nu^*(\tau^{2\nu})]^{m_\nu}d\tau^{2N}. \tag{20}$$

Let us now write the right-hand side of (20) in terms of the cluster integrals. We introduce the cluster functions $\beta_s(\tau^{2n_s})$ defined by

$$\beta_{\boldsymbol{s}}(\tau^{2n_{\boldsymbol{s}}})=\sum\prod g_{\mu\nu}\prod_{\nu}[B_\nu^*(\tau^{2\nu})]^{s_\nu}, \tag{21}$$

Sum over all products of the g-functions connecting $\sum s_\nu$ molecules.

where the expression $[B_\nu^*(\tau^{2\nu})]^{s_\nu}$ has the same meaning as in (13), and $2n_s$ is the total number of fermions composing the cluster functions $\beta_s(\tau^{2n_s})$, that is, $n_s=\sum_\nu \nu s_\nu$.

In particular,

$$\beta_{0\cdots 0 1_\nu 0\cdots}(\tau^{2\nu})=B_\nu^*(\tau^{2\nu}). \tag{22}$$

The cluster represented by the function $\beta_s(\tau^{2n_s})$ will be called a "complex molecule" of species $\boldsymbol{s}$. This complex molecule consists of s_1 molecules of species 1, s_2 molecules of species 2,$\cdots$, s_ν molecules of species $\nu,\cdots$, which are connected to one another by no-less-than one g-functions. In particular, the complex molecule of species $(0,\cdots 0, 1_\nu, 0,\cdots)$ is a single molecule of species ν. Some example of graphs corresponding to the cluster functions $\beta_s(\tau^{2n_s})$ are shown in Fig. 3.

With this definition of the cluster functions, we find that

$$\prod_{(\mu\nu)}(1+g_{\mu\nu})\prod_{\nu}[B_\nu^*(\tau^{2\nu})]^{m_\nu}=\sum\prod\beta_s(\tau^{2n_s}), \tag{23}$$

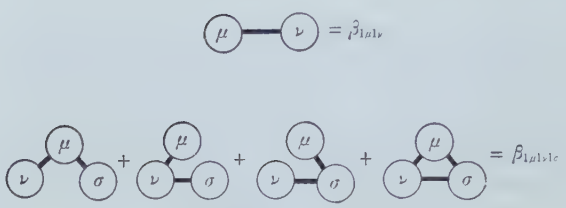

Fig. 3. Some example of graphs representing the cluster function $\beta_s(\tau^{2n_s})$.

where $\sum\Pi$ denotes a sum of products of $\beta_s(\tau^{2n_s})$-functions over all possible division of $\sum m_\nu$ molecules into complex molecules, as in (8). Equation (23) corresponds to the Ursell-Mayer development in the statistical mechanics of an imperfect classical gas. Inserting (23) in (20), we have

$$W_N(\boldsymbol{m}) = \sum\Pi\beta_{\boldsymbol{s}}, \tag{24}$$

where

$$\beta_{\boldsymbol{s}} = \int \beta_{\boldsymbol{s}}(\tau^{2n_{\boldsymbol{s}}})\, d\tau^{2n_{\boldsymbol{s}}}. \tag{25}$$

In particular,

$$\beta_{0\cdots01_\nu0\cdots} = \int B_\nu^*(\tau^{2\nu})\, d\tau^{2\nu}. \tag{26}$$

In (24), one of the products represents $m_{\boldsymbol{s}}$ complex molecules of species $\boldsymbol{s}$, $m_{\boldsymbol{s}'}$ complex molecules of species $\boldsymbol{s}'$,$\cdots$; $\Pi_{\boldsymbol{s}}\beta_{\boldsymbol{s}} = \Pi_{\boldsymbol{s}}(\beta_{\boldsymbol{s}})^{m_{\boldsymbol{s}}}$. Here we need to know how many times the products belonging to the set $\{\boldsymbol{m}_{\boldsymbol{s}}\}$ of complex molecules occur in the summation of (24). This is equal to the number of ways in which the distinguishable $\sum m_\nu$ molecules are divided into groups so that there are $m_{\boldsymbol{s}}$ complex molecules of species $\boldsymbol{s}$, $m_{\boldsymbol{s}'}$ complex molecules of species $\boldsymbol{s}'$,$\cdots$. The total number of those ways of divisions is clearly

$$\frac{\Pi_\nu\, m_\nu!}{\Pi_{\boldsymbol{s}}\, (\boldsymbol{s}!)^{m_{\boldsymbol{s}}}\, m_{\boldsymbol{s}}!}, \tag{27}$$

where the bold-face symbol $\boldsymbol{s}!$ indicates that $\boldsymbol{s}! = s_1!\ s_2!\cdots s_\nu!\cdots$. Using (27), we can write (24) as

$$W_N(\boldsymbol{m}) = \Pi_\nu m_\nu! \sum_{\sum_{\boldsymbol{s}} s_\nu} \Pi_{\boldsymbol{s}} \frac{(\beta_{\boldsymbol{s}}/\boldsymbol{s}!)^{m_{\boldsymbol{s}}}}{m_{\boldsymbol{s}}!}. \tag{28}$$

Inserting (28) into (11) and considering the restrictions that $\sum_{\boldsymbol{s}} s_\nu\, m_{\boldsymbol{s}} = m_\nu$ ($\nu = 1, 2, \cdots$), we get

$$\langle \Psi | \Psi \rangle = 2^N (N!)^2 (2N)! \sum_{\sum_s \sum_\nu \nu s_\nu m_s = N} \prod \frac{(2^{\Sigma(\nu-1)s_\nu} \beta_s / \boldsymbol{\nu}^s \mathbf{s}!)^{m_s}}{m_s!}, \tag{29}$$

where the bold-face symbol $\boldsymbol{\nu}^s$ indicates that $\boldsymbol{\nu}^s = 1^{s_1} 2^{s_2} \cdots \nu^{s_\nu} \cdots$.

Let us now rewrite (29) in a simpler form. In the definition (25) of β_s, we carry out the integration over the spin coordinates $\boldsymbol{s}$. According to (A·14) of A, the cluster integrals β_s can be rewritten as

$$\beta_s = \frac{(-1)^{\Sigma(\nu-1)s_\nu}}{2^{\Sigma(\nu-1)s_\nu}} \int \sum \prod g_{\mu\nu} \prod_\nu [B_\nu^*(\boldsymbol{r}^{2\nu})]^{s_\nu} d\boldsymbol{r}^{2\eta_s}, \tag{30}$$

Sum over all products of the g-functions connecting $\sum s_\nu$ molecules

where

$$B_\nu^*(\boldsymbol{r}^{2\nu}) = \prod_{1 \le i < k \le 2\nu} (1 + h_{j_i j_k}) \psi(r_{j_1 j_2}) \psi(r_{j_2 j_3}) \cdots \psi(r_{j_{2\nu} j_1}). \tag{31}$$

We now define the complex cluster integrals b_s by

$$b_s = \frac{(-1)^{\Sigma(\nu-1)s_\nu}}{\Omega \boldsymbol{\nu}^s \mathbf{s}!} \int \sum \prod g_{\mu\nu} \prod_\nu [B_\nu^*(\boldsymbol{r}^{2\nu})]^{s_\nu} d\boldsymbol{r}^{2\eta_s}. \tag{32}$$

Sum over all products of the g-functions connecting $\sum s_\nu$ molecules

In particular,

$$b_{0 \cdots 1_\nu \cdots} = \frac{(-1)^{\nu-1}}{\Omega \nu} \int B_\nu^*(\boldsymbol{r}^{2\nu}) \, d\boldsymbol{r}^{2\nu}. \tag{33}$$

When $h=0$, $b_{0 \cdots 0 \cdot 1_\nu \cdot 0 \cdots}$ is nothing but the ring-cluster integral defined by (4). Inserting (30) into (29) and using (32), we obtain

$$\langle \Psi | \Psi \rangle = 2^N (N!)^2 (2N)! \sum_{\sum_s \sum_\nu \nu s_\nu m_s = N} \prod \frac{(\Omega b_s)^{m_s}}{m_s!}. \tag{34}$$

Equation (34) is the final result for the normalization integral $\langle \Psi | \Psi \rangle$, which is formally identical with the expression for the normalization integral of the BCS wave function (2), (cf. (6) of A). We, therefore, find that the normalization integral of the wave function (5) is written in terms of the complex cluster integrals b_s and has the same form as that of the partition function in Mayer's theory of an imperfect classical gas, with the exception of the factor $2^N (N!)^2 \times (2N)!$. As seen from the definition (32) of b_s, the complex cluster integrals are constructed with the B_ν^*-functions of various kinds (or molecules), which are connected to one another by no-less-than one g-functions. In this meaning, we may well say that the complex cluster integrals are analogous to the classical cluster integrals for an imperfect classical gas of multicomponents. However, it is to be noted that, in the case of the quantum-mechanical complex cluster integrals b_s, the

effect of the Fermi-Dirac statistics is contained in the factor (-1). The graphs representing the complex cluster integrals b_s are shown in Fig. 3. Therefore, we may well say that the complex cluster integrals are graphically analogous to the ordinary Mayer cluster integrals.

§ 3. Cluster expansion formula for the energy expectation value

In this section, we shall derive the cluster expansion formula for the energy expectation value of the state represented by the wave function (5). This formula will be written in terms of the complex cluster integrals b_s defined by (32).

The Hamiltonian of the system of $2N$ fermions is

$$H=\sum_i \boldsymbol{p}_i^2/2m+\sum_{(i,j)} v(r_{ij}), \tag{35}$$

where the two-body potential $v(r)$ contains a large repulsive core. The problem is the calculation of

$$E=\langle \Psi|H|\Psi\rangle/\langle \Psi|\Psi\rangle. \tag{36}$$

Using the wave function (5), we can write the numerator of (36) as follows,

$$\langle \Psi|H|\Psi\rangle=\langle \Psi_0|F\sum_i \boldsymbol{p}_i^2/2m|\Psi_0\rangle+\langle \Psi_0|\sum_{(i,j)} F_{ij}V_{ij}|\Psi_0\rangle$$
$$+\langle \Psi_0|\sum_{(ijk)} F_{ijk}\ V_{ijk}|\Psi_0\rangle, \tag{37}$$

where

$$F=\prod_{(ij)} f^2(r_{ij}),$$
$$F_{ij}=F/f^2(r_{ij}),$$

and

$$F_{ijk}=F/f^2(r_{ij})f^2(r_{jk})f^2(r_{ki}).$$

Also, V_{ij} and V_{ijk} are defined by

$$V_{ij}=\frac{\hbar^2}{m}[f'(r_{ij})]^2+v(r_{ij})f^2(r_{ij}) \tag{38}$$

and

$$V_{ijk}=-\frac{\hbar^2}{4m}\sum \frac{\partial}{\partial \boldsymbol{r}_{ij}} f^2(r_{ij})\cdot\frac{\partial}{\partial \boldsymbol{r}_{jk}} f^2(r_{jk})\cdot f^2(r_{ki}). \tag{39}$$

Sum over the cyclic permutations of (i, j, k)

respectively, where the prime on $f(r)$ indicates the first derivative with respect to r. V_{ij} is the effective two-body potential, in which the first term is the repulsive part originating from the introduction of the correlation function $f(r)$. Furthermore, the singularity of the two-body potential $v(r)$ can be removed by the choice

of an appropriate form of $f(r)$, as done by Jastrow. Also, V_{ijk} is the effective three-body potential originating from the introduction of $f(r)$.

First of all, we shall calculate the first term of (37). Using the definition (14) of the function h_{ij}, we can write

$$G_1=\langle \Psi_0|F\sum_i \boldsymbol{p}_i^2/2m|\Psi_0\rangle=(2N)!\int \prod_{(ij)}(1+h_{ij})\, W_N'(\tau^{2N})\, d\tau^{2N}, \tag{40}$$

where

$$\begin{aligned} W'(\tau^{2N})=&\{\textstyle\sum_i (\boldsymbol{p}_i^2/2m)\chi(1,2)\chi(3,4)\cdots\chi(2N-1,2N)\}\\ &\times\sum_P(-1)^P P\{\chi(1,2)\chi(3,4)\cdots\chi(2N-1,2N)\}. \end{aligned} \tag{41}$$

We now introduce the kinetic energy function $t_\nu(\boldsymbol{r}^{2\nu})$ of a ring-cluster of species ν defined by

$$\begin{aligned} t_\nu(\boldsymbol{r}^{2\nu})=&\psi(r_{j_2j_3})\psi(r_{j_4j_5})\cdots\psi(r_{j_{2\nu}j_1})\textstyle\sum_i(\boldsymbol{p}_i^2/2m)\ \psi(r_{j_1j_2})\psi(r_{j_3j_4})\cdots\\ &\psi(r_{j_{2\nu-1}\,j_{2\nu}})/\psi(r_{j_1j_2})\psi(r_{j_2j_3})\cdots\psi(r_{j_{2\nu}\,j_1}). \end{aligned} \tag{42}$$

Then, $W_N'(\tau^{2N})$ can be written as follows,

$$W_N'(\tau^{2N})=\sum\Pi B_\nu(\tau^{2\nu})\textstyle\sum_\mu t_\mu(\boldsymbol{r}^{2\mu}). \tag{43}$$

The derivation of (43) will be given in Appendix 2. $\sum_\nu$ denotes a sum of the kinetic energy functions $t_\nu(\boldsymbol{r}^{2\nu})$ over all ring-clusters composing a product $\Pi B_\nu(\tau^{2\nu})$. Also, $\sum\Pi$ denotes the summation over all possible divisions of $2N$ fermions into ring-clusters, as in (8). Of course, a term of this summation is $\Pi B_\nu(\tau^{2\nu})\sum_\mu t_\mu(\boldsymbol{r}^{2\mu})$, and the number of the terms belonging to the set $\{\boldsymbol{m}\}$ of ring-clusters is given by (10). Inserting (43) into (40), therefore, we obtain

$$G_1=2^N(N!)^2(2N)!\sum_{\sum\nu m_\nu=N}\prod_\nu\frac{(2^{\nu-1}/\nu)^{m_\nu}}{m_\nu!}W_N'(\boldsymbol{m}), \tag{44}$$

where

$$W_N'(\boldsymbol{m})=\int\prod_{(ij)}(1+h_{ij})\prod_\nu[B_\nu(\tau^{2\nu})]^{m_\nu}\textstyle\sum_\mu t_\mu(\boldsymbol{r}^{2\mu})\, d\tau^{2N}. \tag{45}$$

Applying the technique in the process of the derivation of (20) from (15) to (45), we can write

$$W_N'(\boldsymbol{m})=\int\prod_{(\mu\nu)}(1+g_{\mu\nu})\prod_\nu[B_\nu^*(\tau^{2\nu})]^{m_\nu}\textstyle\sum_\sigma t_\sigma(\boldsymbol{r}^{2\sigma})\, d\tau^{2N}. \tag{46}$$

Inserting (23) into (46), we have

$$W_N'(\boldsymbol{m})=\sum\int\Pi_s\beta_s(\tau^{2n_s})\textstyle\sum_\sigma t_\sigma(\boldsymbol{r}^{2\sigma})\, d\tau^{2N}. \tag{47}$$

Now, the summation $\sum_\sigma$ can be divided in the sum of the groups working on each complex molecules composing a product $\Pi_s\beta_s(\tau^{2n_s})$. Then, by making

use of the technique in the process of the derivation of (28) from (23), (47) can readily be written as

$$W_N'(\boldsymbol{m}) = \prod_\nu m_\nu ! \sum_{\sum_s \nu_s m_s = m_\nu} \sum \prod_s \frac{(\beta_s / \boldsymbol{s}\,!)^{m_s}}{m_s !}, \tag{48}$$

where

$$t_s = \int \beta_s(\tau^{2n_s}) \sum_\mu t_\mu(\boldsymbol{r}^{2\mu})\, d\tau^{2n_s} \Big/ \int \beta_s(\tau^{2n_s})\, d\tau^{2n_s}. \tag{49}$$

t_s may be interpreted as the kinetic energy of the complex molecule of species s. Inserting (48) into (44), we have

$$G_1 = 2^N (N!)^2 (2N)! \sum_{\sum_s \sum_\nu \nu s_\nu m_s = N} \sum \prod_s \frac{(2^{\Sigma(\nu-1)s_\nu} \beta_s / \boldsymbol{\nu}^s \boldsymbol{s}\,!)^{m_s}}{m_s !}. \tag{50}$$

Using (30) and (32), we get

$$G_1 = 2^N (N!)^2 (2N)! \sum_{\sum_s \sum_\nu \nu s_\nu m_s = N} \sum \prod_s \frac{(\Omega b_s)^{m_s}}{m_s !}, \tag{51}$$

where t_s is written as

$$t_s = \int \beta_s(\boldsymbol{r}^{2n_s}) \sum_\sigma t_\sigma(\boldsymbol{r}^{2\sigma})\, d\boldsymbol{r}^{2n_s} \Big/ \int \beta_s(\boldsymbol{r}^{2n_s})\, d\boldsymbol{r}^{2n_s}, \tag{52}$$

by carrying out the integration over the spin coordinates in (49), where

$$\beta_s(\boldsymbol{r}^{2n_s}) = \sum \prod_{\mu\nu} g_{\mu\nu} \prod_\nu [B_\nu^*(\boldsymbol{r}^{2\nu})]^{s_\nu}. \tag{53}$$

Sum over all products of the g-functions connecting $\sum s_\nu$ molecules

Equation (51) is the final result for G_1.

Secondly, we shall calculate the second term of (37). We can write it as

$$G_2 = \langle \Psi_0 | \sum_{(i,j)} F_{ij} V_{ij} | \Psi_0 \rangle = (2N)! \int \sum_{(ij)} \prod^{(ij)} (1 + h_{mn})\, V_{ij} W_N(\tau^{2N})\, d\tau^{2N}, \tag{54}$$

where the superscript (ij) on $\prod$ denotes the omission of a factor $(1+h_{ij})$ from the product $\prod_{(m,n)} (1+h_{mn})$. We now introduce the projection operator P_{ij} defined by

$$P_{ij} \prod_{(m,n)} (1 + h_{mn}) = \prod^{(ij)} (1 + h_{mn}). \tag{55}$$*

With this definition of the projection operator, we can write as follows,

$$\prod^{(ij)} (1 + h_{mn})\, V_{ij} = \mathcal{V}_{ij} \prod_{(mn)} (1 + h_{mn}), \tag{56}$$

where

* In other words, when P_{ij} works on the sum of the products of h-functions, it plays a role of excluding the terms containing h_{ij} from the sum.

$$\mathcal{V}_{ij} = V_{ij} P_{ij}. \tag{57}$$

Inserting (56) into (54), we have

$$G_2 = (2N)! \int \sum_{(i,j)} \mathcal{V}_{ij} \prod_{(mn)} (1+h_{mn}) W_N(\tau^{2N}) d\tau^{2N} \tag{58}$$

$$= 2^N (N!)^2 (2N)! \sum_{\sum \nu m_\nu = N} \prod \frac{(2^{\nu-1}/\nu)^{m_\nu}}{m_\nu !} W_N''(\boldsymbol{m}), \tag{59}$$

where

$$W_N''(\boldsymbol{m}) = \sum \int \sum_{(ij)} \mathcal{V}_{ij} \prod \beta_s(\tau^{2n_s}) d\tau^{2N}. \tag{60}$$

In the derivation of (59), we have used the technique in the process of the derivation of (47) from (40).

In (60), the summation $\sum_{(ij)}$ can be divided in the sum of the groups working on each complex molecules and the sum of the groups connecting two complex molecules. Then, we can readily write as

$$W_N''(\boldsymbol{m}) = \prod_\nu m_\nu ! \sum_{\sum_s s_\nu m_s = m_\nu} \{ \sum_s m_s v_s^{(2)} + \sum_{(s,s')} m_s m_{s'} v_{ss'}^{(2)}$$

$$+ \tfrac{1}{2} \sum_s m_s (m_s - 1) v_{ss}^{(2)} \} \prod_s \frac{(\beta_s / s!)^{m_s}}{m_s !}, \tag{61}$$

where

$$v_s^{(2)} = \int \sum_{(i,j) \subset s} \mathcal{V}_{ij} \beta_s(\tau^{2n_s}) d\tau^{2n_s} \Big/ \int \beta_s(\tau^{2n_s}) d\tau^{2n_s}; \tag{62}$$

$$v_{ss'}^{(2)} = \int \sum_{i \subset s, j \subset s'} \mathcal{V}_{ij} \beta_s(\tau^{2n_s}) \beta_{s'}(\tau'^{2n_{s'}}) d\tau^{2n_s} d\tau'^{2n_{s'}} \Big/$$

$$\int \beta_s(\tau^{2n_s}) \beta_{s'}(\tau'^{2n_{s'}}) d\tau^{2n_s} d\tau'^{2n_{s'}}. \tag{63}$$

In (62), the summation $\sum$ is taken over all pairs of fermions composing the complex molecule of species s. On the other hand, in the summation $\sum$ of (63), the fermion i belongs to the complex molecule of species s and the fermion j to the other complex molecule of species s', that is, $\mathcal{V}_{ij}$ plays a role of connecting two complex molecules. So, the symbol $i \subset s$ indicates that the fermion i belongs to the complex molecule of species s. Hereafter, this symbol will be employed. $v_s^{(2)}$ and $v_{ss'}^{(2)}$ may be interpreted as the potential energy of a complex molecule and the two-body interaction energy between two complex molecules, respectively originating from the effective two-body potential V_{ij}.

Inserting (61) into (59), we get

$$G_2=2^N(N!)^2(2N)!\sum_{\sum_s\sum_\nu \nu s_\nu m_s=N}\{\sum_s m_s v_s^{(2)}+\sum_{(ss')} m_s m_{s'} v_{ss'}^{(2)}+\tfrac{1}{2}\sum_s m_s(m_s-1)v_{ss}^{(2)}\}$$

$$\times\prod_s \frac{(2^{\Sigma(\nu-1)s_\nu}\beta_s/\boldsymbol{\nu}^s\boldsymbol{s}\,!)^{m_s}}{m_s\,!}. \tag{64}$$

Using (30) and (32), we can write (64) as follows,

$$G_2=2^N(N!)^2(2N)!\sum_{\sum_s\sum_\nu \nu s_\nu m_s=N}\{\sum_s m_s v_s^{(2)}+\sum_{(ss')} m_s m_{s'} v_{ss'}+\tfrac{1}{2}\sum_s m_s(m_s-1)v_{ss}^{(2)}\} \tag{65}$$

$$\times\prod_s \frac{(\Omega b_s)^{m_s}}{m_s\,!},$$

where $v_s^{(2)}$ and $v_{ss'}^{(2)}$ are written as

$$v_s^{(2)}=\int\sum_{(i,j)\subset s}{}^{\nu}\mathcal{V}_{ij}\,\beta_s(\boldsymbol{r}^{2n_s})\,d\boldsymbol{r}^{2n_s}\Big/\int\beta_s(\boldsymbol{r}^{2n_s})\,d\boldsymbol{r}^{2n_s} \tag{66}$$

and

$$v_{ss'}^{(2)}=\int\sum_{i\subset s,\,j\subset s'}V_{ij}\,\beta_s(\boldsymbol{r}^{2n_s})\beta_{s'}(\boldsymbol{r}'^{2n_{s'}})\,d\boldsymbol{r}^{2n_s}\,d\boldsymbol{r}'^{2n_{s'}}\Big/$$

$$\int\beta_s(\boldsymbol{r}^{2n_s})\beta_{s'}(\boldsymbol{r}'^{2n_{s'}})\,d\boldsymbol{r}^{2n_s}\,d\boldsymbol{r}'^{2n_{s'}}, \tag{67}$$

respectively, by carrying out the integration over the spin coordinates in (62) and (63), where in the last equation we have replaced ${}^{\nu}\mathcal{V}_{ij}$ by V_{ij}, noting that the two functions $\beta_s(\boldsymbol{r}^{2n_s})$ and $\beta_{s'}(\boldsymbol{r}'^{2n_{s'}})$ are not connected by the h-functions. The definition of the $\beta(\boldsymbol{r})$-function is given by (53). Equation (65) is the final result for G_2. It will be interesting to compare (65) with (51).

Lastly, we shall calculate the third term of (37). We can write it as follows,

$$G_3=\langle\Psi_0|\sum_{(ijk)}F_{ijk}V_{ijk}|\Psi_0\rangle=(2N)!\int\sum_{(ijk)}\Pi^{(ijk)}(1+h_{mn})\,V_{ijk}W_N(\tau^{2N})\,d\tau^{2N}, \tag{68}$$

where the superscript $(i\,j\,k)$ on Π denotes the omission of a factor $(1+h_{ij})(1+h_{jk})\times(1+h_{ki})$ from the product $\prod_{(m,\,n)}(1+h_{mn})$. Using the projection operator defined by (55), we have

$$\Pi^{(ijk)}(1+h_{mn})\,V_{ijk}={}^{\nu}\mathcal{V}_{ijk}\prod_{(mn)}(1+h_{mn}) \tag{69}$$

where

$${}^{\nu}\mathcal{V}_{ijk}=V_{ijk}\,P_{ij}\,P_{jk}\,P_{kl}. \tag{70}$$

Then, it readily follows that

$$G_3=(2N)!\int\sum_{(ijk)}{}^{\nu}\mathcal{V}_{ijk}\prod_{(mn)}(1+h_{mn})\,W_N(\tau^{2N})\,d\tau^{2N} \tag{71}$$

$$=2^N(N!)^2(2N)!\sum_{\sum_\nu \nu m_\nu=N}\prod_\nu \frac{(2^{\nu-1}/\nu)^{m_\nu}}{m_\nu!}W_N'''(\boldsymbol{m}), \tag{72}$$

where

$$W_N'''(\boldsymbol{m})=\sum\int\sum_{(ijk)}{}^{h}\mathcal{V}_{ijk}\prod_s \beta_s(\tau^{2n_s})\,d\tau^{2N}. \tag{73}$$

Using the technique in the process of the derivation of (65) from (60), we readily obtain

$$G_3=2^N(N!)^2(2N)!\sum_{\sum_s\sum_\nu \nu s_\nu m_s=N}\{\sum_s m_s v_s{}^{(3)}+\sum_{(ss')}m_s m_{s'}v_{ss'}^{(3)}+\tfrac{1}{2}\sum_s m_s(m_s-1)v_{ss}{}^{(3)}$$

$$+\sum_{(s,s',s'')}m_s m_{s'}m_{s''}v_{ss's''}+\frac{1}{2}\sum_{(ss')}m_s(m_s-1)m_{s'}\,v_{sss'}+\frac{1}{3!}\sum_s m_s(m_s-1)(m_s-2)v_{sss}\}$$

$$\times\prod_s\frac{(\Omega b_s)^{m_s}}{m_s!}, \tag{74}$$

where

$$v_s{}^{(3)}=\int\sum_{i,j,k\subset s}{}^{h}\mathcal{V}_{ijk}\beta_s(\boldsymbol{r}^{2n_s})\,d\boldsymbol{r}^{2n_s}\Big/\int\beta_s(\boldsymbol{r}^{2n_s})\,d\boldsymbol{r}^{2n_s}, \tag{75}$$

$$v_{ss'}^{(3)}=\int\sum_{\substack{i,j\subset s;\ k\subset s'\\ i,j\subset s';\ k\subset s}}{}^{h}\mathcal{V}_{ijk}\,\beta_s(\boldsymbol{r}^{2n_s})\beta_{s'}(\boldsymbol{r}'^{2n_{s'}})\,d\boldsymbol{r}^{2n_s}d\boldsymbol{r}'^{2n_{s'}}\Big/$$

$$\int\beta_s(\boldsymbol{r}^{2n_s})\beta_{s'}(\boldsymbol{r}'^{2n_{s'}})\,d\boldsymbol{r}^{2n_s}\,d\boldsymbol{r}'^{2n_{s'}}, \tag{76}$$

$$=\int\sum_{i\subset s,j\subset s',k\subset s''}V_{ijk}\,\beta_s(\boldsymbol{r}^{2n_s})\beta_{s'}(\boldsymbol{r}'^{2n_{s'}})\beta_{s''}(\boldsymbol{r}''^{2n_{s''}})\,d\boldsymbol{r}^{2n_s}d\boldsymbol{r}'^{2n_{s'}}d\boldsymbol{r}''^{2n_{s''}}\Big/$$

$$\int\beta_s(\boldsymbol{r}^{2n_s})\beta_{s'}(\boldsymbol{r}'^{2n_{s'}})\beta_{s''}(\boldsymbol{r}''^{2n_{s''}})\,d\boldsymbol{r}^{2n_s}d\boldsymbol{r}'^{2n_{s'}}d\boldsymbol{r}''^{2n_{s''}}. \tag{77}$$

$v_s{}^{(3)}$, $v_{ss'}{}^{(3)}$ and $v_{ss's''}$ are interpreted as the potential energy of a complex molecule, the two-body interaction energy between two complex molecules, and the three-body interaction energy among three complex molecules, respectively, originating from the effective three-body potential V_{ijk}. In (77), we have replaced ${}^{h}\mathcal{V}_{ijk}$ by V_{ijk}, noting that three complex molecules are not connected by the h-functions. The diagrams representing those interactions are shown in Fig. 4.

Inserting (51), (64) and (74) in (37), we have

$$\langle\Psi|H|\Psi\rangle=2^N(N!)^2(2N)!\sum_{\sum_s\sum_\nu \nu s_\nu m_s=N}\varepsilon(\boldsymbol{m}_s)\prod_s\frac{(\Omega b_s)^{m_s}}{m_s!}, \tag{78}$$

where

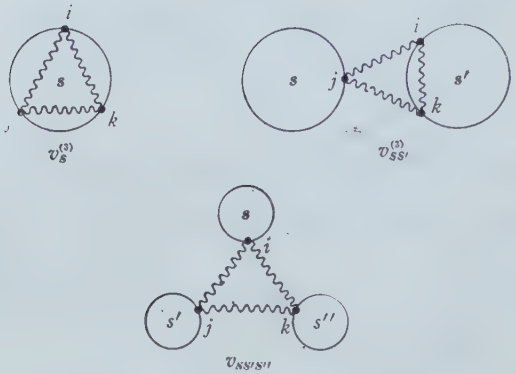

Fig. 4. Diagrammatic representation of $v_s^{(3)}$, $v_{ss'}^{(3)}$ and $v_{ss's''}$; The wavy triangle represents $\overset{h}{\mathcal{V}}_{ijk}$-connection.

$$\varepsilon(\boldsymbol{m}_s)=\sum_s m_s(t_s+v_s)+\sum_{(ss')} m_s m_{s'} v_{ss'}+\tfrac{1}{2}\sum_s m_s(m_s-1)v_{ss}$$
$$+\sum_{(ss's'')} m_s m_{s'} m_{s''} v_{ss's''}+\tfrac{1}{2}\sum_{(ss')} m_s(m_s-1)m_{s'}v_{sss'}$$
$$+\frac{1}{3!}\sum_s m_s(m_s-1)(m_s-2)v_{sss}, \tag{79}$$

in which

$$v_s=\int(\sum_{i,j\subset s}\overset{h}{\mathcal{V}}_{ij}+\sum_{i,j,k\subset s}\overset{h}{\mathcal{V}}_{ijk})\beta_s(\boldsymbol{r}^{2n_s})\,d\boldsymbol{r}^{2n_s}\Big/\int\beta_s(\boldsymbol{r}^{2n_s})\,d\boldsymbol{r}^{2n_s} \tag{80}$$

and

$$v_{ss'}=\int(\sum_{i\subset s,\ j\subset s'}V_{ij}+\sum_{\substack{i,j\subset s;\ k\subset s'\\ i,j\subset s';\ k\subset s}}\overset{h}{\mathcal{V}}_{ijk})\beta_s(\boldsymbol{r}^{2n_s})\beta_{s'}(\boldsymbol{r}'^{2n_{s'}})\,d\boldsymbol{r}^{2n_s}d\boldsymbol{r}'^{2n_{s'}}\Big/$$
$$\int\beta_s(\boldsymbol{r}^{2n_s})\beta_{s'}(\boldsymbol{r}'^{2n_{s'}})\,d\boldsymbol{r}^{2n_s}d\boldsymbol{r}'^{2n_{s'}}. \tag{81}$$

$\varepsilon(\boldsymbol{m}_s)$ may be interpreted as the total energy of the group consisting of complex molecules of the set $\{\boldsymbol{m}_s\}$. Thus, substituting (34) and (78) in (36), we can write the expectation value of the Hamiltonian (35) as follows,

$$E=\sum_{\sum_s\sum_\nu \nu s_\nu m_s=N}\varepsilon(\boldsymbol{m}_s)\prod_s\frac{(\Omega b_s)^{m_s}}{m_s!}\Big/\sum_{\sum_s\sum_\nu \nu s_\nu m_s=N}\prod_s\frac{(\Omega b_s)^{m_s}}{m_s!}. \tag{82}$$

This expression is formally identical with (13) of A. The difference is that (13) of A is expressed in terms of the ring-cluster integrals, while the present expression is in terms of the complex cluster integrals.

Now, it is possibie to rewrite (82) in a simpler form, by using the technique in the theory of an imperfect classical gas. We can readily rewrite (82) as

$$E=\sum_s \langle m_s\rangle \varepsilon_s+\frac{1}{2!}\sum_{\substack{s,s'\\ \text{unrestricted}}}\langle m_s\rangle\langle m_{s'}\rangle v_{ss'}+\frac{1}{3!}\sum_{\substack{s,s',s''\\ \text{unrestricted}}}\langle m_s\rangle\langle m_{s'}\rangle\langle m_{s''}\rangle v_{ss's''}, \tag{83}$$

where

$$\varepsilon_s=t_s+v_s\,, \tag{84}$$

and

$$\langle m_s\rangle=b_s\frac{\partial}{\partial b_s}\ (\log\ Q_N), \tag{85}$$

in which

$$Q_N=\sum_{\sum_s\sum_\nu \nu s_\nu m_s=N}\prod\frac{(\Omega b_s)^{m_s}}{m_s\,!}. \tag{86}$$

Equation (83) is the energy expectation value of the system in the "cluster-representation". $\langle m_s\rangle$ is interpreted as the mathematical average number of complex clusters contributing to the energy, and it corresponds to the canonical ensemble average number of particles in the classical statistical mechanics. Also, in (83), the first-, the second- and the third term express one-cluster energy, two-cluster interaction energy, and three-cluster interaction energy, respectively.

In the systems with the large number of fermions, we can write (83) in the other form. As shown by Born and Fuchs,[10] we can put for large N

$$\langle m_s\rangle=\Omega b_s\, z^{n_s}, \tag{87}$$

where z is a parameter corresponding to the fugacity in the classical statistical mechanics and is determined by

$$\Omega\sum_s n_s b_s z^{n_s}=N. \tag{88}$$

Equation (88) is the condition restricting the number of particles. Then, we get

$$E=\Omega\sum_s \varepsilon_s\, b_s z^{n_s}+\frac{\Omega^2}{2!}\sum_{\substack{ss'\\ \text{unrestricted}}} v_{ss'}\, b_s b_{s'} z^{n_s+n_{s'}}$$

$$+\frac{\Omega^3}{3!}\sum_{\substack{s,s',s''\\ \text{unrestricted}}} v_{ss's''}\, b_s\, b_{s'}\, b_{s''}\, z^{n_s+n_{s'}+n_{s''}}, \tag{89}*$$

with the subsidiary condition (88). This equation is the expression for the energy expectation value of the large system. In the derivation of this formula, the analogy of our method to that of the statistical mechanics is remarkable.

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* It is easy to prove that the three-cluster interaction term is equal to $(4/3)\,\Omega(N/\Omega)^3 v_3(0)$, where $v_3(0)=(1/\Omega)\int V_{ijk}\, d\boldsymbol{r}_i\, d\boldsymbol{r}_j\, d\boldsymbol{r}_k$

§ 4. Discussion

In this paper, we have shown another example of the cluster expansion technique in the quantum-mechanics, and developed its mathematical formalism. Our method corresponds to that used in the theory of an imperfect classical gas of multicomponents. This fact will be clarified more distinctly by looking at the quantum-mechanical complex cluster integrals b_s defined by (32). In our formalism, the system of fermions is replaced by the group of the complex molecules of the set $\{\langle \boldsymbol{m}_s \rangle\}$. This will be easily understood by looking at the formula (83) for the energy expectation value of the system. Here, it is to be noted that $\langle \boldsymbol{m}_s \rangle$ corresponds to the canonical ensemble average number of clusters in the classical statistical mechanics.

The wave function (5), which we have studied in this paper, may be obtained by the extension of the BCS wave function to the systems with singular potentials. Then, it will be understood that $f(r)$ is a function giving the spatial correlation between two pairs. Of course, those correlations are not contained in the BCS wave function. Therefore, it will be interesting to apply our method to the calculation of pairing energy of the systems with singular potentials (nuclear matter and He^3). The discussions in this field will be given in the following paper.

In conclusion, I wish to thank Dr. A. M. Sessler of The Ohio State University and Prof. H. Ichimura of Tokyo Institute of Technology for stimulating discussions. Also I am particularly grateful to Dr. Y. Ishikawa of The Fundamental Research Section for his continued encouragement. Thanks are also due to Dr. Y. Wada of Tokyo University for many comments on this work.

Appendix 1

Putting

$$\varphi = \chi(1, 2)\chi(3, 4)\cdots\chi(2N-1, 2N) \tag{1·1}$$

and

$$P\varphi' = P\{\chi'(1, 2)\chi'(3, 4)\cdots\chi'(2N-1, 2N)\}, \tag{1·2}$$

where $\chi'(i, j)$ is of the same form as $\chi(i, j)$ and shall later be put equal to $\chi(i, j)$, we consider the following equation:

$$W_N(P) = (-1)^p \varphi P \varphi'. \tag{1·3}$$

Hereafter, we shall express $\chi(i, j)$ and $\chi'(i, j)$ by the following graphs (or bonds):

$\chi(i, j) =$

and

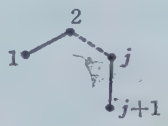
$\chi'(i, j) =$

.

Let us now start with $\chi(1, 2)$ in φ-function and assume that the point 2 appears in $P\varphi'$-function in the form of $\chi'(2, j)$. Noting that the point 2 appears once for all in φ-function, we find that the connection of the following type appears in $\varphi P\varphi'$:

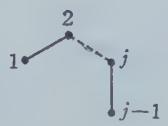
$$=\chi(1, 2)\chi'(2, j). \tag{1·4}$$

Secondly, the point j appears in φ-function in the form of $\chi(j, j+1)$ when j is odd, or in the form of $\chi(j-1, j)$ when j is even. However, we should note that $\chi(j-1, j)$ is replaced by $-\chi(j, j-1)$, according to (3). Then, we find that the connection of the following type appears in $\varphi P\varphi'$ for j=odd number



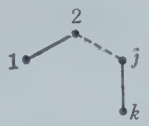
$$=\chi(1, 2)\chi'(2, j)\chi(j, j+1), \tag{1·5}$$

or for j=even number



$$=-\chi(1, 2)\chi'(2, j)\chi(j, j-1). \tag{1·6}$$

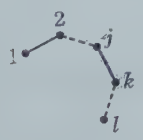
In an abbreviated form, we shall express (1.5) and (1.6) as follows,



$$=\chi(1, 2)\chi'(2, j)(\pm 1)\chi(j, k), \tag{1·7}$$

where the plus factor in front of $\chi(j, k)$ is taken when $k=j+1$, and the minus factor when $k=j-1$.* In other words, the plus factor is taken when the point j appears in the form of $\chi(j, k)$ and the minus factor when the point j appears in the form of $\chi(k, j)$. In the former case, $\chi(j, k)$ shall be said to be of "the original type", and in the latter, $\chi(j, k)$ to be of "the exchange type".

Thirdly, the point k appears in $P\varphi'$-function in the form of $\chi'(k, l)$ or $\chi'(l, k)$. As in the second step, $\chi'(l, k)$ is replaced by $-\chi'(k, l)$. Then, in an abbreviated form, the connection of the following type appears in $\varphi P\varphi'$:



$$=\chi(1, 2)\chi'(2, i)(\pm 1)\chi(i, k)(\pm 1)\chi'(k, l), \tag{1·8}$$

---

* It is to be noted that two coordinates in χ-function appear only in the form of $\chi(j, j+1)$ or $\chi(j-1, j)$. However, this is not the case for $\chi'(i, j)$ in $P\varphi'$-function.

where the plus factor in front of $\chi'(k, l)$ is taken when $\chi'(k, l)$ is of the original type, and the minus factor when $\chi'(k, l)$ is of the exchange type, that is, when the point k appears in $P\varphi'$-function in the form of $\chi'(l, k)$.

Continuing the above process successively, and noting that the two different bonds (the solid line bond and the dotted line bond) emerge from one point, we find that the connection of the ring type appears in $\varphi P\varphi'$:

$$= \chi(1, 2)\chi'(2, j)(\pm 1)\chi(j, k)(\pm 1)\chi'(k, l)\cdots(\pm 1)\chi'(z, 1) \quad (1\cdot 9)$$

$$= (-1)^{n_\chi}(-1)^{n_{\chi'}}\chi(1, 2)\chi'(2, j)\chi(j, k)\chi'(k, l)\cdots\chi'(z, 1), \quad (1\cdot 10)$$

where the polygon has even number of vertices and n_χ and $n_{\chi'}$ are the numbers of χ- and χ'-function of the exchange type contributing to the construction of the polygon respectively.

Now, the parity of the permutation of the points, 1, 2, j, k, l,$\cdots$, and z, in $P\varphi'$-function contributing to the construction of the polygon is clearly

$$(-1)^{n_\chi}(-1)^{n_{\chi'}}(-1),$$

where the last minus factor originates from the fact that the parity of the permutation $(2, j, k, \cdots, z, 1)$ with regard to the permutation $(1, 2, j, k, \cdots z)$* is odd.

Thus, we find that the connection of the following type appears in $(-1)^n\varphi P\varphi'$:

$(-1)^{n_\chi}(-1)^{n_{\chi'}}(-1)\times$

$$= (-1)\chi(1, 2)\chi'(2, j)\chi(j, k)\chi'(k, l)\cdots\chi'(z, 1) \quad (1\cdot 11)$$

Here, when we put $\chi'=\chi$, we find that $(-1)^n\varphi P\varphi'$ is expressed in the product of the ring-cluster functions B_ν defined by (9). Therefore, $W_N(r^{2N})$ can be written in the sum of the products of the ring-cluster functions.

Appendix 2

As in Appendix 1, we consider the following equation:

$$W'_N(P) = \sum_i \frac{p_i^2}{2m}(-1)^n\varphi P\varphi', \qquad (2\cdot 1)$$

where $\sum p_i^2/2m$ works on φ-function. As proved in Appendix 1, $(-1)^n\varphi P\varphi'$ can be expressed in the product of the ring-cluster functions:

* It is to be noted that the number of members in this set is even.

$$(-1)^{p}\varphi P\varphi' = B_{\nu}'(\tau^{2\nu})B_{\nu'}'(\tau'^{2\nu'})\cdots, \tag{2·2}$$

where

$$B_{\nu}'(\tau^{2\nu}) = (-1)\chi(j_1, j_2)\chi'(j_2, j_3)\chi(j_3, j_4)\cdots\chi'(j_{2\nu}, j_1). \tag{2·3}$$

Now, the summation $\sum p_i^2/2m$ can be rearranged in the sum of groups working on each ring-cluster function contributing to $(-1)^{p}\varphi P\varphi'$. Then, $W_N'(P)$ can be written as follows,

$$\begin{aligned} W_N(P) = &\{\sum_{i\subset\nu}(p_i^2/2m)\ B_{\nu}'(\tau^{2\nu})\}B_{\nu'}'(\tau'^{2\nu'})\times\cdots \\ &+B_{\nu}'(\tau^{2\nu})\{\sum_{i'\subset\nu'}(p_{i'}^2/2m)\ B_{\nu'}'(\tau'^{2\nu'})\}\times\cdots \\ &+\cdots\cdots\cdots\cdots, \end{aligned} \tag{2·4}$$

where $i\subset\nu$ denotes that the fermion i belongs to the groups of fermions contributing to the construction of B_{ν}' and the summation $\sum_i$ is over the members of the group. Now, we can write

$$\begin{aligned} \sum_{i\subset\nu}(p_i^2/2m)B_{\nu}'(\tau^{2\nu}) &= (-1)\sum_{i\subset\nu}(p_i^2/2m)\chi(j_1, j_2)\chi'(j_2, j_3)\chi(j_3, j_4)\cdots\chi'(j_{2\nu}, j_1) \\ &= (-1)S(j_1, j_2)S(j_2, j_3)\cdots S(j_{2\nu}, j_1) \\ &\times\psi'(r_{j_2j_3})\psi'(r_{j_4j_5})\cdots\psi'(r_{j_{2\nu}j_1}) \\ &\times\sum_{i\subset\nu}(p_i^2/2m)\psi(r_{j_1j_2})\psi(r_{j_3j_4})\cdots\psi(r_{j_{2\nu-1}j_{2\nu}}), \end{aligned} \tag{2·5}$$

where $S(i, j) = \{\alpha(i)\beta(j) - \alpha(j)\beta(i)\}/\sqrt{2}$. Then, introducing $t_{\nu}'(\tau^{2\nu})$-function defined by

$$\begin{aligned} t_{\nu}'(\tau^{2\nu}) = &\psi'(r_{j_2j_3})\psi'(r_{j_4j_5})\cdots\psi'(r_{j_{2\nu}j_1}) \\ &\times\sum_{i\subset\nu}(p_i^2/2m)\ \psi(r_{j_1j_2})\psi(r_{j_2j_3})\cdots\psi(r_{j_{2\nu-1}\,j_{2\nu}})/ \\ &\psi(r_{j_1j_2})\psi'(r_{j_2j_3})\psi(r_{j_3j_4})\cdots\psi'(r_{j_{2\nu}j_1}), \end{aligned} \tag{2·6}$$

we can write (2.5) as follows,

$$\sum_{i\subset\nu}(p_i^2/2m)\ B_{\nu}'(\tau^{2\nu}) = B_{\nu}'(\tau^{2\nu})t_{\nu}'(\tau^{2\nu}). \tag{2·7}$$

(2.6) corresponds to (42). Therefore, substituting (2.7) in (2.4), we get

$$\begin{aligned} W_N(P) = &B_{\nu}'(\tau^{2\nu})\cdot t_{\nu}'(\tau^{2\nu})\cdot B_{\nu'}'(\tau'^{2\nu'})\cdots \\ &+B_{\nu}'(\tau^{2\nu})\cdot B_{\nu}'(\tau'^{2\nu'})\cdot t'_{\nu'}(\tau'^{2\nu'})\cdots \\ &+\cdots\cdots \end{aligned} \tag{2·8}$$

$$= B_{\nu}'(\tau^{2\nu})\cdot B_{\nu'}'(\tau'^{2\nu'})\cdots\{t_{\nu}'(\tau^{2\nu}) + t'_{\nu'}(\tau'^{2\nu'}) + \cdots\}. \tag{2·9}$$

Again, putting $\chi' = \chi$, we shall find that $W_N'(\tau^{2N})$ is written in the form of (43).

References

1) R. Jastrow, Phys. Rev. **98** (1955), 1479.
2) F. Iwamoto and M. Yamada, Prog. Theor. Phys. **17** (1957), 543.
3) C. D. Hartogh and H. A. Tolhoek, Physica **24** (1958), 721, 875, 896.
4) J. W. Clark and E. Feenberg, Phys. Rev. **113** (1959), 388.
5) K. Nakamura, Prog. Theor. Phys. **21** (1959), 713.
6) J. E. Mayer and M. G. Mayer, *Statistical Mechanics* (Wiley, 1940).
7) W. G. McMillan, Jr. and J. E. Mayer, J. Chem. Phys. **13** (1945), 276.
8) G. E. Uhlenbeck and E. Beth, Physica **3** (1936), 729.
9) B. Kahn, *Dissertation* (Utrecht, 1938).
10) M. Born and K. Fuchs, Proc. Roy. Soc. London **A166** (1938), 391.

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On the Possibility of the Existence of a New Meson†

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A new meson, λ^0, which interacts only with the Λ particle is introduced to explain the mass difference between the Λ particle and the nucleon. The requirement that the self-energy of the Λ particle be positive restricts the type of λ^0 meson to three possibilities, $V(v)$, $V(t)$ or $PS(ps)$. Some properties of the λ^0 meson as well as ideas for possible experimental verification of its existence are discussed.

§ 1. Introduction

For several years it has been established that there exist various kinds of elementary particles known as strange particles. In particular, the strongly interacting particles among them have been successfully classified by means of the introduction of the concept of strangeness.

No one, however, had any idea of the physical meaning of strangeness. Sakata[1] suggested a new approach to the analysis of the structure of elementary particles based upon the composite particle model. In his scheme strangeness is replaced by a more physically realistic quantity, the number of Λ particles. Consequently the conservation of strangeness in strong interactions is reduced to the conservation of Λ particles.

Ogawa et al. and Yamaguchi[2] discussed mathematical aspects of Sakata's model, assuming a symmetry property which leaves theory invariant under the interchange of the nucleon and the Λ particle as long as the mass difference between the nucleon and the Λ particle is neglected. If we assume this symmetry (πK) currents are necessarily added to the usual strangeness violating baryon vector currents in the conventional Yukawa type interaction. By taking the (πK) currents into account, the transition probabilities for K_{e3}, $K_{\mu 3}$ and beta- and muon-decays of Λ are shown to be consistently described with the same coupling constant as that obtained from $K^+_{\mu 2}$ decay.[3] The introduction of a new symmetry property such as this enriches the content of the theory, but it also makes important the

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elucidation of the distinction between the fundamental particles Λ, p and n. First of all the main differences between these fundamental particles are characterized by strangeness and electric charge. Another remarkable distinction between them is the mass difference, which we will try to account for.

In the case of the neutron and proton, we can explain the order of magnitude of the mass difference by taking into account the electromagnetic interaction in addition to choosing a particular value of the cutoff (of order of magnitude of the nucleon mass). In the case of the nucleon-Λ mass difference, however, a satisfactory theory has not yet been given. There are various possibilities for the explanation of the mass difference of elementary particles. The following two approaches seem to be most plausible: (1) the existence of an internal structure for elementary particles, (2) the existence and interaction of new particles.

In discussing the problem of the mass difference of leptons Taketani et al.[4] introduced form factors which are responsible for the non-locality of interactions. It seems to us that they tried from the former point of view to find a clue to the dynamics which governs the inside regions of elementary particles.

This approach, however, is too ambiguous to discuss the structure of elementary particles directly through form factors, because there exists scarcely any concrete knowledge about the dynamical laws of the inside region.

Although one may think that the second approach is impractical at first sight, it has advantages making possible the discussion of interactions or the restriction of their types in the framework of the existing theory. It seems to us necessary and useful that we consider the latter possibility first before making a frontal attack on the problems of the structure of elementary particles.

In this paper we shall discuss the nucleon-Λ mass difference along the line of the ideas mentioned above. A new neutral boson which interacts only with the Λ particle is introduced. Just as the charge of an electron is characterized by its interaction with the photon, the strangeness of the Λ particle is in our scheme characterized by its interaction with a neutral boson with mass $m_{\lambda 0}$ which we shall call the λ^0 meson. The nucleon and the Λ should have the same mass before the interaction between the λ^0 meson and the Λ particle is switched on.

The physical mass of the Λ, therefore, arises from the self-energy due to the interaction with the λ^0 meson. The possible mass, type or coupling constant of the λ^0 meson will be determined consistently with existing experiments in the following sections. The experimental observability of the λ^0 meson is also examined.

Fig. 1.

§ 2. Properties of λ^0-meson

(i) *The type and the coupling constants*

For simplicity the spin of the λ^0 meson is assumed to be zero or one. The

mass renormalization constant δM can be easily calculated in such a lowest order perturbation theory as Fig. 1 shows. The requirement that the self-energy of the Λ particle should be positive restricts the λ^0-Λ interaction to three possibilities: $PS(ps)$, $V(v)$ or $V(t)$.

Assuming the mass of the λ^0 meson and the cutoff momentum K, one can determine coupling constants under the requirement that $\delta M/M$ should give the experimental result. Table I shows numerical values of the coupling constants for $m_{\lambda^0}=200m_e$ and $550m_e$ and for Feynman cutoff parameter $K=M/2$, M and $2M$.

Table I. Types and coupling constants (f and g defined in Appendix)

m_{λ^0} / K		$200\,m_e$			$550\,m_e$		
type	Coupling	$M/2$	M	$2M$	$M/2$	M	$2M$
S	s	×	×	×	×	×	×
S	v	×	×	×	×	×	×
V	v	0.556	0.277	0.131	0.788	0.369	0.194
V	t	0.787	0.131	0.025	7.071	1.069	0.195
PS	ps	6.550	2.842	1.335	8.061	3.309	1.588
PS	pv	×	×	×	×	×	×
A	a	×	×	×	×	×	×
A	t	×	×	×	×	×	×

(ii) *Decay processes*

If a neutral meson of such a type and coupling constant exists in nature, in what process one can observe it ? Decay processes are first of all examined in detail. Table II gives possible competing processes for the decay interactions, where × means an energetically forbidden process.

Table II. Decay and competing processes

processes / m_{λ^0}	m_K	$3m_\pi$	m_K-m_π	$2m_\pi$	m_π
$\lambda^0\to 3\pi$	0	0	×	×	×	×
$\lambda^0\to 2\pi$	0	0	0	0	×	×
$\lambda^0\to 2\pi+\gamma$	0	0	0	0	×	×
$\lambda^0\to \pi^0+\gamma$	0	0	0	0	0	×
$\lambda^0\to 2\gamma$	0	0	0	0	0	0
$\pi^0\to \lambda^0+\gamma$	×	×	×	×	×	0
$\pi^0\to \lambda^0+2\gamma$	×	×	×	×	×	0
$K_1^0\to \lambda^0+\gamma$	×	0	0	0	0	0
$K_1^0\to \lambda^0+\pi^0$	×	×	×	0	0	0
$K^+\to \lambda^0+\pi^+$	×	×	×	0	0	0

(a) *Processes competing with $\pi^0\to 2\gamma$ decay*

Table III gives selection rules for processes competing with $\pi^0\to 2\gamma$ decay, (where L and F mean processes forbidden by the conservation of angular momenta and Furry's theorem, respectively). Since the decay probability for $\pi^0\to\lambda^0+2\gamma$ is small compared with $\pi^0\to 2\gamma$, there would be no inconsistency even if it could not be detected by present experiments. However, since the probability of the process $\pi^0\to\lambda^0+\gamma$ is of the same order of magnitude as the probability of $\pi^0\to 2\gamma$, one must require that this experimentally unobserved process is forbidden by some selection rule.

Table III. Selection rules for processes competing with $\pi^0\to 2\gamma$ decay

	$S(s)$	$S(v)$	$PS(ps)$	$PS(pv)$	$V(v)$	$V(t)$	$A(a)$	$A(t)$
$\pi^0\to\lambda^0+\gamma$	L	L	L	L	O	O	F	O
$\pi^0\to\lambda^0+\gamma+\gamma$	O	F	O	O	F	F	O	F

O: O.K.
L: forbidden by the conservation of augular momentum
F: forbidden by Furry's theorem

(b) *Process competing with $K\to 2\pi$ decay*

Since the coupling constant between the λ^0 meson and the Λ-particle is small, there is no serious problem for $K_1^0\to\lambda^0+\pi^0$. On the other hand, one must take care of processes $K^+\to\lambda^0+\pi^+$ competing with $K^+\to\pi^0+\pi^+$. According to experiments,

$$\frac{\tau(K^+)}{\tau(K_1^0)}\approx 100,\quad \frac{w(K^+)}{w(K_1^0)}\approx\frac{1}{500}.$$

The lifetime of K^+ is much longer than K_1^0's. Thus it may be thought that $K^+\to\pi^+\pi^0$ is forbidden in the first approximation. In other words one can rea-

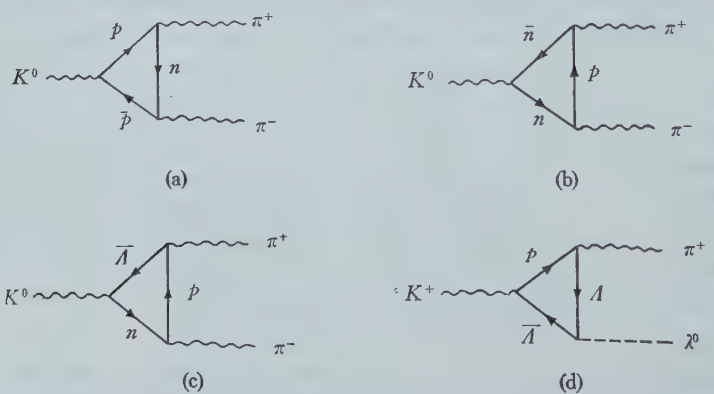

Fig. 2.

sonably argue that $K^+\to\pi^++\pi^0$ results from the deviation from the $\Delta I=1/2$ rule.

Therefore $w(K^+\to\lambda^0+\pi^+)$ would not necessarily be much smaller than $w(K^+\to\pi^0+\pi^+)$ even if $w(K^+\to\lambda^0+\pi^+)$ is very small compared with $w(K_1^0\to\pi^++\pi^-)$. These situations can be understood by examining $w(K^+\to\lambda^0+\pi^+)/w(K^+\to\pi^0+\pi^+)$ using such diagrams as Fig. 2 which contain only Λ, p or n. Let g_π, g_K and F be coupling constants for πNN(ps-coupling), $K\Lambda N$ and weak interactions, respectively, and let f be the coupling constant for $\lambda^0\Lambda\Lambda$ as defined in the Appendix. Then we have

$$w(K^+\to\lambda^0+\pi^+)=a\frac{f^2}{4\pi}\frac{F^2}{4\pi}\frac{g_K^2}{4\pi},$$

$$w(K_1^0\to\pi^++\pi^-)=b\frac{F^2}{4\pi}\left(\frac{g_\pi^2}{4\pi}\right)^2.$$

Here a and b, respectively, depend upon the number of diagrams which contribute to relevant processes, phase space volume and matrix elements.

Assuming the matrix elements to be constant, we have

$$\frac{a}{b}\simeq\left(\sim\frac{1}{3}\right)^2\left(\sim\frac{1}{2}\right).$$

The former comes from the ratio of number of diagrams and the latter from phase space volume assuming λ^0 to be a vector meson with mass 550 m_e.

$$\frac{f^2}{4\pi}\Big/\frac{g_\pi^2}{4\pi}\sim 0.3/15,$$

$$\frac{g_K^2}{4\pi}\Big/\frac{g_\pi^2}{4\pi}\sim 1/15.$$

Consequently, we have

$$\frac{w(K^+\to\lambda^0+\pi^+)}{w(K_1^0\to\pi^++\pi^-)}=\frac{a}{b}\left(\frac{f^2/4\pi}{g_\pi^2/4\pi}\right)\left(\frac{g_K^2/4\pi}{g_\pi^2/4\pi}\right)\sim 10^{-4}.$$

Comparing this with the ratio mentioned earlier,

$$\frac{w(K^+)}{w(K_1^0)}\sim\frac{1}{500},$$

we obtain

$$\frac{w(K^+\to\lambda^0+\pi^+)}{w(K^+\to\pi^0+\pi^+)}\sim\frac{1}{20}.$$

λ^0, therefore, will be observed experimentally through very careful examination of $K^+\to\pi^+$ decay processes.

Hiida[5] has pointed out that

$$K^+\to\pi^++\text{ (a neutral boson with mass 550 }m_e)$$

may possibly occur among anomalous K^+ decay processes and this may be found by analyzing the spectrum for emitted pions. Our λ^0 meson may be identified with this neutral meson giving mass $550m_e$.

(c) *Decay of the λ^0 meson*

Table IV summarizes all possible decay processes for the λ^0 meson. The most favorable processes which are consistent with the types determined in (i) are the following.

$$\lambda^0 \to \pi^0 + \gamma \quad \text{for } V(v) \text{ and } V(t)$$
$$\to 2\gamma \quad \text{for } PS(ps).$$

These would be very difficult to observe because of short lifetimes, but it may be possible to check them experimentally by most careful investigation.

Table IV. Selection rules for decay processes of λ^0 meson

	$S(s)$	$S(v)$	$V(v)$	$V(t)$	$PS(ps)$	$PS(pv)$	$A(a)$	$A(t)$
$\lambda^0 \to \pi^+ + \pi^- + \pi^0$	L	L	O	O	F	F	F	O
$\pi^0 + \pi^0 + \pi^0$	L	L	O	O	F	F	F	O
$\pi^+ + \pi^-$	◎	F	F	F	L	L	L	L
$\pi^0 + \pi^0$	◎	F	F	F	L	L	L	L
$\pi^+ + \pi^- + \gamma$	O	O	O	O	O	O	◎	O
$\pi^0 + \pi^0 + \gamma$	F	O	O	O	F	F	F	O
$\pi^0 + \gamma$	L	L	◎	◎	L	L	F	◎
$\pi^0 + \gamma + \gamma$	O	F	F	F	O	O	◎	F
$\gamma + \gamma$	◎	F	F	F	◎	◎	×	F

O : O.K.
◎ : most probable
F : forbidden by the Furry's theorem
L : forbidden by the conservation of angular momentum

(iii) *Production of the λ^0 meson*

We can expect only a very small production rate for the λ^0 meson, because λ^0 does not interact with any field other than the Λ with which it has a small coupling constant. For production of the λ^0 meson the following reaction may be favorable.

$$d + d \to \lambda^0 + He^4$$

This is forbidden for the π^0 due to isospin conservation, whereas λ^0 with zero isospin can occur. We can, therefore, observe the λ^0 meson, if we can detect a γ or π^0 from this reaction.

On the other hand some people anticipate the existence of the π_0^0 which may also similarly appear through

$$d+d\rightarrow\pi_0^0+He^4.$$

The magnitude of the cross section is of the order of magnitude:[6]

$$\sigma_{\pi_0^0}<10^{-31}\ \mathrm{cm}^2.$$

Experimentally the answer is still negative. In order to produce λ^0 meson, however, the nucleon has to be changed to Λ particle virtually emitting K meson. In addition the smallness of the λ^0-Λ coupling constant will make considerably lower the production rate for λ^0 (roughly an order of magnitude). Therefore it seems to be very difficult to check through production process.

§ 3. Discussion and concluding remarks

We have proposed a new particle λ^0 which interacts only with the Λ particle and which is responsible for the nucleon-Λ mass difference. It had been established that the properties of λ^0 will be restricted to $V(v)$, $V(t)$ or $PS(ps)$ and its mass will be greater than that of the neutral pion. Decay schemes are

$$\lambda^0\rightarrow\pi^0+\gamma \quad \text{for } V(v) \text{ and } V(t)$$
$$\rightarrow 2\gamma \quad \text{for } PS(ps).$$

It should have a smaller coupling constant than that of the pion, and a very short lifetime. Although it may be difficult to check experimentally because of the smaller coupling constant as well as the short lifetime for the above decay processes, careful examination will make its detection possible. For this purpose

$$d+d\rightarrow\lambda^0+He^4$$

and

$$K^+\rightarrow\lambda^0+\pi^+$$

may be useful.

We cannot immediately decide which is most favorable for the λ^0 among the three possibilities, $V(v)$, $V(t)$ and $PS(ps)$. Remembering that the introduction of the λ^0 is motivated by analogy with the photon in the case of the neutron-proton mass difference, it seems plausible that the λ^0 belongs to the same family as the photon. In this case we might obtain the interaction of the Λ by replacing ∂_μ, the differential operator for the Λ by $\partial_\mu-if\phi_\mu$.

For direct Fermi interactions the magnitude of the interactions is different from the ordinary one when Λ particles take part in interactions. In other words, although not yet established, the following might hold:[7]

(a) The coupling constant for $\Lambda Ne\nu$ is smaller than that for $NNe\nu$.

(b) The coupling constant for $\Lambda\Lambda NN$ is smaller than that for $NNNN$.

The introduction of the λ^0 meson may explain the change of these coupling constants qualitatively, not quantitatively, because of the smallness of the λ^0 coupling constant. The first thing to do is to search for experimentally.

Finally we would like to remark briefly on another aspect of the λ^0 meson. Since the λ^0 meson belongs to strongly interacting particles because of its coupling constant as determined here, it may have some influence on strong interactions, for instance, nuclear forces or pion production processes.

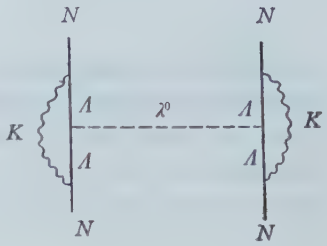

Fig. 3.

Breit[8] recently discussed empirical evidence for considering the repulsive core of nucleon-nucleon interactions and the spin-orbit interaction as originating in a vector field coupling. He determined its mass to be 9 to 11 times the pion mass compared with the spin-orbit coupling introduced phenomenologically. In our model also there is a possibility that the spin-orbit interaction may arise from such a diagram as Fig. 3. Quantitatively, however, it seems to be difficult to obtain the required magnitude. We shall examine this problem on another occasion.

Appendix

We shall evaluate the self-energy of the Λ in the lowest order of perturbation theory. The interaction Hamiltonians are as follows,

$$H_I=f\bar{\psi}(x)O\psi(x)\phi(x)+\delta M\bar{\psi}(x)\psi(x) \qquad \text{for direct coupling,}$$

$$H_I=\frac{g}{m_{\lambda^0}}\bar{\psi}(x)O\psi(x)\partial_\mu\phi(x)+\delta M\bar{\psi}(x)\psi(x) \qquad \text{for derivative coupling,}$$

where $\psi(x)$, $\phi(x)$ represent Λ and λ^0, and M, m_{λ^0} their mass respectively. δM is the renormalization constant for Λ. O generally means the interaction operator between λ^0 and Λ, i.e.

		direct	derivative
Scalar	O_S	1	$i\gamma_\mu$
Vector	O_V	$i\gamma_\mu$	$i\gamma_{\mu\nu}$
Pseudoscalar	O_P	$i\gamma_5$	$i\gamma_5\gamma_\mu$
Pseudovector	O_A	$i\gamma_5\gamma_\mu$	$\gamma_{\mu\nu}\gamma_5$,

where

$$\gamma_{\mu\nu}=\frac{i}{2}(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu).$$

Using the Feynman cutoff, we obtain the following results

(i) *Direct coupling*

$$\frac{\delta M_S}{M}=-\frac{f^2}{4\pi}\frac{1}{4\pi}\frac{K^2}{2}\int_0^1 dx\int_0^1 2y\,dy\,\frac{2-xy}{D^2}.$$

$$\frac{\delta M_V}{M}=\frac{f^2}{4\pi}\frac{1}{4\pi}K^2\int_0^1 dx\int_0^1 2y\,dy\,\frac{1+xy}{D^2},$$

$$\frac{\delta M_P}{M}=\frac{f^2}{4\pi}\frac{1}{4\pi}\frac{K^2}{2}\int_0^1 dx\int_0^1 2y\,dy\,\frac{xy}{D^2},$$

$$\frac{\delta M_A}{M}=-\frac{f^2}{4\pi}\frac{1}{4\pi}K^2\int_0^1 dx\int_0^1 2y\,dy\frac{3-xy}{D^2},$$

where

$$D^2=M^2x^2y^2+K^2(1-y)+m_{\lambda 0}^2(1-x)y.$$

(ii) *Derivative coupling*

$$\frac{\delta M_S}{M}=-\frac{g^2}{4\pi}\frac{1}{4\pi}\left(\frac{M}{m_{\lambda 0}}\right)^2\left(\frac{K}{M}\right)^4\int_0^1 x\,dx\int_0^1 y^2\,dy\left[\frac{2-3y}{L^2}+\frac{M^2(1+y)(1-y)^2}{L^4}\right],$$

$$\frac{\delta M_V}{M}=\frac{g^2}{4\pi}\frac{1}{4\pi}\left(\frac{M}{m_{\lambda 0}}\right)^2\left(\frac{K}{M}\right)^4\int_0^1 x\,dx\int_0^1 y^2\,dy\left[\frac{15-9y}{L^2}+\frac{3M^2(1-y)^3}{L^4}\right],$$

$$\frac{\delta M_P}{M}=-\frac{g^2}{4\pi}\frac{1}{4\pi}\left(\frac{M}{m_{\lambda 0}}\right)^2\left(\frac{K}{M}\right)^4\int_0^1 x\,dx\int_0^1 y^2\,dy\left[\frac{6-3y}{L^2}-\frac{M^2(1-y)^3}{L^4}\right],$$

$$\frac{\delta M_A}{M}=\frac{g^2}{4\pi}\frac{1}{4\pi}\left(\frac{M}{m_{\lambda 0}}\right)^2\left(\frac{K}{M}\right)^4\int_0^1 x\,dx\int_0^1 y^2\,dy\left[\frac{3-9y}{L^2}+\frac{3M^2(1+y)(1-y)^2}{L^4}\right],$$

where

$$L^2=M^2(1-y)^2+m_{\lambda 0}^2(1-x)y+K^2xy.$$

Note added in proof : Sakurai also proposed the neutral vector meson theory in connection with the spin-orbit coupling. Employing p-p data at 310 Mev he arrived at the conclusion that the mass of the neutral vector meson must be close to $3m_\pi$. J. J. Sakurai, Phys. Rev. **119** (1960), 1784. See. also J. J. Sakurai, Ann. Phys. **11** (1960), 1.

References

1) S. Sakata, Prog. Theor. Phys. **16** (1956), 686.
2) S. Ogawa, Prog. Theor. Phys. **21** (1959), 209.
   M. Ikeda, S. Ogawa and Y. Ohnuki, Prog. Theor. Phys. **22** (1959), 715.
   Y. Yamaguchi, Prog. Theor. Phys. Supplement No. 7 (1958).
3) C. Iso, M. Kawaguchi and Y. Miyamoto, Prog. Theor. Phys. **24** (1960), 97.
4) Y. Katayama, M. Taketani and J. L. Ferreira, Prog. Theor. Phys. **21** (1959), 818.
   J. L. Ferreira and Y. Katayama, Prog. Theor. Phys. **23** (1960), 776.
5) K. Hiida, Nuovo Cimento **13** (1959), 1117.
6) 1959 Annual International Conference on High Energy Physics at Kiev.
7) L. B. Okun', 1958 Annual International Conference on High Energy Physics at CERN.
   C. Iso, M. Kawaguchi and Y. Miyamoto, Prog. Theor. Phys. **24** (1960), 97.
8) G. Breit, Proc. Natl. Acad. of Sci. (U. S.) **46** (1960), 746; Phys. Rev. **120** (1960), 287.
   One of us (T. K.) wishes to express his thanks to Professor S. Ohnuma for showing him these papers before publication and calling his attention to this point.

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On Hugenholtz-Pines' Theory about Ground State Energy of Interacting Bosons

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In their recently published paper[1] Hugenholtz and Pines have developed a new method to evaluate the ground state energy of a system of interacting bosons, the merit of which consists in the elimination of the zero-momentum state and accordingly the applicability of the linked cluster expansion. Their theory, however, does not give correct results because of the undue treatment of the zero-momentum state; this will be proved in a simple example. Up to third order in coupling parameter of the ground state energy we compare the result of the usual Rayleigh-Schrödinger perturbation method with that obtained according to Hugenholtz-Pines' theory and find that the latter does not agree with the former.

§ 1. Introduction

As is well known, in the case of fermion gas the ground state energy can be described by the famous "linked cluster expansion" of Goldstone[2] by virtue of the fact that an unoccupied single-particle state in the Fermi sphere is read as a hole which is subjected to Fermi statistics and that the unperturbed ground state (the Fermi sphere) is characterized by the state with no excited particles and no holes.

On the other hand, in the boson case the situation is quite different. An unoccupied zero-momentum state cannot be considered as a hole obeying the Bose statistics and therefore the unperturbed state (all particles are sinking into the zero-momentum state) is not specified by no holes. From this fact naturally follows the inapplicability of the linked cluster expansion. As is seen instantly, the usual Rayleigh-Shrödinger perturbation formula for the perturbed energy contains, in more than second order in coupling parameter, various possible disconnected diagrams, each of which has such a singular character that the contribution to the energy per particle becomes infinite with the volume of the system at a constant number density. After all diagrams are properly taken into account following the R-S perturbation formula, the above physically absurd terms cancel each other, leaving only the terms dependent upon the density alone.

To treat correctly the contribution to the energy from the disconnected diagrams is rather a complicated problem. In the case of weak repulsive interaction, if we simply replaced the creation and annihilation operators for the zero-momentum particles in the interaction Hamiltonian by c-number, $N_0^{1/2}$ (N_0 being number

of the zero-momentum particles), considering the fact that a great majority of the particles is lying in the zero-momentum state and followed the usual perturbation formula to evaluate the energy, we would fall into two errors: One is ignoring the contribution arisen from the disconnected diagrams and the other is the neglect of the so-called " depletion effect " (after H. P.), i.e. the effect associated with the difference between the total number N and N_0. To avoid these difficulties H. P. have deduced a new method by introducing to the Hamiltonian a parameter similar to the chemical potential in considering grand ensemble. However, their theory is based on an unproved theorem and it is due to this that we cannot answer definitely where they made error. It is, however, very probable that the replacement of the zero-particles operators by c-number is not admissible, even though a Lagrangian multiplier method is introduced, in a sence that either the effect of the disconnected diagrams or the depletion effect is not rightly treated.

In § 2 we list the result of the ground state energy following the usual R-S perturbation up to third order in coupling parameter. The lowest order including the contribution from the disconnected diagrams is nothing but the third order. In §3 we evaluate the energy following the Hugenholtz-Pines' formalism and find that the contribution from the disconnected diagram obtained here is twice as much as that obtained in § 2 in the third order. In § 4, we make some comments on these problems.

§ 2. Usual perturbation method

Let us consider a system of N interacting bosons enclosed in a cubic box of volume Ω. The Hamiltonian may be written, in the unit $\hbar=2m=1$, as

$$H=H_0+V, \tag{2·1}$$

$$H_0=\sum_k k^2 a_k^\dagger a_k, \tag{2·2}$$

$$V=\frac{1}{2\Omega}\sum_{1,2,3,4} v(1\,2\,;\,4\,3)\, a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_{k_4}. \tag{2·3}$$

The operators a_k and $a_k^\dagger$ are ordinary ones obeying the commutation relations for the Bose statistics and v is defined as

$$v(12\,;\,43)=\delta(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{k}_3-\boldsymbol{k}_4)\, v\left(\frac{\boldsymbol{k}_1-\boldsymbol{k}_2}{2},\ \frac{\boldsymbol{k}_4-\boldsymbol{k}_3}{2}\right), \tag{2·4}$$

where δ means the Kronecker delta function and

$$v(\boldsymbol{k},\boldsymbol{k}')=\int e^{-i\boldsymbol{k}\boldsymbol{x}} v(\boldsymbol{x})\, e^{i\boldsymbol{k}'\boldsymbol{x}}\, d\boldsymbol{x}. \tag{2·5}$$

Here we assume the two-body potential $v(\boldsymbol{x})$ to be non-singular and so weak as to be able to apply the perturbation method, which does not mean any loss of generality in the standpoint of this paper. Though $v(\boldsymbol{k},\boldsymbol{k}')$, in (2·5), is a func-

tion dependent only upon the difference between its arguments $\boldsymbol{k}-\boldsymbol{k}'$ and accordingly $v(\boldsymbol{k}, \boldsymbol{k})=v(0, 0)$, we do distinguish $v(\boldsymbol{k}, \boldsymbol{k})$ from $v(0, 0)$, because from observing the arguments we may appreciate the pattern of the scatterings.

With the above notation the unperturbed ground state $|\Phi_0\rangle$ is given by

$$|\Phi_0\rangle=\frac{1}{\sqrt{N!}}(a_0^\dagger)^N|0\rangle, \tag{2·6}$$

$|0\rangle$ being the vacuum state with no particles, and as the excited states we have only to consider such a type as

$$|\Phi_k\rangle=\frac{1}{\sqrt{(N-2)!}}a_k^\dagger a_{-k}^\dagger(a_0^\dagger)^{N-2}|0\rangle, \tag{2·7}$$

so far as the energy up to third order in coupling parameter is concerned. In (2·7) $\boldsymbol{k}$ must be restricted to a half region in the momentum space, since $|\Phi_{\boldsymbol{k}}\rangle=|\Phi_{-\boldsymbol{k}}\rangle$; for instance, we take the states with $k_x>0$.

By using the equations from (2·3) to (2·7), the matrix elements of V are evaluated as

$$\begin{aligned}
&2\Omega V_{00}=N(N-1)v(0, 0),\\
&2\Omega V_{0k}=2\sqrt{N(N-1)}\,v(0, \boldsymbol{k}),\\
&2\Omega V_{km}(\boldsymbol{k}\neq\boldsymbol{m})=2(v(\boldsymbol{k}, \boldsymbol{m})+v(\boldsymbol{k}, -\boldsymbol{m})),\\
&2\Omega V_{kk}=2(v(\boldsymbol{k}, \boldsymbol{k})+v(\boldsymbol{k}, -\boldsymbol{k}))+4(N-2)\left(v\left(\frac{\boldsymbol{k}}{2}, \frac{\boldsymbol{k}}{2}\right)+v\left(\frac{\boldsymbol{k}}{2}, -\frac{\boldsymbol{k}}{2}\right)\right)\\
&\qquad+(N-2)(N-3)v(0, 0).
\end{aligned} \tag{2·8}$$

When inserting these into the well-known perturbation formula for the ground state energy,

$$E_0=\varepsilon_0+V_{00}-\sum_k{}'\frac{V_{0k}V_{k0}}{\varepsilon_k-\varepsilon_0}+\sum_{m,k}{}'\frac{V_{0k}V_{km}V_{m0}}{(\varepsilon_k-\varepsilon_0)(\varepsilon_m-\varepsilon_0)}-\sum_k{}'\frac{V_{00}V_{0k}V_{k0}}{(\varepsilon_k-\varepsilon_0)^2}, \tag{2·9}$$

$$\varepsilon_k=\langle\Phi_k|H_0|\Phi_k\rangle=2k^2,\ \varepsilon_0=0,$$

we finally get*

$$\begin{aligned}
E_0=&\frac{N(N-1)}{2\Omega}v(0, 0)-\frac{N(N-1)}{(2\Omega)^2}\sum_{\boldsymbol{k}}{}'\frac{v(0, \boldsymbol{k})v(\boldsymbol{k}, 0)}{k^2}\\
&+\frac{N(N-1)}{(2\Omega)^3}\left[\sum_{\boldsymbol{k},\boldsymbol{k}'}{}'\frac{v(0, \boldsymbol{k})v(\boldsymbol{k}, \boldsymbol{k}')}{k^2k'^2}v(\boldsymbol{k}', 0)+\sum_{\boldsymbol{k}}{}'\frac{v(0, \boldsymbol{k})v(\boldsymbol{k}, 0)}{k^4}\right.\\
&\left.\times\left\{2(N-2)\left(v\left(\frac{\boldsymbol{k}}{2}, \frac{\boldsymbol{k}}{2}\right)+v\left(\frac{\boldsymbol{k}}{2}, -\frac{\boldsymbol{k}}{2}\right)\right)-(2N-3)v(0, 0)\right\}\right].
\end{aligned} \tag{2·10}$$

If we make the volume Ω become infinite at a constant density, (2·10) is reduc-

* Hereafter the primed summation symbol means that $\boldsymbol{k}=0$ is excluded from the summation.

ed to

$$E_0=\frac{N^2}{2\Omega}v(0,0)-\frac{N^2}{(2\Omega)^2}\sum_k{}'\frac{v(0,k)\,v(k,0)}{k^2}$$
$$+\frac{N^2}{(2\Omega)^3}\sum_{k,k'}{}'\frac{v(0,k)\,v(k,k')\,v(k',0)}{k^2k'^2}$$
$$+\frac{2N^3}{(2\Omega)^3}\sum_k{}'\frac{v(0,k)\,v(k,0)}{k^4}\left(v\left(\frac{k}{2},\frac{k}{2}\right)+v\left(\frac{k}{2},-\frac{k}{2}\right)\right)$$
$$-\frac{2N^3}{(2\Omega)^3}\sum_k{}'\frac{v(0,0)\,v(0,k)\,v(k,0)}{k^4}, \qquad (2\cdot11)$$

Fig. 1. Disconnected diagram in third order. Excited and unexcited particles are indicated by solid and dashed lines, respectively.

only the terms proportional to N in (2·10) remaining. The last term containing a term $v(0,0)$ in (2·11) is nothing but the contribution from the disconnected diagram as shown in Fig. 1.

§ 3. Hugenholtz-Pines Method

Let us begin to follow their paper, particularly § 3, and list the essentials in it.

One replaces the operators $a_0^\dagger$ and a_0 by c-number $N_0^{1/2}$ and eliminate the zero-momentum state. The Hamiltonian becomes

$$H(N_0)=H_0+V(N_0). \qquad (3\cdot1)$$

Now the number of particles is no longer conserved, since $V(N_0)$ does not commute with the operator $N'=\sum_k{}'a_k^\dagger a_k$. N' is, however, still approximately a good quantum number, in that it commutes with H to order Ω^{-1}. Then one imposes the following subsidiary condition:

$$\langle N'\rangle=N-N_0, \qquad (3\cdot2)$$

and under this condition one seeks for the least eigenvalue of (3·1) referring to N_0. To avoid difficulties arising from (3·2) one follows the method of the undetermined multiplier. The ground state of the Hamiltonian (3·1) with the subsidiary condition (3·2) is also the ground state of the Hamiltonian

$$H'=H(N_0)-\mu N', \qquad (3\cdot3)$$

without any subsidiary condition. The ground state wave function $|\psi_0(N_0,\mu)\rangle$ and thus also the expectation values $E_0'(N_0,\mu)$, $E_0(N_0,\mu)$ and $N'(N_0,\mu)$ of H', H and N', respectively, depend on a parameter μ, which is determined by

$$N'(N_0,\mu)=N-N_0. \qquad (3\cdot4)$$

This relation expresses, for fixed N, the parameter μ in terms of N_0. N_0 is determined by the condition that, for N fixed,

$$\frac{dE_0}{dN_0}=0. \qquad (3\cdot5)$$

From (3·4) and (3·5) one obtains

$$\mu=\frac{\partial E_0'}{\partial N_0}. \tag{3·6}$$

Finally, since the zero-momentum state completely disappears from the problem, one may use the linked cluster expansion[2] for the ground state energy:

$$E_0'=\langle 0|\left\{V-V\frac{1}{H_0'}V+V\frac{1}{H_0'}V\frac{1}{H_0'}V-\cdots\right\}|0\rangle, \tag{3·7}$$

with the Hamiltonian:

$$H'=H_0'+V(N_0),\ H_0'=\sum_k{}'(k^2-\mu)a_k^\dagger a_k \tag{3·8}$$

Following the preceding paragraph, we first evaluate E_0', Eq. (3·7). This is simply obtained from (2·11) by dropping the last term (the contribution from the unlinked diagram as shown in Fig. 2) and replacing N and the energy denominator k^2 by N_0 and $k^2-\mu$, respectively:

Fig. 2. Unlinked diagram in third order.

$$E'_0=\frac{N_0^2}{2\Omega}v(0,0)-\frac{N_0^2}{(2\Omega)^2}\sum_k{}'\frac{v(0,\boldsymbol{k})v(\boldsymbol{k},0)}{k^2-\mu}+\frac{N_0^2}{(2\Omega)^3}\sum_{k,k'}{}'\frac{v(0,\boldsymbol{k})v(\boldsymbol{k},\boldsymbol{k}')v(\boldsymbol{k}',0)}{(k^2-\mu)(k'^2-\mu)}$$
$$+\frac{2N_0^3}{(2\Omega)^3}\sum_k{}'\frac{v(0,\boldsymbol{k})v(\boldsymbol{k},0)}{(k^2-\mu)^2}\left(v\left(\frac{\boldsymbol{k}}{2},\frac{\boldsymbol{k}}{2}\right)+v\left(\frac{\boldsymbol{k}}{2},-\frac{\boldsymbol{k}}{2}\right)\right). \tag{3·9}$$

Differentiating this with respect to N_0, we get

$$\mu=\frac{N_0}{\Omega}v(0,0)-\frac{N_0}{2\Omega^2}\sum_k{}'\frac{v(0,\boldsymbol{k})v(\boldsymbol{k},0)}{k^2-\mu}+\cdots, \tag{3·10}$$

which may be solved by iteration and up to first order in coupling parameter gives

$$\mu=\frac{N_0}{\Omega}v(0,0)+O(v^2). \tag{3·11}$$

Inserting (3·11) into (3·9) and neglecting terms beyond the third order, we have

$$E_0'=\frac{N_0^2}{2\Omega}v(0,0)-\frac{N_0^2}{(2\Omega)^2}\sum_k{}'\frac{v(0,\boldsymbol{k})v(\boldsymbol{k},0)}{k^2}+\frac{N_0^2}{(2\Omega)^3}\sum_{k,k'}{}'\frac{v(0,\boldsymbol{k})v(\boldsymbol{k},\boldsymbol{k}')v(\boldsymbol{k}',0)}{k^2k'^2}$$
$$+\frac{2N_0^3}{(2\Omega)^3}\sum_k{}'\frac{v(0,\boldsymbol{k})v(\boldsymbol{k},0)}{k^4}\left(v\left(\frac{\boldsymbol{k}}{2},\frac{\boldsymbol{k}}{2}\right)+v\left(\frac{\boldsymbol{k}}{2},-\frac{\boldsymbol{k}}{2}\right)\right)$$
$$-\frac{2N_0^3}{(2\Omega)^3}\sum_k{}'\frac{v(0,0)v(0,\boldsymbol{k})v(\boldsymbol{k},0)}{k^4}. \tag{3·12}$$

This just agrees with (2·11) except that N is here replaced by N_0. However, at the same time, we must consider the depletion effect, i.e. the difference between N and N_0. For that purpose we consider the first order ground state wave

function

$$|\psi_0(N_0, \mu)\rangle = \left\{1 - \frac{N_0}{4\Omega}\sum_k{}' \frac{v(0, k)}{k^2 - \mu} a_k^\dagger a_{-k}^\dagger\right\}|0\rangle. \tag{3·13}$$

The expectation value of H' with respect to (3·13) rightly gives (3·9) up to second order. The expectation value of N' gives

$$N'(N_0, \mu) = \frac{N_0^2}{(2\Omega)^2}\sum_k{}' \frac{v(0, k)v(k, 0)}{(k^2 - \mu)^2}, \tag{3·14}$$

and thus (3·4) becomes

$$N_0 = N - \frac{N_0^2}{(2\Omega)^2}\sum_k{}' \frac{v(0, k)v(k, 0)}{(k^2 - \mu)^2}.$$

Solving this with regard to N_0 by iteration and using (3·11), we have

$$N_0 = N - \frac{N^2}{(2\Omega)^2}\sum_k{}' \frac{v(0, k)v(k, 0)}{k^4} + O(v^3). \tag{3·15}$$

If we insert (3·15) into (3·12), N_0^2 in the first term of (3·12) becomes

$$N_0^2 = N^2 - \frac{2N^3}{(2\Omega)^2}\sum_k{}' \frac{v(0, k)v(k, 0)}{k^4} + O(v^3),$$

and for the ground state energy we finally get

$$\begin{aligned} E_0 = {} & \frac{N^2}{2\Omega}v(0, 0) - \frac{N^2}{(2\Omega)^2}\sum_k{}' \frac{v(0, k)v(k, 0)}{k^2} + \frac{N^2}{(2\Omega)^2}\sum_{k,k'}{}' \frac{v(0, k)v(k, k')v(k', 0)}{k^2 k'^2} \\ & + \frac{2N^3}{(2\Omega)^3}\sum_k{}' \frac{v(0, k)v(k, 0)}{k^4}\left(v\left(\frac{k}{2}, \frac{k}{2}\right) + v\left(\frac{k}{2}, -\frac{k}{2}\right)\right) \\ & - \frac{4N^3}{(2\Omega)^3}\sum_k{}' \frac{v(0, 0)v(0, k)v(k, 0)}{k^4}. \end{aligned} \tag{3·16}$$

Thus the last term due to the unlinked diagram does not agree with (2·11); this is twice as much as that.

§ 4. Concluding remarks

As is seen in the previous sections, Hugenholtz-Pines' formalism fails to give correct value even in the third order in coupling parameter, whereas, when collecting the energy in terms of scattering length, their result agrees with Sawada's[3] and Wu's[4] up to the order of Lenz's value times $na^3\log(na^3)$ in terms of hard sphere (n and a being the number density and the diameter of the sphere). However, it is never strange that such a situation happens, since the depletion effect does not affect the terms evaluated already, as was discussed in H. P. In our example, also, if we neglect the depletion effect (3·15), Eq. (3·12) obtained according to H.

P. agrees with the correct expression (2·11). Accordingly, if one proceeds to the evaluation of corrections up to higher orders of na^3 associated with the depletion effect, one cannot get the correct values any more.

Note added in proof: The third order unlinked diagram, the last term in Eq. (2·11) or (3·16), would give a term of order n^2 to the energy per particle, if the $\boldsymbol{k}$-integration converged. Actually, the integration exhibits ultra-violet divergence for the short range potential and hence contributes to the energy a term with the branch point singularity at $n=0$, which has been known empirically as $n^{3/2}$ (Lenz's value times $(na^3)^{1/2}$). Judging from the fact that the result of H. P. concerning this term agrees with Sawada's, would any substitution have been done in the theory?

References

1) N. M. Hugenholtz and D. Pines, Phys. Rev. **116** (1959), 489.
2) J. Goldstone, Proc. Roy. Soc. (London) **A239** (1957), 267.
3) K. Sawada, Phys. Rev. **116** (1959), 1344.
4) T. T. Wu, Phys. Rev. **115** (1959). 1390.

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On the Structure of Extensive Air Showers. II

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The structure of extensive air showers (EAS) is examined in detail, in particular in its dependence upon the characteristics of the high-energy nuclear interactions and the primary energy spectrum. The following effects are taken into account: (1) The fluctuation in the depth at which the primary particles make their first interaction. (2) The fluctuations in the first interaction made by heavy primary particles.

The main results are summarized as follows: (i) Shower curves are derived which are consistent with experiments, by a proper choice of the parameters involved in our model of the high-energy interaction. (ii) Under the assumptions that the relative abundance of various groups of primary nuclei at the top of the atmosphere is the same as measured at lower energies and that all groups have the same energy spectra as the proton spectrum, the calculated shower size spectrum is not inconsistent with the observed one even if there is a cutoff in the primary energy at about $5 \cdot 10^8$ Mc^2 per nucleon. (iii) Under the same assumptions, the fluctuation in the ratio between numbers of muons and of electrons is, at sea level, almost entirely governed by the fluctuation in the depth of the first interaction of protons. At mountain altitudes the fluctuation in the ratio is governed nearly equally by both the fluctuations due to protons and that due to heavy nuclei. (iv) Under the same assumptions, the shower rate due to primary nuclei heavier than protons is, at mountain altitude, equal to or greater than twice that due to protons. (v) There is a possibility that high-energy nuclear active particles ($\gtrsim 10^{12}$ ev) in EAS initiated by a proton are as abundant as those in an EAS initiated by a heavy nucleus.

Additional remarks which would be useful for further investigations are also given.

§ 1. Introduction

Extensive air showers (EAS) have been studied for the purposes of getting information on the primary energy spectrum on the one hand and on the chracteristics of extremely high-energy nuclear interactions on the other. The energy of a primary particle in the former type of investigation has been evaluated from the number of electrons in an EAS near sea level. In the latter analyses have been made to investigate the behaviours of energetic nucleons, muons, the structure of shower cores and so on. In both studies, however, the conventional assumption has been made that fluctuations in the longitudinal development would not cause any serious error in the analyses.

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Recently Kraushaar[1] and Miyake[2] pointed out independently the importance of the fluctuation in the depth at which the first nuclear interaction occurs. Kraushaar showed that the conventional conversion of the size spectrum into a primary energy spectrum led to an overestimation of the energy by a factor of 1.4. A more refined analysis made by Miyake showed that the attenuation length of the shower rate is intimately related to the interaction mean free path of primary particles with air nuclei,* and that the intensity of the primary particles decreases more rapidly at high energies than does the intensity hitherto considered and that the practical upper limit of the energy spectrum would be about $5 \cdot 10^{17}$ ev. Recent experiments made by the Japanese[5] and Russian groups[6] also show that the lateral distribution of charged particles and the ratio between numbers of muons and of all charged particles fluctuates rather widely from shower to shower.

The purposes of the present paper are to examine quantitatively the effect of the fluctuations on various quantities, taking into account the shape of the primary energy spectrum and the characteristics of the high-energy interactions and to get at least a first approximation to a consistent picture of EAS.

The important sources of the fluctuations are certainly in the very early stages of the longitudinal development, in particular, the first nuclear interaction is most significant.[7] The fluctuations in this first interaction can be classified into three types. The first one is, as mentioned above, the depth at which the first interaction occurs and the second one is the possibility of fluctuations in multiplicity, inelasticity and energy distribution of the particles produced. The third one is due to heavy particles. It arises from consideration of how many nucleons in the heavy nucleus, in the range from helium to iron, participate in the first interaction.

The second type of fluctuations could be relatively unimportant as discussed by Kraushaar.[1] Its effects could be interpreted as due to an apparent fluctuation in the depth. Suppose, for example, that the first interaction is very elastic and the second one inelastic. An EAS initiated by such a sequence of interactions would be interpreted at low altitudes as being initiated at the depth at which the second interaction occurred.

The fluctuation due to heavy particles has in some sense a similar feature to the second type of fluctuation. But it is more important than the latter, owing to the dependence on the mass number. Fluctuations other than those in the first interaction cannot be of major importance on account of the great number of participant particles. We shall therefore consider only the first and third types of fluctuations.

There is one important effect to be considered. If an EAS is initiated at large depth, the geometrical distances travelled by the nuclear active unstable particles from production to interaction are so shortened due to the change of air density that they can make more nuclear interactions. This effect has the result that for a given atmospheric thickness through which EAS develops, EAS initiated

* This fact was also suggested modestly by Greisen[3] and Cranshaw et al.[4]

at larger depths involve more nuclear active particles and less muons than do EAS initiated near the top of the atmosphere even if their primary energies are the same. This effect will be taken into account approximately to solve the nucleonic cascade equation.

The magnitude of the fluctuation in the depth is sensitive to the shape of the shower curve, i. e. the variation of shower size with atmospheric thickness through which an EAS develops. The shape of the shower curve depends strongly on the characteristics of the high-energy nuclear interactions as emphasized in our previous calculation.[8] Therefore we shall first examine in detail EAS initiated by protons. This will enable us as a first approximation to estimate the energy spectrum of primary protons and the characteristics of the high-energy interactions.

From the experimental point of view, the magnitude of the fluctuation in the depth could be estimated from a positive correlation between various measurable quantities in each EAS. Apart from the fluctuation in the depth, however, heavy primaries might cause similar effects. Therefore we shall also examine EAS initiated by heavy particles, comparing their characteristics with those produced by protons.

As a model of the high-energy nuclear interaction we shall use the same one as in I and review it here:

(1) The collision cross section is independent of the energy of a colliding particle and is also independent of the kind of colliding particle, i. e. a nucleon or a charged pion.

(2) The number of secondary particles produced by a particle of energy* E in the laboratory system is given by

$$n=2(\eta E)^{1/4}.$$

η plays the role of the inelasticity. An energy of amount ηE is transferred to secondary particles and $(1-\eta)E$ is carried away by the colliding particle.

(3) The energy transferred to the secondary particles, ηE, is equipartitioned among them in the laboratory system.

(4) Two of the secondary particles are nucleons. Of the $(n-2)$ particles, $\Delta(n-2)$ particles are charged pions and $(1-\Delta)(n-2)$ particles are neutral pions. The fraction Δ is independent of energy. The production of nucleon pairs and of heavy mesons is disregarded.

(5) Nuclear active particles of energy lower than the critical value E_c do not contribute to the further development of the nucleonic cascade process.

In the calculation we shall fix $\Delta=0.7$ and $E_c=10$, and regard η and the interaction mean free path of nuclear particles L_s as parameters. In particular the η-dependence of various quantities will be examined.

At first sight our model seems to be rather crude. As far as shower curves

* In the present paper energy is measured in unit of the nucleon rest energy, Mc^2.

are concerned, however, it will be seen that by a proper choice of the parameters, shower curves can be derived which are consistent with experimental results. A detailed discussion on this point will be made in connection with results of jet showers.

In § 2 we shall give a qualitative discussion on the fluctuation in the depth of the first interaction in order to make clear the relationship between the various quantities affecting this fluctuation. The calculated results on EAS initiated by protons will be given in § 3. In § 4 the model of the first interaction of heavy particles will be proposed and the calculated results will be given. In § 5 comparison of the calculated results with experimental results will be discussed.

The calculations reported here were carried out on the electronic computer SILLIAC of the Basser Computing Department in the School of Physics.

§ 2. A qualitative argument on the fluctuation in depth

We shall treat EAS initiated by different kinds of primary particles separately and call them A-particles according to their mass numbers.

We use simplified expressions for some quantities to make it easy to understand how they are related to the fluctuation. This simplification does not change the following argument. We suppose the differential energy spectrum of A-particles of energy* between E_0 and E_0+dE_0 per nucleon to be expressed by

$$I_A(E_0)\,dE_0=k_A E_0^{-(\gamma_A+1)}\,dE_0, \tag{1}$$

where γ_A, the exponent of the energy spectrum, is assumed tentatively to be independent of E_0, and the number of particles in an EAS at depth x' from the top of the atmosphere, initiated by an A-particle of energy E_0, to be represented by

$$N_A(E_0, x)=N_A E_0^{s}\exp\left[-\frac{x'-x_1}{\lambda_A(x'-x_1)}\right]=N_A E_0^{s}\exp\left[-\frac{x}{\lambda_A(x)}\right], \tag{2}$$

where x_1 is the depth at which the A-particle makes the first interaction and $x(=x'-x_1)$ is the thickness through which the EAS develops. s has a value close to one at the lower atmosphere though being a function of x and depending on the model of the cascade. $\lambda_A(x)$, which governs the main shape of the shower curve, tends to the attenuation length λ_A of the shower size at large depth.

From Eqs. (1) and (2) the frequency with which A-particles making their first collisions at a given depth x_1 give rise to EAS with shower sizes between N and $N+dN$ at x' is given by

$$I_A(E_0(N, x_1))\left(\frac{\partial E_0(N, x_1)}{\partial N}\right)_{x_1} dN, \tag{3}$$

where $E_0(N, x_1)=(N/N_A)^{1/s}\exp[x/s\lambda_A(x)]$ is an inverse of the function $N_A(E_0, x)$.

* Primary energies are represented by energy per nucleon, if not specially mentioned.

Since the probability that an A-particle makes the first collision at depth between x_1 and x_1+dx_1 is given by

$$e^{-x_1/L_A}\,dx_1/L_A, \tag{4}$$

where L_A is the interaction mean free path of the A-particle, the frequency with which A-particles give rise to EAS with sizes between N and $N+dN$ at x' is given by

$$F_A(N)\,dN=dN\int f_A(N, x_1)\,dx_1/L_A, \tag{5}$$

where

$$f_A(N, x_1)=I_A(E_0(N, x_1))\left(\frac{\partial E_0}{\partial N}\right)_{x_1} e^{-x_1/L_A}$$

$$=\frac{k_A}{sN_A}\left(\frac{N}{N_A}\right)^{-(\gamma_A/s)-1}\exp\left[-\frac{\gamma_A x'}{s\lambda_A(x)}\right]\exp\left[\left(\frac{\gamma_A}{s\lambda_A(x)}-\frac{1}{L_A}\right)x_1\right] \tag{6}$$

is the differential rate of showers for a given size N at x', initiated at a depth between x_1 and x_1+dx_1. The integral domain of Eq. (5) must be determined in such a way that for a given size N, E_0 and x_1 satisfy Eq. (2). With increasing x_1 from the top of the atmosphere, E_0 decreases correspondingly until the minimum energy and then increases to the maximum energy which may still satisfy Eq. (2). The particle of the minimum energy initiating at the corresponding depth produces a shower whose maximum development is at x'.

The shower size spectrum corresponding to the experimentally observed one is a superposition of Eq. (5) on EAS initiated by all kinds of primary particles. On the other hand the customarily assumed spectrum is of the form[9]

$$\mathcal{J}(N)\,dN=KN^{-(\gamma/s)-1}\exp\left[-\frac{\gamma x'}{s\lambda}\right], \tag{7}$$

where the attenuation length λ is used instead of $\lambda(x)$. K, γ/s and λ have been measured experimentally.

At an observation point EAS may be classified into three types according to their stages of development, i. e. growing-up stage, near and at the maximum development and decreasing stage. For EAS in the former two stages, their contribution to the integral (5) in the lower half of the atmosphere is relatively unimportant, because $\lambda_A(x)$ in the former stages has a negative or large absolute value. The larger x', the more EAS are observed in the decreasing stage.

Suppose an observation is made at so large a depth, say sea level, that EAS are observed almost entirely in the decreasing stage and their $\lambda_A(x)$ can be replaced by λ_A. In this case the integral (5) is reduced to

$$\int\exp\left[\left(\frac{\gamma_A}{s\lambda_A}-\frac{1}{L_A}\right)x_1\right]dx_1/L_A \tag{8}$$

neglecting the dependence of s and λ_A on E_0 and x_1. If

$$\gamma_A/s\lambda_A \ll 1/L_A, \tag{9}$$

the integral (8) tends to unity and the formula (5) coincides with the one corresponding to (7). Then the attenuation of the shower rate Λ_A is related to λ_A with the well known formula[9]

$$\Lambda_A = s\lambda_A/\gamma_A = \lambda_A/\delta_A, \tag{10}$$

where δ_A is the exponent of the size spectrum of EAS initiated by A-particles. On the other hand, if the integral domain of (8) has an appreciable domain of x_1 satisfying

$$\gamma_A/s\lambda_A \gtrsim 1/L_A, \tag{11}$$

the integral (8) becomes much larger than unity. That is, EAS initiating at rather large depths contribute to (8) much more than those initiating at the top of the atmosphere and the fluctuation cannot be neglected.

In general Λ_A is given by the relation derived by Miyake:[2]

$$\Lambda_A = L_A(1+r_A), \tag{12}$$

where r_A is approximately represented by the ratio of shower rate initiating at the top to that initiating at larger depths, and gives a measure of the magnitude of the fluctuation. The larger the fluctuation, the smaller is r_A. As shown above, if the fluctuation is small, Eq. (12) can be rewritten as Eq. (10).

As one can see from the integral (8), the magnitude of r_A depends essentially on λ_A, L_A, γ_A and the domain of x_1 satisfying the condition (11). That is (i) the smaller λ_A, (ii) the larger L_A, (iii) for a given size, the larger x', and (iv) independent of the shape of shower curves, the steeper the primary energy spectrum, the smaller is r_A. And if λ_A is an increasing function of E_0, the smaller the size, the smaller is r_A for given x' and γ_A. This is the case in our model.

It is necessary to remark on a relation between L_A and λ_A in connection with the condition (9) or (11). λ_A depends sensitively on the interaction mean free path, L_s, of the secondary nuclear particles which produce the main contribution to the shower size. Let their average L_s over the energies concerned be $\bar{L}_s$. λ_A is roughly proportional to $\bar{L}_s$, since the attenuation length of these nuclear particles is proportional to $\bar{L}_s$. If $L_A \simeq \bar{L}_s$, λ_A varies proportionally to L_A.* In this case, therefore, the condition (9) or (11) is roughly kept irrespective of the magnitude of L_A. This will happen in the case where the primary particle is a proton, unless its interaction mean free path varies drastically with energy. On the other hand, if L_A is small as compared with $\bar{L}_s$, as for example for a heavy primary particle, the fluctuation is small.

* This relation holds roughly as far as L_s is larger than the radiation length. In the limit of $\bar{L}_s \to 0$, λ_A tends to the absorption length of a pure electromagnetic shower.

§ 3. Calculated results. I

—— EAS initiated by protons ——

In this section we give the results for EAS initiated by protons. The calculations (see the Appendix) were made for $E_0=10^6$, 10^7, 10^8 and $10^9\,Mc^2$, and for the following combinations of η and L_s:

$$a)\quad \eta=1,\quad L_s=90,$$
$$b)\quad \eta=0.7,\quad L_s=90,$$
$$c)\quad \eta=0.3,\quad L_s=90,$$
$$c')\quad \eta=0.3,\quad L_s=70.$$

Moreover to obtain the characteristics of EAS initiated by heavy primaries, the averaged EAS over the depth of the first interaction was also calculated for the case *b* (see § 4). The method is the same as the one given in I.

As energy spectrum of the primary protons, three forms were chosen:

$$\text{I.}\quad \gamma_p=2.0 \qquad \text{for all energies,}$$

$$\text{II.}\quad \gamma_p=\begin{cases}1.8 & \text{for } E_0<10^7,\\ 2.1 & \text{for } 10^7<E_0<5\cdot10^8,\\ \infty & \text{for } E_0>5\cdot10^8,\end{cases}$$

$$\text{III.}\quad \gamma_p=\begin{cases}1.6 & \text{for } E_0<10^7,\\ 2.1 & \text{for } E_0>10^7.\end{cases}$$

At the present time it is not clear whether there is or is not a high-energy cut-off in the primary energy. The results derived with the spectrum II have some features, as will be seen, which may provide a possible method of testing for the existence of a cut-off. The knees in the spectra II and III have no special meaning. In the actual spectrum γ_p would vary gradually and continuously as a function of E_0. The knees, however, make it easy to see how the fluctuation is related to the primary spectrum.

3·1 *Dependence of the shower curves on η, L_s and E_0*

The shower curves initiated at the top of the atmosphere are shown in Figs. 1a and 1b. The maximum sizes, their depths and attenuation lengths, λ_p's, p meaning a proton primary, are given in Table I. λ_p's were evaluated from the numbers of electrons at depths of 24 and 28 radiation lengths. The maximum sizes coincide with each other within a factor of less than 2, irrespective of the values of η and L_s. As emphasized in I the η-dependence of λ_p is significant. Furthermore λ_p is sensitively dependent on L_s. These two facts show that we can get similar shower curves, using two alternative combinations of η and L_s, one

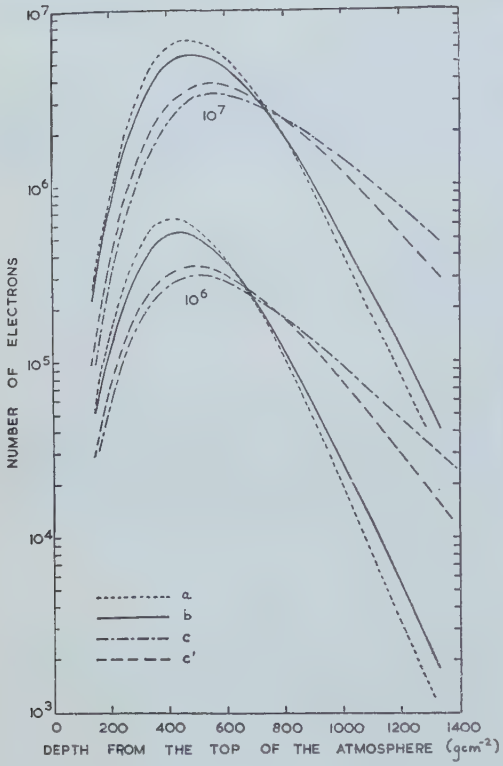

Fig. 1a. Altitude variations of electrons in EAS initiated at the top of the atomosphere for $E_0=10^6$ and 10^7 Mc^2.

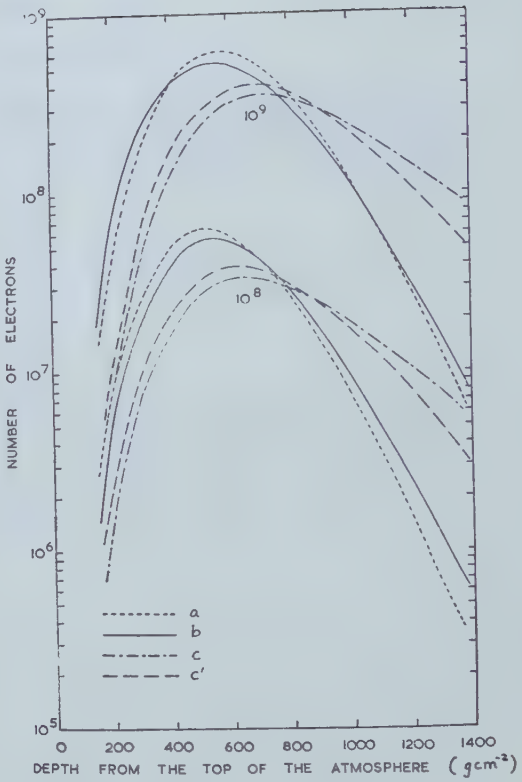

Fig. 1b. The same as Fig. 1a for $E_0=10^8$ and 10^9 Mc^2.

Table 1

Case	E_0	Max. size	Depth of max. size	λ_p
a	10^6	$6.4\cdot10^5$	420	107
b		$5.2\cdot10^5$	430	127
c		$3.1\cdot10^5$	500	314
c'		$3.5\cdot10^5$	480	233
a	10^7	$6.4\cdot10^6$	470	123
b		$5.4\cdot10^6$	490	141
c		$3.2\cdot10^6$	570	334
c'		$3.7\cdot10^6$	560	261
a	10^8	$6.3\cdot10^7$	530	136
b		$5.5\cdot10^7$	540	157
c		$3.3\cdot10^7$	650	395
c'		$3.9\cdot10^7$	640	290
a	10^9	$6.1\cdot10^8$	590	159
b		$5.2\cdot10^8$	570	180
c		$3.4\cdot10^8$	720	482
c'		$4.0\cdot10^8$	700	337

being large η and large L_s and the other small η and small L_s.* It should be noted, however, that the magnitude of the fluctuation is large in the former case as compared with that in the latter case, because L_p may be correspondingly larger in the former. This point we shall discuss again in a later section. λ_p is an increasing function of E_0. This would be a general tendency irrespective of the model of the high-energy interaction, unless any peculiar interaction does happen at higher energies.

3·2 *Features of EAS initiated at large depths*

Nuclear active unstable particles can make more nuclear interaction at lower altitude than they do at higher altitude, because the geometrical distances from

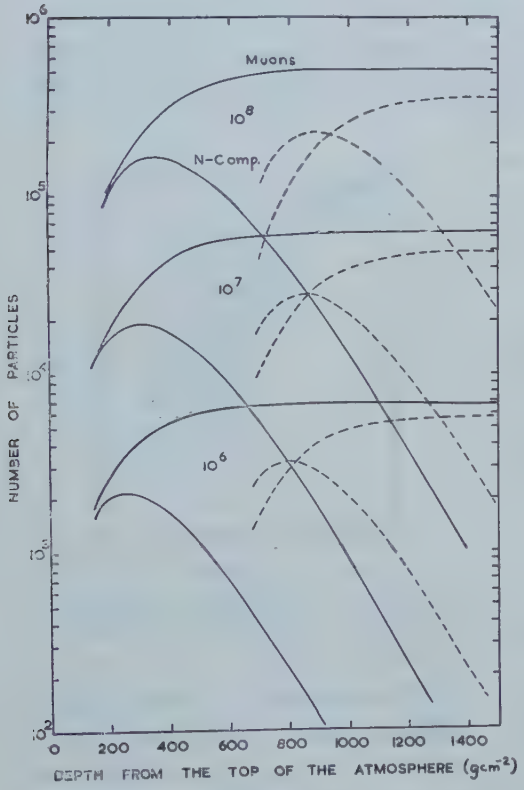

Fig. 2a. Altitude variations of nucleonic components and muons ($E \geq 10\ Mc^2$) in EAS initiated at the top of the atmosphere (solid curves) and at the depth of 540 gcm^{-2} (broken curves) for the case *b*.

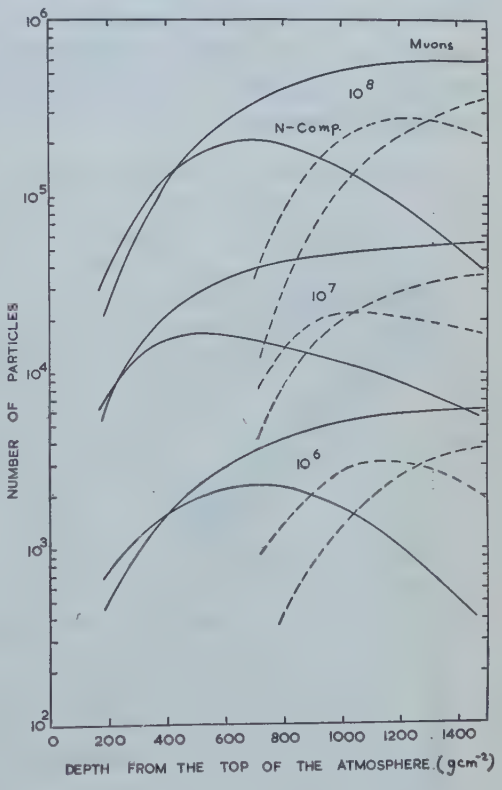

Fig. 2b. The same as Fig. 2a for the case *c*.

* For example, similar shower curves to *c'* could be obtained by using a combination of $L_s=90$ and $\eta \simeq 0.5$.

their production- to interaction-points become shorter at lower altitude due to the higher density of the air. EAS initiated at larger depth, therefore, involve more nuclear active particles and less muons than do EAS initiated near the top of the atmosphere, for a given atmospheric thickness through which they develop. In the calculation this effect was taken into account as a modification of the decay constant of charged pions as shown in the Appendix.

Since electrons are mainly produced by neutral pions of which parental nuclear particles have much higher energies than the decay constant, the effect does not much affect the shapes of shower curves irrespective of initiating depths. However, numbers of nuclear particles and muons are affected, as they are mainly occupied by particles of energies lower than the decay constant. Altitude variations of the nucleonic component and muons in EAS initiated at the top of the atmosphere and at the depth of 540 gcm^{-2} are shown in Figs. 2a and 2b for the cases *b* and *c*, respectively. The numbers of electrons in both EAS as a function of depth below the points of the first interaction coincided with each other within a difference of less than 10 percent.

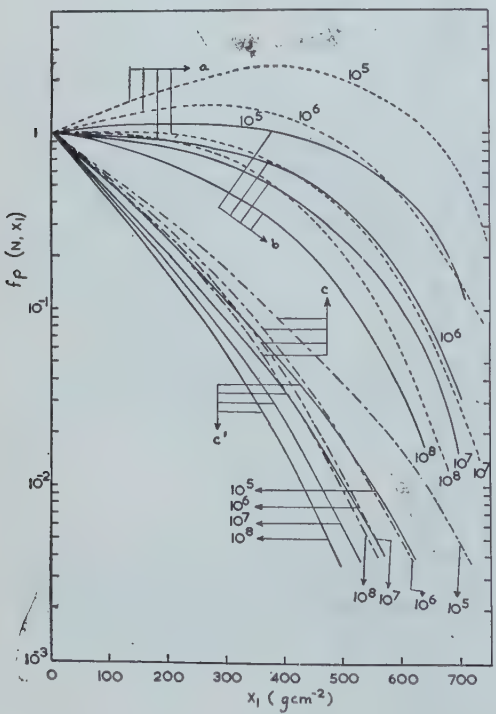

Fig. 3a. Relative shower rates for a given size N at sea level as a function of the initiating depth x_1 for the spectrum I. Sizes are attached to the curves. $f_\rho(N, x_1)$ is normalized to 1 at $x_1=0$.

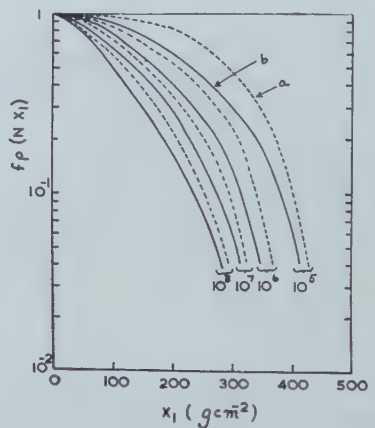

Fig. 3b. The same as Fig. 3a. Mountain altitude (700 gcm^{-2}).

If the fluctuation in the depth is negligibly small, this effect is unimportant because most EAS are initiated at the top of the atmosphere. However, the greater the magnitude in the possible fluctuations in the depth of the first interaction, the greater is the magnitude of the resulting fluctuations in the numbers of nuclear active particles and of

muons. An important fact one can see from the figures is that for a given depth x' the number of muons fluctuates in the opposite sense to the other components. Therefore the ratio between numbers of muons and of electrons in each EAS gives an important measure of the fluctuation experimentally.

Table IIa. The case a.

Size	x_1	Sea Level				Mountain (700 gcm^{-2})			
		E_0	$f_p(N, x_1)$			E_0	$f_p(N, x_1)$		
			I	II	III		I	II	III
10^5	0	$4.2\cdot10^6$	1	1	1	$5.1\cdot10^5$	1	1	1
	100	$2.2\cdot10^6$	1.4	1.2	1.1	$3.0\cdot10^5$	0.96	0.91	0.82
	200	$1.1\cdot10^6$	1.8	1.3	1.1	$1.9\cdot10^5$	0.84	0.70	0.58
	300	$6.0\cdot10^5$	2.1	1.5	1.0	$1.5\cdot10^5$	0.49	0.39	0.30
	400	$3.8\cdot10^5$	2.3	1.3	0.80	$2.1\cdot10^5$	0.11	0.086	0.071
	500	$2.2\cdot10^5$	2.0	1.1	0.67				
	600	$1.5\cdot10^5$	1.4	0.71	0.35				
	700	$3.2\cdot10^5$	0.60	0.31	0.13				
10^6	0	$2.5\cdot10^7$	1	1	1	$3.6\cdot10^6$	1	1	1
	100	$1.4\cdot10^7$	1.2	1.2	1.2	$1.8\cdot10^6$	0.83	0.75	0.69
	200	$7.2\cdot10^6$	1.4	1.3	1.3	$1.6\cdot10^6$	0.53	0.46	0.40
	300	$4.5\cdot10^6$	1.4	1.3	1.1	$1.8\cdot10^6$	0.24	0.18	0.15
	400	$2.7\cdot10^6$	1.2	1.0	0.80	$2.7\cdot10^6$	0.033	0.020	0.015
	500	$1.9\cdot10^6$	0.86	0.80	0.50				
	600	$1.5\cdot10^6$	0.44	0.33	0.22				
	700	$2.0\cdot10^6$	0.14	0.085	0.063				
10^7	0	$1.7\cdot10^8$	1	1	1	$2.7\cdot10^7$	1	1	1
	100	$8.6\cdot10^7$	1.0	1.1	1.1	$1.8\cdot10^7$	0.69	0.71	0.71
	200	$5.0\cdot10^7$	1.0	1.2	1.2	$1.6\cdot10^7$	0.43	0.56	0.56
	300	$2.9\cdot10^7$	0.87	1.1	1.1	$1.9\cdot10^7$	0.090	0.10	0.10
	400	$2.1\cdot10^7$	0.64	0.80	0.80	$3.1\cdot10^7$	0.0099	0.015	0.015
	500	$1.6\cdot10^7$	0.38	0.43	0.43				
	600	$1.6\cdot10^7$	0.14	0.18	0.18				
	700	$2.1\cdot10^7$	0.028	0.030	0.030				
10^8	0	$1.0\cdot10^9$	1		1	$2.1\cdot10^8$	1	1	1
	100	$5.8\cdot10^8$	0.97		1.0	$1.6\cdot10^8$	0.60	0.61	0.61
	200	$3.5\cdot10^8$	0.90		1.0	$1.6\cdot10^8$	0.22	0.25	0.25
	300	$2.5\cdot10^8$	0.72		0.82	$2.2\cdot10^8$	0.041	0.043	0.043
	400	$1.8\cdot10^8$	0.46		0.49	$4.4\cdot10^8$	0.0028	0.0035	0.0035
	500	$1.5\cdot10^8$	0.22		0.26				
	600	$1.6\cdot10^8$	0.057		0.056				
	700	$2.5\cdot10^8$	0.0056		0.0056				

Table IIb. The case b.

Size	x_1	Sea level				Mountain (700 gcm^{-2})			
		E_0	$f_p(N, x_1)$			E_0	$f_p(N, x_1)$		
			I	II	III		I	II	III
10^5	0	$3.3\cdot10^6$	1	1	1	$5.0\cdot10^5$	1	1	1
	100	$1.9\cdot10^6$	1.1	0.97	0.87	$3.2\cdot10^5$	0.89	0.82	0.74
	200	$1.1\cdot10^6$	1.1	0.89	0.72	$2.3\cdot10^5$	0.61	0.51	0.44
	300	$6.0\cdot10^5$	1.1	0.82	0.58	$1.9\cdot10^5$	0.31	0.25	0.21
	400	$3.7\cdot10^5$	0.98	0.70	0.45	$2.5\cdot10^5$	0.063	0.050	0.040
	500	$2.9\cdot10^5$	0.79	0.50	0.30				
	600	$2.3\cdot10^5$	0.47	0.27	0.16				
	700	$2.4\cdot10^5$	0.14	0.10	0.051				
10^6	0	$2.0\cdot10^7$	1	1	1	$3.6\cdot10^6$	1	1	1
	100	$1.2\cdot10^7$	0.96	1.0	1.0	$2.4\cdot10^6$	0.73	0.64	0.63
	200	$7.0\cdot10^6$	0.91	0.92	0.84	$2.0\cdot10^6$	0.39	0.35	0.31
	300	$4.2\cdot10^6$	0.80	0.75	0.61	$1.9\cdot10^6$	0.14	0.12	0.11
	400	$2.9\cdot10^6$	0.65	0.54	0.42	$3.3\cdot10^6$	0.018	0.013	0.010
	500	$2.2\cdot10^6$	0.40	0.29	0.23				
	600	$2.4\cdot10^6$	0.16	0.13	0.091				
	700	$3.0\cdot10^6$	0.034	0.029	0.025				
10^7	0	$1.3\cdot10^8$	1	1	1	$2.7\cdot10^7$	1	1	1
	100	$7.8\cdot10^7$	0.91	0.95	0.95	$2.0\cdot10^7$	0.61	0.63	0.63
	200	$4.5\cdot10^7$	0.76	0.83	0.83	$1.8\cdot10^7$	0.31	0.32	0.32
	300	$3.2\cdot10^7$	0.57	0.66	0.66	$2.2\cdot10^7$	0.065	0.068	0.068
	400	$2.4\cdot10^7$	0.37	0.44	0.44	$4.6\cdot10^7$	0.0051	0.0056	0.0056
	500	$1.8\cdot10^7$	0.19	0.23	0.23				
	600	$2.3\cdot10^7$	0.057	0.060	0.060				
	700	$3.7\cdot10^7$	0.010	0.011	0.011				
10^8	0	$1.0\cdot10^9$	1		1	$2.3\cdot10^8$	1	1	1
	100	$6.1\cdot10^8$	0.86		0.90	$1.9\cdot10^8$	0.50	0.52	0.52
	200	$3.8\cdot10^8$	0.65		0.70	$1.9\cdot10^8$	0.17	0.18	0.18
	300	$2.6\cdot10^8$	0.49		0.57	$2.5\cdot10^8$	0.029	0.030	0.030
	400	$2.0\cdot10^8$	0.30		0.41	$4.8\cdot10^8$	0.0026	0.0030	0.0030
	500	$1.9\cdot10^8$	0.13		0.18				
	600	$2.6\cdot18^8$	0.030		0.037				
	700	$4.0\cdot10^8$	0.0039		0.0049				

3·3 *Dependence of the fluctuation on λ_p, γ_p and L_p*

Fig. 3a shows the differential shower rate $f_p(N, x_1)$ defined by Eq. (6), as a function of an initiating depth x_1, for the cases a, b and c with $L_p=90$ gcm^{-2}. The cases a and b satisfy the condition (11) and c the condition (9) irrespective of the choice of primary spectra I, II or III. To see the dependence of the

fluctuation of L_p in connection with λ_p, in Fig. 3a we also show the case c' with $L_p=70$ gcm^{-2}. As compared with the case c, λ_p is shortened by a factor of about 90/70 as given in Table I, therefore the condition (9) is still satisfied in this case.

Fig. 4a shows the ratio q of the shower rate $F_p(N)$ given by Eq. (5) to the rate $\mathcal{J}_p(N)$ expected on the assumption that all EAS are initiated at the top of the atomosphere. q is approximately given by $(1+r_p)/r_p$ with r_p defined by Eq. (12) as $r_p \cong \mathcal{J}_p(N)/(F_p(N)-\mathcal{J}_p(N))$. One can easily see how the magnitude of the fluctuation depends upon γ_p. In the cases a and b, as $r_p \ll 1$, q depends sensitively on γ_p. The minimum values of q for the spectra II and III at size about $5\cdot10^5$ are due to the knees of their spectra at $E_0=10^7\ Mc^2$. On the contrary, in the cases c and c' such a dependence on γ_p does not appear as can be seen from the condition (9).

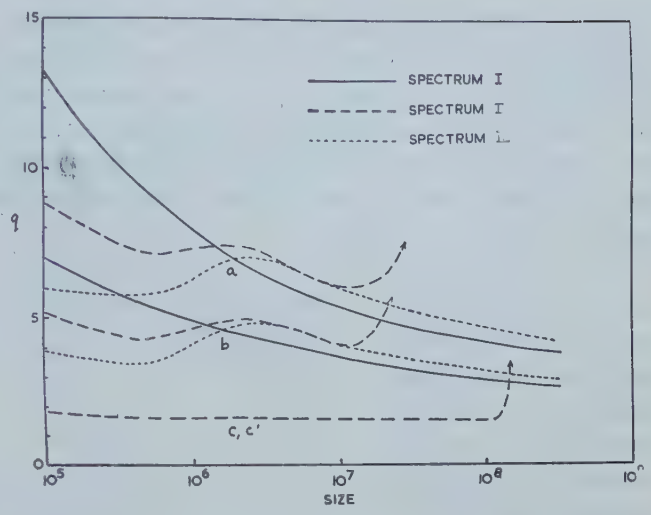

Fig. 4a. The ratio q of the shower rate $F_p(N)$ to the rate expected on the assumption that all EAS are initiated at the top of the atmosphere as a function of size. Sea level.

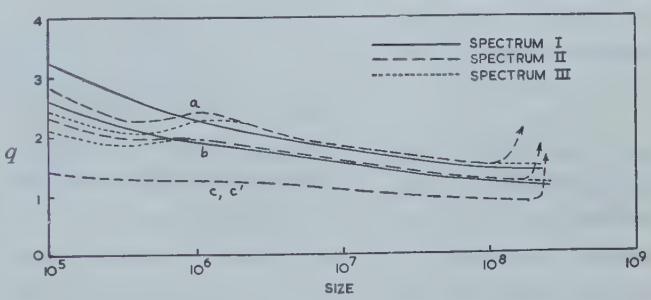

Fig. 4b. The same as Fig. 4a. Mountain altitude (700 gcm^{-2}).

From these figures we see that for the cases c and c' the fluctuation is quite small and shower sizes correspond well to their primary energies. For the spectrum II, therefore, q shown in Fig. 4a increases quite sharply at the size corresponding to the cutoff energy as compared with the cases a and b, where q starts at a rather smaller size to increase gradually.

For the cases a and b, however, the shower size gives information about only possible upper and lower values of primary energies as shown in Tables 2a and 2b where a primary energy corresponding to an initiating depth x_1 is given with $f_p(N, x_1)$. At sea level a large proportion of the observed EAS of size 10^5 are initiated at large depths. For the case a with the spectrum I, the domain of x_1

giving rise $f_\gamma(N, x_1)$ greater than one half of $f_\gamma(N, 0)$ is from $x_1=0$ to 700 gcm^{-2} and the corresponding range of E_0 is from $4\cdot10^6$ to $1.5\cdot10^5$ Mc^2, the difference being a factor of more than 20. In other words, for 10^5 size more than half of the showers have been initiated at depths greater than 300 gcm^{-2} and more than 1/3 at depths greater than 500 gcm^{-2}. The fluctuation becomes smaller at larger size. However, even for size 10^8 the corresponding figures are $x_1=0\sim400$ gcm^{-2} and $E_0=1.0\cdot10^9\sim1.8\cdot10^8$ Mc^2.

At mountain altitude the situation is rather different from that at sea level. Since EAS observed there are almost near or at the maximum development, the proportion of EAS satisfying the condition (9) is much less than that at sea level. This fact can be seen in Tables IIa and IIb and Figs. 3b and 4b.

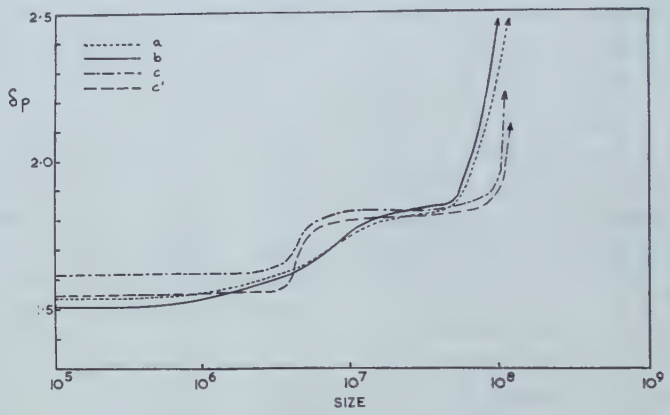

Fig. 5a. The exponent of the power law size spectra for the spectrum II as a function of size. Sea level.

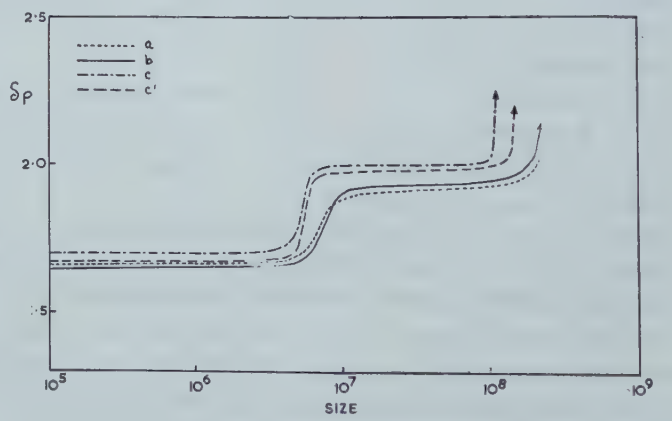

Fig. 5b. The same as Fig. 5a. Mountain altitude (700 gcm^{-2}).

It should be remembered that the results shown above were derived for EAS developing vertically. If we take into account inclined EAS, the fluctuation might become larger.

3·4 *The size spectrum and the attenuation length*

As has been emphasized, the shower curves are essentially related to both the fluctuation and the characteristics of the high-energy interaction. On the other hand experimental quantities connected with the shower curves are the size spectrum and the attenuation length of shower rate Λ_γ. The results are shown in Figs. 5a and 5b for the spectrum II and Fig. 6.

The variation of δ_γ, the exponent of the power-law size spectrum, due to the

knees or the cut-off energy in the spectra, appears more clearly in the cases c and c' than in a and b. Except for this point* δ_p has a value insensitive to the shape of the shower curves. δ_p is slightly larger at mountain altitude than at sea level.

The relation between λ_p and Λ_p is well approximated by Eq. (10) for the cases c and c'. One can see in Fig. 6 that for c and c' Λ_p varies clearly according to the change of γ_p. On the contrary the variation of Λ_p is well smeared out for the cases a and b even if there is a cutoff in the primary spectrum. Λ_p is lengthened in these cases as compared with a value derived from Eq. (10) assuming that all EAS are initiated at the top of the atmosphere. This means that if λ_p is evaluated by Eq. (10) from experimental values of δ_p and Λ_p in spite of the fluctuation being large, then it is overestimated.

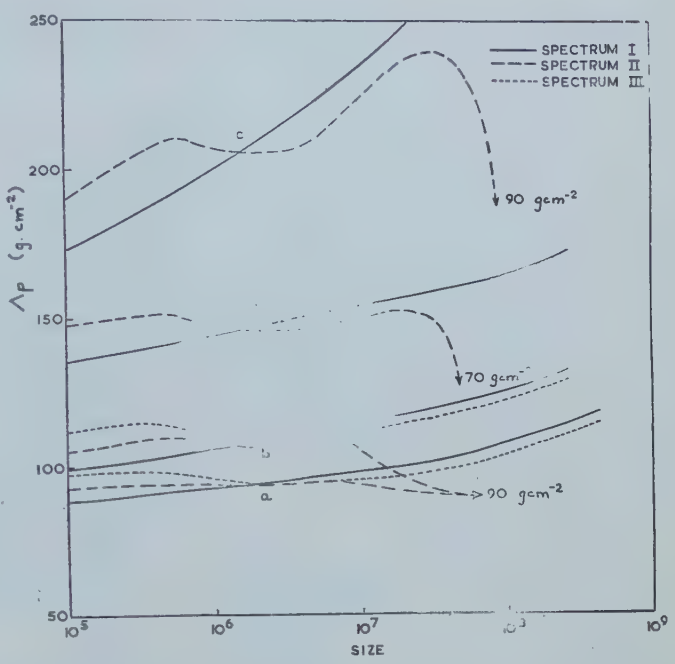

Fig. 6. The attenuation lengths of shower rates near sea level as a function of size.

We shall return to discussion of the size spectrum in § 4 in connection with EAS initiated by particles heavier than protons.

3·5 *The nucleonic component and muons*

The number of nucleonic component as a function of size are shown in Figs. 7a and 7b. Some irregularity in the curves seems to be accidental owing to the assumed energy distribution—equipartition—of secondary particles in our model.** The curves attached "top" show the numbers involved in EAS initiated at the top of the atmosphere so that their shower sizes correspond one to one to their primary energies. The "average" numbers, $\bar{n}_p(N)$, were derived by the expression

* The actual energy spectrum would vary gradually and continuously with energy. The difference of δ_p between the cases of large and of small fluctuations, therefore, would appear only near a cut-off if it existed.

** This irregularity is so large in the case a that we cannot say anything about the nucleonic component and muons.

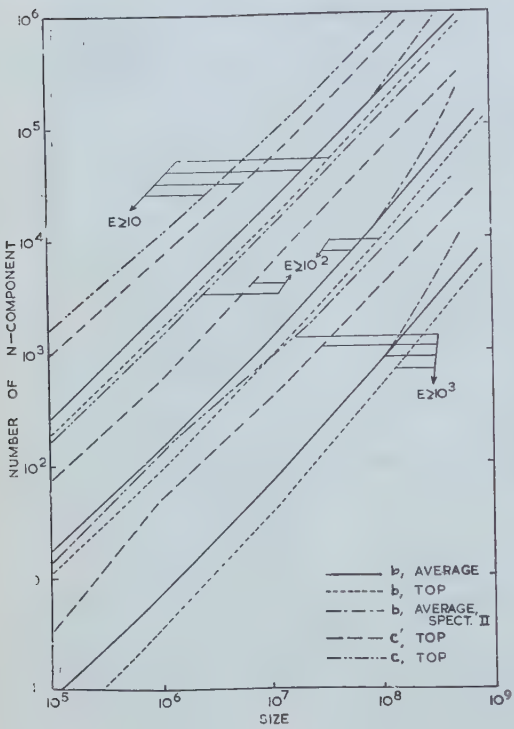

Fig. 7a. Numbers of nucleonic components having energy greater than 10, 10^2 and 10^3 Mc^2, respectively; as a function of size. Sea level. "Top" means the numbers in EAS initiated at the top of the atmosphere.

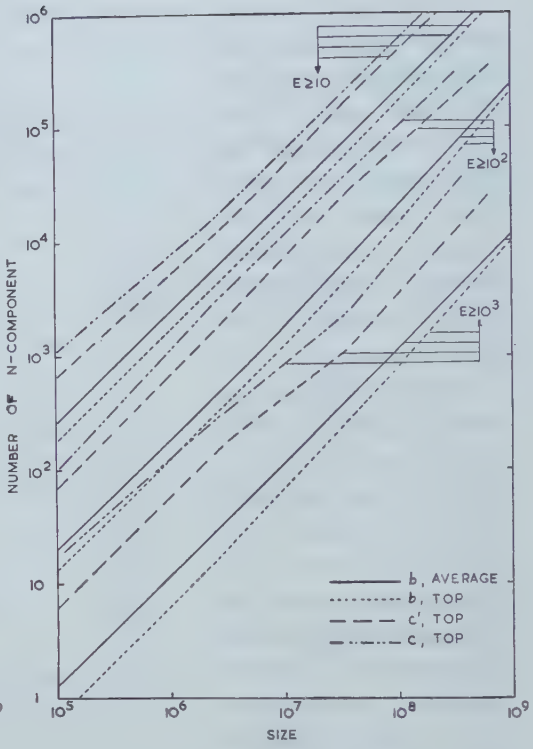

Fig. 7b. The same as Fig. 7a. Mountain altitude (700 gcm^{-2})

$$\bar{n}_p(N) = \int n_p(N, x_1) f_p(N, x_1)\, dx_1 / \int f_p(N, x_1)\, dx_1, \qquad (13)$$

where $n_p(N, x_1)$ is the number involved in EAS initiated at x_1. Then they were averaged over the depth of the first interaction and over the primary energy spectrum as can be seen from the definition of $f_p(N, x_1)$, Eq. (6).

Let us consider the case b at sea level. The average numbers are larger by a factor of 1.3 to 2 than those in EAS initiated at the top of the atmosphere. This factor increases with increasing energy of the nucleonic component, because the higher the energy the more quickly the nucleonic component attenuates. For the spectrum II the average number increases abruptly at size of about 10^8 due to the cut-off. Except for this point the average numbers are not much affected by the shapes of the primary spectra, because the numbers of electrons and of nucleonic component fluctuate in the same way.

For the cases c and c' the average numbers almost coincided with those in EAS initiated at the top and the abrupt increase of the numbers for the spectrum

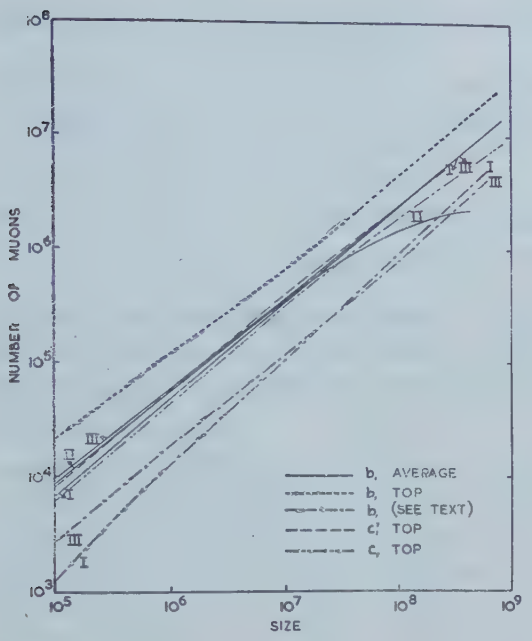

Fig. 8a. Numbers of muons as a function of size at sea level. —·—·—: Numbers in EAS of which shower rates are one halves of those of EAS initiated at the top of the atmosphere, for the spectra I and III.

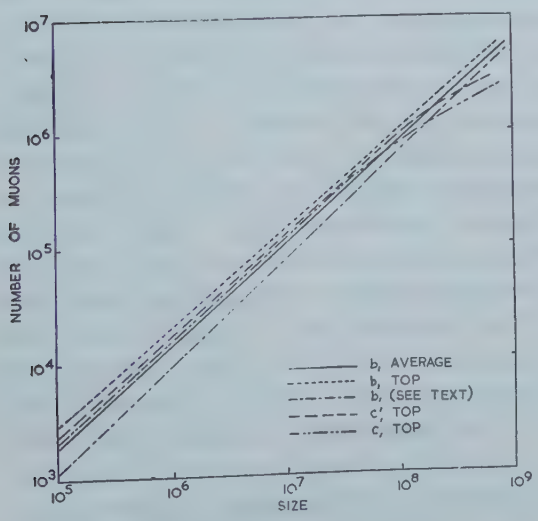

Fig. 8b. The same as Fig. 8a. Mountain altitude (700 gcm^{-2}).

II did not appear. Curves for these average numbers are not, therefore, included in the figures.

Except for the irregularity mentioned above, it could be stated that the numbers of the nucleonic component both at sea level and mountain altitude are linearly proportional to size, independently of η and L_s in our model.

Similar figures for the muons of which parental charged pions have energies greater than 10 Mc^2 are shown in Figs. 8a and 8b. For the case *b*, their dependence on the primary spectra appears slightly as compared with the numbers of the nucleonic component, because the numbers of electrons and of muons fluctuate in an opposite sense. The dotted-broken curves show the numbers in EAS whose frequency, $f_p(N, x_1)$, is one half of that of EAS initiated at the top. As can be seen from Table IIb, initiating depths of such EAS observed at sea level are 400～600 gcm^{-2} for size 10^5 and about 300 gcm^{-2} for size 10^8, respectively, depending on the primary spectra. It should, therefore, be remembered that EAS initiated between the top and the depth of several hundreds gcm^{-2} contribute to the shower rate with *nearly equal frequency* and furthermore the ratio between the numbers of muons and electrons in such EAS fluctuates by as much as a factor of 5 to 10. At mountain altitude, though the fluctuation of the ratio becomes small, it still gives the

same order as the one due to heavy primary particles, as will be shown in the next section.

For all the cases b, c and c', the average numbers of muons are proportional to N^{α}, where $\alpha=0.80$ at sea level and 0.85 at mountain altitude, N being the size.

§ 4. Calculated results. II

——*EAS initiated by heavy particles*——

As will be discussed in § 5, the case b gives a good agreement with the experiment so that we shall confine ourselves mainly to this case and additionally to the case c' which is in some sense related to the results (iii′) and (iv′) in § 5.

4·1 *Fluctuation due to heavy particles and the model of their first interaction*

Since the interaction mean free path of heavy particles is short enough to satisfy the condition (9), the fluctuation in the depth of the first interaction is relatively unimportant (see Fig. 10). An important fluctuation due to heavy particles arises from their dependence on mass number A and from how they break up at the first interaction and how many nucleons composing them participate in the interaction.

From the analysis[10] of a jet shower induced by a heavy nucleus, it is shown that nucleons composing the heavy nucleus, which do not participate in the collision with an emulsion nucleus come out individually or as fragments by the evaporation process so that they have almost the same energy as they have before the collision. A similar collision may occur in the collision of a heavy particle with an air nucleus. Nucleons and fragments emerging from the collision make further interactions after penetrating through the atmosphere. An EAS initiated by a heavy nucleus is, thus, a superposition of EAS initiated by these particles and an EAS initiated at the first interaction. The number of nucleons composing a heavy nucleus which participate in the first interaction may fluctuate even for a given A. An EAS initiated by an A-particle would have, therefore, a characteristic intermediate between the following two extreme cases. One (hereafter referred to as an E-shower) is an EAS for which at the first collision all A nucleons interact in the same way and the other (referred to as an F-shower) is a fictitious EAS for which at the first collision all A nucleons come out without any energy loss and make their real interaction individually and with *the interaction mean free path of nucleons*. An E-shower initiated by an A-particle of total energy E_T can be replaced by a superposition of EAS initiated by A protons of energy E_T/A. In an F-shower, the depths at which A nucleons make their real interactions fluctuate from nucleon to nucleon. Since to take into account this fluctuation would be rather complex, we shall approximate an F-shower by a superposition of EAS by A protons of energy E_T/A, *averaged over depths of their first real interactions*. This approximation would not be too far from the actual one for a

large A. The calculation for an F-shower is just the same as the one given in I. It should be noted that the fluctuation in the depths of the real interaction is quite different from the one hitherto considered and has no connection with γ_A and L_A, and also that according to the definition of E- and F-showers the calculated results for both showers would give the upper and lower limits of the fluctuation due to A-particles.

As an example, the altitude variations of the numbers of electrons in E- and F-showers, respectively, initiated at the top of the atmosphere by an α-particle are shown in Fig. 9. F-showers were only calculated for the case b.

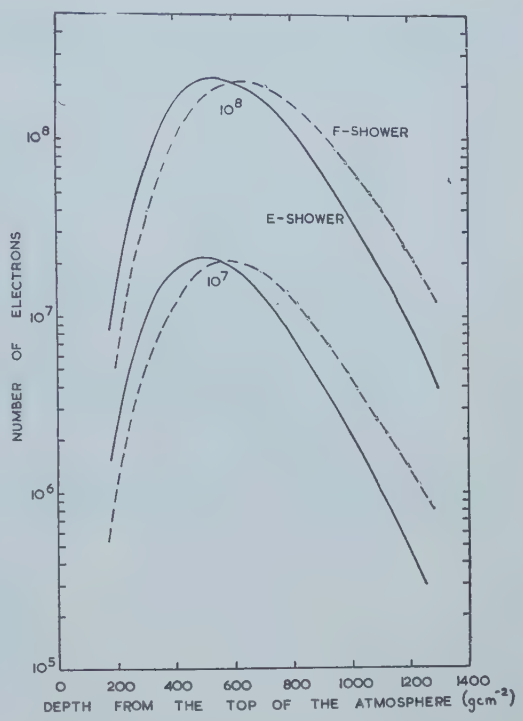

Fig. 9. Altitude variations of electrons in EAS initiated at the top of the atmosphere by α-particles of energy 10^7 and 10^8 Mc^2 per nucleon, respectively.

4·2 *The size spectrum and the relative shower rates due to various groups of primary nuclei*

To calculate quantities concerned with EAS initiated by heavy particles, it is assumed that all groups of primary nuclei have the same spectrum of energy per nucleon as the proton spectrum and that the relative abundance of those groups at the top of the atmosphere is the same as at energies below 10 Gev[11] as listed in Table 3. The interaction mean free paths of the representative nuclei given in Table 3 were evaluated with the empirical formula given by Peters.[10] Precise values of L_A's are not necessary, because the fluctuation in the depth of the first interaction is small.

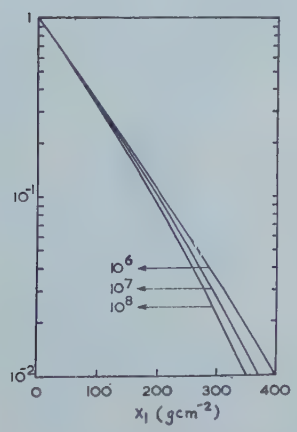

Fig. 10. The same as Fig. 3a for E-showers initiated by α-particles for the spectrum I. Sea level.

The calculation was made with the spectrum II in order to examine to what extent one can get a smooth size spectrum.

The size spectra for the case b are shown in Fig. 11a for E-showers and in Fig. 11b for F-showers. To indicate the intensity of inclined EAS, the size spectra at the depth of 1300 gcm^{-2} are also shown.

Table III

group		A	L_A	rel. abund.
p	p	1		100
α	He	4	45	10
L	Li, Be, B	9	25	0.18
M	C, N, O, Ne	16	2[illegible]	0.64
H	Mg, Si, S, A	30	21	0.10
Fe	Fe	56	16	0.04

The mass thickness of 1300 gcm^{-2} corresponds to the zenith angle of 38° at sea level. From both figures, one can see that up to the size of about $3 \cdot 10^9$ an appreciable change of the spectrum does not appear even if the primary spectrum has a cut-off at $5 \cdot 10^8$ Mc^2.

The case c' is similar to the case b, as shown in Fig. 11c. Though the size spectrum produced by each group of primaries drops rather suddenly at the sizes

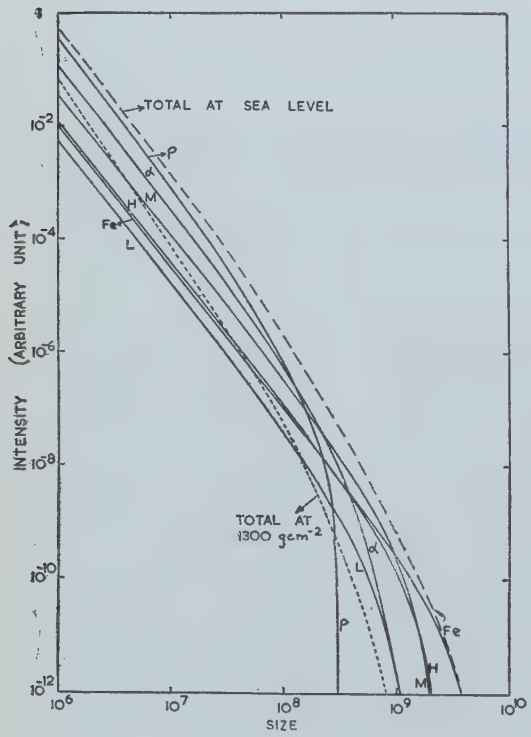

Fig. 11a. The differential size spectra at sea level of EAS initiated by various groups of primary nuclei. The broken and dotted curves are the total size spectra at sea level and at the depth of 1300 gcm^{-2} respectively. The case b for E-showers.

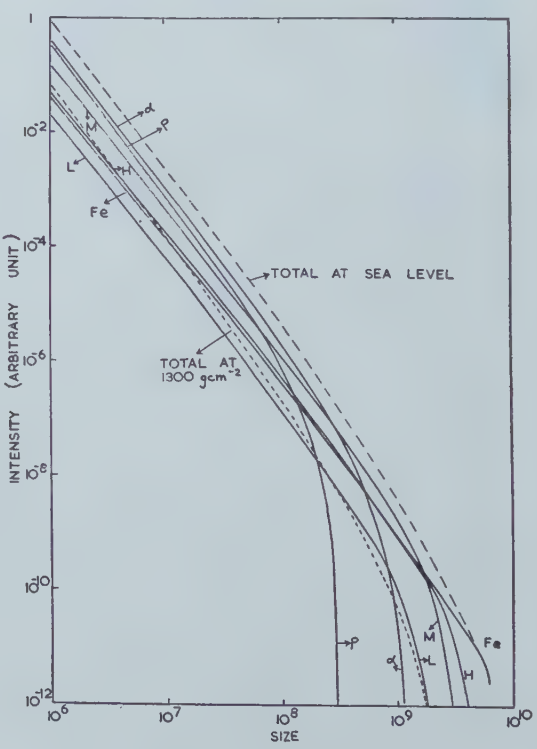

Fig. 11b. The same as Fig. 11a. The case b for F-showers.

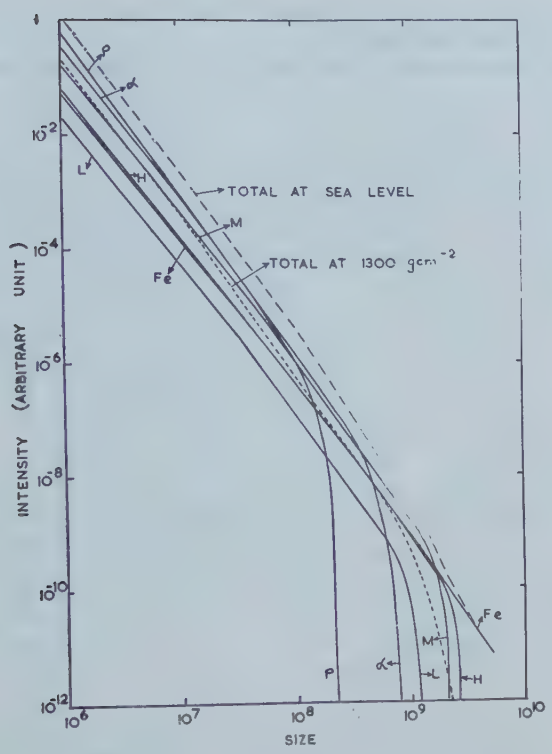

Fig. 11c. The same as Fig. 11a. The case c' for E-showers.

corresponding to the cut-off energy per nucleon, the total intensity is quite smooth up to the same size as the one in the case b. This is due to the fact that for a given primary energy EAS in the case c' have larger sizes than those in the case b at sea level.

A remarkable fact is that the shower rates of E-showers initiated by heavy nuclei relative to that of protons are significantly different at sea level and at mountain altitude for the case b as compared with c', as shown in Table IV. At mountain altitude, since even for the case b the fluctuation in the depth of the first interaction is small, the relative shower rates are quite similar to the case c' up to a size of about 10^8 above which the effect of the cut-off appears. On the other hand, since, for the case b, EAS initiated by protons at large depths give rise to a considerable contribution to the shower rate, the relative shower rate of protons to heavier nuclei increases by a factor of 2 for E-showers* in going from mountain altitude to sea level where the effect of the cut-off appears above a size of about $5 \cdot 10^7$ (see also Table IIb).

Since the relative shower rates at mountain altitude are insensitive to η and L_s, it is concluded that the shower rates due to heavy nuclei are greater than or equal to twice that due to protons at mountain altitude under the assumptions given at the beginning of this subsection. From this fact it is reasonably understood that EAS having double or more cores are observed frequently at mountain altitude.

One can also see from Table IV that with increasing size the relative rates due to nuclei heavier than protons increases gradually. In other words δ_A, the exponent

* For F-showers the shower rate due to protons seems small. This is due to the fact that the fluctuation in the first interaction of protons similar to the one discussed in § 4·1 is not taken into account. If this fluctuation is considered, the relative shower rate due to protons will increase and become closer to that in the case of E-showers.

Table IV. The relative shower rates due to various groups of primary nuclei at depths of 700, 1035 and 1300 gcm^{-2}.

case	*b*								*c'*			
	E-shower				*F*-shower				*E*-shower			
size	10^6	10^7	10^8	10^9	10^6	10^7	10^8	10^9	10^6	10^7	10^8	10^9
Mount.												
p	34	27	16		28	24	16		35	27	17	
α	32	34	29	26	35	36	29	27	32	35	29	
L	2	2	3	3	2	2	3	3	2	2	2	4
M	18	20	27	34	19	21	27	33	17	20	26	44
H	7	8	12	16	8	8	12	16	7	8	13	23
Fe	7	9	13	21	8	9	13	21	7	8	13	29
Sea level												
p	67	57	36		35	28	15		46	35	27	
α	21	24	34	0.5	39	41	40	1	30	33	29	
L	1	1	2	0.5	2	2	2	3	2	3	2	2
M	7	11	16	24	15	18	23	50	13	17	22	49
H	2	4	6	35	5	6	9	22	5	6	10	23
Fe	2	3	6	40	4	5	10	24	4	6	10	26
1300 gcm^{-2}												
p	75	69	55		39	33	15		47	38	5	
α	16	19	8	4	39	37	37	2	31	30	44	
L	1	1	1	1	2	2	3	1	2	2	3	
M	5	7	20	60	13	16	26	2	12	17	26	1
H	2	2	8	25	4	6	10	5	4	7	11	48
Fe	1	2	8	10	3	6	9	90	4	6	11	51

of the shower size spectrum decreases with increasing A, though being not appreciable (see Figs. 11a, b and c). This fact is attributed to an increasing γ_A with energy—the knee at $10^7\,Mc^2$ in the spectrum II.

4·3 *The attenuation length of the shower rate*

The calculated Λ_A's of EAS initiated by α-particles and iron nuclei are shown in Fig. 12. Even for the case *b*, since the fluctuation in the depth of the first interaction is small, Λ_A's are approximately represented by Eq. (10). For *E*-showers Λ_A's are shorter than that of EAS initiated by protons, due to the short L_A's. Λ_A's of *F*-showers are larger than those of *E*-showers as expected. An average Λ_α of α-particles which could contribute to the shower rate with a frequency comparable with protons, is about 95 gcm^{-2}. The difference between Λ_α and Λ_{Fe} is mainly due to the fact that for a given shower size, an α-particle has an energy per nucleon $\sim$14 times that of an iron nucleus. Λ corresponding to the one observed experimentally is the one averaged over the relative shower rates due to various

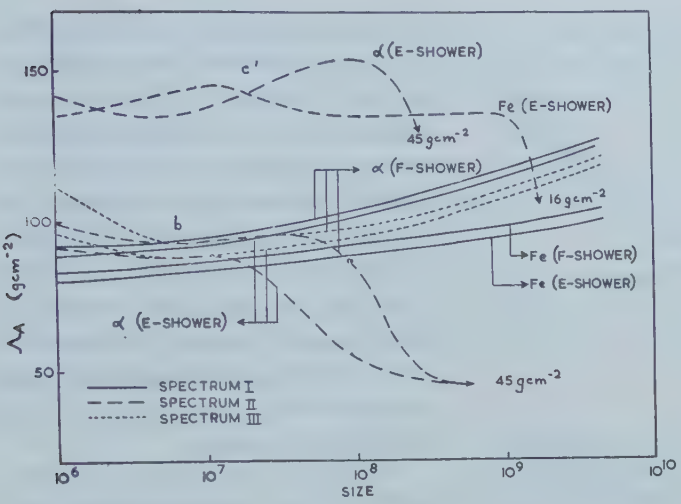

Fig. 12. The attenuation lengths of shower rates near sea level, for EAS initiated by α-particles and iron nuclei, as a function of size.

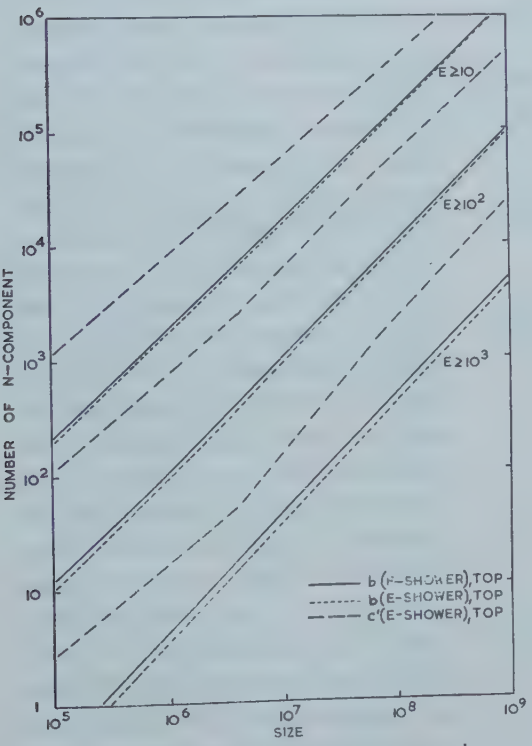

Fig. 13. Numbers of nucleonic components having energy greater than 10, 10^2 and 10^3 Mc^2, respectively, in EAS initiated at the top of the atmosphere by particles heavier than protons as a function of size. The dependence on the mass number almost disappears.

groups of primary nuclei. From Figs. 6 and 12, the calculated value is 100~110 gcm^{-2} for the case *b*. This value is maintained up to a size of 10^8 for the spectrum II and up to about 10^9 for the spectra I and III.

For the case *c'*, Λ_A's are not very different from Λ_p, because the fluctuation in the depth is small for all EAS including those due to protons. Hence Λ averaged over the relative rate is close to Λ_p.

4·4 *The nucleonic component and muons*

Generally, if the number of the nucleonic component or muons in EAS initiated by protons is represented by $n_p = kN^\alpha$ as a function of size, the corresponding number in EAS initiated by *A*-particles is roughly given by $n_A = kA(N/A)^\alpha = kA^{1-\alpha}N^\alpha$. If $\alpha = 1$, the *A*-dependence of n_A disappears. That is, for a given size an EAS initiated by even an iron nucleus involves almost the same n_A as an EAS by a proton. This is just the case of the nucleonic component in our model. The result is shown in Fig. 13. Since the fluctuation in the depth is small, the number of the nucleonic component involved in *E*-showers due to heavy particles almost coincides with the one involved in EAS initiated at the top of the atmosphere

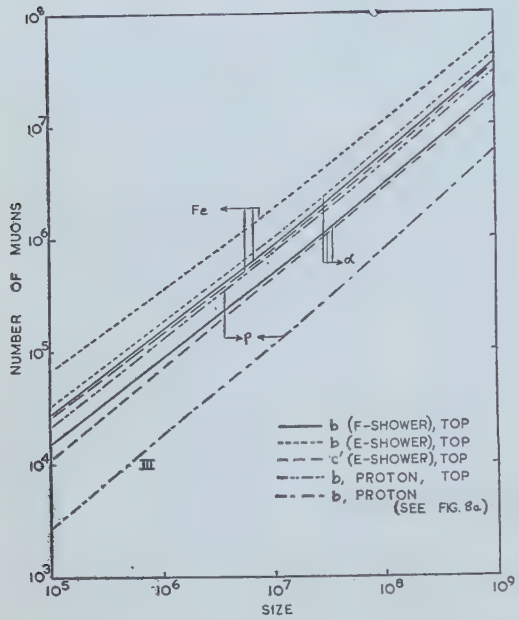

Fig. 14a. Numbers of muons in E- and F-showers initiated at the top of the atmosphere by α-particles and iron nuclei, as a function of size. For comparison the numbers in EAS due to protons for the case b are also given.
—·—·—: Numbers of muons in EAS due to protons of which shower rates are one halves of those in EAS initiated at the top of the atmosphere for the spectrum III. Sea level.

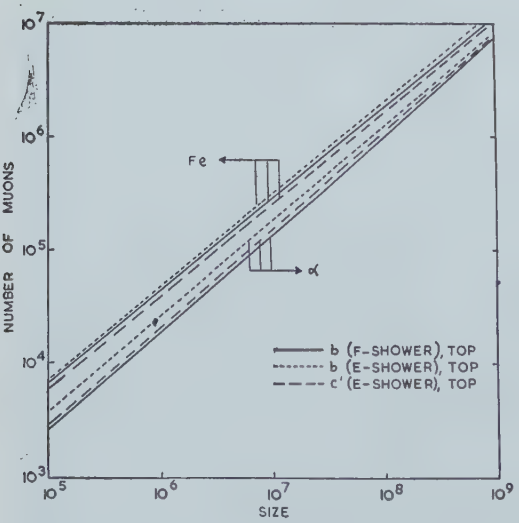

Fig. 14b. The same as Fig. 14a. Mountain altitude (700 gcm^{-2}).

by protons (compare with Fig. 7a). A slight difference arises between E-and F-showers. It should be noted that EAS initiated by protons contain a number of nuclear active particles of energy even greater than $10^3\,Mc^2$ comparable with those in EAS initiated by heavy particles. This fact would make it difficult to distinguish between EAS initiated by protons and those by heavy particles from observations of the nucleonic component.

For the muons the A-dependence of n_A appears as $\alpha \cong 0.8$. The results are shown in Figs. 14a and 14b where the results of EAS initiated by protons is also presented for the case b. The A-dependence and the difference between E- and F-showers is appreciable so that the ratio of the number of muons to that of electrons at sea level fluctuates by a factor of 3∼4 between F-showers initiated by α-particles and E-showers initiated by iron nuclei. Comparing the shower rate initiated by protons at large depths shown in Table 2b with Table 4, we see that this factor of 3∼4, which would be an upper value according to the definition of E- and F-showers, is however about 3 times smaller than the one due to protons. Taking into account the fluctuations due to protons and due to heavy nuclei, the ratio between numbers of muons and of electrons fluctuates by a factor of ten or more for more than 10 percent of EAS of a given size at sea level. Since the fluctuation in the depth of the first interac-

tion of protons increases with increasing atmospheric thickness, the larger the zenith angle the more the ratio fluctuate. From these considerations we see that for the case b the fluctuation in various measurable quantities at sea level are mainly attributable to the proton primaries.

At mountain altitude the fluctuation in the ratio of muons to electrons, the factor being not more than 5 is nearly equally contributed to by protons and by heavy nuclei as primaries.

For the case c', the fluctuation in the ratio is mainly due to heavy nuclei. It should be remembered that the larger the zenith angle the less the ratio fluctuates, because the shower rate due to heavy nuclei forms a smaller proportion of the total rate because of the short L_A's.

§ 5. Comparison with experiments and discussion

There are three experimental results which are directly related to the shower curves :

(i) The integral size spectrum is well represented by a power function of size N and the exponent δ at sea level is given by[5),12),13),15)]

$$\delta = \begin{cases} 1.5 \pm 0.2 & \text{for } N \lesssim 10^6, \\ 1.9 \pm 0.2 & \text{for } N \gtrsim 10^6. \end{cases}$$

Table V

Observer	Size	Λ (gcm^{-2})	Reference No.
Clark	$\gtrsim 10^5$	107 ± 11	(14)
Greisen	$10^5 \sim 10^8$	140 ± 10	(17)
Harwell group	$3 \cdot 10^5$	114	(4)
	$8 \cdot 10^5$	110	(4)
	$2 \cdot 10^6$	111	(4)
MIT group	$3 \cdot 10^5 \sim 3 \cdot 10^6$	98 ± 5	(15)
	$3 \cdot 10^6 \sim 3 \cdot 10^7$	90 ± 5	(15)
	$10^5 \sim 10^7$	113 ± 9	(16)
Sydney group	$3 \cdot 10^5$	$125 \pm {}^{19}_{17}$	(18)
	$8.5 \cdot 10^5$	$107 \pm {}^{16}_{14}$	(18)
Tokyo group	$10^5 \sim 10^7$	100 ± 15	(5)

(ii) The attenuation length of shower rate Λ near sea level is nearly independent of N. All values shown in Table V are in agreement with each other,[19] except the rather large value derived by Greisen.[17]

(iii) EAS of size $\gtrsim 10^7$ are observed at sea level.[15),17)]

If all EAS having sizes $\gtrsim 10^9$ at sea level are initiated by iron nuclei, the sizes are $\gtrsim 2 \cdot 10^7$ per nucleon, corresponding to $E_0 \gtrsim 10^8 Mc^2$. It is, therefore, rea-

conable to assume that the energy spectrum of primary protons extends *at least* p to about $10^8 Mc^2$, so that EAS initiated by protons satisfy the results (i) and (ii) up to a size of about $2 \cdot 10^7$.

As easily seen from Figs. 5a, 5b and 6, the case *b* is in good agreement with (i) and (ii), below such a size irrespective of the spectra I, II and III. This result is not altered by a contribution of EAS initiated by heavy nuclei as shown in § 4·2 and 4·3. Therefore, our model of high-energy interaction for $\eta=0.7$ and $L_s=L_p=90\ \mathrm{gcm}^{-2}$ gives a good fit with (i) and (ii) for an energy spectrum of protons such that γ_p is 1.8 at about $10^6 Mc^2$, increasing gradually with E_0, and is 2.1~2.2 near $10^8 Mc^2$. It is noted that shower curves for the case *b* are quite similar to that for a primary energy of about 10^{15} ev derived by Miyake[2] analysing the experimental data.

One could get other shower curves quite similar to the case *b*, choosing other combinations of η and L_s properly, —more generally choosing inelasticity, interaction mean free path, energy distribution of secondary particles and their energy dependences properly and so on. It should be noted, however, that the shower curves derived in such a way could not fit the experiments, unless $L_p \cong 90\ \mathrm{gcm}^{-2}$, because Λ_p is intimately related to L_p.

As far as the results (i) and (ii) are concerned, there is another possible way to explain them. For instance, as can be guessed from the results for the case *c'*, our model for $\eta=0.3$ and $L_s=60\sim50\ \mathrm{gcm}^{-2}$, though the choice be artificial, could explain them, because Λ_η is well approximated by Eq. (10). An essential difference between this case and *b* is that in the former case the fluctuation in various quantities would be quite small as compared with the case *b*. According to the results of II, the ratio between numbers of muons and of electrons at sea level fluctuates by at least a factor ten for vertical EAS and several times this for steeply inclined EAS (see Table VI). The case *b* can explain this large fluctuation as discussed in § 4·4. On the other hand, the other case cannot explain it.

Finally, we shall discuss the relationship between our model for the case *b* and the experimental results from the study of jet showers which could govern appreciably the shape of shower curves. The results are as follows:

Table VI. The numbers of observed EAS for given ranges of size N and of the number of muons n_μ. θ is the zenith angle (cited from II).

	$\theta \leq 25°$					$\theta \geq 45°$			
100~300	0	0	3	4	8	1	0	1	1
30~100	0	1	21	21	2	0	2	6	0
10~ 30	0	16	43	6	0	0	9	7	2
3~ 10	2	21	19	2	0	0	8	11	0
1~ 3	1	6	0	0	0	0	4	0	0
n_μ / $N/10^5$	0.4~0.9	1~3	3~10	10~30	30~100	100~300	0.4~0.9	1~3	3~10

(i′) The energy spectrum of high-energy γ-rays which are attributed to the decay of neutral pions becomes suddenly steep at about 10^{12} ev.[20),21)] The energy of the nuclear interactions in which such neutral pions are produced would be about 10^{14} ev or more.

(ii′) About 20 percent of secondary particles in the interactions of energy $10^{11}\sim 10^{13}$ ev are heavier than pions.[22),23)] Though there are no data on the proportion above 10^{13} ev, there might be a tendency for the proportion to increase with increasing energy, because the higher the energy the more important may be the core of the nucleon. Perhaps (i′) and (ii′) are correlated with each other.

(iii′) The inelasticity K of primary jets seems to have a decreasing tendency with increasing energy. That of secondary jets is close to unity.[22)]

(iv′) The interaction mean free path seems to decrease with increasing energy.[21)]

The results (i′) and (ii′) act to shorten λ_p independent of L_p. (iv′) acts not only to shorten λ_p, but also to decrease the fluctuation in the depth. (iii′) acts to lengthen λ_p and also to decrease the fluctuation. Hayakawa and Ogita,[24)] taking into account (i′) and (ii′) roughly with $L_s=90\,\mathrm{gcm}^{-2}$, showed that the calculated shower curves are quite similar to the ones for the case *a*. This fact shows that the rather large value of η in the case *b* has the same effect as would be obtained by a lower K together with the effects (i′) and (ii′). That is, $\overline{K}$ averaged over energies at which the main contribution to the number of electrons arises might be 0.4∼0.5, but the effects of (i′) and (ii′) could raise the effective figure up to 0.6∼0.7.

Since (iii′) and (iv′) make the fluctuation small at higher energies, the fluctuation in the ratio between numbers of muons and of electrons decreases with increasing size as compared with the case *b*. If the effects of (iii′) and (iv′) are already appreciable within the size range of $10^5\sim 10^6$, that is, within the energy range of nuclear active particles which contribute mainly to the number of electrons in EAS of the size of $10^5\sim 10^6$, the observed fluctuation in the ratio could not be explained as shown in § 4·4. Therefore, we could conclude that the fluctuation in the ratio within the size range of $10^5\sim 10^6$ is mainly governed by that in the depth of the first interaction of protons as given in the case *b*.*

For the size spectrum, under the assumptions given in § 4·2 a consistent spectrum can be derived insensitive to η and L_s, even if there is a cut-off at about $5\cdot 10^8\,Mc^2$ per nucleon in the primary energy spectrum. If the fluctuation in the depth due to protons is as appreciable as in the case *b* even at larger sizes, there is a possibility that the number of the nucleonic component or muons in EAS

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* Since the number of muons has the A-dependence of $A^{1-\alpha}$, if $\alpha\lesssim 0.6$, the fluctuation in the ratio might be equally attributed to heavy nuclei as well as to protons. Experimental values of α, however, distribute widely from 0.4 to 1.[25)] Therefore it is desirable to measure α more precisely, though our evaluated value 0.8 at sea level is in agreement with values derived by Cranshaw et al.[4)] (0.75±0.18) and Kraushaar[25)] (0.75±0.09).

which are in early stages of development, increases or decreases at some size as shown in Fig. 7a and 8a, though EAS initiated by heavy nuclei might smear out this tendency.

On the nucleonic component and muons the experimental results do not agree at the present time. And also our model is rather crude to discuss these components in detail, because they are mainly occupied by low energy particles. Thus we shall make only a few remarks. Under the assumptions given in § 4·2 one third of EAS at sea level and two-third or more at mountain altitude are initiated by nuclei heavier than protons so that it is quite probable that multiple cores will be observed very near the shower axes. These cores could be attributed to survivals of nucleons composing their primary particle. On the other hand, if the number of high-energy nuclear active particles is proportional to size, those particles in EAS initiated by protons are as abundant as in EAS by heavy nuclei as shown in § 4·4. The difference between these two cases might be that in the latter case cores have *identical or at least comparable energies* with each other. Furthermore the latter case can give a high ratio between numbers of muons and electrons as compared with the former. These facts would give a possible clue to distinguish between EAS due to protons and due to heavy nuclei, but a high statistical accuracy would be required. It should be remembered that the smaller the ratio of muous to electrons, the more abundant are the high-energy particles in EAS initiated by protons. This tendency is opposite to EAS initiated by heavy nuclei. Events having double or more cores have been observed.[26] In order to confirm whether they are attributed to heavy nuclei, however, more refined analyses would be required.

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Appendix

The method of calculation

The diffusion equations of a nucleonic cascade initiated at the top of the atmosphere are given by

$$\frac{\partial N(E, x')}{\partial x'} = -N(E, x') + \int S_{NN}(E, E')\, N(E', x')\, dE' + \int S_{N\pi}(E, E')\, \pi(E', x')\, dE'$$

$$\frac{\partial \pi(E, x')}{\partial x'} = -\left(1+\frac{B}{Ex'}\right)\pi(E, x') + \int S_{\pi N}(E, E')N(E', x')\,dE' + \int S_{\pi\pi}(E, E')\pi(E', x')\,dE'. \quad (A\cdot 1)$$

Here $N(E, x')\,dE$ and $\pi(E, x')\,dE$ are numbers of nucleons and charged pions, respectively, having energy between E and $E+dE$ at depth x' in interaction mean free path L_s units,* measured from the top of the atmosphere. $S_{\pi N}(E, E')$ is the production spectrum of charged pions produced by a nucleon of energy E', and so on. The decay constant B of a charged pion is defined by

$$B=\frac{mc}{\tau}\left(\frac{x'}{\rho(x')}\right)_{Av}, \quad (A\cdot 2)$$

where $\rho(x')$ is the density of air at depth x', and m and τ are the mass and life time of the charged pion respectively. Since $x'/\rho(x')$ is almost constant over all altitudes, the value $(x'/\rho(x'))_{Av}$ averaged over depth from the top of the atmosphere to sea level is used to define B.

In the case of a nucleonic cascade initiated at depth x_1, the term representing the decay loss of charged pions in Eq. (A·1) must be replacd by, if no averaging procedure is made,

$$\frac{B}{Ex'}\pi(E, x') \to \frac{1}{E\rho(x_1+x)}\frac{mc}{\tau}\pi(E, x) = \frac{1}{Ex}\pi(E, x)\frac{mc}{\tau}\frac{x}{\rho(x_1+x)}, \quad (A\cdot 3)$$

where $x(=x'-x_1)$ is an atmospheric thickness through which the nucleonic cascade develops. Though $x/\rho(x_1+x)$ is zero at $x=0$ as it should be, it is nearly constant for a given x_1, from sea level to $x'=x_1+1$, the average depth at which the second interaction occurs. One can, therefore, define a new decay constant $B(x_1)$ as a function of x_1, i. e.

$$B(x_1)=\frac{mc}{\tau}\left(\frac{x}{\rho(x_1+x)}\right)_{Av} \quad (A\cdot 4)$$

averaged over the range from $x'=x_1+1$ to sea level. By this approximation the number of pions belonging to the first generation is underestimated. But this is not significant because their energies are very large compared with $B(x_1)$. Several values of $B(x_1)$ for $L_A=L_s=90\ \mathrm{gcm}^{-2}$ are shown in Table VII. To evaluate them the graph cited by Rossi[27] was used.

Table VII

x_1	$B(x_1)$
0	148
1	120
2	102
3	88
4	74
5	63
6	52

The diffusion equations of the nucleonic cascade initiated at depth x_1 become the same as Eq. (A·1)

* In the Appendix the depth is measured in interaction mean free path units, except Eqs. (A·8) and (A·9).

simply by replacing B and x' by $B(x_1)$ and x respectively, and can be solved in a similar manner given in I. The only difference is as follows. Since the fluctuation in the depth of the first nuclear interaction is concerned, $N(E, x)$ and $\pi(E, x)$ must be expanded as

$$\begin{aligned} N(E, x) &= \sum_{l=1}^{\infty} \frac{e^{-x} x^{l-1}}{(l-1)!} N_l(E), \\ \pi(E, x) &= \sum_{l=1}^{\infty} \frac{e^{-x} x^{l-1}}{(l-1)!} \pi_l(E). \end{aligned} \tag{A·5}$$

Thus the recurrence formulae for each generation turn out to be for $l \geqq 2$

$$N_l(E) = \int S_{NN}(E, E') N_{l-1}(E')\, dE' + \int S_{N\pi}(E, E') \pi_{l-1}(E')\, dE', \tag{A·6}$$

$$\pi_l(E) = \frac{(l-1)E}{(l-1)E + B(x_1)} \left[\int S_{\pi N}(E, E') N_{l-1}(E')\, dE' + \int S_{\pi\pi}(E, E') \pi_{l-1}(E')\, dE' \right].$$

$N_1(E)$ and $\pi_1(E)$, the secondary nucleons and charged pions produced in the first interaction respectively, are fixed by the model of high-energy nuclear interaction, i.e.

$$N_1(E) = S_{NN}(E, E_0), \quad \pi_1(E) = S_{\pi N}(E, E_0), \tag{A·7}$$

where E_0 is an energy of a primary proton.

The total number of electrons at depth t, initiated by neutral pions which are produced in the l-th nuclear interactions and which have the energy spectrum $\pi_l^0(E)\, dE$ is given by

$$\pi_l(E, t)\, dE = \pi_l^0(E)\, dE \left(\frac{2}{\pi}\right)^{1/2} \frac{M(\bar{s}) K_1(\bar{s}, -\bar{s})}{(\bar{s}+1)[\bar{s} g''(\bar{s})]^{1/2}} \left(\frac{E}{\varepsilon_c}\right)^{\bar{s}} \exp[\lambda_1(\bar{s}) t] \frac{f_{l-2}(\bar{\xi} t)}{(l-2)! \bar{\xi}^{l-1}}, \tag{A·8}$$

where

$$f_l(\bar{\xi} t) = \begin{cases} 1 & \text{for } l = 1, \\ \displaystyle\int_0^{\bar{\xi} t} \frac{e^{-y} y^l}{l!}\, dy & \text{for } l \geqq 2, \end{cases} \tag{A·9}$$

and t in radiation units is measured from the depth of the first interaction. The notations are the same as given in I.

The number of muons at depth x which are the decay products of charged pions of the l-th generation is for $l \geqq 2$

$$\mu_l(E_\mu)\, dE_\mu = \pi_l(E)\, dE \int_0^x \frac{B(x_1)}{Ey} \frac{e^{-y} y^{l-1}}{(l-1)!}\, dy = \pi_l(E)\, dE \frac{B(x_1)}{(l-1)E} \int_0^x \frac{e^{-y} y^{l-2}}{(l-2)!}\, dy. \tag{A·10}$$

The integral diverges for $l=1$. This divergence arises in connection with the

definition of $B(x_1)$. If the integral for $l=1$ is evaluated in a more correct manner, however, it gives a very small value, because the energy of the charged pions produced in the first interaction is so high that they almost always collide with air nuclei before decaying into muons. Therefore muons which are the decay product of the charged pions of the first generation are neglected in our calculation.

References

1) W. Kraushaar, Nuovo Cimento, Suppl. **8** (1958), 649.
2) S. Miyake, Prog. Theor. Phys. **20** (1958), 844.
3) K. Greisen, *Progress in Cosmic Ray Physics* III (1955) edited by J. G. Wilson.
4) T. E. Cranshaw, J. E. de Beer, W. Galbraith. A. M. Hillas, S. Norris and N. A. Porters, Phil. Mag. **3** (1958), 811.
5) S. Fukui, H. Hasegawa, T. Matano, I. Miura, N. Ogita, K. Suga, G. Tanahashi and Y. Tanaka, reported at Moscow Conference (1959), referred to as II.
S. Miyake, K. Hinotani, I. Katsumata and T. Kaneko, reported at Moscow Conference (1959).
T. Kameda, T. Maeda and Y. Toyoda, reported at Moscow Conference (1959).
6) Kristiansen et al., reported at Moscow Conference (1959).
7) L. Janossy and H. Messel, Proc. Phys. Soc. A **63** (1950), 1101.
8) A. Ueda and N. Ogita, Prog. Theor. Phys. **18** (1957), 269, referred to as **I.**
9) For example, W. Galbraith, *Extensive Air Shower.*
10) For example, B. Peters, *Progress in Cosmic Ray Physics* I (1952) edited by J. G. Wilson.
11) Cited from figures given by S. Hayakawa, K. Ito and Y. Terashima, Prog. Theor. Phys. Suppl. No. **6** (1958), 1.
12) T. E. Cranshaw, J. E. de Beer, W. Galbraith and N. A. Porters, Phil. Mag. **3** (1958), 377.
13) M. H. Brennan, D. D. Millar and C. S. Wallace, Nature **132** (1958), 905.
14) G. Clark, Phys. Rev. **108** (1957), 454.
15) G. Clark, J. Earl, W. Kraushaar, J. Linsley, B. Rossi and F. Sherb, Nature **180** (1957), 353.
16) MIT Group, reported at Moscow Conference (1959).
17) K. Greisen, reported at Moscow Conference (1959).
18) J. Malos, reported at Moscow Conference (1959).
19) See reference 4) where the values of previous works are also summarized.
20) Y. Fujimoto, S. Hasegawa, J. Nishimura, K. Niu, M. Kazuno and N. Ogita, reported at Moscow Conference (1959).
21) Perkins, Duthie, Fisher, Kaddoura and Pinkau, reported at Moscow Conference (1959).
22) B. Edwards, J. Losty, D. H. Perkins, K. Pinkau and J. Reynolds, Phil. Mag. **3** (1958), 237.
23) F. A. Erisbout, C. Dahanayake, A. Engler, Y. Fujimoto and D. H. Perkins, Phil. Mag. **1** (1956), 605.
24) S. Hayakawa and N. Ogita, reported at Moscow Conference (1959).
25) Andronikashvili et al., reported at Moscow Conference (1959).
G. Cocconi, V. Tongiorgi and K. Greisen, Phys. Rev. **75** (1949), 1063.
W. L. Kraushaar, reported at Varenna Conference (1957).
J. A. Lehane, D. D. Millar and M. H. Rathgeber, Nature **182** (1958), 1699.
C. Milone, Nuovo Cimento **9** (1952), 637.
Vernov et al., reported at Moscow Conference (1959).
26) For example, N. Dobrotin, O. Dovzhenko, V. Zatsepin, E. Murzina, S. Nikolsky, I. Rakobolskaya and E. Tukish, Nuovo. Cimento, Suppl. 8 (1958), 612.
Nikolsky, and Pomansky, J. Exptl. Theor. Phys. **35** (1958), 618.
N. N. Goryunov, A. D. Erlykin, A. B. Kamnev and G. T. Zatsepin, reported at Moscow Conference (1959).
27) B. Rossi, Rev. Mod. Phys. **20** (1948), 537.

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Pion-Nucleon Scattering and Pion-Pion Interaction

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S-wave pion-nucleon scattering is investigated from the viewpoint that pion-pion interaction is important. Starting from the interaction Lagrangian L_I

$$-L_I=\frac{G^2}{2M}\int\rho_c(r)\phi^2(\boldsymbol{r})d^3\boldsymbol{r}+\left(\frac{G}{2M}\right)^2\tau\cdot\int\rho_c(r)\phi(\boldsymbol{r})\times\dot{\phi}(\boldsymbol{r})d^3\boldsymbol{r}+V(0)\int\rho_\pi(r)\phi^2(\boldsymbol{r})d^3\boldsymbol{r},$$

(where the third term corresponds to the effective potential obtained from the original pion-pion interaction $H_I^{\pi-\pi}=\lambda(\phi\cdot\phi)^2$,) we solve directly the Klein-Gordon equation on the matrix-elements of the meson field operator between the nucleon state and the nucleon plus one-meson state. Choosing Yukawa type for the form factors $\rho_i(r)=(4\pi r)^{-1}\Lambda_i^2\exp[-\Lambda_i r]$, we can obtain the best fit to the experimental behavior of the S-wave phase shifts for the following value of the parameters: $\Lambda_G\approx5\mu$, $\Lambda_\pi\approx3.5\mu$, $(2/4\pi)(G^2/2M+V(0))=(2/4\pi)(G^2/2M)(2/10)\sim0=(0.4\sim0)\mu^{-1}$. This choice of $V(0)$ corresponds to the original pion-pion coupling constant $\lambda/4\pi=0.3\sim0.5$ and is consistent with the value $|\lambda|/4\pi\approx1$ determined from the $(\pi$-$2\pi)$ process at 1.4 Bev. On the basis as obtained above, the qualitative nature of the small P-wave phase shifts is discussed. And we obtain the results which coincide with the ones of analysis by Lomon and Chiu.

§ 1. Introduction and summary

The characteristic feature of the S-wave pion-nucleon scattering, though more detailed knowledge has recently been acquired about it, is described by Orear's line: $\delta_1^\circ=9.2\,k$ and $\delta_3^\circ=-6.3\,k$ for the iso-spin $I=\frac{1}{2}$ state and the $I=\frac{3}{2}$ state respectively. As is well known, for the iso-odd part of the phase shifts $(\delta_1-\delta_3)$ various theories can give the right order of magnitude, but for the iso-even part $(\delta_1+2\delta_3)$ no satisfactory explanations have yet been given (e.g. perturbational calculation of the pseudoscalar coupling theory gives the larger value for it by a factor $10\sim100$).

In this paper we shall call our attention, among the many investigations[1] so far made, to the theories based on the Tani-Foldy transformation from the pseudoscalar coupling:

$$H_I=\lambda_1\left(\int\rho(r)\phi(\boldsymbol{r})\,d^3\boldsymbol{r}\right)^2+\lambda_2\left(\int\rho(r)\phi(\boldsymbol{r})\,d^3\boldsymbol{r}\right)\times\left(\int\rho(r)\dot{\phi}(\boldsymbol{r})\,d^3\boldsymbol{r}\right)\cdot\tau, \quad (1)$$

where $\lambda_1=G^2/2M$ and $\lambda_2=(G/2M)^2$. And we shall reconsider them from the viewpoint that pion-pion interaction takes an important role also in the low energy pion-nucleon scattering. Recently, the importance of pion-pion interaction has been very frequently emphasized in the problems of nucleon structure,[2] of the second

and third maxima[3] of the pion-nucleon scattering and of single pion production.[4),5] Our thought of line, as has already been tried by some authors,[6),9] consists in that the original large iso-even part which plays the mischief should be cancelled out with the effective potential resulted from the pion-pion interaction $H_I^{\pi-\pi}=\lambda(\phi\cdot\phi)^2$, i.e.

$$-L_I=\frac{G^2}{2M}\left(\int\rho_c(r)\phi(\boldsymbol{r})\,d^3\boldsymbol{r}\right)^2+\left(\frac{G}{2M}\right)^2\tau\cdot\left(\int\rho_c(r)\phi(\boldsymbol{r})\,d^3\boldsymbol{r}\right)$$
$$\times\left(\int\rho_c(r)\dot{\phi}(\boldsymbol{r})\,d^3\boldsymbol{r}\right)+V(0)\left(\int\rho_\pi(r)\phi(\boldsymbol{r})\,d^3\boldsymbol{r}\right)^2*, \qquad (2)$$

where we assume $V(0)$ is negative following the above thought of line. Then we solve directly the Klein-Gordon equation on the matrix-elements of the meson field operator between the nucleon state and the nucleon plus one-meson incoming state and ask the phase shifts. This method is frequently used,[7] and in the previous paper[10] the author showed that the phase shifts defined from the above mentioned matrix-elements as in the usual quantum mechanics are really identified with the scattering phases. Taking Yukawa type for the form factors

$$\rho_c(r)=(4\pi r)^{-1}\Lambda_c^2\exp[-\Lambda_c r],\quad \rho_\pi(r)=(4\pi r)^{-1}\Lambda_\pi^2\exp[-\Lambda_\pi r],$$

we shall try to reproduce the experimental behavior of the phase shifts for the three parameters, Λ_c, Λ_π and $V(0)$.

In § 2, i), we shall calculate the S-wave phase shifts according to the method mentioned above and try to fit the theory to the experimental behavior for the following six cases,

(A) $\Lambda_c=\Lambda_\pi=2\mu$**; (B) $\Lambda_c=\Lambda_\pi=3\mu$; (C) $\Lambda_c=\Lambda_\pi=4\mu$;

(D) $\Lambda_c=3\mu$, $\Lambda_\pi=2\mu$; (E) $\Lambda_c=4\mu$, $\Lambda_\pi=3\mu$; (F) $\Lambda_c=5\mu$, $\Lambda_\pi=3.5\mu$.

The results are shown in Fig. 1a (A), (B), (C), and Fig. 1b (D), (E), (F) respectively (in each case $(2/4\pi)(G^2/2M+V(0))=(0.4\sim0)\mu^{-1}$ is most profitable). From the comparison of them, we conclude that the best fit to the experimental behavior can be obtained for the case (F) (i.e. $\Lambda_c=5\mu$ and $\Lambda_\pi=3.5\mu$) which gives the reasonable figure to the nucleon form factor consistent with the one obtained thus far.

In § 2, ii), we discuss the relation between the third term of Eq. (2) and the effective pion-pion interaction $H_I^{\pi-\pi}=\lambda(\phi\cdot\phi)^2$. We calculate the effective potential through the graphs, Fig. 2 (a) and (b), and determine the value $\lambda/4\pi=0.3\sim0.5$ corresponding to the above mentioned value of $V(0)$. This value for pion-pion coupling constant is consistent with the value $|\lambda|/4\pi\approx1$ determined from the analyses[4),8] of $(\pi\text{-}2\pi)$ process at 1.4 Bev.

* As usual we make the non-local separable approximation for L_I in the treatment of the S-wave scattering.

** We shall put $\hbar=c=1$ and μ (μ is pion mass) $=1$ in the places where no misleading may occur

In § 3 we shall discuss qualitative behavior of the P-wave phase shifts of the pion-nucleon scattering on the basis obtained above. As a result we shall see the following points: i) If we do not consider the effect of pion-pion interaction, we see that the term $(G^2/2M)\phi^2$ plays the mischief for the phase shift for the (3, 3) state as well as for the S-wave phase shifts. And ii) we obtain for small P-wave phase shifts the results which coincide qualitatively with the ones of analysis by Lomon and Chiu.[18]

Finally in this section we shall briefly review and discuss the works which are related to our thought of line. Along the Chew-Low formalism, Drell, Friedman and Zachariasen[11] analyzed the interaction Hamiltonian (1) in the one-meson approximation. Choosing the straight cut for the nucleon form factor and treating λ_1, λ_2 and cutoff energy ω_{max} as parameters, they determined values for them $\lambda_1=0.4\mu^{-1}$, $\lambda_2=0.4\mu^{-2}$ and $\omega_{max}=4.5\mu$, which give reasonable agreement with the experiments of the S-wave phase shifts up to $\simeq$100 Mev kinetic energy. (Afterwards Goebel[12] pointed out that such values for parameters should lead to a resonance at momentum $k=2.2\mu$ for iso-spin $I=\frac{1}{2}$ state.) Their work can be considered to correspond to our case (C) (i.e. $\Lambda=\Lambda_\pi=4\mu$) (see Fig. 1a(C)).

Lomon[13] treated also the interaction Hamiltonian (1) and succeeded in diagonalizing the Hamiltonian obtained from (1).

At a glance his solution seems to be exact, but afterwards Kobayashi and Klein[14] pointed out that Lomon's solution is exact only in a certain approximation, in which the time dependence of the iso-spin is neglected (our solution is also in this approximation).

Recently Miyazawa and Kawarabayashi[9] proposed an interesting model in which they assumed as follows:

i) The source of S-wave pion is extended to a range $\approx(2\mu)^{-1}$.

ii) Attractive pion-pion interaction $(\lambda(\phi\cdot\phi)^2:\lambda<0)$ is important for S-wave pion nucleon scattering.

Starting from (2) and using the Tamm-Dancoff approximation, they succeeded to reproduce main features of the S-wave phase shifts in the low energy region. Their work can be considered to correspond to our case (A) (i.e. $\Lambda_c=\Lambda_\pi=2\mu$) (see Fig. 1a(A)).

With regards their assumption i), we cannot immediately agree with it and its necessity for the explanation of the experiment is not so clear (this point has been the main motivation for this work). They pointed out that such an extended core is consistent with the conclusion from the electromagnetic structure of the nucleon and also from the analysis of the nucleon-antinucleon interaction by Koba and Takeda. On the other hand, we know the analysis of the nucleon-antinucleon interaction by Ball, Chew[15] and Fulco. They assumed that the absorbing region for nucleon-antinucleon annihilation is inside the rather small region $\sim(3\mu)^{-1}$ which is consistent with the repulsive hard core assumed in the analysis of nuclear force. Up to now, their theory seems to be based on the most acceptable founda-

tions and can explain low energy nucleon-antinucleon experiments very well. We shall see in § 2 that we do not need such a drastic change for the figure of nucleon form factor that the source of S-wave pion is extended to a range $r_0 \approx (2\mu)^{-1}$.

With regards their assumption ii), it is in agreement with our standpoint that pion-pion interaction is important for the S-wave pion-nucleon scattering. But in § 2, ii), we shall see that pion-pion interaction should be repulsive ($\lambda(\phi\cdot\phi)^2 : \lambda > 0$) against their result that it is attractive ($\lambda < 0$), if we consider the graph Fig. 2(b) as well as Fig. 2(a).

Recently, Sugano[16] has shown that " in the case of pseudoscalar coupling the second order matrix-elements are almost canceled by some of the fourth order ones which appear in the reduced Chew-Low equation as modified Born terms ". In the present paper we shall not enter into the problem of the higher order correction to the interaction Hamiltonian (1), but we only remark here on the work by Goebel,[17] which examines the higher order correction only due to the two-vertex interaction and concludes that " the effective S-wave two-vertex interaction consists very nearly of simply the lowest order Foldy terms " (1).

§ 2. S-wave pion-nucleon scattering

i) *Calculation of the phase shifts*

As mentioned in § 1 we solve directly the Klein-Gordon equation on the matrix-elements of the meson field operator between the physical nucleon state $|N\rangle$ and the nucleon plus one-meson incoming state $|N, \omega_k\rangle$ (ω_k is the energy of incident meson) and ask the phase shifts defined as in usual quantum mechanics.

From (2) we obtain

$$(\Box-\mu^2)\phi(\boldsymbol{r}) = \frac{2G^2}{2M}\rho_c(r)\int\rho_c(r')\phi(\boldsymbol{r}')\,d^3\boldsymbol{r}' + \left(\frac{G}{2M}\right)^2\rho_c(r)\left(\int\rho(r')\dot{\phi}(\boldsymbol{r}')\,d^3\boldsymbol{r}'\right)\times\tau$$
$$+\left(\frac{G}{2M}\right)^2\rho_c(r)\frac{\partial}{\partial t}\left(\int\rho_c(r')\phi(\boldsymbol{r}')\,d^3\boldsymbol{r}'\times\tau\right)+2V(0)\rho_\pi(r)\int\rho_\pi(r')\phi(\boldsymbol{r}')\,d^3\boldsymbol{r}', \tag{3}$$

$$(\nabla^2+k^2)\phi_k(\boldsymbol{r}) = \frac{2G^2}{2M}\rho_c(r)\int\rho_c(r')\phi_k(\boldsymbol{r}')\,d^3\boldsymbol{r}' +$$
$$2\left(\frac{G}{2M}\right)^2 Z(-i\omega_k)\rho_c(r)\left(\int\rho_c(r')\phi(\boldsymbol{r}')\,d^3\boldsymbol{r}'\right)\times\tau+2V(0)\rho_\pi(r)\int\rho_\pi(r')\phi_k(\boldsymbol{r}')\,d^3\boldsymbol{r}', \tag{4}$$

where $$\phi_k(\boldsymbol{r}) \equiv \langle N|\phi(\boldsymbol{r})|N, \omega_k\rangle.$$

Here we made such an approximation,* as

$$\langle N|\tau\times\dot{\phi}(\boldsymbol{r})|N, \omega_k\rangle = \langle N|\tau|N\rangle\times\langle N|\dot{\phi}(\boldsymbol{r})|N, \omega_k\rangle, \tag{5}$$

* This approximation, which we call the "zero-meson approximation", corresponds to the one pointed out by Kobayashi-Klein[14] and does not coincide with the usual Tamm-Dancoff method.

and defined new renormalization constant Z

$$\langle N|\tau|N\rangle \equiv Z\tau. \tag{6}$$

Decomposing Eq. (4) into the eigenstates of the iso-spin I and transforming it into the momentum representation, we obtain

$$(\boldsymbol{p}^2-\boldsymbol{k}^2)\phi_k^I(\boldsymbol{p}) = -(\lambda_1^I v_1(\boldsymbol{p})C_1+\lambda_2 v(\boldsymbol{p})C_2), \tag{7}$$

where we defined

$$\phi_k^I(\boldsymbol{p}) = \int e^{-i\boldsymbol{p}\cdot\boldsymbol{r}}\, d^3\boldsymbol{r}\langle N|\phi(\boldsymbol{r})|N,\ \omega_k^I\rangle,$$

$$v_{1,2}(\boldsymbol{p}) = (2\pi)^{-3/2}\int \rho_{c,\pi}(r)\, e^{-i\boldsymbol{p}\cdot\boldsymbol{r}}\, d^3\boldsymbol{r},$$

$$\lambda_1^I = \left(\frac{2G^2}{2M}+2\begin{pmatrix} G \\ 2M\end{pmatrix}^2 Z\omega_k\xi^I\right),\quad \lambda_2 = 2V(0),\quad \xi^I = \begin{pmatrix} -2 \\ 1\end{pmatrix} \text{ for } I=\begin{pmatrix} 1/2 \\ 3/2\end{pmatrix},$$

$$C_{1,2} = \int v_{1,2}(\boldsymbol{p})\phi_k(\boldsymbol{p})\, d^3\boldsymbol{p}. \tag{8}$$

Considering the incoming-wave boundary condition, we can directly write the solution for Eq. (7) :

$$\phi_k(\boldsymbol{p}) = \delta(\boldsymbol{p}-\boldsymbol{k}) + \frac{-(\lambda_1 v_1(\boldsymbol{p})C_1+\lambda_2 v_2(\boldsymbol{p})C_2)}{\boldsymbol{p}^2-\boldsymbol{k}^2-i\epsilon}. \tag{9}$$

To determine C_i we derive the equation

$$C_1 = v_1(\boldsymbol{k}) - \int \frac{\lambda_1 v_1^2(\boldsymbol{p})C_1+\lambda_2 v_1(\boldsymbol{p})v_2(\boldsymbol{p})C_2}{\boldsymbol{p}^2-\boldsymbol{k}^2-i\epsilon}\, d^3\boldsymbol{p},$$

$$C_2 = v_2(\boldsymbol{k}) - \int \frac{\lambda_1 v_1(\boldsymbol{p})v_2(\boldsymbol{p})C_1+\lambda_2 v_2^2(\boldsymbol{p})C_2}{\boldsymbol{p}^2-\boldsymbol{k}^2-i\epsilon}\, d^3\boldsymbol{p}, \tag{10}$$

by using Eq. (8), the definition of C. We then obtain

$$C_i = N_i/D, \tag{11}$$

where

$$D = \begin{vmatrix} 1+\lambda_1\int \dfrac{v_1^2(\boldsymbol{p})\, d^3\boldsymbol{p}}{\boldsymbol{p}^2-\boldsymbol{k}^2-i\epsilon}, & \lambda_2\int \dfrac{v_1(\boldsymbol{p})v_2(\boldsymbol{p})\, d^3\boldsymbol{p}}{\boldsymbol{p}^2-\boldsymbol{k}^2-i\epsilon} \\ \lambda_1\int \dfrac{v_1(\boldsymbol{p})v_2(\boldsymbol{p})\, d^3\boldsymbol{p}}{\boldsymbol{p}^2-\boldsymbol{k}^2-i\epsilon}, & 1+\lambda_2\int \dfrac{v_2^2(\boldsymbol{p})\, d^3\boldsymbol{p}}{\boldsymbol{p}^2-\boldsymbol{k}^2-i\epsilon} \end{vmatrix},$$

$$N_1 = \begin{vmatrix} v_1(\boldsymbol{k}), & \lambda_2\int \dfrac{v_1(\boldsymbol{p})v_2(\boldsymbol{p})\, d^3\boldsymbol{p}}{\boldsymbol{p}^2-\boldsymbol{k}^2-i\epsilon} \\ v_2(\boldsymbol{k}), & 1+\lambda_2\int \dfrac{v_2^2(\boldsymbol{p})\, d^3\boldsymbol{p}}{\boldsymbol{p}^2-\boldsymbol{k}^2-i\epsilon} \end{vmatrix},\quad N_2 = \begin{vmatrix} 1+\lambda_1\int \dfrac{v_1^2(\boldsymbol{p})\, d^3\boldsymbol{p}}{\boldsymbol{p}^2-\boldsymbol{k}^2-i\epsilon}, & v_1(\boldsymbol{k}) \\ \lambda_1\int \dfrac{v_1(\boldsymbol{p})v_2(\boldsymbol{p})\, d^3\boldsymbol{p}}{\boldsymbol{p}^2-\boldsymbol{k}^2-i\epsilon}, & v_2(\boldsymbol{k}) \end{vmatrix}.$$

Then we can obtain the scattering amplitude $f(\theta) = e^{i\delta}\sin\delta/k$ from the relation

$f(\boldsymbol{p})=-2\pi^2(\lambda_1 v_1 c_1+\lambda_2 v_2 c_2)$. From now we choose Yukawa type $\rho_c=(4\pi r)^{-1}\Lambda_c^2\exp[-\Lambda_c r]$ and $\rho_\pi=(4\pi r)^{-1}\Lambda_\pi^2\exp[-\Lambda_\pi r]$ for the form factors.

For the comparison we first write the scattering length in the case of no pion-pion interaction (i.e. $V(0)=0$)

$$a^I=\frac{-\frac{2}{4\pi}\frac{G^2}{2M}\left(1+\xi^I\frac{1}{2M}\right)}{1+\frac{G^2}{2M}\left(1+\xi^I\frac{1}{2M}\right)\frac{\Lambda_c}{4\pi}}. \tag{12}$$

Here the numerator is the Born term and the denominator is the order of magnitude $\approx 1+\Lambda_c$. From this we see that we can obtain the right order of magnitude for the iso-even part of the phase shift by taking $\Lambda_c=M\sim 2M$ (M: nucleon mass) but this leads to the corresponding reduction of the iso-odd part. So in the case of no pion-pion interaction, it is the iso-odd part rather than the iso-even part that is difficult to explain.

Next we write down the results of calculation for $\tan\delta/k$,

$$\tan\delta^I/k=N^I/D,$$

$$N^I=-\frac{1}{4\pi}\left\{\frac{\lambda_1{}^I\Lambda_c^4}{(\Lambda_c^2+k^2)^2}+\frac{\lambda_2\Lambda_\pi^4}{(\Lambda_\pi^2+k^2)^2}+\frac{\lambda_1{}^I\lambda_2}{4\pi}\frac{\Lambda_c^3\Lambda_\pi^3}{(\Lambda_c^2+k^2)^2(\Lambda_\pi^2+k^2)^2}\right.$$
$$\left.\times\left[\frac{\Lambda_c(\Lambda_\pi^2-k^2)+\Lambda_\pi(\Lambda_c^2-k^2)}{2}-\frac{2\Lambda_c\Lambda_\pi(\Lambda_c\Lambda_\pi-k^2)}{\Lambda_c+\Lambda_\pi}\right]\right\}, \tag{13}$$

$$D=1+\lambda_1{}^I\frac{1}{4\pi}\frac{1}{(\Lambda_c^2+k^2)^2}\frac{\Lambda_c^3(\Lambda_c^2-k^2)}{2}+\lambda_2\frac{1}{4\pi}\frac{1}{(\Lambda_\pi^2+k^2)^2}\frac{\Lambda_\pi^3(\Lambda_\pi^2-k^2)}{2}$$
$$+\frac{\lambda_1{}^I}{4\pi}\frac{\lambda_2}{4\pi}\frac{\Lambda_c^3\Lambda_\pi^3}{(\Lambda_c^2+k^2)^2(\Lambda_\pi^2+k^2)^2}\left[\frac{1}{4}(\Lambda_c^2-k^2)(\Lambda_\pi^2-k^2)-\frac{\Lambda_c\Lambda_\pi(\Lambda_c\Lambda_\pi-k^2)^2}{(\Lambda_c+\Lambda_\pi)^2}\right],$$

and for the scattering length

$$a^I=\frac{-\frac{1}{4\pi}\left(\lambda_1{}^I+\lambda_2+\lambda_1{}^I\lambda_2\frac{1}{8\pi}\frac{(\Lambda_c-\Lambda_\pi)^2}{(\Lambda_c+\Lambda_\pi)}\right)}{1+\lambda_1{}^I\frac{1}{8\pi}\Lambda_c+\lambda_2\frac{1}{8\pi}\Lambda_\pi+\frac{\lambda_1{}^I}{8\pi}\frac{\lambda_2}{8\pi}\Lambda_c\Lambda_\pi\frac{(\Lambda_c-\Lambda_\pi)^2}{(\Lambda_c+\Lambda_\pi)^2}}. \tag{14}$$

Considering that the iso-even part of the first term in the numerator of Eq. (14) is too large and the iso-odd part of it has the right order of magnitude, we see that such a prescription as

$$\text{iso-even part of } \lambda_1^I\approx-\lambda_2,$$
$$\Lambda_c\gtrapprox\Lambda_\pi, \tag{15}$$

may reproduce the experimental behavior of the phase shifts.

To see the qualitative behavior we take the value for the coupling constant $(1/4\pi)(G/2M)^2=0.08$ and $Z=1$ and show the results of calculation for the next

six cases of parameters in Fig. 1:

(A) $\Lambda_c=\Lambda_\pi=2\mu$; (B) $\Lambda_c=\Lambda_\pi=3\pi$; (C) $\Lambda_c=\Lambda_\pi=4\mu$;

(D) $\Lambda_c=3\mu$, $\Lambda_\pi=2\mu$; (E) $\Lambda_c=4\mu$, $\Lambda_\pi=3\mu$; (F) $\Lambda_c=5\mu$, $\Lambda_\pi=3.5\mu$.

In each case $(2/4\pi)(G^2/2M+V(0))=(2/4\pi)(G^2/2M)(2/10\sim0)=(0.4\sim0)\mu^{-1}$ is most profitable corresponding to the first condition of (15). From the comparison of them we conclude that the best fit to the experimental behavior can be obtained for the case (F) (i.e. $\Lambda_c=5\mu$, $\Lambda_\pi=3.5\mu$) which gives the reasonable figure to the nucleon form factor, and we need no such drastic change as the case in the theory of Miyazawa and Kawarabayashi[9] to explain the experimental behavior of the S-wave phase shifts.

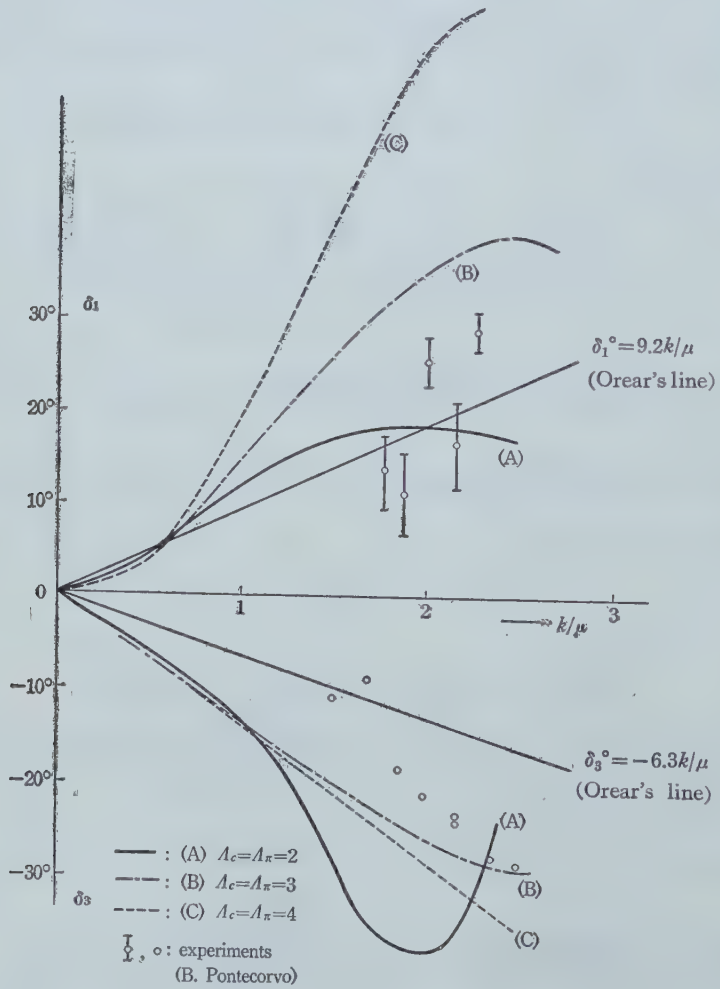

Fig. 1a. S-wave phase shifts (A, B, C)

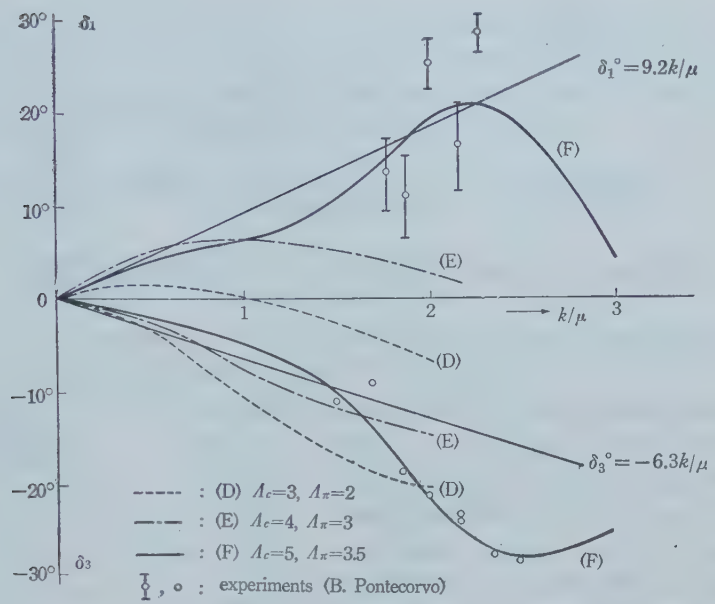

Fig. 1b. *S*-wave phase shifts (D, E, F)

Now we add some remarks about the results of Fig. 1. In case (A) which can be considered to correspond to the case of the theory of Miyazawa and Kawarabayashi, the magnitude of the phase shifts damps down rapidly at the region $\omega \approx 2$ corresponding to the choice of $\Lambda_c = \Lambda_\pi = 2$. On the other hand, the experimental value for $I = \frac{3}{2}$ phase shift is rather enlarged for this energy region (see Fig. 1).

Case (C) can be considered to correspond to the case of the theory of Drell, Friedman and Zachariasen,[11] and we can see that the resonance for the $I = \frac{1}{2}$ state pointed out by Goebel[12] will occur really in this case. As for cases (A), (B) and (C), i.e. for $\Lambda_c = \Lambda_\pi$, the third term in (4) cancel out the first term irrespective of the energy, but as for cases (D), (E) and (F), i.e. for $\Lambda_c > \Lambda_\pi$, the third term damps down earlier than the first term does and the phase shift δ_1 becomes negative at certain energy corresponding to the dominance of the first term.

ii) *Effective coupling constant of pion-pion interaction*

In the last section we obtained the value for the parameter $V(0)$ in the third term of (2) $(2/4\pi)(G^2/2M + V(0)) = (2/10 \sim 0)(2/4\pi)(G^2/2M)$ in order to fit the theory to the experiment.

In this section we discuss the relation between $V(0)$ and the original effective pion-pion coupling constant λ. This task is very difficult, so the following result should be only of a qualitative nature. Up to the second order in the original coupling constant G, we obtain[9]

$$V(q^2)=\int V(r)\,e^{-iq\cdot r}\,d^3\boldsymbol{r}$$

$$=\frac{-i5\lambda}{2\pi^3}\int\frac{A(k_0)+2B(k_0)+C(k_0)}{(k_0^2-\omega_k^2)\,((k+q)_0^2+\omega_{k+q}^2)}\;k^4\,dk\cdot dk_0^{*}$$

$$+\frac{G^2}{2M}\frac{i5\lambda}{4\pi^2}\int\frac{1}{(k_0^2-\omega_k^2)\,((k+q)_0^2+\omega_{k+q}^2)}\,d^3\boldsymbol{k}\cdot dk_0\,,$$

$$V(0)=\frac{\lambda}{4\pi}\frac{f^2}{4\pi}80\int\frac{k^4}{\omega_k^4}\,dk-\frac{\lambda}{4\pi}\frac{1}{4\pi}\frac{G^2}{2M}\times 40\int\frac{k^2}{\omega_k^3}\,dk\,, \qquad (16)$$

where we take $(f^2/4\pi)=(G/2M)^2(1/4\pi)=0.08$.

The first term in Eq. (16) is due to virtual P-wave meson (Fig. 2 (a)), which only is considered in the M. K. theory, and the second term is due to virtual S-wave meson (Fig. 2 (b)). Taking straight-cut for the integral in Eq. (16) ($\omega_{\max}=6$ for the first term, $\omega_{\max}=5$ for the second) we obtain** from Eq. (16) and the above mentioned value for $V(0)$

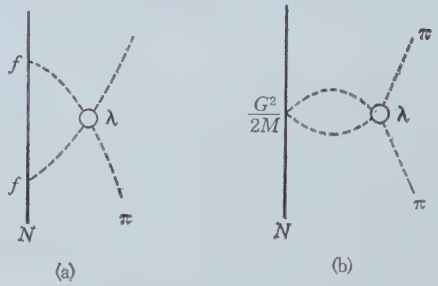

Fig. 2. Potential due to pion-pion interaction

$$V(0)=\left(-1\sim-\frac{8}{10}\right)\frac{G^2}{2M}=\frac{\lambda}{4\pi}\left\{\frac{f^2}{4\pi}80\times 4.5-\frac{1}{4\pi}\frac{G^2}{2M}40\times 1.4\right\},$$

$$\frac{\lambda}{4\pi}=0.36\sim 0.45.$$

This value for λ is consistent*** with the value $|\lambda|/4\pi\approx 1$ determined by the analysis[5] of (π-2π) process at 1.4 Bev. The difference about the sign of λ from the M. K. theory comes from the contribution due to the graph Fig. 2(b) which is larger than the one due to the graph Fig. 2(a) by factor 2 and opposite in sign.

§ 3. *P*-wave pion-nucleon scattering

In § 2 we showed that one can explain the experimental behavior of S-wave phase

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* The definition of A, B and C is given in the work by H. Miyazawa, Phys. Rev. **104** (1956), 1741.

** The author should like to express his thanks to Dr. Y. Fujii and to Dr. K. Igi for pointing out the error made in the calculation of Eq. (16) in the previous manuscript of this work.

*** This conclusion seems to differ from the one by Ito and Minami[6] who estimated the effect of pion-pion interaction (with the coupling constant determined from the analysis of (π-2π) process) to the S-wave pion-nucleon scattering by use of relativistic perturbation and concluded this effect is negligible. But to our regret, their works are mistaken in the following two points.

i) Their Eq. (12) should be multiplied by factor 4.

ii) Pion-pion coupling constant[8] determined by (π-2π) process is $|\lambda|/4\pi\approx 1.6$ for the interaction Hamiltonian $H_I=\lambda(\phi\cdot\phi)^2$, and not $(\lambda/4\pi)^2\approx 1.6$ for $H_I=(\lambda/16)(\phi\cdot\phi)^2$.

shifts from the viewpoint that pion-pion interaction is important.

In this section we discuss the qualitative behavior of P-wave phase shifts on the basis obtained in § 2. Instead of (2) we take the original (without the non-local separable approximation) interaction Hamiltonian

$$H_I=\frac{G^2}{2M}\int\rho_c(r)\phi^2(\boldsymbol{r})\,d^3\boldsymbol{r}+\left(\frac{G}{2M}\right)^2\tau\cdot\int\rho_c(r)\phi(\boldsymbol{r})\times\dot{\phi}(\boldsymbol{r})\,d^3\boldsymbol{r}$$
$$+V(0)\int\rho_\pi(r)\phi^2(\boldsymbol{r})\,d^3\boldsymbol{r}+\left(\frac{G}{2M}\right)^2\int\rho(r)\,(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\,(\tau\cdot\phi(\boldsymbol{r}))\,d^3\boldsymbol{r}, \qquad (17)$$

obtained by the Tani-Foldy transformation from the *ps-ps* coupling theory with the addition of attractive potential term due to pion-pion interaction.

From (17) we obtain* up to the second order in the original pion-nucleon coupling constant G

$$(\boldsymbol{\nabla}^2+\boldsymbol{k}^2)\phi_k(\boldsymbol{r})=-\Big\{\frac{2G^2}{2M}\rho_c(r)\phi_k(\boldsymbol{r})-2i\omega_k\left(\frac{G}{2M}\right)^2Z\phi_k(\boldsymbol{r})\times\tau\rho_c(r)$$
$$+2V(0)\rho_\pi(r)\phi_k(\boldsymbol{r})-\frac{2}{i\omega_k}\left(\frac{G}{2M}\right)^2Z\boldsymbol{\nabla}\rho\cdot\int[\boldsymbol{\nabla}'\tau\times\phi_k(\boldsymbol{r}')+\boldsymbol{\sigma}\times\boldsymbol{\nabla}'\phi_k(\boldsymbol{r}')]\rho(r')\,d^3\boldsymbol{r}'\Big\}. \qquad (18)$$

where we defined $\phi_k(\boldsymbol{r})\equiv\langle N|\phi(\boldsymbol{r})|N,\,\omega_k\rangle$. Here we made the "zero-meson approximation" as was done in § 2. Transforming Eq. (18) into momentum representation and decomposing it into the eigenstates of iso-spin and angular momentum

$$(\boldsymbol{p}^2-\boldsymbol{k}^2)\phi_k(\boldsymbol{p})=-\frac{2G^2}{2M}\int v_1(\boldsymbol{p}-\boldsymbol{p}')\phi_k(\boldsymbol{p}')\,d^3\boldsymbol{p}'$$
$$-2\left(\frac{G}{2M}\right)^2\omega_k\xi^I\int v_1(\boldsymbol{p}-\boldsymbol{p}')\phi_k(\boldsymbol{p}')\,d^3\boldsymbol{p}'$$
$$-2V(0)\int v_2(\boldsymbol{p}-\boldsymbol{p}')\phi_k(\boldsymbol{p}')\,d^3\boldsymbol{p}'+\frac{3}{\omega_k}4\pi\lambda_\alpha\boldsymbol{p}v(\boldsymbol{p})\int\boldsymbol{p}'v(\boldsymbol{p}')\phi_k(\boldsymbol{p}')\,d^3\boldsymbol{p}', \qquad (19)$$

where $\lambda_\alpha=\frac{2}{3}\frac{1}{4\pi}\left(\frac{G}{2M}\right)^2\begin{pmatrix}-4\\-1\\2\end{pmatrix}$ for $\begin{pmatrix}11\\13,\ 31\\33\end{pmatrix}$ state as usual.

Then we solve Eq. (19) only in the Born approximation (this coincides with the Born approximation in the usual method). This solution may not be correct in quantitative nature, but it would be correct in qualitative nature (especially about the sign of the phase shifts).

* We here made the following prescription: $\langle N|\tau|N\rangle=Z\tau$, $\langle N|\sigma|N\rangle=Z'\sigma$ and $Z=Z'$ (really Z and Z' differ from each other in the order of $(G/2M)^4$). There is some ambiguity about the above definition of Z. On this point and on the derivation of Eq. (18), see reference 10). From now we set $Z=1$.

Choosing Yukawa type $\rho_t(r)=(4\pi r)^{-1}\Lambda_t^2 \exp[-\Lambda_t r]$ for the form factors as in § 2, we obtain the scattering amplitude $f(\theta)$ in this approximation

$$f(\theta)\equiv\sum_t (2l+1)\,e^{i\delta}\sin\delta\,P_l(\cos\theta)/k$$
$$=-\frac{2}{4\pi}\left(\frac{G^2}{2M}+\omega_k\left(\frac{G}{2M}\right)^2\xi^I\right)\frac{\Lambda_1^2}{\Lambda_1^2+(\boldsymbol{p}-\boldsymbol{k})^2}-\frac{2}{4\pi}V(0)\frac{\Lambda_2^2}{\Lambda_2^2+(\boldsymbol{p}-\boldsymbol{k})^2}$$
$$+\frac{3}{\omega_k}\lambda_\alpha k^2\left(\frac{\Lambda^2}{\Lambda^2+k^2}\right)^2\cos\theta, \tag{20}$$

where $(\theta=\widehat{\boldsymbol{p},\boldsymbol{k}})$ and $p=k$.

The first term and the second term on the right-hand of Eq. (20) contribute to all the partial waves and the third term contribute only to the P-wave.

The contribution to each partial wave from the first and the second terms can be easily calculated using the following formula:

$$\frac{\Lambda^2}{\Lambda^2+(\boldsymbol{p}-\boldsymbol{k})^2}=\frac{\Lambda^2}{\Lambda^2+2k^2-2k^2\cos\theta}=\frac{\Lambda^2}{2k^2}\frac{1}{(\Lambda^2/2k^2+1)-\cos\theta}$$
$$=\frac{\Lambda^2}{2k^2}\sum_l (2l+1)Q_l\left(1+\frac{\Lambda^2}{2k^2}\right)P_l(\cos\theta), \tag{21}$$

where $Q_l(x)$'s are Legendre's polynomials of the second kind and we used the formula $1/(x-t)=\sum(2l+1)\,P_l(t)Q_l(x)$ for $|x|>1>|t|$. For the low energy we may expand in $\cos\theta$ the first equation in Eq. (21), and then we write the results of the expansion only for P-wave

$$f_v(\theta)/P_1(\cos\theta)=3e^{i\delta}\sin\delta/k$$
$$=-\frac{2}{4\pi}\frac{G^2}{2M}\frac{2\Lambda_1^2k^2}{(\Lambda_1^2+2k^2)^2}-\frac{2}{4\pi}\left(\frac{G}{2M}\right)^2\omega_k\frac{2\Lambda_1^2k^2}{(\Lambda_1^2+2k^2)^2}\begin{pmatrix}-2\\-2\\1\\1\end{pmatrix}$$
$$-\frac{2}{4\pi}V(0)\frac{2\Lambda_2^2k^2}{(\Lambda_2^2+2k^2)^2}+\frac{2}{4\pi}\left(\frac{G}{2M}\right)^2\frac{1}{\omega_k}\left(\frac{\Lambda^2}{\Lambda^2+k^2}\right)^2k^2\begin{pmatrix}-4\\-1\\-1\\2\end{pmatrix},\ \text{for}\ \begin{pmatrix}11\\13\\31\\33\end{pmatrix}\ \text{state.} \tag{22}$$

We remark here factor ω_k of the second term in the right-hand side of Eq. (22) and factor $1/\omega_k$ of the fourth term. From Eq. (22) we can see the following points.

i) For the (3·3) state. The first term has the magnitude $\approx 1/3$ of the one of the fourth term at zero momentum and has opposite sign to it, and reaches the equal order of magnitude at the energy $\omega=2$. So, the term $(G^2/2M)\phi^2$ plays an undesirable role for explaining the experimental behavior of the (33) state phase shift from the *ps-ps* theory, if we do not consider the effect of pion-pion interaction.

ii) For other states: The Chew-Low theory predicts that small P-wave phase shifts are always negative. On the other hand, from (22) we can see that the $(1, 3)$ state phase shift δ_{13} becomes positive at a certain energy, and the $(1, 1)$ state phase shift δ_{11} may become positive at higher energies.

Taking for the parameters the set (F) which gave the best fit to the experimental behavior of the S-wave phase shifts, we show the results of the calculation of (20) for P-wave in Fig. 3(a).

These results coincide qualitatively with the type (i) solution (Fig. 3(b)) by Lomon and Chiu[18] who analyzed the data between 100 Mev and 220 Mev. They

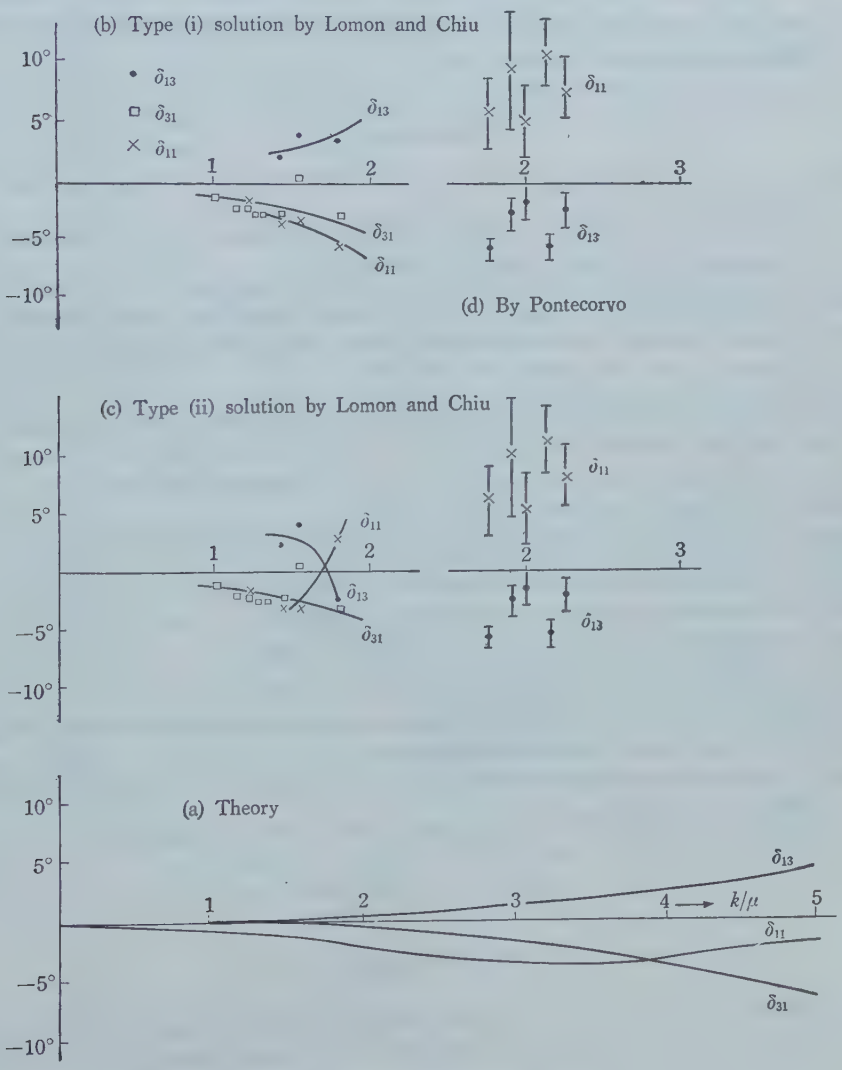

Fig. 3. Small P-wave phase shifts

obtained the two sets of solutions related to the ambiguity as to the sign of δ_{13} at 220 Mev, and preferred the type (i) solution, which gives δ_{13} the positive sign at that energy, from certain reasons. But the results (Fig. 3(d)) by Pontecorvo,[19] who analyzed the data between 220 Mev and 333 Mev, seem to be connected smoothly to the type (ii) solution (Fig. (3c)) by Lomon and Chiu. Before any definite conclusion is mentioned, more accurate analysis of the small P-wave phase shifts must be made. We here remark the results of dispersion-theoretical approach[20] which predict that δ_{13} is positive.

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References

1) For example, O. Hara, Prog. Theor. Phys. **24** (1960), 495 and reference 16). Other works are listed in these papers.
2) W. R. Frazer and J. R. Fulco, Phys. Rev. Letters **2** (1958), 356; Phys. Rev. **117** (1960), 1609. Y. Miyamoto, Prog. Theor. Phys. **24** (1960), 840.
3) P. Carruthers and H. A. Bethe, Phys. Rev. Letters **4** (1960), 536. K. Itabashi, M. Kato, K. Nakagawa and G. Takeda, Prog. Theor. Phys. **24** (1960), 529.
4) D. Ito, M. Yamazaki, T. Kobayashi and K. Mori, Prog. Theor. Phys. **22** (1959), 448.
5) T. Sakuma, Prog. Theor. Phys. **23** (1960), 810.
6) D. Ito and S. Minami, Prog. Theor. Phys. **16** (1956), 548.
7) For example, reference 14) and I. Umemura, Prog. Theor. Phys. **23** (1960), 1216.
8) M. Yamazaki, private communication.
9) K. Kawarabayashi and H. Miyazawa, Prog. Theor. Phys. **23** (1960), 490.
10) S. Ishida, Prog. Theor. Phys. **24** (1960), 1166.
11) S. D. Drell, M. H. Friedman and F. Zachariasen, Phys. Rev. **104** (1956), 236.
12) C. J. Goebel, Phys. Rev. Letters **1** (1958), 337.
13) E. L. Lomon, Nuovo Cimento **4** (1956), 106.
14) T. Kobayashi and A. Klein, Nuovo Cimento **8** (1958), 850.
15) J. S. Ball and G. F. Chew, Phys. Rev. **109** (1958), 1385.
16) R. Sugano, Prog. Theor. Phys. **22** (1959), 381.
17) C. J. Goebel, Phys. Rev. **101** (1956), 468.
18) Lomon and Chiu, *Proceedings of the 8th Conference on High Energy Nuclear Physics* at CERN (1958); Annals of physics **6**(1959), 50.
19) B. Pontecorvo, *Proceedings of the 9th Conference on High Energy Nuclear Physics* at Kiev (1959).
20) K. Ishida, A. Takahashi and Y. Ueda, Prog. Theor. Phys. **23** (1960), 731.

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On the Validity of Multiple Dispersion Representations

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The validity of multiple dispersion representations are investigated in perturbation theory. The lowest-order vertex function is analysed as functions of two and three variables. A conjecture is proposed for the possibility of double dispersion representations. As an application of the double dispersion representation, the Bethe-Salpeter equation in ladder approximation is solved in the case of total mass zero.

§ 1. Introduction

Five years ago, Nambu[1] proposed multiple dispersion representations for the three-body and four-body Green functions without any justification. It is known, however, that these representations break down even in the non-trivial lowest order of perturbation theory.[2]

But recently Mandelstam[3] has proposed a double dispersion representation for the two-particle scattering amplitude. Its validity has been checked in many cases, though its proof is not yet given.

In this paper, we investigate in what cases the multiple dispersion representation is possible. In § 2, we summarize properties of the general term of perturbation theory and present a general necessary condition for the possibility of multiple dispersion representations. In § 3, the lowest-order vertex function is investigated as concrete examples of the double dispersion and Nambu representations. In § 4, we analyse the curious complex singularity found in § 3, and propose a conjecture for the validity of the double dispersion representation. In § 5, we discuss the relations between the double dispersion representation and the Deser-Gilbert-Sudarshan-Ida integral representation for the vertex function with two variables. In § 6, as an application of the double dispersion representation, the Bethe-Salpeter equation in ladder approximation is solved in the case in which the total mass is zero but the exchanged meson mass is non-vanishing.

§ 2. General consideration

The matrix element, $M(k)$, corresponding to an arbitrary Feynman graph G is given by the following parametric integral[4] after all momentum integrations are carried out by means of the generalized Feynman identity:

$$M(k)=\int \frac{F(k,x)}{\{V(k,x)-i\varepsilon\}^{\alpha-2n}}dx^{(\alpha-1)}, \tag{2·1}$$

where

α : number of internal lines,

n : number of momentum integrations,

$dx^{(\alpha-1)}=\delta(1-\sum_{i=1}^{\alpha}x_i)\prod_{i=1}^{\alpha}dx_i,\ x_i\geqq 0,$

k_j : external momentum,

$F(k,x)$: a polynomial of k_j's whose coefficients are rational functions of x_i's. Their denominators are $U^l(l>0)$, where U is an nth order homogeneous polynomial of x_i's.

$V(k,x)$: a linear function of scalar products k_ik_j whose coefficients are homogeneous rational functions of x_i's.

The analyticity of $M(k)$ is determined by the function $V(k,x)$, whose explicit form is already derived in a previous work.[4] $V(k,x)$ is continuous with respect to x in the domain $x_i\geqq 0$ up to its boundary.[5]* The singularity of $M(k)$ is given as follows.[5)~7)]

Divide the internal lines of G into two parts, A and B, and put

$$x_j=0 \quad \text{for } j\in A. \tag{2·2}$$

Then V is reduced to $V^{(B)}$ that is defined as the V-function for the graph B^* which is obtained from G by shrinking each line of A to one point. There appears a singularity of $M(k)$ for such a set of k_j's that a *minimum* (but not a saddle point) of $V^{(B)}$ with respect to $x_i(i\in B)$ happens just at a zero point of $V^{(B)}$.

We call the singularity for which B represents just an intermediate state the "ordinary threshold", and that for which B contains an intermediate state as its proper subset the "anomalous threshold".**

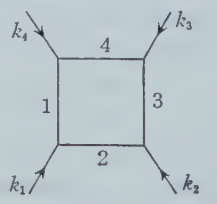

Fig. 1

As an example, we consider the lowest-order graph of the two-particle scattering shown in Fig. 1. Here we write

$$\begin{aligned}&-k_i^2=M_i^2, \quad (i=1,2,3,4)\\ &-(k_1+k_2)^2=s, \quad -(k_1+k_4)^2=t, \quad -(k_1+k_3)^2=u.\end{aligned} \tag{2·3}$$

Then an identity

$$s+t+u=\sum_{i=1}^{4}M_i^2 \tag{2·4}$$

---

* But it should be remarked that $\partial V/\partial x_i$ is not well defined at $x_i=0$ (for $n\geqq 2$) as was noticed previously.[5]

** If B contains no intermediate state no corresponding singularity appears.[5]

holds on account of momentum conservation $\sum_{i=1}^{4} k_i=0$. The V-function for this graph is

$$V(s,t;x)=(\sum_{i=1}^{4}x_i)(\sum_{i=1}^{4}x_i m_i^2)-x_1x_2M_1^2-x_2x_3M_2^2 \\ -x_3x_4M_3^2-x_4x_1M_4^2-x_1x_3s-x_2x_4t. \qquad (2\cdot5)$$

Singularities are classified into the following three types:

(a) $B=\{1,3\}$ or $\{2,4\}$.

$$V^{(B)}=(x_1+x_3)(x_1m_1^2+x_3m_3^2)-x_1x_3s, \quad \text{for } B=\{1,3\}. \qquad (2\cdot6)$$

(b) $B=\{1,2,3\}$ or $\{1\ 2,4\}$ or $\{1,3,4\}$ or $\{2,3,4\}$.

$$V^{(B)}=(\sum_{i=1}^{3}x_i)(\sum_{i=1}^{3}x_i m_i^2)-x_1x_2M_1^2-x_2x_3M_2^2-x_1x_3s, \\ \text{for } B=\{1,2,3\}. \qquad (2\cdot7)$$

(c) $B=\{1,2,3,4\}$.

$$V^{(B)}=V. \qquad (2\cdot8)$$

The type (a) corresponds to ordinary thresholds, while the types (b) and (c) correspond to anomalous thresholds, which do not always appear.

According to Mandelstam,[3] this matrix element can generally be represented as the following double dispersion form:

$$\int ds' \int dt' \frac{\varphi(s',t')}{(s'-s-i\varepsilon)(t'-t-i\varepsilon)}. \qquad (2\cdot9)$$

If there are only ordinary thresholds, this representation is naturally valid. Even when there appear anomalous thresholds, he has conjectured it will be still valid if they are of type (b) only. But the double dispersion representation will become impossible when a singularity of type (c) appears.

Now we call, in general, the singularity for which $V^{(B)}$ contains only one among variables $s, t, \cdots$, the "one-variable singularity", and that for which $V^{(B)}$ contains two or more variables the "many-variable singularity". In the above example, the types (a) and (b) are one-variable singularities, and the type (c) is a many-variable singularity. In general, the location of the singularity is obtained by eliminating x_i's from

$$\frac{\partial V^{(B)}}{\partial x_i}=0, \quad (i\in B). \qquad (2\cdot10)$$

For instance, the singularity of Fig. 1 is determined by the equation

$$\det\left(\frac{\partial^2 V^{(B)}}{\partial x_i \partial x_k}\right)=0, \quad (i,k\in B). \qquad (2\cdot11)$$

Therefore the one-variable singularity is always given as $s=$const. (or $t=$const.,

or$\cdots$), while the location of the many-variable singularity is generally determined by an equation

$$F(s, t, \cdots)=0 \tag{2·12}$$

which contains two or more variables. Since (2·12) gives a complex surface singularity when analytically continued, *the matrix element which contains many-variable singularity cannot be represented in such a multiple dispersion form as* (2·9). So it is very natural that Mandelstam representation breaks down in the case in which an anomalous threshold of type (c) appears.

Conversely, there arises a question: Is the multiple dispersion representation always possible for the matrix element which contains one-variable singularities alone? This will be examined by concrete examples in the next section.

But, in this place, we should remark that the concept of the many-variable singularity depends on the choice of variables. For example, the Mandelstam representation for the general graph is

$$\int ds' \int dt' \frac{\varphi_1(s', t')}{(s'-s-i\varepsilon)(t'-t-i\varepsilon)} + \int dt' \int du' \frac{\varphi_2(t', u')}{(t'-t-i\varepsilon)(u'-u-i\varepsilon)} + \int du' \int ds' \frac{\varphi_3(u', s')}{(u'-u-i\varepsilon)(s'-s-i\varepsilon)}, \tag{2·13}$$

where u is not independent of s and t as seen in (2·4). So, $u=$const. will be a many-variable singularity if we adopt s and t only as the variables. However, it becomes a one-variable singularity if we add u as a new variable. By introducing some of such extra variables, the multiple dispersion representation in a wide sense such as (2·13) may become possible to a considerable extent.* But this problem is not treated in the present paper (it will be an interesting problem whether or not it is necessary that $F(s, t, \cdots)$ is linear when we adopt F as a new variable.).

§ 3. Examples in the lowest-order vertex function

In this section we investigate the possibility of the multiple dispersion representation by some concrete examples. As the simplest ones, we take the non-trivial lowest-order vertex function shown in Fig. 2. For simplicity, we assume all particles are scalar.

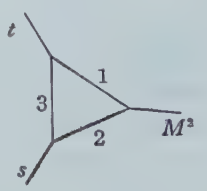

Fig. 2

[A] *Double dispersion representations*

We consider the vertex function as a function of two variables, s and t. Then the V-function is given by

* Prof. S. Hitotsumatsu pointed out that the following result was obtained in the theory of several complex variables. There are three closed sets $V_i (i=1, 2, 3)$ in a domain D. D ane $D-V_i$ are domains of regularity. Given an analytic function f regular in $D-V_1-V_2-V_3$, we can represent f as

$$f=f_{12}+f_{23}+f_{31},$$

where f_{ij} is regular in $D-V_i-V_j$.

$$V(s,t;x)=(\sum_{i=1}^{3}x_i)(\sum_{i=1}^{3}x_i m_i^2)-x_2x_3s-x_1x_3t-x_1x_2M^2. \qquad (3\cdot1)$$

Since, in the present case, the many-variable singularity appears only when an anomalous threshold appears, we first consider the condition for the absence of the anomalous threshold.

As was proved previously,* the anomalous threshold appears only when the stability conditions

$$\begin{aligned}&(m_2-m_3)^2<s<(m_2+m_3)^2,\\&(m_1-m_3)^2<t<(m_1+m_3)^2,\\&(m_1-m_2)^2<M^2<(m_1+m_2)^2\end{aligned} \qquad (3\cdot2)$$

are satisfied and the inequality

$$m_1^2<\frac{m_2t+m_3M^2}{m_2+m_3}-m_2m_3 \quad \text{for the spectrum of } s$$

or (3·3)

$$m_2^2<\frac{m_1s+m_3M^2}{m_1+m_3}-m_1m_3 \quad \text{for the spectrum of } t$$

holds. Therefore the condition for the absence of the anomalous threshold (for arbitrary s and t) is

$$\begin{aligned}&\text{i)} \quad |m_1-m_2|\geqq M,\\ \text{or}\quad &\text{ii)} \quad m_1+m_2\leqq M,\\ \text{or}\quad &\text{iii)} \quad m_3=0.\end{aligned} \qquad (3\cdot4)$$

(3·4) is obtained also as the condition of the impossibility of drawing such a dual diagram[1] as Fig. 3 for any positive values of s and t. In order to investigate the possibility of the double dispersion representation, it suffices to consider only the case in which one of (3·4) holds. Calculations are all presented in the Appendix.

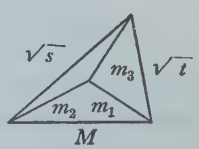

Fig. 3

i) In the case $M=0$, we can easily represent the matrix element (apart from some constant coefficient) as

$$M(s,t;M=0)=\int_{(m_2+m_3)^2}^{\infty}ds'\int_{(m_1+m_3)^2}^{\infty}dt'\frac{\varphi(s',t')}{(s'-s-i\varepsilon)(t'-t-i\varepsilon)}, \qquad (3\cdot5)$$

where

$$\varphi(s',t')\equiv\int_0^1 dx\;\delta(x(s'-t')+m_1^2-m_2^2)\,\theta\left(s'-\frac{m_2^2}{x}-\frac{m_3^2}{1-x}\right)$$

* See (6·13) of reference 5). The inequalities (3·2) follow from $\beta_{ii}>0$.

$$= \int_0^1 dx\ \delta(x(s'-t')+m_1^2-m_2^2)\,\theta\left(t'-\frac{m_1^2}{x}-\frac{m_3^2}{1-x}\right). \tag{3·6}$$

ii) When $M \geqq m_1+m_2$, the external particle becomes unstable, so an imaginary part appears even for $s<0$ or $t<0$. We consider only the case of $m_1=m_2=m_3=0$:

$$M(s, t\,;\, m_i=0) = \int \frac{dx^{(2)}}{-[x_2x_3s+x_1x_3t+x_1x_2M^2]-i\varepsilon} \tag{3·7}$$

since calculation generally become troublesome. We then obtain

$$M(s, t\,;\, m_i=0) = \int_0^\infty\int_0^\infty \frac{ds'\,dt'}{(s'-s-i\varepsilon)\,(t'-t-i\varepsilon)}\left[\frac{\theta(|\sqrt{s'}-\sqrt{t'}|-M)}{\sqrt{(s'+t'-M^2)^2-4s't'}}\right.$$

$$\left.+i\frac{\theta(M-|\sqrt{s'}-\sqrt{t'}|)\,\theta(\sqrt{s'}+\sqrt{t'}-M)}{\sqrt{4s't'-(s'+t'-M^2)^2}}-\frac{\theta(M-\sqrt{s'}-\sqrt{t'})}{\sqrt{(M^2-s'-t')^2-4s't'}}\right]. \tag{3·8}$$

If we want to avoid the imaginary part in the spectral function, there will appear spectra of $s<0$ or $t<0$.

iii) In the case of $m_3=0$, in order to avoid unnecessary complexity, we consider

$$N(s, t) \equiv -\left(\frac{\partial}{\partial m_1^2}+\frac{\partial}{\partial m_2^2}\right) M(s, t\,;\, m_3=0). \tag{3·9}$$

If $|m_1-m_2| \geqq M$, we obtain

$$N(s, t) = \int_{m_2^2}^\infty ds' \int_{m_1^2}^\infty dt' \frac{s'-m_2^2-t'+m_1^2}{(s'-s-i\varepsilon)\,(t'-t-i\varepsilon)}\,\epsilon(m_1(s'-m_2^2)-m_2(t'-m_1^2))$$

$$\times\delta(m_1^2(s'-m_2^2)^2+(M^2-m_1^2-m_2^2)(s'-m_2^2)(t'-m_1^2)+m_2^2(t'-m_1^2)^2), \tag{3·10}$$

where $\epsilon(x)=\theta(x)-\theta(-x)$. If $|m_1-m_2|<M<m_1+m_2$, the double dispersion representation becomes impossible. This circumstance will be investigated in detail in the next section.

[B] *Triple dispersion representation*

Putting $M^2=u$ in (3·1), we examine the possibility of the Nambu representation as a function of three variables s, t and u.

The condition for the absence of many-variable singularity is $m_1m_2m_3=0$, because (3·4) must hold for arbitrary M. It turns out, however, that the Nambu representation is allowed only in the case

$$m_1=m_2=m_3=0. \tag{3·11}$$

Then

$$M(s, t, u\,;\, m_i=0) = \int_0^\infty ds' \int_0^\infty dt' \int_0^\infty du' \qquad \frac{\pi^{-1}\Delta^{-1/2}\theta(\Delta)}{\cdots i\varepsilon)\,(t'-t-i\varepsilon)\,(u'-u-i\varepsilon)}, \tag{3·12}$$

where

$$\Delta \equiv (\sqrt{s'}+\sqrt{t'}+\sqrt{u'})\,(\sqrt{s'}+\sqrt{t'}-\sqrt{u'})\,(\sqrt{s'}-\sqrt{t'}+\sqrt{u'}) \times (-\sqrt{s'}+\sqrt{t'}+\sqrt{u'}). \tag{3·13}$$

Namely, the spectral function is non-vanishing only when a triangle can be composed by $\sqrt{s'}$, $\sqrt{t'}$ and $\sqrt{u'}$, and its value is inversely proportional to the area of this triangle.

§ 4. Curious complex singularity

We have pointed out in the last section [A] iii) that the double dispersion representation is impossible when $m_3=0$ and $|m_1-m_2|<M<m_1+m_2$ in spite of the absence of many-variable singularity. In this section we investigate the reason and condition of the impossibility of the double dispersion representation in the absence of many-variable singularity.

$N(s, t)$ in (3·9) is rewritten as

$$N(s,t)=\int_0^1 \frac{dx}{f(x)\cdot[(1-x)(m_2^2-s)+x(m_1^2-t)-i\varepsilon]}. \tag{4·1}$$

where

$$f(x)\equiv(1-x)m_2^2+xm_1^2-x(1-x)M^2. \tag{4·2}$$

The difficulty is caused by the fact that $1/f(x)$ has complex poles with respect to x when $|m_1-m_2|<M<m_1+m_2$. If we divide $1/f(x)$ into partial fractions, the problem is essentially reduced to the integral

$$I(s,t)=\int_0^1 \frac{dx}{\{x+e^{i\theta}(1-x)\}\{(1-x)(-s)+x(-t)\}} \quad (0<\theta<\pi)$$

$$=\frac{1}{s-e^{i\theta}t}\left[\log\frac{(-t)}{(-s)}+i\theta\right], \tag{4·3}$$

where we have rewritten $s-m_2^2$ by s and $t-m_1^2$ by t for simplicity. For $s=e^{i\theta}\cdot t$ $I(s, t)$ is analytic if

$$\arg s=\theta+\arg t. \tag{4·4}$$

Since the physical sheet of t is of course $0<\arg t<2\pi$, s must belong to the domain

$$\theta<\arg s<2\pi+\theta \tag{4·5}$$

in order to satisfy (4·4). But $2\pi<\arg s<2\pi+\theta$ belongs to an unphysical sheet. Conversely, if we put s on the physical sheet, the domain $-\theta<\arg t<0$ becomes necessary. Thus though $I(s, t)$ is analytic in the domain of a direct product of two cut-planes, the double dispersion representation in the ordinary sense becomes

impossible because these two cut-planes are relatively shifted by the angle θ.

If we allow the operation of taking out a pole on an unphysical sheet by using "complex distribution" which was previously introduced by the present author,[8] we can formally represent $I(s, t)$ as

$$I(s,t)=\int_0^\infty \frac{dt'}{t'^{(-)}-t}\int_0^\infty ds'\,\frac{\delta(s'-e^{i\theta}t')}{s'^{(-)}-s}=\int_0^\infty \frac{ds'}{s'^{(-)}-s}\int_0^\infty dt'\,\frac{e^{-i\theta}\delta(t'-e^{-i\theta}s')}{t'^{(-)}-t}. \tag{4·6}$$

But in the present paper such an extension is not treated.

In the following we consider in what cases the double dispersion representation in the ordinary sense becomes impossible by the appearance of the above curious complex singularity due to the relative shift of cut-planes. We propose the following conjecture for this problem.

CONJECTURE: The double dispersion representation will generally become impossible when the denominator function V satisfies the following condition:

The simultaneous equations

$$\begin{aligned} &x_j=0 \quad \text{for } j\in A,\\ &\frac{\partial V}{\partial x_i}=0 \quad \text{for } i\in B \end{aligned} \tag{4·7}$$

have *definite*[6),7)] (complex) solutions with respect to the ratios of x_i's, and the equation

$$F(s,t)=0 \tag{4·8}$$

which is obtained by eliminating x_i's from (4·7) has *complex* solutions with respect to s(or t) when t(or s) takes some values on the cut-line.

We examine this conjecture in the following. For simplicity we omit the superscript (B) of V.

i) We apply this conjecture to the example (4·1) in which we put $x_1=x$, $x_2=1-x$ and

$$V=f(x_1, x_2)[x_2(-s)+x_1(-t)], \tag{4·9}$$

where $f(x_1, x_2)$ may be any arbitrary mth order homogeneous polynomial. We know directly that the double dispersion representation is impossible only when $1/f$ has complex poles. Now, from (4·7) we have

$$\begin{aligned} &\frac{\partial V}{\partial x_1}=\frac{\partial f}{\partial x_1}[x_2(-s)+x_1(-t)]+f\cdot(-t)=0,\\ &\frac{\partial V}{\partial x_2}=\frac{\partial f}{\partial x_2}[x_2(-s)+x_1(-t)]+f\cdot(-s)=0. \end{aligned} \tag{4·10}$$

As $(m+1)V=\sum_{i=1}^{2} x_i\partial V/\partial x_i=0$, from (4·9) we get

$$f(x_1, x_2)=0 \tag{4·11}$$

or

$$x_2(-s)+x_1(-t)=0. \tag{4·12}$$

When (4·11) holds, from (4·10) we get

$$\partial f/\partial x_1=\partial f/\partial x_2=0 \tag{4·13}$$

or (4·12), but (4·13) generally has no solution since a ratio $x_1:x_2$ must satisfy two equations. Therefore it suffices to consider the case in which both (4·11) and (4·12) hold. So, if (4·11) has complex roots with respect to $x_1:x_2$, then from (4·12) s becomes complex for any real t $(\neq 0)$.

ii) In higher order graphs the denominator consists of two factors $V^m(1\leqq m\leqq\alpha-2n)$ and $U^l(l\geqq 1)$. From our conjecture, we can show that the above complex singularity is not caused by U.

We apply our conjecture to the denominator

$$V'\equiv U^l V^m. \tag{4·14}$$

We then have

$$\frac{\partial V'}{\partial x_i}=lU^{l-1}\frac{\partial U}{\partial x_j}V^m+mU^l V^{m-1}\frac{\partial V}{\partial x_i}=0. \tag{4·15}$$

From $V'(\propto\sum_i x_i\partial V'/\partial x_i)=0$, we get $U=0$ or $V=0$. For $U=0$, from (4·15) we find

$$\text{when}\quad l>1,\quad U=0\quad\text{only}; \tag{4·16}$$

$$\text{when}\quad l=1,\quad \partial U/\partial x_i=0, \tag{4·17}$$

$$\text{or}\quad U=0,\quad V=0. \tag{4·18}$$

Since U involves at least three x_i's except for some trivial cases, the ratios of x_i's are undetermined in the cases (4·16) and (4·18). In the case (4·17) the number of the equations is more than the number of the independent ratios of x_i's. Thus, after all, (4·15) is reduced to the equations

$$\partial V/\partial x_i=0. \tag{4·19}$$

iii) For lowest-order graphs Eq. (4·8) is given by such a form as (2·11). For the vertex function of Fig. 2 it is

$$F(s,t)\equiv -1/2\cdot\begin{vmatrix} 2m_1^2 & m_1^2+m_2^2-M^2 & m_1^2+m_3^2-t \\ m_1^2+m_2^2-M^2 & 2m_2^2 & m_2^2+m_3^2-s \\ m_1^2+m_3^2-t & m_2^2+m_3^2-s & 2m_3^2 \end{vmatrix}=0. \tag{4·20}$$

The discriminant of this quadratic equation with respect to s is

$$D=\{M^2-(m_1-m_2)^2\}\{M^2-(m_1+m_2)^2\}\{t-(m_1-m_3)^2\}\{t-(m_1+m_3)^2\}. \tag{4·21}$$

For the values on the cut-line of $t(t>(m_1+m_3)^2)$, the third and fourth factors of (4·21) are positive definite, hence it is necessary and sufficient for $D<0$ that

$$|m_1-m_2|<M<m_1+m_2. \tag{4·22}$$

As was stated in the last section, the double dispersion representation is not possible when (4·22) holds.

It will be worthwhile to notice that $F(s, t)$ in (4·20) is just the denominator function, Φ, of the explicit expression of the lowest-order vertex function (modified) which was given by Källén and Wightman,[2] hence the matrix element cannot have any complex singularity other than the solutions of (4·20).

iv) In order that the Nambu representation is possible, it is necessary that

(a) One of the virtual-particle masses is zero, say $m_3=0$, for the absence of many-variable singularity.

(b) One of the remaining virtual-particle masses is zero, say $m_2=0$, because it must be $D\geqq 0$ for any real $u=M^2$ when $t>(m_1+m_3)^2=m_1^2$.

Then Eq. (4·20) becomes

$$F(s, t)=s[m_1^2 s+(t-m_1^2)(u-m_1^2)]=0. \tag{4·23}$$

If the Nambu representation holds, a double dispersion representation is to hold when t and u are identified ($t\equiv u$). From our conjecture, therefore, (4·23) must not have any complex root also for $t\equiv u$ when $s>0$. Accordingly,

(c) The last one of the virtual-particle masses must also be zero.

Thus (3·11) is only the case in which the Nambu representation is valid.

v) We now apply our conjecture to the lowest-order two-particle scattering amplitude (cf. Fig. 1). For simplicity we make use of the notation of Karplus, Sommerfield and Wichmann:[9]

$$V=\sum_{i,j=1}^{4} y_{ij}x_i x_j, \tag{4·24}$$

where

$$y_{ij}=y_{ji}=\frac{m_i^2+m_j^2+(q_i-q_j)^2}{2m_i m_j}, \tag{4·25}$$

$$\begin{aligned} &-(q_1-q_2)^2=M_1^2, \quad -(q_2-q_3)^2=M_2^2,\\ &-(q_3-q_4)^2=M_3^2, \quad -(q_4-q_1)^2=M_4^2,\\ &-(q_1-q_3)^2=s, \quad -(q_2-q_4)^2=t. \end{aligned} \tag{4·26}$$

We assume the stability conditions

$$|y_{12}|<1,\ |y_{23}|<1,\ |y_{34}|<1,\ |y_{14}|<1. \tag{4·27}$$

Eq. (4·8) is written as

$$F(s, t)\equiv\det(y_{ij})=0. \tag{4·28}$$

The discriminant of this quadratic equation with respect to y_{13} is

$$D=\begin{vmatrix} y_{24} & y_{12} & 1 \\ y_{34} & 0 & y_{23} \\ 1 & y_{14} & y_{24} \end{vmatrix}^2 - \begin{vmatrix} 1 & y_{24} \\ y_{24} & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & y_{12} & 0 & y_{14} \\ y_{12} & 1 & y_{23} & y_{24} \\ 0 & y_{23} & 1 & y_{34} \\ y_{14} & y_{24} & y_{34} & 1 \end{vmatrix}, \tag{4·29}$$

which can be rewritten as

$$D=\begin{vmatrix} 1 & y_{23} & y_{24} \\ y_{23} & 1 & y_{34} \\ y_{24} & y_{34} & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & y_{12} & y_{14} \\ y_{12} & 1 & y_{24} \\ y_{14} & y_{24} & 1 \end{vmatrix} \tag{4·30}$$

because of Jacobi's theorem, hence it becomes $D<0$ when these two factors have opposite signs. They are positive for

$$\begin{aligned} &y_{23}y_{34}-\sqrt{(1-y_{23}^2)(1-y_{34}^2)} < y_{24} < y_{23}y_{34}+\sqrt{(1-y_{23}^2)(1-y_{34}^2)}, \\ &y_{12}y_{14}-\sqrt{(1-y_{12}^2)(1-y_{14}^2)} < y_{24} < y_{12}y_{14}+\sqrt{(1-y_{12}^2)(1-y_{14}^2)}, \end{aligned} \tag{4·31}$$

respectively. When there is no anomalous threshold for t, it becomes $y_{24}<-1$ for t on the cut-line $(t>(m_2+m_4)^2)$. Then each factor of (4·30) becomes negative definite, hence $D>0$. Thus it should be possible to write down the double dispersion representation when no anomalous threshold is present. Indeed, this was given explicitly by Mandelstam[3] and Kibble.[10] But, if there appears an anomalous threshold (of type (b)) for t, e.g.

$$y_{23}+y_{34}<0, \tag{4·32}$$

then the cut-line becomes

$$y_{24}<y_{23}y_{34}-\sqrt{(1-y_{23}^2)(1-y_{34}^2)} \tag{4·33}$$

Therefore if

$$y_{23}y_{34}-\sqrt{(1-y_{23}^2)(1-y_{34}^2)} > y_{12}y_{14}-\sqrt{(1-y_{12}^2)(1-y_{14}^2)}, \tag{4·34}$$

it can occur $D<0$ on the cut-line, so the double dispersion representation will break down.

vi) The condition for the impossibility of the double dispersion representation is (from the above argument)

(a) $y_{23}+y_{34}<0$

and $y_{23}y_{34}-\sqrt{(1-y_{23}^2)(1-y_{34}^2)} > y_{12}y_{14}-\sqrt{(1-y_{12}^2)(1-y_{14}^2)}$,

or (b) $y_{12}+y_{14}<0$

and $y_{12}y_{14}-\sqrt{(1-y_{12}^2)(1-y_{14}^2)} > y_{23}y_{34}-\sqrt{(1-y_{23}^2)(1-y_{34}^2)}$.

If we interchange s and t in the above argument, we obtain the condition

(c) $y_{12}+y_{23}<0$

and $y_{12}y_{23}-\sqrt{(1-y_{12}^2)(1-y_{23}^2)} > y_{14}y_{34}-\sqrt{(1-y_{14}^2)(1-y_{34}^2)}$.

or (d) $y_{14}+y_{34}<0$

and $y_{14}y_{34}-\sqrt{(1-y_{14}^2)(1-y_{34}^2)}>y_{12}y_{23}-\sqrt{(1-y_{12}^2)(1-y_{23}^2)}$.

The condition (a) or (b) should be equivalent to the condition (c) or (d), since the double dispersion representation is symmetric with respect to s and t. We will demonstrate that either (c) or (d) follows from (a) (other cases are similar).

We put*

$$y_{ij}=\sin\varphi_{ij},\quad |\varphi_{ij}|<\pi/2,\ (|i-j|\neq 2) \tag{4·35}$$

Then (a) can be rewritten as

$$\sin\varphi_{23}<-\sin\varphi_{34}$$

and (4·36)

$$\cos(\varphi_{23}+\varphi_{34})<\cos(\varphi_{12}+\varphi_{14}),$$

namely

$$-\pi<\varphi_{23}+\varphi_{34}<0$$

and (4·37)

$$|\varphi_{23}+\varphi_{34}|>|\varphi_{12}+\varphi_{14}|,$$

hence

$$\varphi_{23}+\varphi_{34}+\varphi_{12}+\varphi_{14}<0. \tag{4·38}$$

Now if

$$|\varphi_{12}+\varphi_{23}|>|\varphi_{14}+\varphi_{34}|, \tag{4·39}$$

from (4·38) we obtain

$$-\pi<\varphi_{12}+\varphi_{23}<0$$

and (4·40)

$$\cos(\varphi_{12}+\varphi_{23})<\cos(\varphi_{14}+\varphi_{34}),$$

i.e. the condition (c). If the inequality in (4·39) is reverse, we get the condition (d). Finally, if the equality holds, we must have

$$\varphi_{12}+\varphi_{23}=\varphi_{14}+\varphi_{34}<0 \tag{4·41}$$

on account of (4·38). If $\underset{|i-j|\neq 2}{\mathrm{Max}}\,\varphi_{ij}=\varphi_{12}$ or φ_{23},

$$\varphi_{12}+\varphi_{23}+\varphi_{34}+\varphi_{14}=2(\varphi_{12}+\varphi_{23})>2\underset{|i-j|\neq 2}{\mathrm{Max}}\,\varphi_{ij}-\pi \tag{4·42}$$

because of (4·35). Likewise, if $\mathrm{Max}\,\varphi_{ij}=\varphi_{14}$ or φ_{34},

$$\varphi_{12}+\varphi_{23}+\varphi_{34}+\varphi_{14}=2(\varphi_{14}+\varphi_{34})>2\underset{|i-j|\neq 2}{\mathrm{Max}}\,\varphi_{ij}-\pi. \tag{4·43}$$

We thus obtain from (4·38), (4·42) and (4·43)

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* $\varphi_{ij}=\pi/2-\theta_{ij}$ where θ_{ij} is the notation of reference 9).

$$2\underset{|i-j|\neq 2}{\text{Max}}\,\varphi_{ij}-\pi<\varphi_{12}+\varphi_{23}+\varphi_{34}+\varphi_{14}<0, \tag{4·44}$$

which is just the condition for the presence of the anomalous threshold of type (c) given by Karplus, Sommerfield and Wichmann.[9] Namely, in the absence of many-variable singularity, the condition (c) or (d) follows from (a).

There are of course the cases in which (4·38) is satisfied but (4·44) is not. For example, put

$$-\pi/2<\varphi_{12}=\varphi_{23}=\varphi_{34}<-\pi/3,\quad \varphi_{14}=0. \tag{4·45}$$

§ 5. Relations with the Deser-Gilbert-Sudarshan-Ida integral representation

An integral representation has been derived for the vertex function with two variables or the Bethe-Salpeter amplitude by Deser, Gilbert and Sudarshan[11] and independently by Ida.[12] It is (see Fig. 4)

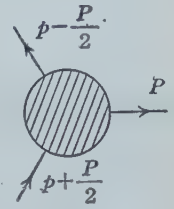

Fig. 4

$$\int_{-1}^{1}d\zeta\int_{0}^{\infty}d\kappa\frac{\Phi(\zeta,\kappa)}{(p-\zeta\cdot P/2)^2+\kappa-i\varepsilon}. \tag{5·1}$$

When we write

$$-\left(p+\frac{P}{2}\right)^2=s,\quad -\left(p-\frac{P}{2}\right)^2=t,\quad -P^2=M^2, \tag{5·2}$$

and transform

$$\zeta=1-2z$$
$$\kappa=\alpha-\frac{1-\zeta^2}{4}M^2=\alpha-z(1-z)M^2, \tag{5·3}$$

(5·1) becomes

$$M(s,t)=\int_{0}^{1}dz\int_{0}^{\infty}d\alpha\frac{\Psi(z,\alpha)}{\alpha-zs-(1-z)t-i\varepsilon}. \tag{5·4}$$

In this section we investigate the relations between (5·4) and the double dispersion representation,

$$M(s,t)=\int_{0}^{\infty}ds'\int_{0}^{\infty}dt'\frac{\varphi(s',t')}{(s'-s-i\varepsilon)(t'-t-i\varepsilon)}, \tag{5·5}$$

considered in § 3.

The representation (5·5) is naturally always converted into the form (5·4) whose spectral function is given by

$$\Psi(z,\alpha)=\int_{0}^{\infty}ds'\int_{0}^{\infty}dt'\,\varphi(s',t')\,\delta'(\alpha-zs'-(1-z)t'), \tag{5·6}$$

provided that the convergence condition is satisfied.

Conversely, of course, (5·4) cannot in general be rewritten as (5·5), but the following conversion is formally possible.

First, we notice the identity

$$\int_0^1 \frac{dz}{\alpha - zs - (1-z)t - i\varepsilon} = \int_\alpha^\infty ds' \int_\alpha^\infty dt' \frac{\delta(s'-t')}{(s'-s-i\varepsilon)(t'-t-i\varepsilon)}, \tag{5·7}$$

or more generally,

$$\begin{aligned}\int_0^1 \frac{z^n dz}{\alpha - zs - (1-z)t - i\varepsilon} &= \left[\int_\alpha^\infty d\alpha \frac{\partial}{\partial s}\right]^n \int_0^1 \frac{dz}{\alpha - zs - (1-z)t - i\varepsilon} \\ &= \int_\alpha^\infty ds' \int_\alpha^\infty dt' \frac{(n!)^{-1}(t'-\alpha)^n \delta^{(n)}(s'-t')}{(s'-s-i\varepsilon)(t'-t-i\varepsilon)}.\end{aligned} \tag{5·8}$$

If $\Psi(z, \alpha)$ can be expanded into a power series of z:

$$\Psi(z, \alpha) = \sum_{n=0}^\infty \psi_n(\alpha) z^n, \tag{5·9}$$

then using (5·8) we have formally

$$\begin{aligned}M(s, t) &= \sum_{n=0}^\infty \int_0^\infty d\alpha \psi_n(\alpha) \int_0^1 \frac{z^n dz}{\alpha - zs - (1-z)t - i\varepsilon} \\ &= \int_0^\infty ds' \int_0^\infty dt' \frac{\sum_{n=0}^\infty \int_0^{t'} d\alpha \psi_n(\alpha) (n!)^{-1} (t'-\alpha)^n \delta^{(n)}(s'-t')}{(s'-s-i\varepsilon)(t'-t-i\varepsilon)},\end{aligned} \tag{5·10}$$

namely

$$\varphi(s', t') = \sum_{n=0}^\infty \int_0^{t'} d\alpha \psi_n(\alpha) \frac{(t'-\alpha)^n}{n!} \delta^{(n)}(s'-t'). \tag{5·11}$$

An infinite series of $\delta^{(n)}$ is not well defined in the ordinary sense of Schwartz's distribution. But if we admit such a formal series as

$$\delta(x+z) = \sum_{n=0}^\infty \frac{z^n}{n!} \delta^{(n)}(x), \tag{5·12}$$

there will be the case in which (5·11) gives a meaningful result. In fact, the difference between the spectra of s' and t' must be caused by such a situation.

As an example, we will consider the matrix element of Fig. 2,

$$M(s, t) \equiv \int \frac{dx^{(2)}}{\sum_{i=1}^3 x_i m_i^2 - x_1 x_2 M^2 - x_2 x_3 s - x_1 x_3 t - i\varepsilon}. \tag{5·13}$$

Multiplying it by

$$1=\int_0^\infty d\alpha\,\delta\left(\alpha-\frac{\sum x_i m_i^2-x_1x_2M^2}{(x_1+x_2)x_3}\right)\int_0^1 dz\,\delta\left(z-\frac{x_2}{x_1+x_2}\right) \tag{5·14}$$

where $M<m_1+m_2$, we can easily convert (5·13) into the form of (5·4). Then its spectral function is

$$\Psi(z,\alpha)=\int\frac{dx^{(2)}}{(x_1+x_2)x_3}\,\delta\left(\alpha-\frac{\sum x_i m_i^2-x_1x_2M^2}{(x_1+x_2)x_3}\right)\delta\left(z-\frac{x_2}{x_1+x_2}\right), \tag{5·15}$$

which is rewritten as

$$\Psi(z,\alpha)=\int_0^1(1-x)\,dx\cdot\delta(x(1-x)\alpha+(1-x)^2z(1-z)M^2 -(1-x)\{(1-z)m_1^2+zm_2^2\}-xm_3^2) \tag{5·16}$$

by putting $x_1=(1-x)(1-y)$, $x_2=(1-x)y$ and $x_3=x$.

In particular, when $M=0$, $\Psi(z,\alpha)$ can be expanded into the power series (5·9) by the aid of (5·12). Then

$$\psi_n(\alpha)=\frac{(m_1^2-m_2^2)^n}{n!}\int_0^1(1-x)^{n+1}dx\cdot\delta^{(n)}(x(1-x)\alpha-(1-x)m_1^2-xm_3^2). \tag{5·17}$$

Therefore from (5·10) the spectral function of the double dispersion representation is

$$\varphi(s',t')=\sum_{n=0}^\infty\frac{(m_1^2-m_2^2)^n}{n!}\int_0^{t'}d\alpha\int_0^1(1-x)^{n+1}dx \times\delta^{(n)}(x(1-x)\alpha-(1-x)m_1^2-xm_3^2)\frac{(t'-\alpha)^n}{n!}\delta^{(n)}(s'-t')$$

$$=\sum_{n=0}^\infty\frac{(m_1^2-m_2^2)^n}{n!}\int_0^1\frac{dx}{x^{n+1}}\theta\left(t'-\frac{m_1^2}{x}-\frac{m_3^2}{1-x}\right)\delta^{(n)}(s'-t'). \tag{5·18}$$

Making use of (5·12) again, we obtain

$$\varphi(s',t')=\int_0^1 dx\,\delta(x(s'-t')+m_1^2-m_2^2)\,\theta\left(t'-\frac{m_1^2}{x}-\frac{m_3^2}{1-x}\right), \tag{5·19}$$

which surely coincide with (3·6).

When $M\neq0$, the expansion of $\Psi(z,\alpha)$ in (5·16) becomes much more complicated than (5·12). So, we will consider $N(s,t)$ in (4·1) for which the spectral function is given by

$$\cdot\delta(\alpha-(1-z)m_2^2-zm_1^2)/f(z). \tag{5·20}$$

Dividing $1/f(z)$ into partial fractions and expanding them into power series by means of (5·12), we obtain the correct result (3·10) if $M \leqq |m_1-m_2|$. If $|m_1-m_2|<M<m_1+m_2$, we shall formally obtain "complex distributions" as presented in (4·6).

§ 6. Application to the Bethe-Salpeter equation

The Bethe-Salpeter equation for the system in which two scalar particles (mass m_1, m_2) are bound by exchanging scalar mesons (mass μ) is

$$f(p, P)=\frac{1}{(p+P/2)^2+m_1^2-i\varepsilon}\cdot\frac{1}{(p-P/2)^2+m_2^2-i\varepsilon} \times\frac{\lambda}{\pi^2 i}\int\frac{d^4q}{(p-q)^2+\mu^2-i\varepsilon}f(q, P), \tag{6·1}$$

in ladder approximation. Here P is the total momentum, p the internal momentum, and λ the square of coupling constant. This equation was solved in the case $\mu=0$ by Wick and Cutkosky.[13]

For $\mu\neq 0$, it has recently been shown by the present author[14] and Ida[15] that when $P_\mu=0$ the Bethe-Salpeter equation (6·1) can be solved by an iteration method with respect to the spectral function, as the integral representation (5·1) then reduces to a single dispersion. In this section we extend the above method to the case $M=0$ under the assumption that Bethe-Salpeter amplitude, $f(p, P)\equiv f(s, t)$ (see (5·2)), can be represented in a double dispersion form,

$$f(s, t)=\int_0^\infty ds'\int_0^\infty dt'\frac{\varphi(s', t')}{(s'-s-i\varepsilon)(t'-t-i\varepsilon)}, \tag{6·2}$$

where subtraction terms will be unnecessary on account of bound state problem.

Using this representation, we have

$$\frac{1}{\pi^2 i}\int\frac{d^4q}{(p-q)^2+\mu^2-i\varepsilon}f(q, P)=\int_0^\infty ds''\int_0^\infty dt''\,M(s, t; s'', t'')\varphi(s'', t''), \tag{6·3}$$

where

$$M(s, t; s'', t'')\equiv\frac{1}{\pi^2 i}\int\frac{d^4q}{\{(p-q)^2+\mu^2-i\varepsilon\}\{(q+P/2)^2+s''-i\varepsilon\}\{(q-P/2)^2+t''-i\varepsilon\}}. \tag{6·4}$$

$M(s, t; s'', t'')$ is just the matrix element of Fig. 2 with

$$m_1=\sqrt{t''},\quad m_2=\sqrt{s''},\quad m_3=\mu. \tag{6·5}$$

Therefore from (3·5)

$$M(s, t; s'', t'')=\int_{(\sqrt{s''}+\mu)^2}^\infty ds'\int_{(\sqrt{t''}+\mu)^2}^\infty dt'\frac{K^0(s', t'; s'', t'')}{(s'-s-i\varepsilon)(t'-t-i\varepsilon)}, \tag{6·6}$$

with

$$K^0(s', t'; s'', t'') \equiv \int_0^1 dx \delta(x(s'-t')+t''-s'')\,\theta\left(s'-\frac{s''}{x}-\frac{\mu^2}{1-x}\right). \quad (6\cdot7)$$

Accordingly, (6·3) becomes

$$\int ds' \int dt' \frac{\Phi(s', t')}{(s'-s-i\varepsilon)(t'-t-i\varepsilon)}, \quad (6\cdot8)$$

with

$$\Phi(s', t') \equiv \int_0^\infty ds'' \int_0^\infty dt'' K^0(s', t'; s'', t'')\varphi(s'', t''). \quad (6\cdot9)$$

Now, as is seen from (6·6) or (6·7), $K^0(s', t'; s'', t'')$ is non-vanishing only where

$$\sqrt{s'} \geqq \sqrt{s''}+\mu, \quad \sqrt{t'} \geqq \sqrt{t''}+\mu. \quad (6\cdot10)$$

So, we substitute (6·2) and (6·8) in both sides of (6·1), the continuous spectrum of the right-hand side starts from the value larger by μ than the smallest value of the spectrum of the left-hand side. Therefore in order that both spectra coincide, the smallest value must be supplied from the poles

$$\frac{1}{m_1^2-s-i\varepsilon} \cdot \frac{1}{m_2^2-t-i\varepsilon}. \quad (6\cdot11)$$

By including these factors under the double dispersion integral, the spectral function becomes

$$\begin{aligned}&\delta(s'-m_1^2)\delta(t'-m_2^2)\int dx \int dy \frac{\Phi(x, y)}{(x-m_1^2)(y-m_2^2)} \\ &-\delta(t'-m_2^2)\frac{1}{s'-m_1^2}\int dy \frac{\Phi(s', y)}{y-m_2^2} - \delta(s'-m_1^2)\frac{1}{t'-m_2^2}\int dx \frac{\Phi(x, t')}{x-m_1^2} \\ &+\frac{\Phi(s', t')}{(s'-m_1^2)(t'-m_2^2)}. \end{aligned} \quad (6\cdot12)$$

Then putting*

$$\varphi(s', t') = \delta(s'-m_1^2)\delta(t'-m_2^2)+\varphi_1(s')\delta(t'-m_2^2)+\varphi_2(t')\delta(s'-m_1^2)+\varphi_{12}(s', t'), \quad (6\cdot13)$$

where

$$\begin{aligned}\varphi_1(s') &= \varphi_1(s')\theta(s'-(m_1+\mu)^2), \\ \varphi_2(t') &= \varphi_2(t')\theta(t'-(m_2+\mu)^2), \end{aligned} \quad (6\cdot14)$$

* For simplicity we normalize the coefficient of the first term to unity.

$$\varphi_{12}(s', t') = \varphi_{12}(s', t')\theta(s' - (m_1+\mu)^2)\theta(t' - (m_2+\mu)^2),$$

and comparing the spectral functions of both sides of (6·1), we obtain

$$1 = \lambda \int dx \int dy \frac{\Phi(x, y)}{(x - m_1^2)(y - m_2^2)}, \tag{6·15}$$

$$\varphi_1(s') = -\frac{\lambda}{s' - m_1^2} \int dy \frac{\Phi(s', y)}{y - m_2^2}, \tag{6·16}$$

$$\varphi_2(t') = -\frac{\lambda}{t' - m_2^2} \int dx \frac{\Phi(x, t')}{x - m_1^2}, \tag{6·17}$$

$$\varphi_{12}(s', t') = \frac{\lambda \Phi(s', t')}{(s' - m_1^2)(t' - m_2^2)}. \tag{6·18}$$

Substituting (6·9) and (6·13) in the right-hand side of (6·18), we get an equation for $\varphi_{12}(s', t')$:

$$\varphi_{12}(s', t') = \eta_{12}(s', t') + \int ds' \int dt' K(s', t' ; s'', t'') \varphi_{12}(s'', t''), \tag{6·19}$$

where

$$\eta_{12}(s', t') \equiv K(s', t' ; m_1^2, m_2^2) + \int ds'' K(s', t' ; s'', m_2^2) \varphi_1(s'') + \int dt'' K(s', t' ; m_1^2, t'') \varphi_2(t''), \tag{6·20}$$

$$K(s', t' ; s'', t'') \equiv \frac{\lambda}{(s' - m_1^2)(t' - m_2^2)} K^0(s', t' ; s'', t''). \tag{6·21}$$

Because of (6·10), (6·19) is a Volterra integral equation of the second kind. Therefore the resolvent $R(s', t' ; s'', t'')$ is given by the Neumann series (always convergent) :

$$R(s', t' ; s'', t'') = \sum_{n=1}^{\infty} K^n(s', t' ; s'', t''), \tag{6·22}$$

where

$$K^n(s', t' ; s'', t'') \equiv \int dx \int dy K^{n-1}(s', t' ; x, y) K(x, y ; s'', t'') \tag{6·23}$$

with $K^1 \equiv K$. Thus the solution of (6·19) is

$$\begin{aligned} \varphi_{12}(s', t') &= \eta_{12}(s', t') + \int ds'' \int dt'' R(s', t' ; s'', t'') \eta_{12}(s'', t'') \\ &= R(s', t' ; m_1^2, m_2^2) + \int ds'' R(s', t' ; s'', m_2^2) \varphi_1(s'') \\ &\quad + \int dt'' R(s', t' ; m_1^2, t'') \varphi_2(t'') \end{aligned} \tag{6·24}$$

on account of the property of the resolvent,

$$R(s', t'; s'', t'') = K(s', t'; s'', t'') + \int dx \int dy\, K(s', t'; x, y) R(x, y; s'', t''). \tag{6·25}$$

Next, substituting (6·9), (6·13) and (6·24) in (6·16) and (6·17), and utilizing (6·25) again, we have

$$\varphi_1(s') = -\int dt' R(s', t'; m_1^2, m_2^2) - \int dt' \int ds'' R(s', t'; s'', m_2^2) \varphi_1(s'') - \int dt' \int dt'' R(s', t'; m_1^2, t'') \varphi_2(t''), \tag{6·26}$$

$$\varphi_2(t') = -\int ds' R(s', t'; m_1^2, m_2^2) - \int ds' \int ds'' R(s', t'; s'', m_2^2) \varphi_1(s'') - \int ds' \int dt'' R(s', t'; m_1^2, t'') \varphi_2(t''), \tag{6·27}$$

i. e.

$$\begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} = \begin{pmatrix} K_{11}(x, m_1^2) \\ K_{22}(x, m_2^2) \end{pmatrix} + \int dy \begin{pmatrix} K_{11}(x, y), & K_{12}(x, y) \\ K_{21}(x, y), & K_{22}(x, y) \end{pmatrix} \begin{pmatrix} \varphi_1(y) \\ \varphi_2(y) \end{pmatrix} \tag{6·28}$$

with

$$\begin{aligned} K_{11}(s', s'') &\equiv -\int dt' R(s', t'; s'', m_2^2), \\ K_{12}(s', t'') &\equiv -\int dt' R(s', t'; m_1^2, t''), \\ K_{21}(t', s'') &\equiv -\int ds' R(s', t'; s'', m_2^2), \\ K_{22}(t', t'') &\equiv -\int ds' R(s', t'; m_1^2, t''). \end{aligned} \tag{6·29}$$

(6·28) is again a Volterra integral equation of the second kind. It can likewise be solved by the iteration method. Thus $\varphi_1(s')$ and $\varphi_2(t')$ are obtained.

Finally, we substitute the solutions $\varphi_1(s')$, $\varphi_2(t')$ and $\varphi_{12}(s', t')$ in (6·15). Then (6·15) becomes a transcendental equation for λ. The eigenvalues of λ are determined by this equation.

It is of some interest to notice the relations

$$\varphi_1(s') = -\int dt' \varphi_{12}(s', t'), \quad \varphi_2(t') = -\int ds' \varphi_{12}(s', t'), \tag{6·30}$$

$$\int ds' \int dt' \varphi_{12}(s', t') = 1, \quad \int ds' \int dt' \varphi(s', t') = 0, \tag{6·31}$$

which immediately follow from (6·15) ~ (6·18) and (6·13). (6·30) is checked also by (6·24), (6·26) and (6·27).

The above method cannot be applied to the general case $M \neq 0$, for the double dispersion representation (6·2) then becomes impossible. The Bethe-Salpeter am-

plitude which can be represented as (6·2) does not tend to a solution of the Schrödinger equation in non-relativistic limit. This has kindly been pointed out to the author by Dr. M. Ida.

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Appendix

Calculations in § 3

[A] i)

$$M(s, t; M=0) = \int \frac{ds'}{s'-s-i\varepsilon} I(s', t; M=0)$$

with

$$I(s', t; M=0) \equiv \int dx^{(2)} \delta(x_2 x_3 s' + x_1 x_3 t - \sum_{i=1}^{3} x_i m_i^2).$$

Taking $t<(m_1+m_3)^2$ and putting $x_1=(1-x)y$, $x_2=(1-x)(1-y)$ and $x_3=x$, we obtain

$$\begin{aligned} I(s', t; M=0) &= \int_0^1 dx \int_0^1 dy \cdot \delta\left(x\{(1-y)s'+yt\} - ym_1^2 - (1-y)m_2^2 - \frac{x}{1-x} m_3^2\right) \\ &= \int_0^1 \frac{dx}{x(s'-t')+m_1^2-m_2^2} \theta\left(xs' - m_2^2 - \frac{x}{1-x} m_3^2\right) \\ &= \int \frac{dt'}{t'-t} \int_0^1 dx \, \delta(x(s'-t')+m_1^2-m_2^2)\, \theta\left(s' - \frac{m_2^2}{x} - \frac{m_3^2}{1-x}\right). \end{aligned}$$

By analytic continuation we get (3·5) with (3·6).

ii) (3·8) is immediately obtained from (3·12) by carrying out the integration over u.

iii)

$$N(s, t) = \int \frac{(x_1+x_2)\, dx^{(2)}}{[x_1 m_1^2 + x_2 m_2^2 - x_1 x_2 M^2 - x_2 x_3 s - x_1 x_3 t - i\varepsilon]^2}.$$

Putting $x_1=(1-x)y$, $x_2=(1-x)(1-y)$ and $x_3=x$, and carrying out the integration over x, we have

$$N(s, t) = \int_0^1 \frac{dy}{\{ym_1^2+(1-y)m_2^2-y(1-y)M^2-i\varepsilon\}\{(1-y)(m_2^2-s)+y(m_1^2-t)-i\varepsilon\}}.$$

Taking $t<m_1^2$ and $M<|m_1-m_2|$, we can write

$$\begin{aligned} N(s, t) &= \int_{m_2^2}^{\infty} \frac{ds'}{s'-s-i\varepsilon} \int_0^1 dy \, \frac{\delta((1-y)(m_2^2-s')+y(m_1^2-t))}{ym_1^2+(1-y)m_2^2-y(1-y)M^2} \\ &= \int_{m_2^2}^{\infty} \frac{ds'}{s'-s-i\varepsilon} \cdot \frac{s'-m_2^2-t+m_1^2}{m_1^2(s'-m_2^2)^2+(M^2-m_1^2-m_2^2)(s'-m_2^2)(t-m_1^2)+m_2^2(t-m_1^2)^2}. \end{aligned}$$

This can be easily represented as (3·10).
[B]

$$M(s, t, u\,;\, m_i=0) = \int \frac{ds'}{s'-s-i\varepsilon} I(s', t, u\,;\, m_i=0)$$

with

$$I(s', t, u\,;\, m_i=0) \equiv \int dx^{(2)}\, \delta(x_2x_3s'+x_1x_3t+x_1x_2u)$$

$$=\frac{\theta(s')}{\sqrt{(s'-t-u)^2-4tu}} \log \frac{s'-t-u+\sqrt{(s'-t-u)^2-4tu}}{s'-t-u-\sqrt{(s'-t-u)^2-4tu}}$$

$$=\theta(s') \int_0^\infty \frac{dt'}{(t'-t)\sqrt{(s'+t'-u)^2-4s't'}}$$

where we have taken $t<0$ and $u<0$. We then notice that

$$\frac{1}{\sqrt{(s'+t'-u)^2-4s't'}} = \frac{1}{\pi} \int_{(\sqrt{s'}-\sqrt{t'})^2}^{(\sqrt{s'}+\sqrt{t'})^2} \frac{du'}{(u'-u)\sqrt{-(s'+t'-u')^2+4s't'}}\,.$$

Thus (3·12) is obtained.

Note added in proof : Recently Eden has proposed a "proof" of the Mandelstam representation, but it is not correct because his sufficient condition cannot exclude the possibitity of the complex singularity stated in § 4.

References

1) Y. Nambu, Phys. Rev. **100** (1955), 394.
2) G. Källén and A. Wightman, Mat. Fys. Skv. Dan. Vid. Selsk. **1** No. 6, (1958).
3) S. Mandelstam, Phys. Rev. **112** (1958), 1344; **115** (1959) 1741, 1752.
4) N. Nakanishi, Prog. Theor. Phys. **17** (1957), 401.
5) N. Nakanishi, Prog. Theor. Phys. **22** (1959), 128.
6) N. Nakanishi, Prog. Theor. Phys. **23** (1960), 284.
7) L. D. Landau, Nuclear. Phys. **13** (1959), 181.
8) N. Nakanishi, Prog. Theor. Phys. **19** (1958), 607.
9) R. Karplus, C. M. Sommerfield and E. H. Wichmann, Phys. Rev. **114** (1959), 376.
10) T. W. B. Kibble, Phys. Rev. **117** (1960), 1159.
11) S. Deser, W. Gilbert and E. C. T. Sudarshan, Phys. Rev. **115** (1959), 731.
12) M. Ida, Prog. Theor. Phys. **23** (1960), 1151.
13) G. C. Wick, Phys. Rev. **96** (1954), 1124.
R. E. Cutkosky, Phys. Rev. **96** (1954), 1135.
14) N. Nakanishi, Soryushiron Kenkyu (mimeographed circular in Japanese) **21** (1960), 193.
15) M. Ida, private communication.

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Lattice Vibration and Random Walk Problems

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Mathematical relations between the lattice vibration and the random walk problems are presented for the case of a system of one-dimensional harmonically coupled oscillators. Namely it is shown that the Laplace and Mellin transforms of the frequency spectrum of the lattice vibration can be expressed in terms of the transition probabilities of the random walk problems with the continuous and discrete time parameters respectively.

§ 1. Introduction

In recent years a considerable number of investigations have been concerned with the problem of studying the effects of the impurities on the lattice vibration.[1)~4)] Namely the appearance of the localized vibrational modes and the modifications of the frequency spectrum were investigated by analyzing the time-independent equations of motion. Montroll and his coworkers suggested the utility of a random walk problem to calculate the moments of the frequency spectrum.[5)] As a preliminary account of these problems, therefore, it seems to be interesting to present the exact formal relations between the problem of the lattice vibration and the random walk problem. In this note we mainly investigate such relations. In the following sections, we show that the Laplace and Mellin transforms of the frequency spectrum can be obtained by solving the random walk problems with the continuous and discrete time parameter respectively.

§ 2. Lattice vibration

We shall consider the system of one-dimensional harmonically coupled oscillators which consists of N identical particles of mass m. The equations of motion can be written in the form

$$\frac{1}{\alpha}\frac{d^2u_n}{dt^2}=\mathfrak{D}(n)\cdot u_n, \tag{1}$$

where u_n is the deviation of the coordinate of nth particle from its equilibrium position and $\mathfrak{D}(n)$ denotes the difference operator defined by

$$\mathfrak{D}(n)\cdot u_n=-\alpha(u_n-u_{n+1})-\alpha(u_n-u_{n-1})=\alpha(u_{n+1}+u_{n-1}-2u_n). \tag{2}$$

The parameter α in the above equations is $\sqrt{K/m}$, where K denotes the force

constant of the interaction between the neighbouring particles of mass m.

The equations of motion (1) can, of course, be easily solved under certain boundary conditions. In this paper, however, we are interested in studying the mathematical relations which connect the characteristic properties of the lattice vibration with the random walk problems.

If we use the coordinates of the particles in equilibrium configuration, instead of the subscript n of the displacement u, the difference operator (2) can be rewritten in a form

$$\mathfrak{D}(X)\cdot u(X)=\alpha\{u(X+a)+u(X-a)-2u(X)\}=2\alpha\left\{\cos\left(a\frac{d}{dX}\right)-1\right\}u(X), \tag{3}$$

where the equilibrium distance of neighbouring particles is denoted by a and the coordinate X takes only the discrete values na ($n=1, 2, \cdots, N$). When the lattice spacing a is sufficiently small, the difference operator (3) obviously takes the form

$$\mathfrak{D}(X)\to\alpha a^2\frac{d^2}{dX^2}. \tag{4}$$

The differential operator (4) is clearly the one which describes the vibration of a continuous elastic string* and also appears in the diffusion equation which describes the motion of free Brownian particle. Therefore, in our case of the discrete lattice space, it is quite natural to associate a random walk, in which only steps of the length a are allowed, with the lattice vibration which is characterized by the difference operator (2).

§ 3. Characteristic matrix and transition probabilities

Before specifying a random walk problem, we shall consider the eigenvalue problem

$$\mathfrak{D}(n)\cdot u_n=-\lambda u_n \tag{5}$$

under a certain boundary condition. We denote the eigenvalues and the components of eigenvectors of Eq. (5) by λ_ν and $u_n^{(\nu)}$ respectively. Using these eigenvalues and eigenvectors we define the characteristic matrix of Laplace type as

$$p(n, m; \tau)=\sum_\nu u_m^{(\nu)}\exp(-\lambda_\nu\tau)u_n^{(\nu)}. \tag{6}$$

The characteristic matrix (6) obviously satisfies the equation

$$\begin{aligned}\frac{\partial}{\partial\tau}p(n, m; \tau)&=\mathfrak{D}(n)\cdot p(n, m; \tau)\\&=\alpha\{p(n+1, m; \tau)+p(n-1, m; \tau)-2p(n, m; \tau)\}\end{aligned} \tag{7}$$

* In this limit, the equation of motion (1) leads to the equation $d^2u/dt^2=(T/\rho)d^2u/dX^2$, where the tension T is given by Ka and the linear density ρ by m/a.

and also, if we assume that the eigenvectors are normalized, it should satisfy the initial condition

$$p(n, m; o) = \sum_{\nu} u_m^{(\nu)} u_n^{(\nu)} = \delta_{m,n}. \tag{8}$$

Now let us consider the one-dimensional random walk problem in which only the steps of length a are permitted and both the probabilities that a variable point makes a step to the right and left in the time interval $\Delta\tau$ are equal to $\alpha(\Delta\tau)$, so the probability of staying at the same position in the time interval $\Delta\tau$ is $\beta(\Delta\tau) = 1-2\alpha(\Delta\tau)$. In this case, if we denote the probability of finding the variable point, which initially started from the lattice point m, at the lattice point n at time $\tau+\Delta\tau$ by $p(n, m; \tau+\Delta\tau)$, it obviously follows that the relation

$$\begin{aligned} p(n, m; \tau+\Delta\tau) = {} & \beta(\Delta\tau) p(n, m; \tau) + \alpha(\Delta\tau) p(n+1, m; \tau) \\ & + \alpha(\Delta\tau) p(n-1, m; \tau) \end{aligned} \tag{9}$$

or

$$\begin{aligned} & p(n, m; \tau+\Delta\tau) - p(n, m; \tau) \\ & = \alpha(\Delta\tau) \{p(n+1, m; \tau) + p(n-1, m; \tau) - 2p(n, m; \tau)\} \end{aligned} \tag{10}$$

holds. Dividing both sides of Eq. (10) by $\Delta\tau$ and assuming that the limit $\alpha(\Delta\tau)/\Delta\tau \to \alpha$ exists as $\Delta\tau \to 0$, we get in the limit

$$\frac{\partial}{\partial\tau} p(n, m; \tau) = \alpha \{p(n+1, m; \tau) + p(n-1, m; \tau) - 2p(n, m; \tau)\}, \tag{11}$$

which is just the equation which holds for the characteristic matrix of the lattice vibration, Eq. (7). Therefore the parameter $\alpha = \sqrt{K/m}$ in Eq. (7) stands for the transition probability rate in the corresponding random walk problem. As already mentioned, it is seen that by the limiting process of $a \to 0$ Eq. (11) is reduced to the diffusion equation with the diffusion constant αa^2.

As for the boundary conditions, we can easily translate the conditions submitted to the lattice vibrations into the boundary conditions for the random walk problem. For instance, the case of the lattice vibration with fixed ends corresponds to a random walk with reflecting barriers one of which is located between the lattice points 1 and 2, and the other between $N-1$ and N. The case of periodic boundary condition can be treated as a random walk on a ring lattice in which the lattice points 1 and N are connected at the distance a.

Thus it has been proved that the characteristic matrix $p(n, m; \tau)$ defined by Eq. (6) is just the probability of finding the variable point at the position n at time τ, provided that it initially started from the point m, in the random walk problem with the continuous time parameter.

§ 4. Solution of the random walk problem with continuous time parameter

Here we shall first consider the random walk problem for the infinitely ex-

tended linear lattice. In this case, Eq. (11) can be easily solved by introducing the generating function defined by

$$F(z, \tau) = \sum_{n=-\infty}^{\infty} z^{n-m} p(n, m; \tau). \tag{12}$$

It is clear that

$$F(1, \tau) = \sum_{n=-\infty}^{\infty} p(n, m; \tau) = 1. \tag{13}$$

Now, multiplying Eq. (11) by z^{n-m} and adding over all n, we obtain the differential equation

$$\frac{\partial}{\partial \tau} F(z, \tau) = \alpha \left(z + \frac{1}{z} - 2 \right) F(z, \tau) \tag{14}$$

The solution of Eq. (14) which satisfies also the subsidiary condition (13) is obtained to be

$$F(z, \tau) = e^{-2\alpha\tau} \exp\left\{ \alpha \left(z + \frac{1}{z} \right) \tau \right\}. \tag{15}$$

The coefficient $p(n, m; \tau)$ of z^{n-m} in Eq. (15) gives us the desired solution of Eq. (11). As is well known, the exponential function

$$\exp\left\{ \alpha \left(z + \frac{1}{z} \right) \tau \right\}$$

is just the generating function of the modified Bessel function $I_l(2\alpha\tau)$. Therefore we have

$$p(n, m; \tau) = e^{-2\alpha\tau} I_{n-m}(2\alpha\tau). \tag{16}$$

This is actually the solution of our random walk problem with a continuous time parameter. Really the relations

$$p(n, m; \tau) \geq 0, \quad \sum_{n=-\infty}^{\infty} p(n, m; \tau) = 1 \tag{17}$$

hold for all τ. Moreover, the probability $p(n, m; \tau)$ depends on the positions n and m only through the distance $|n-m|$, that is,

$$p(n-m; \tau) = p(-n+m; \tau), \tag{18}$$

because of the relation $I_{-l}(x) = I_l(x)$ for the modified Bessel function. It is also verified that the function (16) satisfies the Kolmogorov-Chapman equation

$$p(n, m; \tau) = \sum_{l=-\infty}^{\infty} p(n-l, m; \tau-\tau') p(l, m; \tau'). \tag{19}$$

This follows directly from the addition theorem of the modified Bessel function

$$I_n(x) = \sum_{l=-\infty}^{\infty} I_{n-l}(x-x') I_l(x'). \tag{20}$$

When x is large and much greater than l, we can use the asymptotic expansion formula of $I_l(x)$, that is,

$$I_l(x) \sim \frac{e^x}{\sqrt{2\pi x}}\left\{1-\frac{(4l^2-1)}{8x}+\frac{(4l^2-1)(4l^2-9)}{2!\,(8x)^2}-\cdots\right\}, \tag{21}$$

so that in this case $p(n, m; \tau)$ can be written in an expansion form

$$p(n, m; \tau) \sim \frac{1}{\sqrt{4\pi\alpha\tau}}\left\{1-\frac{(\mu-1)}{16\alpha\tau}+\frac{(\mu-1)(\mu-9)}{2!\,(16\alpha\tau)^2}-\cdots\right\}, \tag{22}$$

where

$$\mu=4(n-m)^2.$$

For the suitably large values of $n-m$, therefore, Eq. (22) can be approximated as

$$p(n, m; \tau) \sim \frac{1}{\sqrt{4\pi\alpha\tau}} \exp\left\{-\frac{(n-m)^2}{4\alpha\tau}\right\}. \tag{23}$$

If we replace the variable $n-m$ by $(X-X_0)/a$ in Eq. (23) we obtain, as we expected, the probability density

$$p(X, X_0; \tau)\,dX=\frac{1}{\sqrt{4\pi\alpha a^2\tau}} \exp\left\{-\frac{(X-X_0)^2}{4\alpha a^2\tau}\right\} dX, \tag{24}$$

which is the solution of the diffusion equation with the diffusion constant αa^2.

The solution of the random walk problem with the periodic boundary condition $p_r(n, m; \tau)=p_r(n+N, m; \tau)$ can be composed of the solutions (16). Namely it is written as

$$p_r(n, m; \tau)=\sum_{k=-\infty}^{\infty} p(n+kN, m; \tau), \tag{25}$$

because, in this case, the linear lattice forms a ring, so that the variable point can reach the position n at time τ after round trips of any number of times. However, when we consider a sufficiently large system, the contributions of terms with $k\neq 0$ are negligibly small.

§ 5. Laplace transform of the frequency spectrum of lattice vibration

The solution of equation of motion (1) can be expressed as a linear combination of the normal modes

$$u_n^{(\nu)}=A_n^{(\nu)} \exp\{i\omega_\nu t\}. \tag{26}$$

Inserting Eq. (26) into Eq. (1), we have the secular equation

$$\mathfrak{D}(n)\cdot u_n^{(\nu)}=-\frac{\omega_\nu^2}{\alpha}u_n^{(\nu)}, \tag{27}$$

by which the characteristic frequencies ω_ν are determined. Comparing this equation with Eq. (5), we find that $\lambda_\nu=\omega_\nu^2/\alpha$.

Now we shall consider the trace of the characteristic matrix (6), that is,

$$\sum_n p(n, n; \tau) = \sum_n \sum_\nu u_n^{(\nu)} e^{-\lambda_\nu \tau} u_n^{(\nu)} = \sum_\nu e^{-\lambda_\nu \tau}. \tag{28}$$

For a ring lattice we must use the periodic solution (25), but when N is sufficiently large we can calculate the trace using only the central term in the sum of Eq. (25). In this case, therefore, it follows that

$$\sum_n p(n, n; \tau) = Ne^{-2\alpha\tau} I_0(2\alpha\tau) = \int_0^\infty e^{-\lambda\tau} \rho(\lambda)\, d\lambda, \tag{29}$$

where the sum over ν in Eq. (28) is replaced by the integral, introducing the spectral density $\rho(\lambda)$ which represents the distribution of ω^2/α.

Thus we can obtain the distribution function of ω^2/α by calculating the Laplace inverse transformation

$$\lim_{T\to\infty} \frac{N}{2\pi i} \int_{c-iT}^{c+iT} e^{\lambda\tau} e^{-2\alpha\tau} I_0(2\alpha\tau) = \rho(\lambda) \tag{30}$$

which gives us the result[6]

$$\rho(\lambda) = \begin{cases} \dfrac{N}{2\pi\alpha} \left\{ 2\left(\dfrac{\lambda}{2\alpha}\right) - \left(\dfrac{\lambda}{2\alpha}\right)^2 \right\}^{-1/2}, & 0 < \lambda < 4\alpha \\ 0, & \lambda > 4\alpha. \end{cases} \tag{31}$$

If we denote the distribution function of ω itself by $f(\omega)\, d\omega$, there is obviously a relation

$$f(\omega)\, d\omega = (2\omega/\alpha)\rho(\omega^2/\alpha)\, d\omega.$$

Therefore, it has been shown that the frequency spectrum of the lattice vibration can be obtained by solving the random walk problem with continuous time parameter and calculating the Laplace inverse transformation.

§ 6. Random walk problem with discrete time parameter

Here we shall introduce another type of the characteristic matrix, which is defined in terms of the eigenvalues and eigenvectors of Eq. (5), by

$$p(n, m; s) = \sum_\nu u_m^{(\nu)} \left(1 - \frac{\lambda_\nu}{\lambda_m}\right)^s u_n^{(\nu)}, \tag{32}$$

where s is a positive integer or zero and λ_m denotes the largest eigenvalue. If we use Eqs. (32) and (5), it follows that

$$\begin{aligned} p(n, m; s+1) &= \sum_\nu u_m^{(\nu)} \left(1 - \frac{\lambda_\nu}{\lambda_m}\right)^s u_n^{(\nu)} - \sum_\nu u_m^{(\nu)} \left(1 - \frac{\lambda_\nu}{\lambda_m}\right)^s \frac{\lambda_\nu}{\lambda_m} u_n^{(\nu)} \\ &= p(n, m; s) + \frac{1}{\lambda_m} \mathfrak{D}(n) \cdot p(n, m; s). \end{aligned} \tag{33}$$

Thus we obtain the difference equation which has to be satisfied by the characteristic matrix (32), that is,

$$p(n, m\,;\, s+1)=\gamma p(n+1, m\,;\, s)+\gamma p(n-1, m\,;\, s)+(1-2\gamma)\,p(n, m\,;\, s), \quad (34)$$

where γ is α/λ_m.

Comparing Eq. (34) with Eq. (9), it can be seen that Eq. (34) has just the same form as Eq. (9) which describes the change of probability distribution of the random walk problem with the transition probability defined for the finite time interval $\Delta\tau$. Consequently, in this case, our problem can be regarded as the random walk problem in which the variable point makes a step to the right or left with equal probability γ or stays in the same position with probability $1-2\gamma$ at each discrete time points $s=0, 1, 2, \cdots$ and $p(n, m\,;\, s)$ is the probability that we find the variable point at the position n just after the time s, provided that it initially started from the point m.

The result obtained in the previous section shows that the largest eigenvalue λ_m is 4α, so that the transition probability γ in our random walk problem should take the value $1/4$.

§ 7. Solution of the random walk problem with discrete time parameter

The solution of this elementary random walk problem can, of course, be constructed as a combinatorial problem, but here we shall solve Eq. (30) by the use of the generating function, in just the same way as in § 4. The generating function is again defined by

$$K(z, s)=\sum_{n=-\infty}^{\infty} z^{n-m}\,p(n, m\,;\, s), \quad (35)$$

which has also to satisfy the condition

$$K(1, s)=\sum_{n=-\infty}^{\infty} p(n, m\,;\, s)=1. \quad (36)$$

If we multiply both sides of Eq. (34) by z^{n-m} and sum up over all n, we have

$$K(z, s+1)=\left\{\gamma\left(z+\frac{1}{z}\right)+(1-2\gamma)\right\}K(z, s). \quad (37)$$

It is readily seen that the generating function, which satisfies the recurrence formula (37) and also the subsidiary condition (36), is given by

$$K(z, s)=\left\{\gamma\left(z+\frac{1}{z}\right)+(1-2\gamma)\right\}^{s}. \quad (38)$$

Now the probability $p(n, m\,;\, s)$ can be obtained by calculating the coefficient of z^{n-m} in Eq. (38). The right-hand side of Eq. (38) can be rewritten in the form

$$\left\{\gamma\left(z+\frac{1}{z}\right)+(1-2\gamma)\right\}^s=\frac{1}{z^s}\{\gamma z^2+(1-2\gamma)z+\gamma\}^s$$

and if we put the value $\gamma=1/4$ into this expression, we have

$$K(z,\ s)=\frac{1}{2^{2s}}\frac{(z+1)^{2s}}{z^s}=\frac{1}{2^{2s}z^s}\sum_{r=0}^{2s}\binom{2s}{r}z^r. \tag{39}$$

Therefore, the probability of finding the variable point at the position n just after the time s is given by

$$\begin{aligned} p(n,\ m\ ;\ s)&=\frac{1}{2^{2s}}\binom{2s}{s+n-m}\\ &=\frac{1}{2^{2s}}\frac{\Gamma(2s+1)}{\Gamma(s+n-m+1)\,\Gamma(s-n+m+1)}, \qquad |n-m|\leq s \qquad (40)\\ p(n,\ m\ ;\ s)&=0, \qquad\qquad |n-m|>s. \end{aligned}$$

The Bernoulli distribution (40) can be derived also from the combinatorial considerations, which we shall discuss in Appendix I. It is also well known that the Gaussian distribution appears as the asymptotic form of Bernoulli distribution, if one applies Stirling's formula to Eq. (40). For the periodic boundary condition, we can follow completely the same discussion that we used in § 4.

§ 8. Mellin transform of the frequency spectrum of lattice vibration

Finally let us consider the characteristic matrix (32) and take its trace, then we have

$$\begin{aligned} \sum_n p(n,\ n\ ;\ s)&=\sum_n\sum_\nu u_n^{(\nu)}\left(1-\frac{\lambda_\nu}{\lambda_m}\right)^s u_n^{(\nu)}=\sum_\nu\left(1-\frac{\lambda_\nu}{\lambda_m}\right)^s\\ &=\int_0^\infty\left(1-\frac{\lambda}{\lambda_m}\right)^s\rho(\lambda)\,d\lambda, \qquad (41) \end{aligned}$$

or writing $\xi=1-\lambda/\lambda_m$ and $\lambda_m=4\alpha$,

$$\sum_n p(n,\ n\ ;\ s)=4\alpha\int_0^\infty\xi^s\bar{\rho}(\xi)\,d\xi \tag{42}$$

where we have defined the function $\bar{\rho}(\xi)$ by $\bar{\rho}(\xi)=\rho(4\alpha(1-\xi))$.

Inserting the solution (40) into Eq. (42), we obtain

$$\int_0^\infty\xi^s\bar{\rho}(\xi)\,d\xi=\frac{N}{4\alpha}\frac{1}{2^{2s}}\frac{\Gamma(2s+1)}{[\Gamma(s+1)]^2}, \tag{43}$$

which is the Mellin transform of function $\bar{\rho}(\xi)$. The inverse transformation of Eq. (43) gives (Appendix II)

$$\bar{\rho}(\xi)=\begin{cases}\dfrac{N}{4\pi\alpha}(\xi-\xi^2)^{-1/2}, & 0<\xi<1,\\ 0, & \xi>1.\end{cases} \tag{44}$$

If we change the variable ξ in Eq. (44) back to λ by the relation $\xi=1-\lambda/4\alpha$, then we obtain again the result (31) which gives us the distribution function of ω^2/α.

Appendix I

Combinatorial calculation of the random walk problem with discrete time parameter

Here we shall show that the random walk problem which has been presented in § 6 can be directly solved as a combinatorial problem. Actually we can treat the problem in two different ways.

The first method is as follows. In our random walk problem (Bernoulli trials), the variable point makes a step of unit length to the right or left with equal probability 1/4 or stays at the same position with probability 1/2, at each discrete time point 1, 2, ···, s. If we assume that, among s trials, i steps are made to the right, j steps to the left and $(s-i-j)$ trials are spent without displacement, then the probability associated with such processes is given by

$$\frac{s!}{i!\,j!(s-i-j)!}\left(\frac{1}{4}\right)^{i+j}\left(\frac{1}{2}\right)^{s-i-j}. \tag{A1}$$

By this process the variable point makes total displacement $j-i$. Therefore the probability of finding the variable point, which has started from the lattice point m, at the point n after s trials can be obtained by adding the probabilities (A1), under the condition that $j-i=n-m$ is fixed. Namely we have

$$\begin{aligned}p(n,m;s)&=\sum_{i=0}^{[(s-n+m)/2]}\frac{s!}{i!(i+n-m)!\,(s-2i-n+m)!}\left(\frac{1}{2}\right)^{s+2i+n-m}\\&=\left(\frac{1}{2}\right)^{s+n-m}\sum_{i=0}^{[(s-n+m)/2]}\binom{s}{2i+n-m}\binom{2i+n-m}{i}\left(\frac{1}{2}\right)^{2i},\end{aligned} \tag{A2}$$

where $[x]$ denotes the integral part of x.

Another method which leads directly to expression (40) is somewhat tricky. We introduce a new random walk problem which relates to the original one in such a way that two successive trials of new random walk problem produce the same result with that of one trial of the original random walk problem. Namely we consider a problem in which the variable point makes a step of length 1/2 to the right or left with equal probability 1/2, at discrete times 1, 2, ···. Then it is clear that by two successive steps the point makes a displacement of distance 1 to the right or left with equal probability 1/4, or returns to the initial position

with probability 1/2. Therefore, we can obtain the required solution by solving this subsidiary random walk problem, and obtaining the probability distribution at the time just after $2s$ trials, that is,

$$p(n, m; s) = P(n, m; 2s) = \frac{1}{2^{2s}} \binom{2s}{s+n-m} \tag{A3}$$

where $P(n, m; 2s)$ is the solution of our subsidiary problem.

Finally we can derive expression (A2) also from the generating function (38), which can be written in the form

$$\begin{aligned}
\left\{\gamma\left(z+\frac{1}{z}\right)+(1-2\gamma)\right\}^s &= \sum_{l=0}^{s} \binom{s}{l} \left(\frac{1}{2}\right)^{s-l} \left(\frac{1}{4}\right)^l \left(z+\frac{1}{z}\right)^l \\
&= \left(\frac{1}{2}\right)^s \sum_{l=0}^{s} \binom{s}{l} \left(\frac{1}{2}\right)^l \left(z+\frac{1}{z}\right)^l \\
&= \left(\frac{1}{2}\right)^s \sum_{l=0}^{s} \sum_{k=0}^{l} \binom{s}{l} \binom{l}{k} \left(\frac{1}{2}\right)^l z^{l-k} z^{-k} \\
&= \left(\frac{1}{2}\right)^s \sum_{k=0}^{s} \sum_{l=k}^{s} \binom{s}{l} \binom{l}{k} z^{l-2k} \left(\frac{1}{2}\right)^l,
\end{aligned} \tag{A4}$$

where we put $\gamma=1/4$. If we pick up the coefficients of z^{n-m} in Eq. (A4), we obtain

$$\left(\frac{1}{2}\right)^{s+n-m} \sum_{i=0}^{[(s+n-m)/2]} \binom{s}{2i+n-m} \binom{2i+n-m}{i} \left(\frac{1}{2}\right)^{2i}. \tag{A5}$$

Therefore, the above discussion gives a proof of the formula

$$\sum_{i=0}^{(s+n-m)/2} \binom{s}{2i+n-m} \binom{2i+n-m}{i} \left(\frac{1}{2}\right)^{2i} = \left(\frac{1}{2}\right)^{s-n+m} \binom{2s}{s+n-m}. \tag{A6}$$

Especially for $n=m$,

$$\sum_{i=0}^{[s/2]} \binom{s}{2i} \binom{2i}{i} \left(\frac{1}{2}\right)^{2i} = \left(\frac{1}{2}\right)^s \binom{2s}{s}. \tag{A7}$$

Appendix II

Mellin transform

Here we shall give the calculation of Mellin transformation of Eq. (44). Substituting Eq. (44) in the left-hand side of Eq. (43) and changing the variable by $x=1/\xi$ and $t=\sqrt{x-1}$, we have

$$\begin{aligned}
\int_0^\infty \xi^s \bar{p}(\xi)\, d\xi &= \frac{N}{4\pi\alpha} \int_0^1 \xi^s\, d\xi / \sqrt{\xi-\xi^2} = \frac{N}{4\pi\alpha} \int_0^1 \xi^{s-1}\, d\xi / \sqrt{1-\xi} \\
&= \frac{N}{4\pi\alpha} \int_1^\infty dx / x^{s+1} \sqrt{x-1} = \frac{2N}{4\pi\alpha} \int_0^\infty dt / (1+t^2)^{s+1}
\end{aligned} \tag{A8}$$

and it follows that

$$I_{s+1}\equiv\int_0^\infty dt(1+t^2)^{-(s+1)}=\int_0^\infty dt(1+t^2)^{-s}-\int_0^\infty dt\ t^2(1+t^2)^{-(s+1)}$$

$$=I_s+\frac{1}{2s}\left[\frac{t}{(1+t^2)^s}\right]_0^\infty-\frac{1}{2s}\int_0^\infty dt(1+t^2)^{-s}=\frac{2s-1}{2s}I_s.$$

Consequently, using $I_1=\pi/2$, we have,

$$I_{s+1}=\frac{\pi}{2^{2s}}\frac{(2s)!}{s!\,s!}=\frac{\pi}{2^{2s}}\frac{\Gamma(2s+1)}{[\Gamma(s+1)]^2}.$$

Thus we finally obtain

$$\int_0^\infty \xi^s\bar{\rho}(\xi)\,d\xi=\frac{N}{4\alpha}\frac{1}{2^{2s}}\frac{\Gamma(2s+1)}{[\Gamma(s+1)]^2}. \qquad \text{(A9)}$$

References

1) E. W. Montroll and R. B. Potts, Phys. Rev. **100** (1955), 525, **102** (1956), 72.
2) F. J. Dyson, Phys. Rev. **92** (1953), 1331.
3) H. Schmidt, Phys. Rev. **105** (1957), 425.
4) J. Hori and T. Asahi, Prog. Theor. Phys. **17** (1957), 523.
5) C. Domb, A. A. Maradudin, E. W. Montroll and G. H. Weiss, Phys. Rev. **115** (1959), 18.
6) H. Bateman, *Tables of Integral Transforms* (McGraw-Hill, 1954). Vols. 1 and 2.

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High Energy Elastic Scattering of Nucleon from Nuclei
—— *Analyses of Experimental Data* ——

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The angular distributions of the nucleon elastically scattered from nuclei are analysed at energies ranging from 95 Mev to 340 Mev. Predictions on the angular distribution put forward by one of the authors (T. S.) are examined. All the predictions seem to be satisfied

§1. Introduction

In the previous paper by one of the authors (T. S.),[1] in the assumption that the short-range correlation between nucleons and the Pauli principle in the target nucleus in its ground state plays an essential role in the high energy elastic scattering of nucleon from heavy nuclei, the following are predicted for the angular distributions:

(1) The positions of maxima and minima coincide respectively with those given by the impulse approximation,

(2) the absolute value of the angular distribution is multiplied by a constant factor $|\lambda|^2$ to that given by the impulse approximation,

(3) this factor is an independent constant of mass number in heavy nuclei and independent of the incident energy, and finally

(4) the magnitude of $\sqrt{|\lambda|^2}$ is order $0.4 \sim 0.5$. This factor results from the Pauli principle in the target nucleus.

There, using the set of phase shifts based on the phenomenological inner potential with the one-pion-exchange tail,[2),3)] the above predictions have been confirmed at 95Mev. In the present paper, we shall report the results of analyses at energies 95 Mev, 135 Mev, 310 Mev, and 340 Mev.

Because, based on such a potential, the analyses of the two-body scattering have not been extended to higher energies, here we shall adopt the phase shifts by Gammel and Thaler[4] listed in the paper by Kerman, McManus and Thaler.[5]

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Equations used for the analyses are all described in the previous paper and here we do not reproduce them.

§ 2. Analyses of the experimental data

The results of analyses are shown in Fig. 1 (95 Mev[6]), Fig. 2 (135 Mev[7]), Fig. 3 (310 Mev[8]) and Fig. 4 (340 Mev[9]). Adopted parameters are listed in Table I, where r_0 is related to the nuclear radius by $R=r_0A^{1/3}$ and a is the parameter of the diffuseness, provided that the nucleon density in the nucleus is given by

$$\rho(r)=\rho_0\left[1+\exp\left(\frac{r-R}{a}\right)\right]^{-1},$$

where ρ_0 is normalized so that $4\pi\int_0^\infty \rho(r)\,r^2dr=1$.

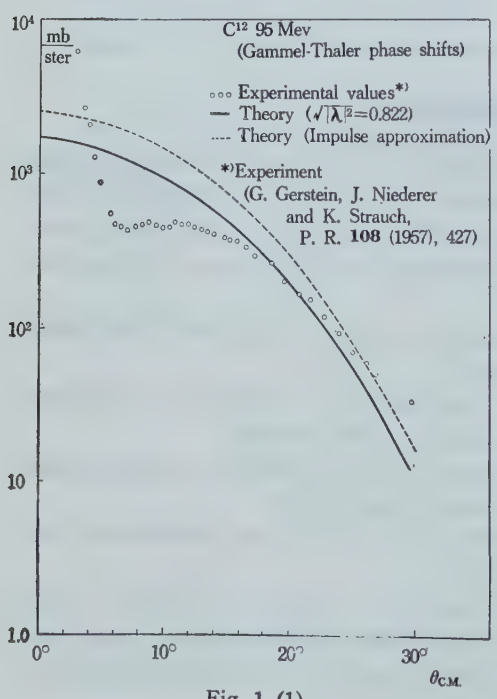


Fig. 1 (1)

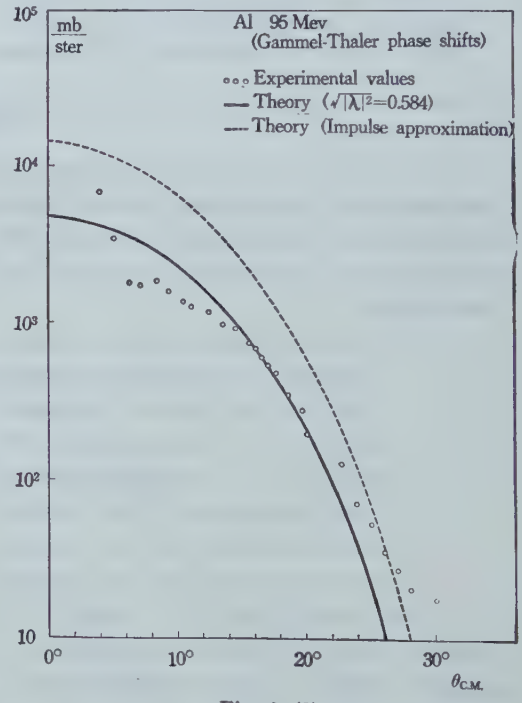
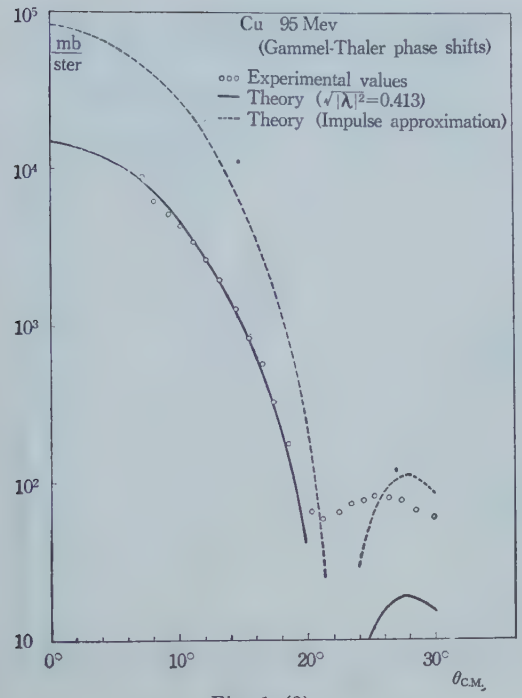


Fig. 1 (2)

Table I

E (Mev)	r_0 (10^{-13}cm)	a (10^{-13}cm)
95	1.32	0.49
135	1.30	0.49
310	1.23	0.49
340	1.23	0.49



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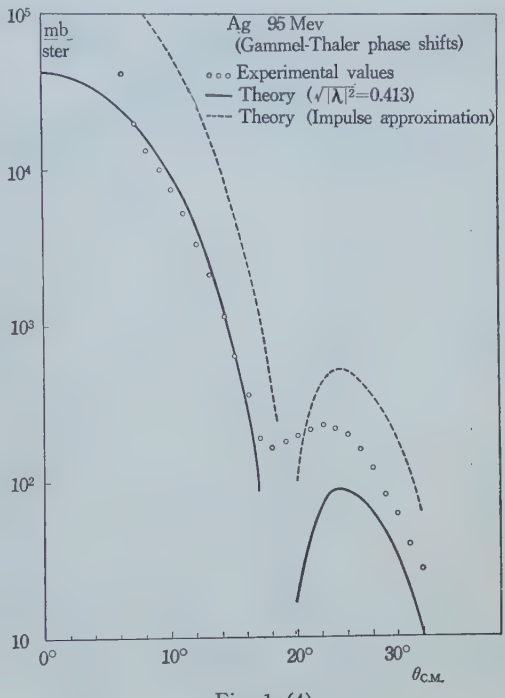

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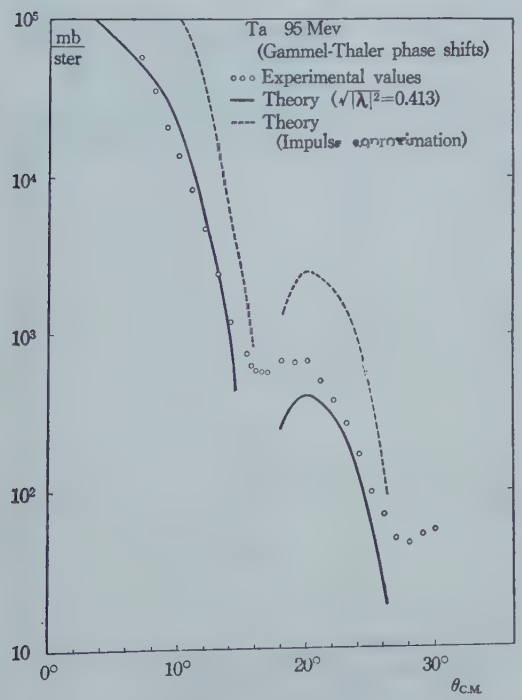

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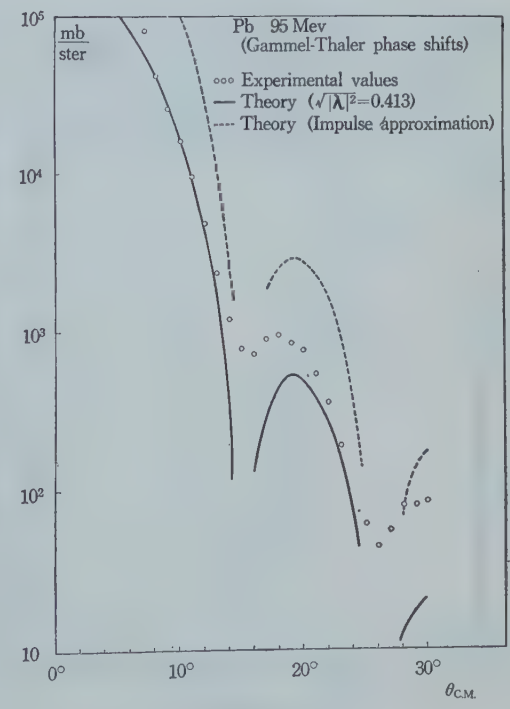

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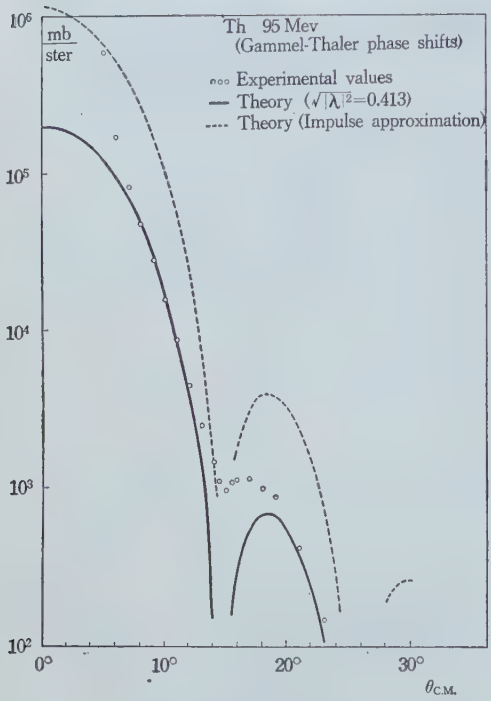

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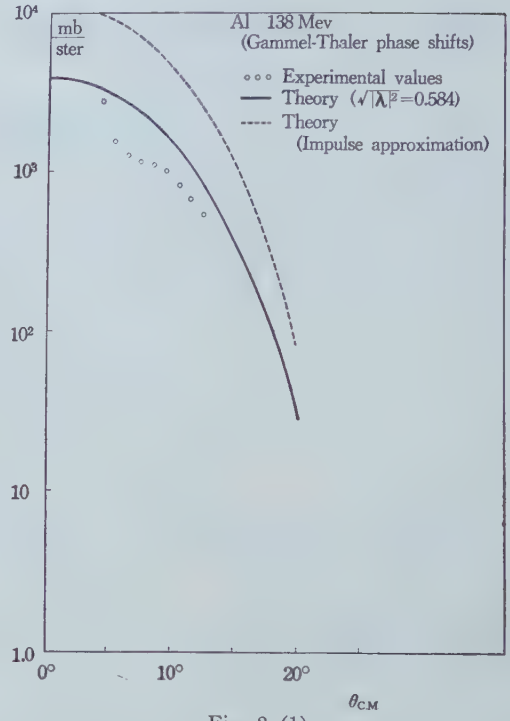

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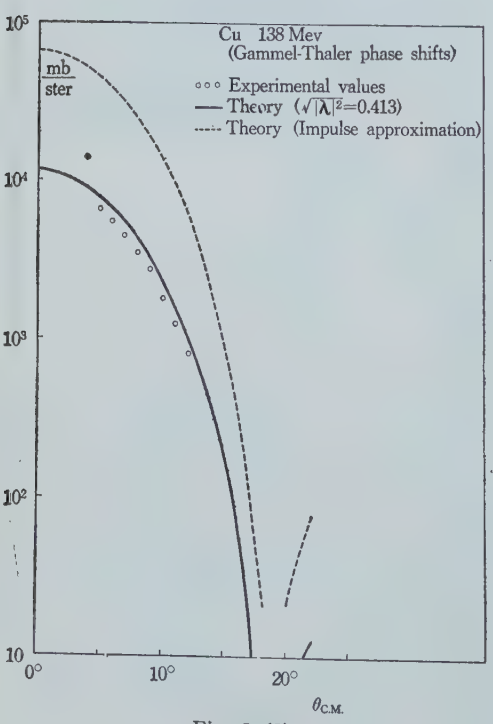

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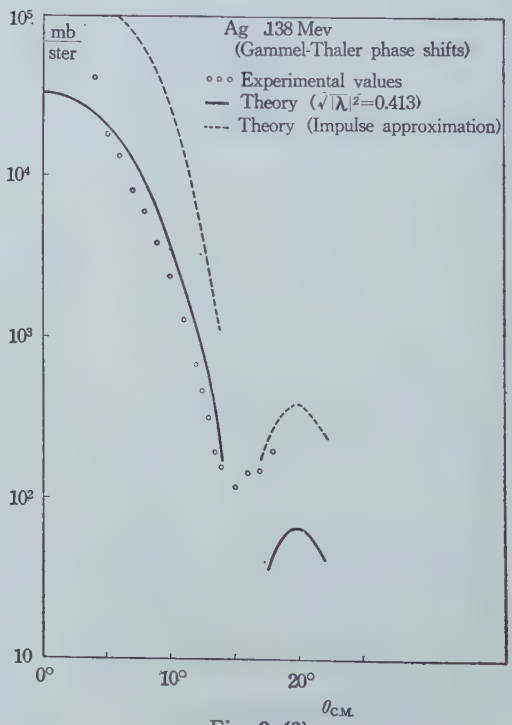


Fig. 2 (3)

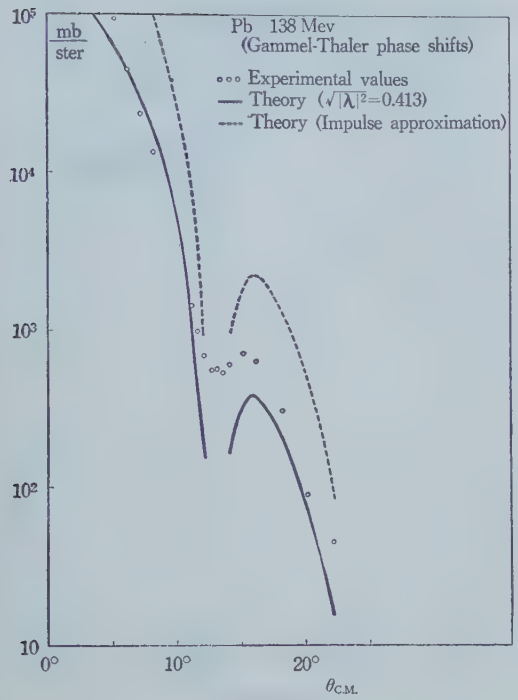
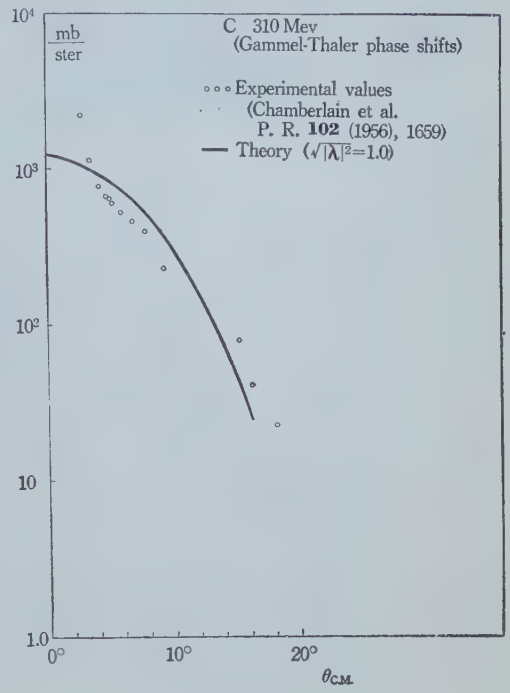


Fig. 2 4)



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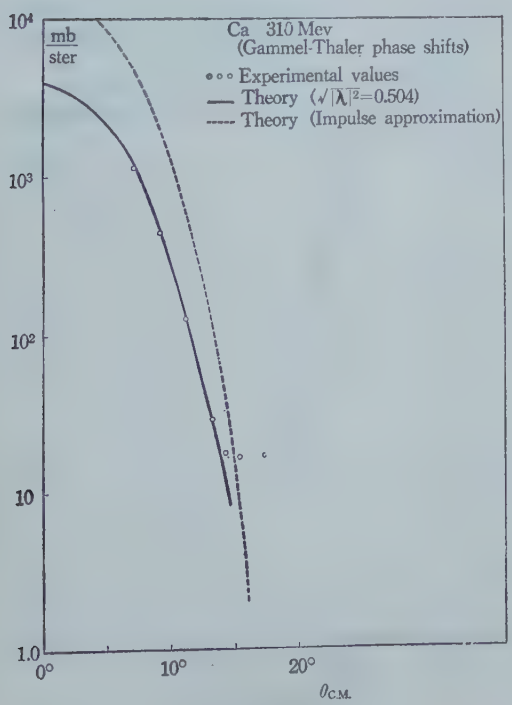

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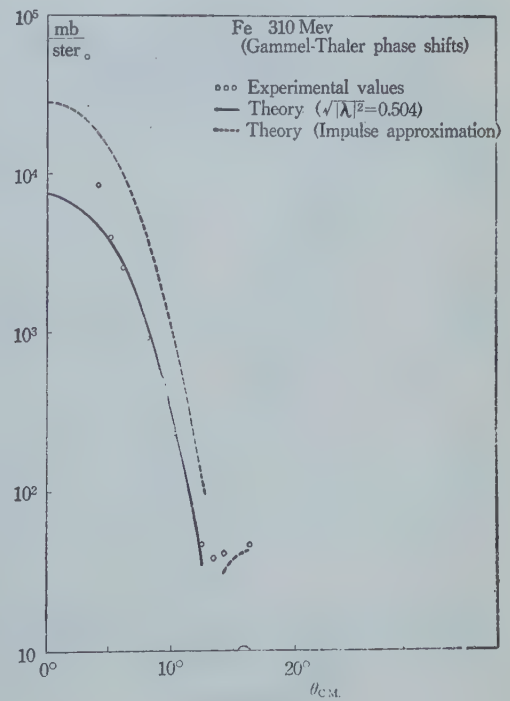

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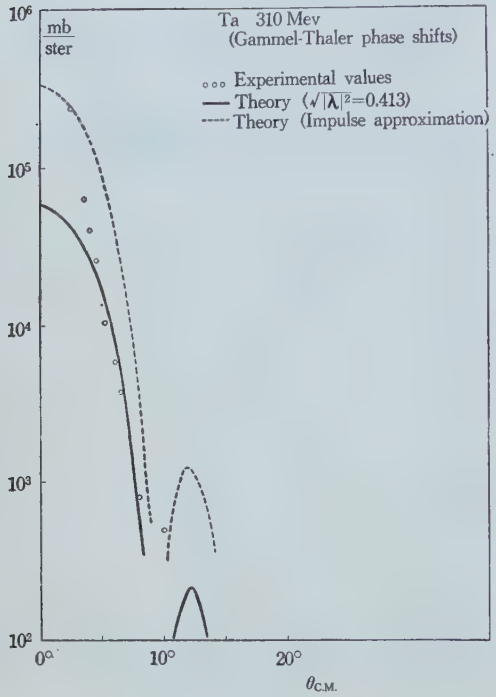

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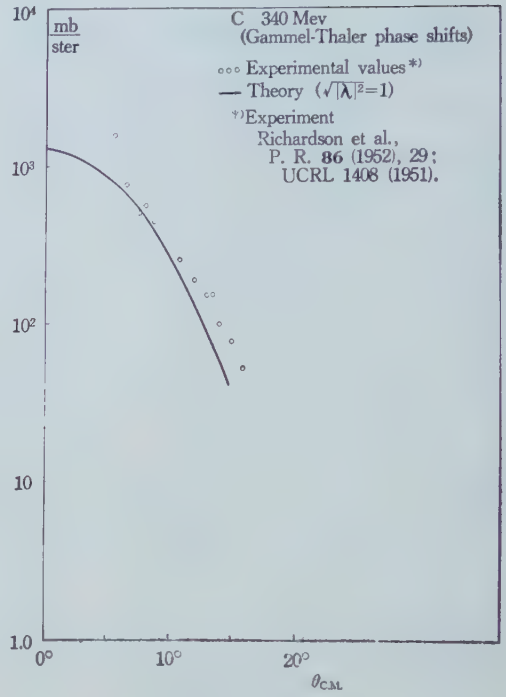

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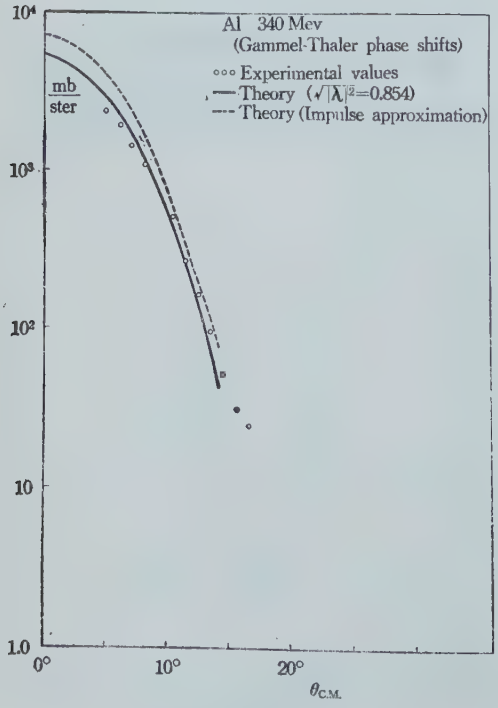

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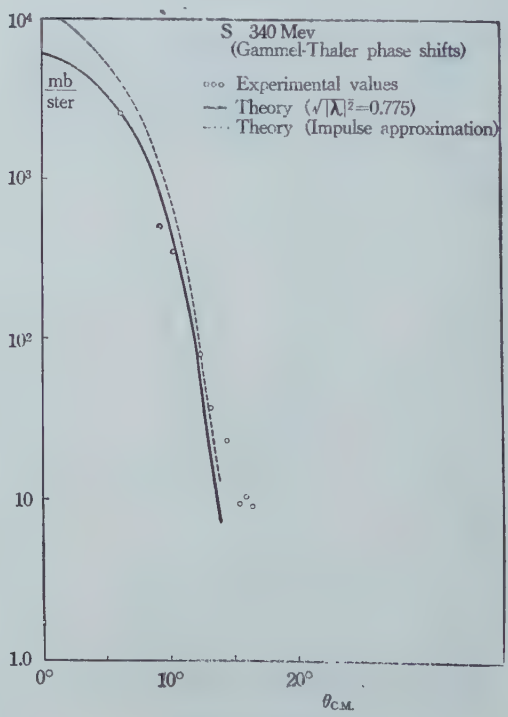

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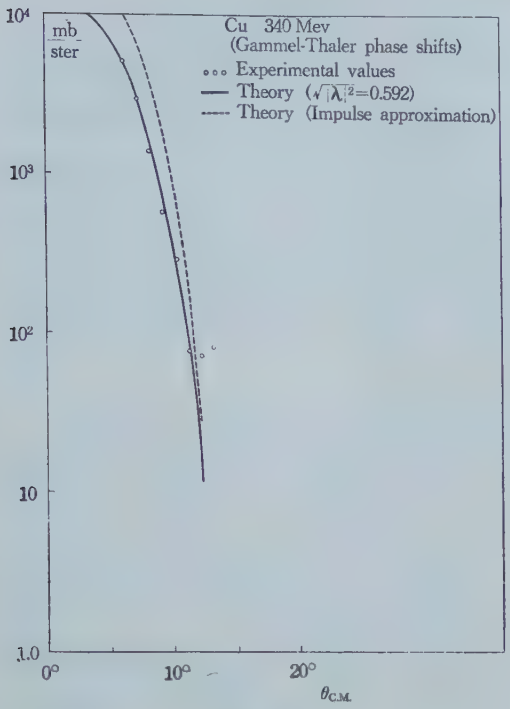

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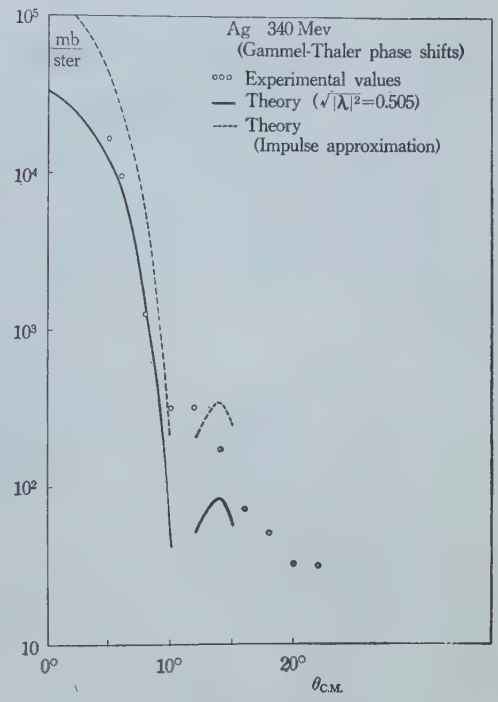

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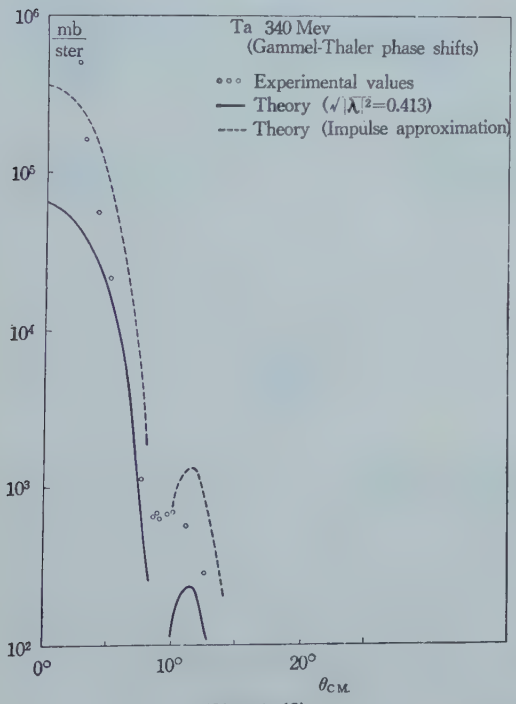

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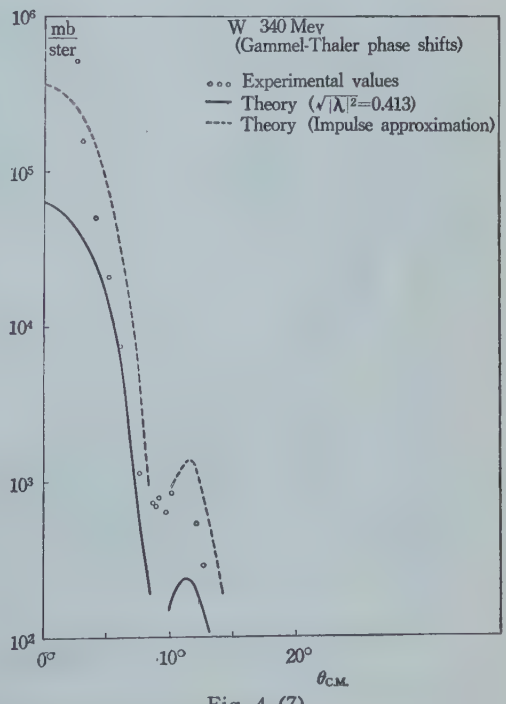

Fig. 4 (7)

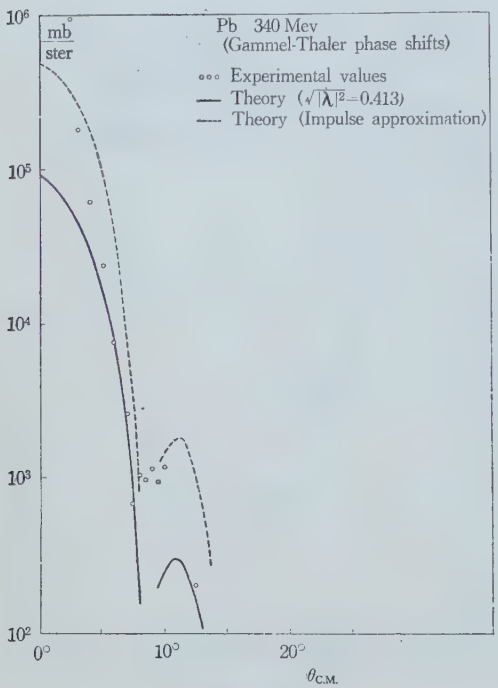

Fig. 4 (8)

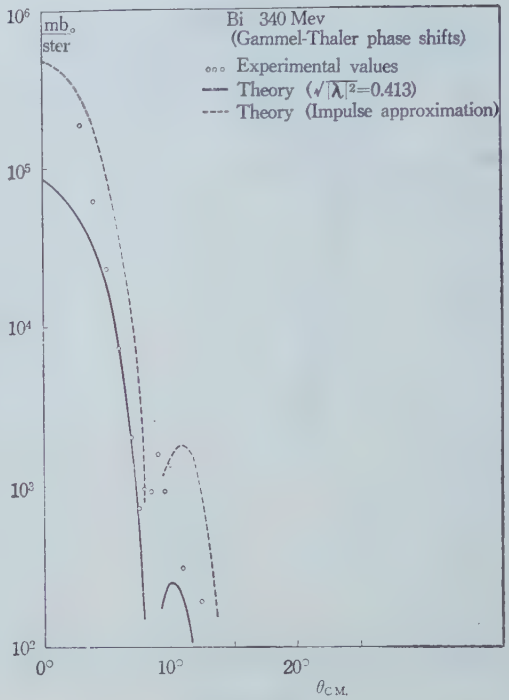

Fig. 4 (9)

Effect of the Coulomb force is neglected. If we take into account the Coulomb force, results are expected to be more satisfactory. To give a measure of this effect, we wish to quote the paper by Cromer.[10] He performed the calculation for C at 310 Mev using the phase shifts by Gammel-Thaler, with the effect of the Coulomb force. He got a satisfactory fit.

If the transparent nucleus theory is assumed, it would not give so satisfactory results as the present ones. For example, see Figs. 7, 12 and 16 in the paper by Richardson et al.[9]

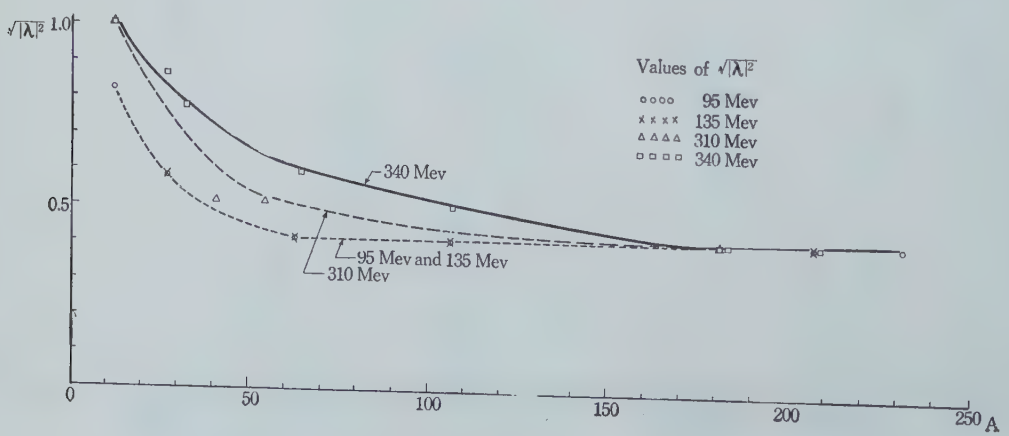

Fig. 5

These facts being considered, the predictions (1) and (2) denoted in § 1 may be said to be true.

The value of $\sqrt{|\lambda|^2}$ is summarized in Fig. 5. The predictions (3) and (4) are satisfied in heavy nuclei ($A \geqq 181$). However, it is slightly increasing with increasing energy in low mass number side. This tendency may have any relation to the one anticipated by Kerman, McManus and Thaler. However, the precise physical meaning as to the point why it appears in low mass number side and not in heavy nuclei is not certain. This requires a more careful theoretical investigation.

§ 3. Comment

In these calculations, we assumed the wave number of a nucleon is not changed inside and outside the nucleus. That is to say, we do not take into account the effect of the binding energy of the target nucleus. To examine this point, we take, as an example, the scattering of 95 Mev protons from Cu. As the depth of the optical potential at these energies is about 20 Mev,[11] we calculated the energy of the impinging nucleon inside the nucleus to be 110 Mev. The result is shown in Fig. 6.

Compare with Fig. 1 (3). The adopted parameters are given in Table II.

Table II

	$r_0(10^{-13}\text{cm})$	$a\,(10^{-13}\text{cm})$	$\sqrt{\lvert\lambda\rvert^2}$
Fig. 1(3)	1.32	0.49	0.452
Fig. 6	1.23	0.49	0.413

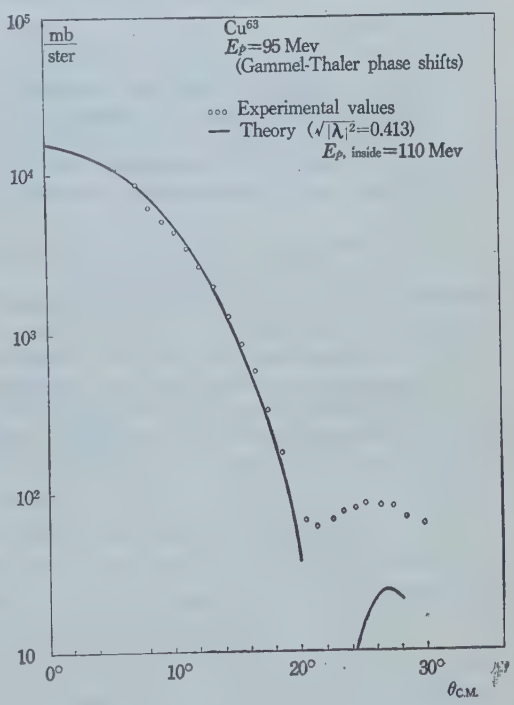

Fig. 6

As for our present analyses, we can not conclude whether we must take into account the effect of the binding energy of the target nucleus. Anyhow, it is of very interesting that the value of $\sqrt{|\lambda|^2}$ is within the predicted value 0.4~0.5, hence we may say that at any rate the most important factor to the scattering mechanism is the short-range correlation and the Pauli principle between the target nucleons, which is embodied in the value of $\sqrt{|\lambda|^2}$.

§ 4. Conclusion

The predictions, put forward in the previous paper[1] and repeated in § 1 of

the present paper, on the angular distribution of high energy nucleon elastically scattered from heavy nuclei are examined. All the predictions (1), (2), (3) and (4) seem to be satisfied as a general feature.

The fact that in light and intermediate nuclei the value of $\sqrt{|\lambda|^2}$ is slightly increasing with increasing energy requires a more careful theoretical investigation.

It is not certain whether we must take into account the effect of the binding energy of the nucleus. At any rate the most important factor in high energy scattering seems to be the short-range correlation and the Pauli principle embodied in $\sqrt{|\lambda|^2}$.

References

1) T. Sasakawa, Prog. Theor. Phys. **24** (1960), 241.
2) S. Otsuki, Prog. Theor. Phys. **20** (1958), 171.
W. Watari, Prog. Theor. Phys. **20** (1958), 181.
3) Hamada, Iwadare, Otsuki, Tamagaki and Watari, Prog. Theor. Phys. **23** (1960), 366
4) J. L. Gammel and R. M. Thaler, Phys. Rev. **107** (1957), 291, 1337.
5) A. K. Kerman and H. McManus and R. M. Thaler, Ann. Phys. **8** (1959), 511.
6) G. Gerstein, J. Niederer and K. Strauch, Phys. Rev. **108** (1957), 427. The authors are grateful for their sending experimental data.
7) J. M. Dickson, private communication. The authors are grateful for his sending experimental data prior to publication.
8) O. Chamberlain, E. Segrè, R. D. Tripp, C. Wiegand and P. Ypsilantis, Phys. Rev. **102** (1956), 1659.
9) R. E. Richardson, W. P. Ball, C. E. Leith and B. J. Moyer, Phys. Rev. **86** (1952), 29. The authors are grateful for their sending experimental data.
10) A. H. Cromer, Phys. Rev. **113** (1959), 1607.
11) A. E. Glassgold and P. J. Kellog, Phys. Rev. **109** (1958), 1291.

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On the Method of the Theory of Nuclear Forces*

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The method of the theory of the nuclear forces proposed by Taketani, Nakamura and Sasaki[1] is developed to clarify the present stage of the theory of the nuclear forces and to discuss the aims, as well as the important results obtained up to present, of the theoretical investigations which are now in progress by our group.

A broad program of theoretical investigation is being carried out by Japanese nuclear force group with the view of proceeding to a new stage of the pion theory of nuclear forces. In this paper we will describe our point of view on the present stage of the theory of nuclear forces and discuss the method that will be used in a series of our papers.

1. Since Taketani, Nakamura and Sasaki[1] (hereafter referred to as TNS) proposed a method in 1951 to attack the problems of nuclear forces, the usefulness of the method was confirmed through investigating many problems which could be treated with the static approximation.

We developed the theory following the point of view that the virtual pion problems, such as the problems of nuclear forces, could be more easily treated without worrying about the very complicated features of quantum mesodynamics than the real pion problems,** especially in the outer region where the distance between two nucleons is larger than the pion Compton wave-length.

2. Most of the results obtained by the above mentioned investigations were summarized in a review article published in 1956[2] (denoted hereafter as Suppl.). The most important results could be summarized as follows:

i) We can divide the region where the nuclear potential acts into three parts, taking the distance between two nucleons, $x=\mu r$ (μ^{-1} is the pion Compton wave-length), as a parameter;

ii) If x is large enough, the nuclear potential can be approximated very well by the static one-pion-exchange potential (denoted hereafter as OPEP), which is derived theoretically without any ambiguity. This region will be called Region I, which could usually be given by $x \gtrsim 1.5$, although it depends on the necessary

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* This work has been done as a part of the 1960–61 Annual Research Project on the Theory of Nuclear Forces organized by the Research Institute for Fundamental Physics, Kyoto University.

** This point of view was contrary to that of most foreign physicists. See, e.g. F.E. Low, Rev. Mod. Phys. **29** (1957), 216.

degree of accuracy for the calculation;

iii) In Region II ($0.7 \lesssim x \lesssim 1.5$), we must take into account dynamical effects and the contributions from two-pion-exchange potential (denoted hereafter as TPEP).

We can rely upon the qualitative features of the static TPEP, which has been calculated by many authors using different methods. However, we have not yet the quantitatively reliable results, because in this region it is expected that the nonstatic effects would contribute appreciably.* So the reliable derivation of the nonstatic potential is very much needed.

iv) In Region III ($x \lesssim 0.7$), many effects, which are difficult to evaluate, such as exchange of three or more pions, exchange of strange particles, relativistic effects and the structure of elementary particles, may give essential contributions. Moreover, it seems that in this Region III the expansion with respect to the number of exchanged pions will not be a good approximation or it may even diverge.[3] Therefore the potential in this region should be treated phenomenologically.

3. The experimental data, which were analyzed in Suppl. to test the vaiidity of the static potential derived from the pion theory, were mainly the electric quadrupole moment of the deuteron and the total and differential cross sections of the *p-p* and *n-p* scatterings below 100 Mev. The results of these analyses still hold good.

4. However, owing to the alteration of the experimental data of the nearly 3 Mev *p-p* scattering and also the necessity of taking into consideration the effects of the vacuum polarization of the Coulomb field,[4] the conclusion previously obtained, that the repulsive central potential in the ^{3}O state in OPFP is *inevitable* to account for the above-mentioned data, must be weakened.** By re-examination we know that there is no definite evidence for the existence of the repulsive central potential in the ^{3}O state in Region I, although the predictions of the OPEP are still within the experimental error.

It is very much desirable both to perform more accurate experiments and to carry out more elaborate theoretical investigations to verify the repulsive central

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* This point was clearly stated by TNS.[1] We read in subsection 11 of TNS "The static potential due to the fourth order calculation has not always meanings, since in the region where the fourth order calculation has sufficient enough magnitude, other nonstatic effects will generally become strong too.....".

Recently, Weisskopf has discussed this point in connection with the problem of the existence of the spin-orbit force in the two-nucleon potential. He says "But does it contain ($L \cdot S$)-force? We do not know. In the western world people say "yes", and in Japan "no"" (Suppl. Prog. Theor. Phys. No. 11 (1959), p. 53). This is unfortunately due to a misunderstanding about the viewpoint of the Japanese nuclear force group. For example, we read in subsection 14 of TNS the following statement: ".....it seems probable that the fourth order perturbations will give $L \cdot S$-coupling from the pseudoscalar meson theory as its nonstatic parts,.....".

** This point has been stressed by T. Ohmura at the Meeting on the Nuclear Force Ploblems held at Institute for Nuclear Study, University of Tokyo, June, 1958.

potential in the ^3O state in Region I.

An analysis of low energy p-p scattering ($0 \sim 20$ Mev) is now in progress by Otsuki, Taketani, Tamagaki, and Watari. According to their preliminary results, the existence of the repulsive central potential in the ^3O state in OPEP seems to be necessary to account for the above-mentioned experimental data.

5. After the publication of Suppl. many experimental data have been published concerning the polarization, depolarization, spin correlation, and so forth. Contrary to the measurements of the total and differential cross sections, which give the observables averaged over the direction of polarization of the spin, these experiments, which give the observables connected with the various polarized states of the nucleons, make it possible to separate the nonstatic contributions from the static one, because considerable parts of the nonstatic contributions appear in combination with the spin operator.

6. So we should take into account the effects of the nonstatic contributions in discussing the experiments on the polarization, depolarization, spin correlation, and so forth.

Especially, to treat the complete or nearly complete experiments[5] theoretically, it will be necessary to start from the completely nonstatic standpoint.

7. The aims of the above-mentioned completely nonstatic investigations will be the following:

A. To make clear to what extent the concept of the two-nucleon potential in x-space holds;

B. To derive, especially, the OPEP and TPEP with full recoil, which will be reliable in Regions I and II, and, as a result, to establish the domain where the static approximation holds good;

C. To clarify the differences between the nonstatic OPEP and TPEP derived from the ps- and pv-coupling respectively, and to compare them with the experiments;

D. To see to what extent the experimental results on the polarization, depolarization, and so forth, may be accounted for from the standpoint of the pion theory.

8. All the calculations of the nonstatic potentials so far published are based on the expansion with respect to the inverse of the rest mass of nucleon, and the actual evaluations have been confined to the terms of the first and the second order with respect to M^{-1} (M being the rest mass of the nucleon) in the TPEP and OPEP respectively.[6]

However, we feel that the expansion with respect to M^{-1} would be a very poor approximation. Especially it seems to indicate the inadequacy of the above-mentioned expansion that the first order corrections with respect to M^{-1} to the static TPEP calculated by I. Sato almost cancel the latter both for the central and tensor part except in the ^3E state.[3]

9. Therefore we must calculate the nonstatic potential without using the M^{-1}-expansion.

Hoshizaki and Machida[7),8)] have derived the nonstatic OPEP and TPEP, the evaluation of which have been carried out in the momentum representation avoiding the M^{-1}-expansion completely. Their main results are the following:

A: The concept of the local potential in the configuration space holds very good for the treatment of the nucleon-nucleon scattering up to nearly 300 Mev.

This is a very important fact which assures the appropriateness of the usual theoretical analyses based on the use of the potential;

B: All the types of the potential, which satisfy the general invariance requirements, appear in the potential in Regions I and II. It is possible that the terms in the potential which show specifically dynamical effects give appreciable contributions to account for the measurements of the polarization of the spin, and so on, even at rather low energies ($\gtrsim$10 Mev);

C: Magnitudes of the central and tensor parts in our nonstatic TPEP with full recoil are completely different from the one obtained by M^{-1}-expansion and are nearly 1/2 of those in the static TPEP,[9)] and the spin-orbit part is rather large in Regions II and III. So the qualitative features of the high energy p-p scattering phase shifts (150 and 310 Mev)[10)] may be understood rather naturally by our potential.[15)] The case of pv-coupling is better than that of ps-coupling in comparison with the experimental data, although one may not definitely conclude it by an analysis at such high energies since the potentials in Region III give appreciable contributions.

D: The nonstatic OPEP includes the following term:

$$\pm\mu\frac{f^2}{4\pi}\frac{1}{2}\left(\frac{\mu}{M}\right)^2(\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)})\left(1+\frac{3}{x}+\frac{3}{x^2}\right)\frac{e^{-x}}{x^3}$$
$$\times[(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)})\boldsymbol{L}^2-(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{L})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{L})], \qquad (1)$$

where $+(-)$ corresponds to the case of $ps(pv)$-coupling among others. This is the first difference appeared in the observable quantities due to the two types of the pion-nucleon coupling established almost rigorously.[7)] So, it is very important to compare this difference with experiment and to decide which of the two coupling types will be main in the low energy pion-nucleon interaction. The determination of this coupling type may give a link between the problems of the pion physics and that of the structure of the elementary particles, because, if the pion has composite structure, the coupling between the pion and the nucleon will generally be a linear combination of the ps- and pv-couplings in the low energy limit, and there is a direct relation between the ratio of the strength of the ps- and pv-couplings and the structure of the interaction kernel for the constituent particles.[11)]

Hamada[12)] has very recently analyzed the p-p scattering experiments at high energies (especially at 150 and 310 Mev) and concluded that the quadratic LS potential of the range μ^{-1} such as just mentioned above is manifestly required in the ^{1}E state where both the tensor and the linear LS potential vanish. Although we must also take into account the other terms than that given in Eq.(1), which

are not included in Hamada's analysis, to discuss the differences in OPEP due to the *ps*- and *pv*-couplings, it would still be interesting to compare Hamada's potential with the corresponding terms in OPEP.* The quadratic LS potential which fits the high energy p-p scattering obtained by Hamada has the following form:

$$+0.0054\mu L(L+1)\frac{e^{-x}}{x^3} \tag{2a}$$

or

$$+0.00155\mu L(L+1)\frac{e^{-x}}{x}\left(1+8\frac{e^{-x}}{x}+6\frac{e^{-2x}}{x^2}\right) \tag{2b}$$

in the ^{1}E state. Eq. (2a) is very similar to the one given in Eq. (1), which may be written as

$$\mp 0.0018\mu L(L+1)\left(1+\frac{3}{x}+\frac{3}{x^2}\right)\frac{e^{-x}}{x^3} \tag{3}$$

in the ^{1}E state. Both the sign and magnitude coincide with the case of *pv*-coupling, while the the case of *ps*-coupling gives the wrong sign and, as a result, too large forward scattering cross section. Hamada has also analyzed the experimental data in the ^{3}O state and found that the fit to the experimental differential cross section at 310 Mev is noticeably improved by including the quadratic LS potential. The sign and the order of magnitude agree with the case of the *pv*-coupling also in this case and the sign given by the *ps*-coupling is wrong, although this result derived from the analysis of the ^{3}O state data is much less conclusive compared with the one derived from the ^{1}E state data.

It should be emphasized that the close examination of the higher partial waves in the ^{1}E states in combination with that of the ^{1}S state will give the most reliable and convenient method to inquire the detailed form of the interaction between the pion and the nucleon at low energies. So it is very much desirable to get more precise information by analyzing the experimental data, and also to establish the applicable domain of the nonstatic OPEP. It is also necessary to examine carefully the effects due to the potentials in Regions II and III.

In this connection, it is also desirable to evaluate the predictions of the nonstatic OPEP using the momentum representation exclusively. This work is now in progress.[13]

10. It is especially interesting to see to what extent the nonstatic pion theoretical potential resembles the phenomenological potentials with large spin-orbit coupling.[14),12]

It should be noted in this connection that not only the magnitude of the spin-

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* Otsuki, Tamagaki and Watari[15] have evaluated scattering phase shifts including all nonstatic terms in OPEP and assuming reasonable potentials in Regions II and III.

The authors are indebted to Drs. Otsuki, Tamagaki and Watari for their helpful discussions on this point.

orbit force but also its relative magnitude compared with the central and tensor potentials are very important to fit the polarization or depolarization experiments.

(a) Actually the nonstatic TPEP derived by Hoshizaki and Machida[8] (denoted hereafter as HM) fit rather well[15] with the experimental phase shifts[10] if one assumes the potentials in Region III appropriately, although the spin orbit potential is of nearly the same order of magnitude with the previously derived ones.[16),17]

The shape of the spin-orbit potential in the HM potential is much more singular than the Gammel-Thaler potential and rather similar to the ones proposed by Bryan[14] and by Hamada.[12]

(b) If we increase the magnitude of the spin-orbit part in the HM potential by multiplying a factor 1.5～2, the fit to the 310 Mev *p-p* scattering phase shifts[10] becomes possible with potentials of reasonable order of magnitude assumed in Region III.[15] It seems probable that this extra factor of 1.5～2 would be obtained by taking into account the effects of (3/2, 3/2)-resonance in the pion-nucleon scattering and of the pion-pion interaction.

So we may think that considerable part of the spin-orbit potential which is needed to reproduce the high-energy nucleon-nucleon scattering data will be given by the two-pion-exchange contribution.

(c) The spin-orbit part in the HM potential is very singular and large in Region III where other processes such as the exchange of three or more pions could contribute appreciably. Therefore, it is rather difficult to separate the contributions of the potentials in Region II from that of the potentials in Region III in the analyses of the experiments at energies higher than nearly 100 Mev.

To compare the potential in Region II with the experimental results in detai and to test the nonstatic pion theory, it would perhaps be necessary to perform the double and triple scattering experiments at energies lower than 100 Mev, e.g. at 50 Mev or so.

(d) Comparison of the HM potential with the experimental phase shifts[10] shows that the *pv*-coupling is much more preferable than the *ps*-coupling.

11. It is very important that the *pv*-coupling is always much better than the *ps*-coupling when the comparison with experiment is possible.

This fact would indicate that the coupling of the pion with the nucleon at low energy would mainly be pseudovector type. The information on this point might give a link between the theory of strong interactions between ordinary particles and the theory of the structure of elementary particles.

12. The corrections to the HM potential which should be taken into consideration will be the following:

(i) *The problem on the validity of the expansion with respect to the coupling constant of the "probability" part* in the general form of the potential derived by Fukuda, Sawada and Taketani.[10] Hoshizaki and Machida[8] have evaluated the alteration of the unexpanded "probability" part due to the nonstatic effects. Comparing the results with that in the static case, this correction seems to be not

so large and have a tendency to make the potentials shallow.

(ii) *Contribution of the (3/2, 3/2)-resonance in the pion-nucleon scattering.* This contribution seems to make the potential deep contrary to the first one. Comparing the HM potential with the TMO potential,[9] which is the static limit of the HM potential, and with KMO potential,[19] which is the TMO potential supplemented by the pion-nucleon rescattering corrections, one may tentatively suppose that the rescattering corrections would not modify the qualitative features of the HM potential, although the depth of the potential might become considerably deeper. It is necessary to perform the evaluation of the contributions from the rescattering corrections to the nonstatic TPEP, for that purpose the dispersion theoretic approach would be most appropriate.

(iii) *Effects of the pion-pion interaction.* Horie and Machida[20] have derived the static potential of the order $g^4\lambda$ due to the interaction Hamiltonian $\lambda\phi^4$. The potential obtained includes only the isospin-independent central force which is repulsive (attractive) when λ is negative (positive), and the magnitude of which is small compared with the corresponding part of the TMO potential.

Fujii and Miyamoto[21] have evaluated a part of the contributions from the state with $I=J=1$ (I and J are the total isotopic spin and angular momentum respectively) in the two-pion system using the phenomenological form factor for the vector current of the nucleon obtained by the electron-nucleon scattering experiments. Their results involve large spin-orbit part which has a similar shape and magnitude with the Gammel-Thaler potential. It will perhaps be necessary to make clear the other part of the contributions from the state with $I=J=1$ in the twopion system, which is not taken into account in their calculation and might alter the results considerably.

13. The difference between the potentials due to the *pv*- and *ps*-couplings that appear in the nonstatic OPEP cannot be treated by a *purely* dispersion theoretic approach, because it depends solely on the subtraction terms.

This fact is an example of the inadequacy of the purely dispersion theoretic approach when one deals with the structure of interactions or elementary particles, although dispersion theoretic approaches are very powerful tool to investigate suitable problems.

14. The nonstatic OPEP in Region I may be derived rigorously in the case of the *ps*-coupling.[7] We think it has a precise meaning also in the case of the *pv*-coupling, despite that it will depend on the structure of the pion and the nucleon.[7]

15. The comparisons of the pion field theory with the experiments so far performed have been restricted mainly to the static approximation both for the real pion phenomena as well as for the virtual pion phenomena.

The nonstatic theory of the pion-nucleon interaction may now be compared with the nucleon-nucleon scattering experiments. Especially, the nonstatic OPEP which seems to be almost free from theoretical ambiguity will give a very con-

venient device to investigate the dynamical properties of the interaction.

It is not accidental that it is the two-nucleon problem where the precise comparison of the theory of the pion-nucleon interaction has become possible and the strong indication concerning the type of the interaction between the pion and the nucleon at low energies has been obtained for the first time. The reasons for this are, firstly, the two-nucleon interaction at far distances is a phenomenon in which the so-called low energy theorem holds exactly, secondly the matrix elements between the states where energy is not conserved contribute to the potential operator, and thirdly the experiments on two-nucleon problems are at present much more accurate than that on real pion processes, and therefore may allow very precise comparison with the theory.

The authors are very much indebted to Drs. S. Otsuki, R. Tamagaki, W. Watari and N. Hoshizaki for their helpful discussions.

References

1) M. Taketani, S. Nakamura and M. Sasaki, Prog. Theor. Phys. **6** (1951), 581. This paper is reprinted in reference 2).
2) Suppl. Prog. Theor. Phys. No.3 (1956).
3) S. Machida and T. Toyoda, reference 2), Part III.
4) L. L. Foldy and E. Eriksen, Phys. Rev. **95** (1954), 1048; **98** (1955), 775.
L. Durand, III, Phys. Rev. **108** (1957), 1597.
5) See, e.g. L. Wolfenstein, Ann. Rev. Nucl. Sci. **6** (1956), 43; L. Puzikov, R. Ryndin and J. Smorodinsky, Nucl. Phys. **3** (1957), 436.
6) A review of the previous calculations may be found in reference 3).
See also S. Okubo and R. E. Marshak, Ann. Phys. **4** (1958), 166; S. Okubo and S. Sato, Prog. Theor. Phys. **21** (1959), 383; M. Sugawara and S. Okubo, Phys. Rev. **117** (1960), 605, 611; N. Tzoar, R. Raphael and A. Klein, Phys. Rev. Letters **2** (1959), 433; **3** (1959), 145.
7) N. Hoshizaki and S. Machida, the following paper, Prog. Theor. Phys. **24** (1960), 1325.
8) N. Hoshizaki and S. Machida, Prog. Theor. Phys., to be published.
9) M. Taketani, S. Machida and S. Ohnuma, Prog. Theor. Phys. **7** (1952), 45
10) P. Cziffra, M. H. MacGregor, M. J. Moravcsik and H. P. Stapp, Phys. Rev. **114** (1959), 880.
M. H. MacGregor, M. J. Moravcsik and H. P. Stapp, Phys. Rev. **116** (1959), 1248.
11) S. Machida, Prog. Theor. Phys. **18** (1957), 467.
12) T. Hamada, Prog. Theor. Phys. **24** (1960), 220, 1033.
13) J. Goto and S. Machida, Prog. Theor. Phys. **25** (1961), No. 1.
14) J. L. Gammel and R. M. Thaler, Phys. Rev. **107** (1957), 291, 1337.
P. S. Signell and R. E. Marshak, Phys. Rev. **109** (1958), 1229.
P. S. Signell, R. Zinn and R. E. Marshak, Phys. Rev. Letters **1** (1958), 416.
R. A. Bryan, Nuovo Cimento, to be published.
15) S. Otsuki, R. Tamagaki and W. Watari, to be published.
16) S. Sato, Prog. Theor. Phys. **13** (1955), 457.
17) M. Sugawara and S. Okubo, Phys. Rev. **117** (1960), 605, 611.
18) N. Fukuda, K. Sawada and M. Taketani, Prog. Theor. Phys. **12** (1954), 156.
19) H. Miyazawa, Phys. Rev. **104** (1956), 1741.
M. Konuma, H. Miyazawa and S. Otsuki, Prog. Theor. Phys. **19** (1958), 17.
20) M. Horie and S. Machida, reported at the Meeting of the Japan Physical Society, Oct., 1960,
21) Y. Fujii and Y. Miyamoto, Prog. Theor. Phys., to be published.

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Two Nucleon Potential with Full Recoil. I

—— *General Formalism and One-Pion-Exchange Potential** ——

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Nonstatic nuclear potential due to one-pion-exchange process is derived in the momentum space without using the expansion with respect to the inverse of the mass of the nucleon. Close examinations of its properties show that it can be expressed with good accuracy by a local potential in x-space at low energies (below nearly 300 Mev). Nonstatic parts of the potential have different signs according as whether the assumed coupling between the pion and the nucleon is pseudoscalar or pseudovector. Comparison with the experimental phase shifts is briefly discussed. It is shown in the case of the pseudoscalar coupling that the nonstatic one-pion-exchange potential derived in this paper is almost exact from the theoretical point of view when the distance between two nucleons is far enough (large compared with one third of the pion Compton wavelength). It is also argued on some physical assumptions that the potential will in fact be exact in the case of the pseudovector coupling too.

§ 1. Introduction

In the preceding paper[1] (referred to hereafter as TM), Taketani and Machida discussed on the problems of the nuclear forces at the present stage and suggested a plan for theoretical investigations which will be followed in this and the subsequent papers.

As has been stressed in TM, one of the central problems at the present stage of the theory of nuclear forces will be to derive the quantitatively reliable nonstatic OPEP (one-pion-exchange potential) and TPEP (two-pion-exchange potential). This calculation will be necessary for interpreting the experimental data concerning the polarization of the nucleon (polarization, depolarization, spin correlation, and so forth), to which the nonstatic interactions will give essential contributions contrary to the observables averaged over the direction of the spin (total and differential cross sections). It is also necessary for establishing the potentials in

* This work has been done as a part of the 1960–61 Annual Research Project on the Theory of the Nuclear Forces organized by the Research Institute for Fundamental Physics, Kyoto University

Regions I ($x \gtrsim 1.5$) and II ($0.7 \lesssim x \lesssim 1.5$).* Especially it will be necessary to exploit, so to speak, the complete potential, which includes all the possible types of the two nucleon potential for the analyses of the complete or nearly complete experiments.

All the evaluations of the nonstatic potentials so far published have been based on the expansion with respect to M^{-1}(M being the rest mass of the nucleon), most of them being confined to the evaluation of the first order correction with respect to M^{-1} for the g^4-order static potential (g is the pion-nucleon coupling constant).[2),3)] Okubo and S. Sato[4)] and Tzoar, Raphael and Klein[5)] have evaluated the spin-orbit potential due to the two-pion-exchange process including the contributions from the (3/2, 3/2) resonance in the pion-nucleon scattering, and Okubo and Marshak[6)] have derived the second order correction with respect to M^{-1} for the g^2-order potential. Unfortunately, however, the validity of the expansion with respect to M^{-1} is very doubtful, or it may even diverge. For example, I. Sato has evaluated the first order corrections with respect to M^{-1} to the fourth order static potential to show that they almost completely cancel the corresponding terms in the static potential both for the central and the tensor parts respectively except in the ^3E state (see Table 5-1 of reference 2)).

The reason for the use of the expansion with respect to M^{-1} in the previous calculations is to carry out practically the Fourier transformation of the potential to the x-space from the momentum space. One of the powerful methods by which one may avoid the use of the M^{-1}-expansion will be the dispersion theoretic approach, which has several advantages such as the relativistic invariance throughout the calculation and unnecessity of the renormalization. There seems to be, however, several difficulties in the course of the evaluation of the observable quantities by dispersion theoretic approaches at present. Moreover, this method is not suitable to treating the problems concerning the structure of the interaction or the elementary particles, such as the determination of the type of the coupling between the pion and the nucleon (this point will be discussed in § 4).

We will develop, in this and the subsequent papers, a method of calculation in the momentum representation, the principal steps of which will be the following:

i) We use the most general forms of the nuclear potential in the momentum representation, which may be expressed as a nonlocal potential in the x-representation. We will evaluate the nonstatic OPEP and TPEP in the momentum representation without using M^{-1}-expansion, and will separate them into invariant forms.

ii) We will insert this potential in the momentum representation into the integral equation** which is the Fourier transform of the Schrödinger equation or

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* x is the distance between two nucleons in the unit of the pion Compton wavelength.

** This integral equation in the momentum representation and the matrix elements of the most general potentials as well as the general types of the potential in the momentum representation will be discussed in detail in a separate paper.[7)]

into the equation for K-matrix, Eq. (2·5), to obtain scattering phase shifts.

iii) If the potential with full recoil obtained in the momentum representation were approximated with sufficient accuracy for a certain domain of energies by a local potential, we may use the usual Schrödinger equation to obtain phase shifts. Actually we shall see that for scatterings at energies below nearly 300 Mev the OPEP and TPEP with full recoil may be represented rather accurately by local potentials in x-space.

We will discuss, using the above-mentioned method, in this and the subsequent papers the following problems:

a) To see to what extent the concept of the local two-nucleon potential in x-space is valid by close investigations of the properties of the potentials with full recoil;

b) To derive the nonstatic nuclear potential in Regions I and II, by evaluating the OPEP and TPEP with full recoil, and to make clear the domain where the static approximation holds good;

c) To see differences between the cases of the *ps*- and *pv*-couplings which may appear in the nonstatic OPEP and TPEP, and to compare them with experiment, if possible;

d) To see to what extent the experiments on the high energy nucleon-nucleon scattering, including experiments on the polarization, depolarization, correlation of the spin, and so forth, could be accounted for by the pion theory of nuclear forces.

In § 2 we will briefly discuss the general forms of the two-nucleon potential in the momentum representation and coupled equations which connect the R-matrix or K-matrix with the potential; a derivation of the nonstatic OPEP will be given in § 3; importance of the nonstatic OPEP will be discussed in § 4 in connection with the fact that the nonstatic OPEP may be derived almost unambiguously; in § 5 the nonstatic OPEP will be numerically examined in detail and a local potential in x-space, which is a rather good approximation at energies below nearly 300 Mev. will be found; in § 6 comparison of the nonstatic OPEP with experiment is briefly discussed postponing the detailed discussion to the paper III which will be published by Otsuki, Tamagaki and Watari.[7a] The contributions from a part of the nonstatic OPEP will also be discussed in connection with recent analyses given by Hamada.[8] Other problems in our program will be treated in subsequent papers.

§ 2. General formalism

Most general types of the two-nucleon potential in the momentum representation may be written in the following manner:

$$<\boldsymbol{p}', -\boldsymbol{p}'|V|\boldsymbol{p}, -\boldsymbol{p}> = V_0 + i\frac{(\boldsymbol{S}\cdot\boldsymbol{k}\times\boldsymbol{q})}{\mu^2}V_1$$

$$+\frac{(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{k})\,(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{k})}{\mu^2}V_2+\frac{(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{q})\,(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{q})}{\mu^2}V_3$$
$$+\frac{(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{k}\times\boldsymbol{q})\,(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{k}\times\boldsymbol{q})}{\mu^4}V_4+(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)})\,V_5, \quad (2\cdot1)$$

where μ is the mass of the pion, $\boldsymbol{p}$, $-\boldsymbol{p}$ and $\boldsymbol{p}'$, $-\boldsymbol{p}'$ are momenta of the two nucleons in the initial and final states respectively and

$$\boldsymbol{k}=\boldsymbol{p}-\boldsymbol{p}',$$
$$\boldsymbol{q}=\frac{1}{2}(\boldsymbol{p}+\boldsymbol{p}'), \quad (2\cdot2)$$
$$\boldsymbol{S}=\frac{1}{2}(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)}),$$

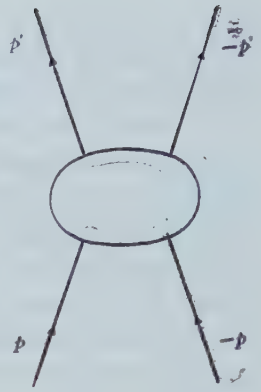

Fig. 1.

and V_i's $(i=0, 1,\cdots5)$ are real functions of $\boldsymbol{k}^2$, $\boldsymbol{q}^2$ and $(\boldsymbol{k}\times\boldsymbol{q})^2$ and may be written explicitly in the isotopic spin space in the following way:

$$V_i=\boldsymbol{1}\cdot V_i^1(\boldsymbol{k}^2,\ \boldsymbol{q}^2,\ (\boldsymbol{k}\times\boldsymbol{q})^2)+(\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)})\,V_i^\tau(\boldsymbol{k}^2,\ \boldsymbol{q}^2,\ (\boldsymbol{k}\times\boldsymbol{q})^2). \quad (2\cdot3)$$

The general form (2·1) has been derived under the following assumptions: invariance under translation, Galilei transformation, interchange of two particles, rotation, space reflection, time reversal, and hermiticity. Okubo and Marshak[6] was the first to derive the general form of the nonstatic nuclear potential, which corresponds to Eq. (2·1) transformed into x-space with respect to $\boldsymbol{k}$. We will give the derivation of Eq. (2·1) in detail in a separate paper,[7] because it will be necessary afterwards (see the Appendix) to obtain the nonstatic potential in x-representation and make clear the properties of operator $\boldsymbol{q}$ in x-representation.

If we expand the potential with respect to M^{-1}, we obtain V_0, V_2 and V_5 only in the static approximation, V_0, V_1, V_2 and V_5 in the first order, and all types in the second and higher orders with respect to M^{-1}.

It should be noted that the conservation of energy does not hold for the matrix elements of the potential operator. Therefore

$$\boldsymbol{p}^2\neq\boldsymbol{p}'^2,$$

and $\boldsymbol{k}^2$, $\boldsymbol{q}^2$ and $(\boldsymbol{k}\times\boldsymbol{q})^2$ are independent variables. When energy conserves we get

$$(\boldsymbol{k}\cdot\boldsymbol{q})=0,\ (\boldsymbol{k}\times\boldsymbol{q})^2=\boldsymbol{k}^2\boldsymbol{q}^2,$$

and V_2, V_3, V_4 and V_5 are not independent of each other because of the identity

$$(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{k})\,(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{k})\,\boldsymbol{q}^2+(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{q})\,(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{q})\,\boldsymbol{k}^2+(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{k}\times\boldsymbol{q})\,(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{k}\times\boldsymbol{q})$$
$$=(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)})\,(\boldsymbol{k}\times\boldsymbol{q})^2+(\boldsymbol{k}\cdot\boldsymbol{q})\,\{(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{k})\,(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{q})+(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{q})\,(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{k})\}. \quad (2\cdot4)$$

The general form (2·1) is also relativistically covariant, because a manifestly covariant expression for the corresponding matrix element written by the Dirac

wave function and γ-matrices may be transformed into the form given in Eq. (2·1) by making use of the Tani-Foldy transformation.[9]

One method to obtain the scattering phase shifts from the potential in the momentum representation is to transform the Schrödinger equation into the integral equation in the momentum space and to solve it. This method will be discussed in a separate paper.[7] Another method will be to solve the following equation[10]

$$<f|\boldsymbol{R}|i> = -<f|\boldsymbol{K}|i> - i\pi\int\delta(E_i-E_n)<f|\boldsymbol{K}|n><n|\boldsymbol{R}|i>,$$

$$<f|\boldsymbol{K}|n> = <f|\boldsymbol{V}|n> + P\int\frac{<f|V|m><m|K|n>}{E_n-E_m}, \qquad (2\cdot5)$$

to obtain the R- or K-matrix, where boldface letter means that the matrix element is taken between two states with equal energy. Inserting the general forms of R- and K-matrices which are similar to Eq. (2·1) into Eq. (2·5), we obtain coupled integral equations for these invariant functions. If one solves these equations to get the invariant functions appearing in R-matrix, one may readily write down the observable quantities by these invariant functions.[11] Or one may solve the second equation in (2·5) to obtain the scattering phase shifts and compare them with the experimental phase shifts, since the eigenvalues of K-matrix are $-\tan\delta/\pi$. This method will be a useful one, because this equation is of the same type as the one tranformed from the ordinary Schrödinger equation, and may be converted to a Fredholm integral equation of the third kind by a simple replacement.[7]

§ 3. Derivation of the nonstatic OPEP

A general formula for the interaction potential between two particles in the quantum theory of fields has been given by Fukuda, Sawada and Taketani[12] (referred to hereafter as FST). This formula may be used equally well in the p-representation as well as in the x-representation. One may calculate the potential up to any desired order of the coupling constant straightforwardly by this formula.

Although the second order nonstatic potential with respect to the coupling constant may be easily obtained by many methods, e.g. by calculating the transition matrix element for finite time according to the old perturbation theory, or by transforming the first order term in the interaction Hamiltonian into the second order one by a unitary transformation, the FST method seems to be more convenient than others to evaluate higher order nonstatic potentials.

When one is going to solve the two nucleon problems using potentials, one must have the off- as well as on-energy shell matrix elements of the potential operator. We will give also the off-energy shell matrix elements by the prescription given below. Our results are the same as those given by many authors[12a),3),3)] although expressed in different forms.

We will evaluate the nonstatic OPEP for the case of the ps- and pv-couplings following the FST formula (the case of a linear combination of the ps- and pv-

couplings will be discussed in a separate paper[13]). We will calculate up to the second order of the coupling constant in the present paper and the results will be shown in § 4 to be in fact rigorous in Region I.

The FST formula may be written in the following form,

$$\begin{aligned} V&=\eta P^{1/2}\eta V_n \eta P^{1/2}\eta-\eta H_0\eta, \\ V_n&=J^\dagger(H_0+H')J, \\ P&=\frac{1}{J^\dagger J}, \end{aligned} \tag{3·1}$$

where V_n may be called the "normal" part and P the "probability" part, and H_0 and H' give free and interaction Hamiltonians respectively. η is a projection operator to a subspace composed of states with two nucleons and no pion. J is a linear operator which transforms the two-nucleon subspace back into the whole Hilbert space, and is actually determined by solving the equation

$$(1-\eta)\{H'J+[H_0, J]-J\eta H'J\}\eta=0, \tag{3·2}$$

under the conditions

$$\begin{aligned} &J=\eta+(1-\eta)J\eta, \\ &J\to\eta, \text{ as } H'\to 0. \end{aligned} \tag{3·3}$$

Expanding Eq. (3·1) with respect to the coupling constant, we obtain the second order potential;

$$V^{(1\pi)}=\eta_2 H'(1-\eta)\frac{(H_0^2+H_0^1)/2-H_0}{(H_0^2-H_0)\ (H_0^1-H_0)}(1-\eta)H'\eta_1 \tag{3·4}$$

where H_0^1 and $H_0^2(=H_0)$ operate on η_1 and η_2 respectively. In Eq. (3·4) we have assumed that H' is of the first order with respect to the coupling constant. When H' includes the second order terms (denoted as H''), which is the case for the *pv*-coupling, the right-hand side of Eq. (3·4) should be supplemented by a term

$$\eta H''\eta\,.$$

Equation (3·4) turns into the following form,

$$V^{(1\pi)}=-[\bar{u}(\boldsymbol{p}')\ O_\alpha u(\boldsymbol{p})]^{(1)}[\bar{u}(-\boldsymbol{p}')O_\alpha u(-\boldsymbol{p})]^{(2)}\Delta(k^2), \tag{3·5}$$

where

$$O_\alpha=\begin{cases} -ig\gamma_5\tau_\alpha & \text{for the } ps\text{-coupling,} \\ -i\dfrac{f}{\mu}\gamma_5\gamma_\kappa\partial_\kappa\tau_\alpha & \text{for the } pv\text{-coupling,} \end{cases} \tag{3·6}$$

where $-i\partial_\mu$ means energy-momentum vector $(\boldsymbol{k}=\boldsymbol{p}-\boldsymbol{p}',\ \omega_k=(\boldsymbol{k}^2+\mu^2)^{1/2})$, and

$$\bar{\Delta}(k^2)=P\frac{1}{k^2+\mu^2},$$
$$k_\mu=(\boldsymbol{p}-\boldsymbol{p}',\ E_p-E_{p'}), \tag{3·7}$$
$$E_p=(\boldsymbol{p}^2+M^2)^{1/2}.$$

$u(\boldsymbol{p})$ is a four component Dirac wave function with momentum $\boldsymbol{p}$ and energy E_p.

The potential (3·5) may be written in the following way by using the Pauli spin matrix:

$$V^{(1\pi)}=\frac{(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{k})\,(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{k})}{\mu^2}V_2+\frac{(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{q})\,(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{q})}{\mu^2}V_3$$
$$+\frac{(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{k}\times\boldsymbol{q})\,(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{k}\times\boldsymbol{q})}{\mu^4}V_4+(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)})\,V_5, \tag{3·8}$$

where

$$V_2=K\left(-A+\frac{\boldsymbol{q}^2}{M^2}C\right),$$
$$V_3=K\left(-B+\frac{\boldsymbol{k}^2}{M^2}C\right),$$
$$V_4=\left(\frac{\mu}{M}\right)^2KC, \tag{3·9}$$
$$V_5=-\left(\frac{\boldsymbol{k}\times\boldsymbol{q}}{M\mu}\right)^2KC,$$

and

$$K=\frac{f^2}{4\pi}(\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)})\frac{4\pi M^4}{E_pE_{p'}(E_p+M)\,(E_{p'}+M)}\bar{\Delta},$$
$$A=\frac{1}{4M^3}(E_p+E_{p'}+2M)^2\left\{M-\theta\frac{(E_p-E_{p'})^2}{E_p+E_{p'}+2M}\right\},$$
$$B=\frac{1}{M^3}(E_p-E_{p'})^2\left\{M-\theta\,(E_p+E_{p'}+2M)\right\}, \tag{3·10}$$
$$C=\frac{E_p+E_{p'}+2M}{M(E_p+E_{p'})}\left\{M-\theta\frac{1}{2}\left(\frac{(E_p-E_{p'})^2}{E_p+E_{p'}+2M}+(E_p+E_{p'}+2M)\right)\right\}.$$

In the above equations, $\theta=0$ corresponds to the *ps*-coupling and $\theta=1$ to the *pv*-coupling, and

$$\frac{f^2}{4\pi}=\left(\frac{\mu}{2M}\right)^2\frac{g^2}{4\pi}\approx 0.08.$$

As is well known, the *ps*-coupling may be transformed into the *pv*-coupling plus additive terms, and vice versa, by appropriate unitary transformations, and

by doing so one will be able to make the second order potentials, due to the two types of couplings, equal with each other, except the contact interaction terms.[14] Therefore it will be possible to transform the nonstatic OPEP given by Eqs. (3·8)～(3·10) derived from the $ps(pv)$-coupling to the one due to the $pv(ps)$-coupling plus higher order terms with respect to the coupling constant. Actually, Sugawara and Okubo[3] transformed a part of the terms of order g^2/M^2 into the terms of order g^4/M to obtain the same second order potential. However, by such a unitary transformation we will have a physically different potential from the original one, because they will give different answers for properties of bound states. So we prefer not to make use of any unitary transformation in this paper. The differences between the case of the ps- and pv-couplings appeared in Eq. (3·10) are physically meaningful, whether one transforms them into other forms by some unitary transformation or not.

The static limit of Eqs. (3·8)～(3·10) reduces to the well-known formula

$$V_{static}^{(1\pi)}=-\frac{f^2}{\mu^2}(\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)})\frac{(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{k})\ (\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{k})}{\boldsymbol{k}^2+\mu^2}. \tag{3·11}$$

It is also readily seen that if one assumes the conservation of energy, the terms V_3, V_4 and V_5 disappear.*

§ 4. Rigorous derivation of the nonstatic OPEP

Before going into detailed investigation of the properties of the nonstatic OPEP, we will show that the nonstatic OPEP given by Eqs. (3·8)～(3·10) is the exact expression for small momentum transfer ($|k^2|\ll(3\mu)^2$) or for large distances ($x\gg 1/3$) in the case of the ps-coupling and that it will also be an exact one in the case of the pv-coupling if we put some physical assumptions on the nature of the interactions between pion and nucleon. The following discussion is a generalization of a paper by Hiida, Iwadare and Machida,[16] where it has been shown that the static OPEP is exactly given by the second order perturbation calculation.

First, we consider the case of the ps-coupling. The nonstatic OPEP including all possible corrections may be written in the following form,

$$V=-[\bar{u}(\boldsymbol{p}')\Gamma_\alpha u(\boldsymbol{p})]^{(1)}[\bar{u}(-\boldsymbol{p}')\Gamma_\alpha u(-\boldsymbol{p})]^{(2)}\bar{\Delta}_c(k^2), \tag{4·1}$$

where

$$\bar{\Delta}_c(k^2)=P\frac{1}{k^2+\mu^2}+P\int_{(3\mu)^2}^{\infty}\frac{\sigma(\kappa^2)\,d\kappa^2}{k^2+\kappa^2},$$

$$\sigma(\kappa^2)\geqq 0, \tag{4·2}$$

* Very recently Gupta has applied the S-matrix method to evaluating the nonstatic corrections in the OPEP and TPEP to obtain just a term included in V_2 in our notation in the case of OPEP.[15]

$$k_\mu = p_\mu - p_\mu' = (\mathbf{p}-\mathbf{p}',\ E_p - E_{p'}),$$

and $\mathbf{p}^2 \neq \mathbf{p}'^2$.

Γ_α in Eq. (4·1) is given by

$$\left(\frac{p_0' p_0}{M^2}\right)^{1/2} <p'|-(\Box^2-\mu^2)\phi_\alpha(0)|p>$$

$$=[\mu^2+(p-p')^2]\Delta_{FC}[(p-p')^2]\bar{u}(p')\Gamma_\alpha(p', p)u(p), \tag{4·3}$$

and may be written in general in the following way,

$$\Gamma_\alpha(p', p) = -ig\gamma_5\tau_\alpha\{1+[(p-p')^2+\mu^2]F_1+(p'^2+M^2)F_2+(p^2+M^2)F_3\}$$
$$+(i\gamma p'+M)i\gamma_5\tau_\alpha(i\gamma p+M)F_4+(i\gamma p'+M)i\gamma_5\tau_\alpha F_5+i\gamma_5\tau_\alpha(i\gamma p+M)F_6, \tag{4·4}$$

where

$$F_i = F_i(p'^2+M^2,\ p^2+M^2,\ (p-p')^2),$$

and g is the observed coupling constant defined by

$$\left(\frac{p_0' p_0}{M^2}\right)^{1/2} <p'|-(\Box^2-\mu^2)\phi_\alpha|p>\Big|_{(p-p')^2+\mu^2=0} = -ig\bar{u}(p')\gamma_5\tau_\alpha u(p). \tag{4·5}$$

nserting Eq. (4·4) into Eq. (4·3), we obtain

$$\bar{u}(p')\Gamma_\alpha u(p) = -ig\bar{u}(p')\gamma_5\tau_\alpha u(p)[1+(k^2+\mu^2)F_1(k^2)]. \tag{4·6}$$

We assume the following spectral representation for $F_1(k^2)$,

$$F_1(k^2) = \frac{1}{\pi}\int_{(3\mu)^2}^{\infty} d\sigma^2 \frac{ImF_1(-\sigma^2)}{k^2+\sigma^2-i\epsilon}. \tag{4·7}$$

The essential point for our discussion is that $F_1(k^2)$ does not have a singularity at $k^2=-\mu^2$. Although we use the unsubtracted form, Eq. (4·7), it will be easily seen in the course of the following discussion that the existence of the subtraction terms in the spectral representation for $F_1(k^2)$ does not alter our conclusions.

Using Eqs. (4·2), (4·6) and (4·7), Eq. (4·1) turns into

$$V = g^2[\bar{u}(p')\gamma_5\tau_\alpha u(p)]^{(1)}[\bar{u}(-p')\gamma_5\tau_\alpha u(-p)]^{(2)}$$
$$\times\left[1+(k^2+\mu^2)\int_{(3\mu)^2}^{\infty} d\lambda^2 \frac{Im\,F_1(-\lambda^2)}{k^2+\lambda^2-i\epsilon}\right]^2\left[P\frac{1}{k^2+\mu^2}+P\int_{(3\mu)^2}^{\infty} d\kappa^2\frac{\sigma(\kappa^2)}{k^2+\kappa^2}\right]$$

$$=g^2[\bar{u}(p')\gamma_5\tau_\alpha u(p)]^{(1)}[\bar{u}(-p')\gamma_5\tau_\alpha u(-p)]^{(2)}$$
$$\times\left\{P\frac{1}{k^2+\mu^2}+2\int_{(3\mu)^2}^{\infty} d\kappa^2\frac{ImF_1(-\kappa^2)}{k^2+\kappa^2-i\epsilon}+P\int_{(3\mu)^2}^{\infty} d\kappa^2\frac{\sigma(\kappa^2)}{k^2+\kappa^2}\right.$$

$$+(k^2+\mu^2)\int_{(3\mu)^2}^{\infty}d\kappa^2\frac{ImF_1(-\kappa^2)}{k^2+\kappa^2-i\epsilon}\Big[2P\int_{(3\mu)^2}^{\infty}d\kappa^2\frac{\sigma(\kappa^2)}{k^2+\kappa^2}$$

$$+\int_{(3\mu)^2}^{\infty}d\kappa^2\frac{ImF_1(-\kappa^2)}{k^2+\kappa^2-i\epsilon}+(k^2+\mu^2)\int_{(3\mu)^2}^{\infty}d\kappa^2\frac{ImF_1(-\kappa^2)}{k^2+\kappa^2-i\epsilon}\cdot P\int_{(3\mu)^2}^{\infty}d\lambda^2\frac{\sigma(\lambda^2)}{k^2+\lambda^2}\Big]. \qquad (4\cdot8)$$

The first term in the right-hand side of Eq. (4·8) has a pole at $-k^2=\mu^2$, while the other terms have singularities only at $-k^2\geqq(3\mu)^2$. So the contributions from the latter may be neglected compared with that from the former for the values of $|k^2|$ which are small compared with $(3\mu)^2$. This fact corresponds in the x-space to the fact that the first term in the right-hand side of Eq. (4·8) gives a term the exponential factor of which is e^{-x}, while the others give terms the exponential factors of which are $e^{-ax}(a\geqq3)$ at large distances. Therefore the exact form of the nonstatic OPEP at large distances ($x\gg1/3$) is given by the first term of the right-hand side of Eq. (4·8), which is just the one given by Eq. (3·5), the order of magnitude of the correction factor being e^{-2}.

It should be remarked that it is essential to define the observed coupling constant at $k^2+\mu^2=0$. If one uses the other coupling constant defined at $k^2=0$, the nonstatic OPEP may not be given by the second order perturbation, as is seen easily.

Now let us consider the case of the *pv*-coupling. The *pv*-coupling in the local field theory is well known to be unrenormalizable and no systematic method is known to extract reliable predictions from that. However, it is plausible that the elementary particles and the interactions between them will have some sort of structures and the interactions will be nonlocal. The interaction between the pion and the nucleon at low energies would, then, have to be considered as the local limit of an interaction with a structure. Actually, if the pion has a composite structure as in the Sakata model,[18] the interaction between the pion and the nucleon at low energies will be a linear combination of the *ps*- and *pv*-couplings and there is a direct connection between the type of the coupling and the properties of the interaction kernel between constituent particles.[19] Then it would be physically natural to assume the same spectral representation in the cases of the *ps*- and *pv*-couplings for the propagation function of the pion and for the vertex operators. So we may consider Eq. (3·5) or Eqs. (3·8)$\sim$(3·10) to be exact for small momentum transfer $|k^2|\ll(3\mu)^2$ although some physical assumptions are made for the case of the *pv*-coupling.

The differences between the nonstatic OPEP obtained by the *ps*-coupling and by the *pv*-coupling appearing in Eqs. (3·8)$\sim$(3·10) are very important, because they are the first observable difference derived almost unambiguously, and they might give an insight into the relation between the theory of the strong interaction of the elementary particles and the theory of the structure of them.

It is not accidental that these differences have first been established in the

investigation of the nuclear potential and not in the phenomena involving real pions. The reasons are: i) There is a physical region (Region I) in the nuclear potential where the so-called low energy theorem holds exactly, contrary to the real pion phenomena where the same theorem holds only in approximation (the mass of the pion is put equal to zero); ii) The matrix elements of the potential between two states with different energies also give observable effects as well as those between two states with equal energy.

It should be noted here that the differences due to two types of couplings obtained above cannot be inferred from *purely* dispersion theoretic approaches. In dispersion theoretic approaches one will evaluate first the absorptive part of the scattering amplitude and then obtain the dispersive part minus the subtraction terms using the dispersion relation. Since the absorptive parts are determined by energy conserving processes with real incoming and outgoing particles, the differences due to the assumed types of couplings will appear only in the subtraction terms, concerning which purely dispersion theoretic approaches say nothing. This is an example of the inappropriateness of purely dispersion theoretic approaches when one deals with the structure of the elementary particles or the interaction between them, although they are very powerful when applied to suitable problems.

§ 5. Detailed properties of the nonstatic OPEP

Because the nonstatic OPEP have evry clear theoretical meaning as have been discussed in the preceding section, it is interesting to investigate their properties in detail.

Our potentials given by Eqs. (3·8)$\sim$(3·10) are functions of $\boldsymbol{k}^2$, $\boldsymbol{q}^2$ and $(\boldsymbol{k}\times\boldsymbol{q})^2$, where $\boldsymbol{q}^2$-dependence gives the degree of nonlocality of the potential in x-space and $(\boldsymbol{k}\times\boldsymbol{q})^2$ corresponds to the square of the orbital angular momentum (see the Appendix). We will investigate the $\boldsymbol{q}^2$-and θ-dependences of our OPEP, θ being the angle between $\boldsymbol{k}$ and $\boldsymbol{q}$:

$$\cos\theta=\frac{(\boldsymbol{k}\cdot\boldsymbol{q})}{kq},$$

$$k=|\boldsymbol{k}|,\ q=|\boldsymbol{q}|.$$

Figs. 2(a), (b) and (c) give q-dependence of V_2, V_3 and V_4 at $\theta=60°$. It is seen that the V_i's become slowly shallow when q increases,*) and may be approximated by a constant in some domain. V_5 coincides with V_4 except for a factor $-(\boldsymbol{k}\times\boldsymbol{q}/\mu^2)^2$.

Figs. 3(a)$\sim$(e) show the θ-dependences of V_i. It is seen that the V_i's hardly depend on θ as long as q is not too large.

Therefore, we must be able to transform the nonstatic OPEP given by Eqs. (3·8)$\sim$(3·10) into the local potential in x-space in a good approximation as long

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* This fact has also been recognised by Signell.[20]

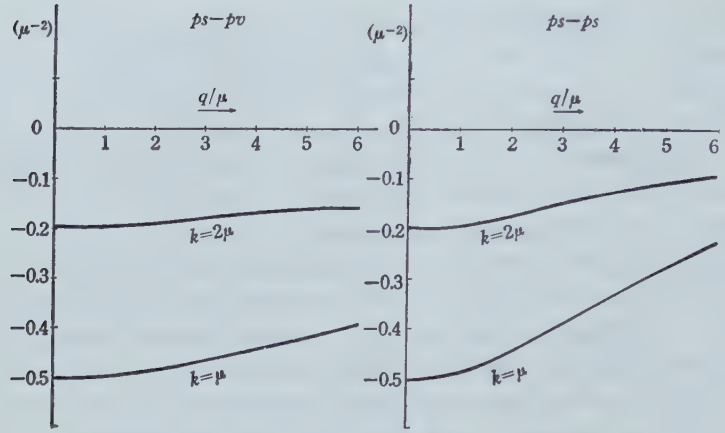

Fig. 2a. q-dependence of V_2 ($\theta=60°$).

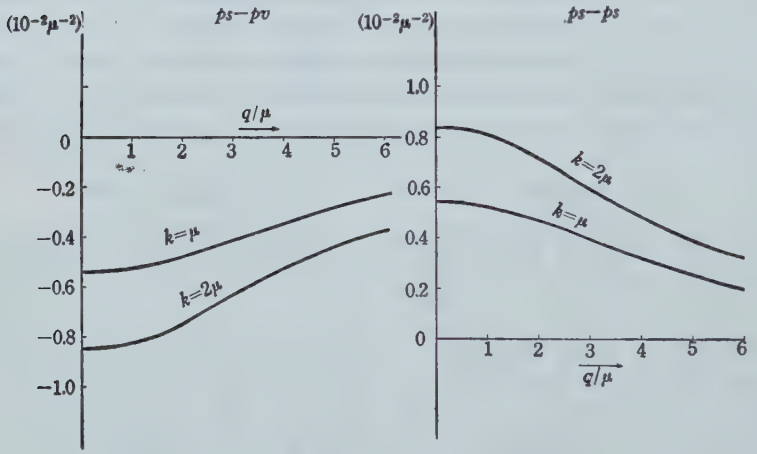

Fig. 2b. q-dependence of V_3 ($\theta=60°$).

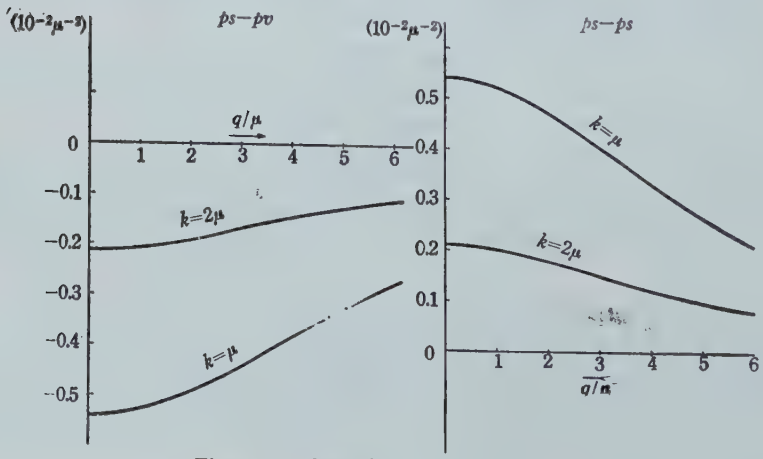

Fig. 2c. q-dependence of V_4 ($\theta=60°$).

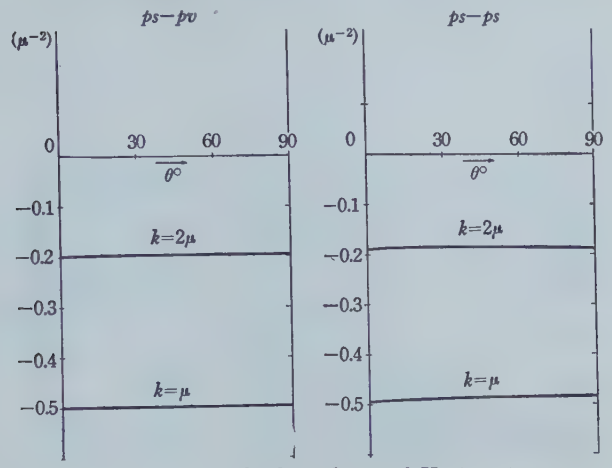

Fig. 3a. Angle dependence of V_2 $(q=\mu)$.

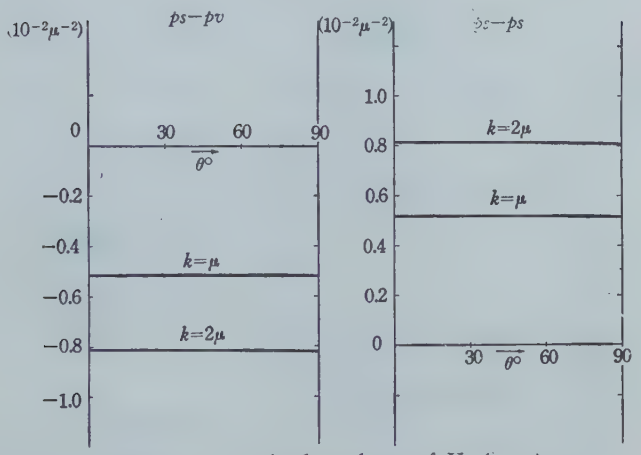

Fig. 3b. Angle dependence of V_3 $(q=\mu)$.

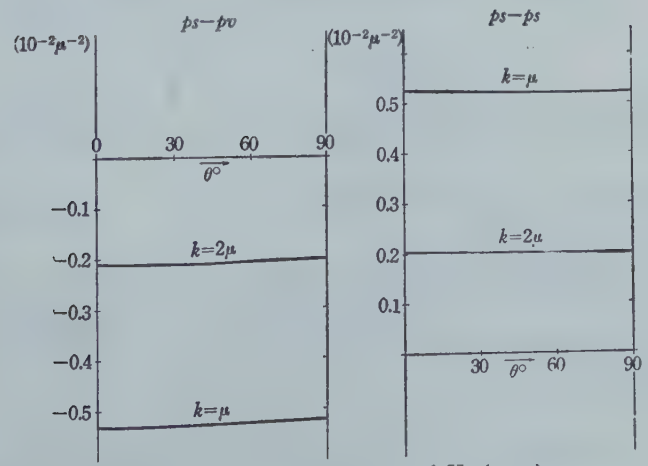

Fig. 3c. Angle dependence of V_4 $(q=\mu)$.

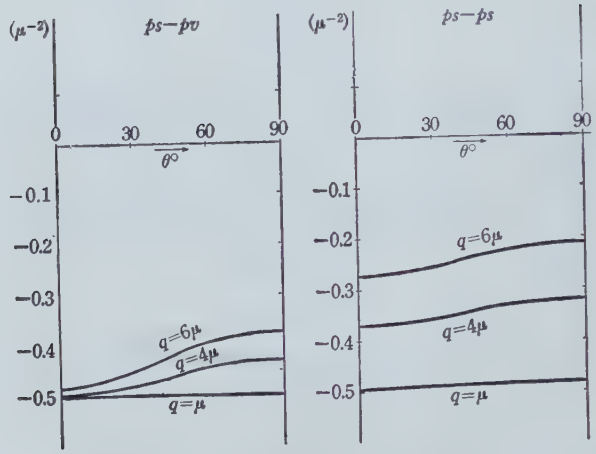

Fig. 3d. Angle dependence of V_2 $(k=\mu)$.

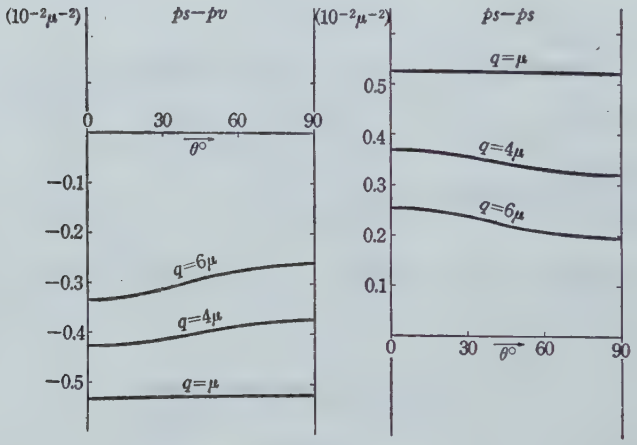

Fig. 3e. Angle dependence of V_4 $(k=\mu)$.

as q is not large (i.e. for nucleon-nucleon scatterings at energies lower than nearly 300 Mev).

k-dependences of V_i's are given in Figs. 4(a)$\sim$(c). It is seen that V_2 is very close, at $q=\mu$, to the static one.

Now we will transform the nonstatic OPEP into x-space when q is not large. V_i's in Eqs. (3·9) and (3·10) are given by terms of first nonvanishing order in $\boldsymbol{q}/M$ sufficiently accurately when q is small compared with M.* Our potential, then, may be written as

* For V_2 it might be necessary to use more accurate form. This point will be investigated in another occasion.

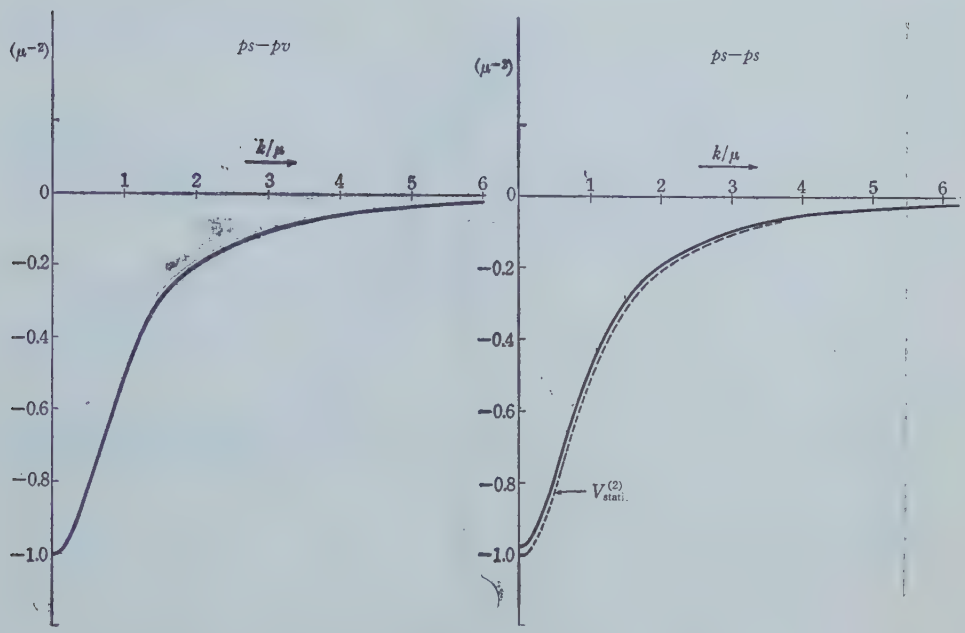

Fig. 4a. k-dependence of V_2 ($\theta=90°$, $q=\mu$)

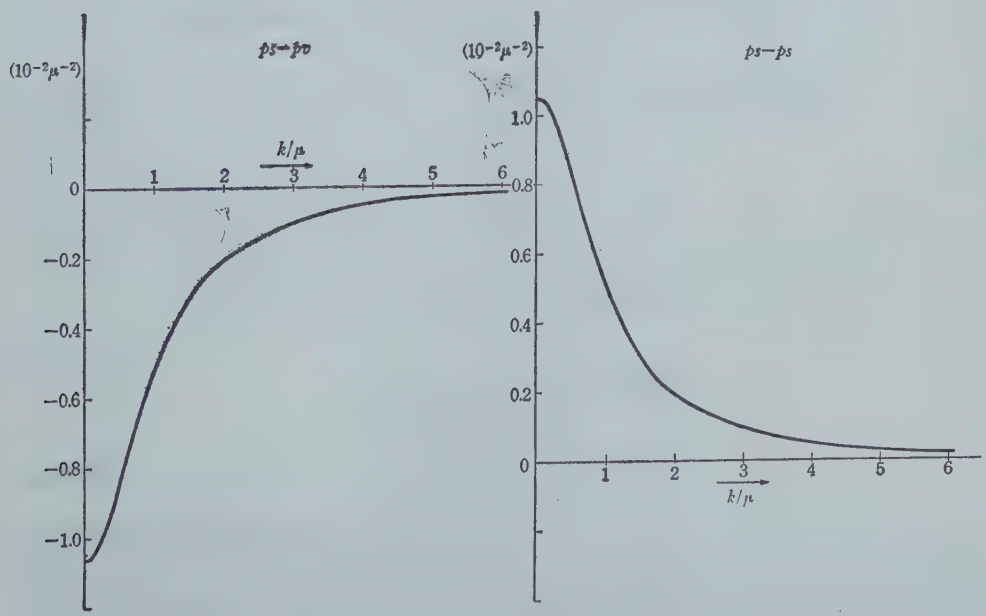

Fig. 4b. k-dependence of V_3/k^2 ($\theta=90°$, $q=\mu$)

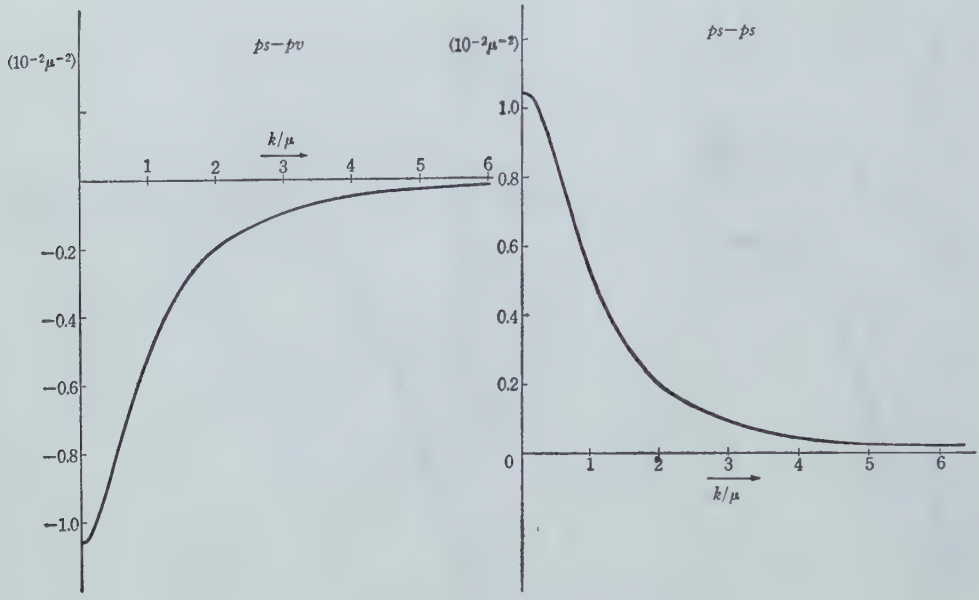

Fig. 4c. k-dependence of V_4 ($\theta=90°$, $q=\mu$).

$$V^{(1\pi)}(\boldsymbol{k},\boldsymbol{q})=(\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)})\Big[\frac{(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{k})\ (\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{k})}{\mu^2}\Big\{V_2(k)+\frac{\boldsymbol{q}^2}{M^2}\Delta V_2(k)\Big\}$$
$$+\frac{(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{q})\ (\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{q})}{\mu^2}V_3(k)+\frac{(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{k}\times\boldsymbol{q})\ (\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{k}\times\boldsymbol{q})}{\mu^4}V_4(k)$$
$$-(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)})\Big(\frac{\boldsymbol{k}\times\boldsymbol{q}}{\mu^2}\Big)^2V_4(k)\Big], \tag{5·1}$$

where

$$V_2(k)=-\frac{f^2}{4\pi}\frac{4\pi}{\omega^2},$$
$$\Delta V_2(k)=\pm\frac{f^2}{4\pi}\frac{2\pi}{\omega^2},$$
$$V_3(k)=\pm\frac{f^2}{4\pi}\frac{k^2}{M^2}\frac{2\pi}{\omega^2},$$
$$V_4(k)=\pm\frac{f^2}{4\pi}\Big(\frac{\mu}{M}\Big)^2\frac{2\pi}{\omega^2}, \tag{5·2}$$

and $\omega=(k^2+\mu^2)^{1/2}$; the upper (lower) sign gives the case of the $ps(pv)$-coupling. Eqs. (5·1) and (5·2) reproduce Figs. 4(a)$\sim$(c) very accurately. We can easily see that the nonstatic terms in Eq. (5·1) vanish as a whole in the first Born approximation because of the identity (2·4).

The potentials in momentum space, Eqs. (5·1) and (5·2), may be transformed

into x-space by

$$V(\boldsymbol{x}, \boldsymbol{q}) = \int \frac{d\boldsymbol{k}}{(2\pi)^3} e^{i\boldsymbol{k}\boldsymbol{r}} V(\boldsymbol{k}, \boldsymbol{q}), \tag{5·3}$$

where $\boldsymbol{q}$ in $V(\boldsymbol{x}, \boldsymbol{q})$ means a differential operator on the two-nucleon wave function (see the Appendix),

$$\psi^*(\boldsymbol{r}) \boldsymbol{q} \psi(\boldsymbol{r}) = \frac{-i}{2} \psi^*(\boldsymbol{r}) (\overleftarrow{\nabla} + \vec{\nabla}) \psi(\boldsymbol{r}).$$

Making use of the following formulas,

$$\int \frac{d\boldsymbol{k}}{(2\pi)^3} e^{i\boldsymbol{k}\boldsymbol{r}} (\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{k}) (\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{k}) V_2(k)$$

$$= -\left[\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^2 V_2(r)\right](\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{r}) (\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{r}) - \left[\frac{1}{r}\frac{\partial}{\partial r} V_2(r) - \frac{1}{3}\delta(\boldsymbol{r})\right](\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)}), \tag{5·4a}$$

$$\int \frac{d\boldsymbol{k}}{(2\pi)^3} e^{i\boldsymbol{k}\boldsymbol{r}} (\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{k}\times\boldsymbol{q}) (\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{k}\times\boldsymbol{q}) V_4(k) = -\left[\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^2 V_4(\boldsymbol{r})\right](\boldsymbol{\sigma}^{(1)} \cdot \widetilde{\boldsymbol{L}}) (\boldsymbol{\sigma}^{(2)} \cdot \widetilde{\boldsymbol{L}})$$

$$+\left[\frac{1}{r}\frac{\partial}{\partial r} V_4(r) - \frac{1}{3}\delta(\boldsymbol{r})\right]\left\{(\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{q}) (\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{q}) - (\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)}) \boldsymbol{q}^2\right\}, \tag{5·4b}*$$

where

$$(\boldsymbol{\sigma}^{(1)} \cdot \widetilde{\boldsymbol{L}}) (\boldsymbol{\sigma}^{(2)} \cdot \widetilde{\boldsymbol{L}}) = (\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{L}) (\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{L}) - \frac{1}{4} (\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)}) (\boldsymbol{x} \cdot \vec{\nabla} - \overleftarrow{\nabla} \cdot \boldsymbol{x})$$

$$+\frac{1}{8}\{(\boldsymbol{\sigma}^{(1)} \cdot \vec{\nabla}) (\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{x}) - (\boldsymbol{\sigma}^{(1)} \cdot \overleftarrow{\nabla}) (\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{x}) + (1) \leftrightarrow (2)\},$$

and

$$\int \frac{d\boldsymbol{k}}{(2\pi)^3} e^{i\boldsymbol{k}\boldsymbol{r}} (\boldsymbol{k}\times\boldsymbol{q})^2 V_4(k)$$

$$= -\left[\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^2 V_4(r)\right]\widetilde{\boldsymbol{L}}^2 - 2\left[\frac{1}{r}\frac{\partial}{\partial r} V_4(r) - \frac{1}{3}\delta(\boldsymbol{r})\right]\boldsymbol{q}^2, \tag{5·4c}$$

where

$$\widetilde{\boldsymbol{L}}^2 = \boldsymbol{L}^2 - \frac{1}{2} (\boldsymbol{x} \cdot \vec{\nabla} - \overleftarrow{\nabla} \cdot \boldsymbol{x}),$$

and

$$V_i(r) = \int \frac{d\boldsymbol{k}}{(2\pi)^3} e^{i\boldsymbol{k}\boldsymbol{r}} V_i(k),$$

---

* In this and following equations, $\boldsymbol{L}$ means the orbital angular momentum operator, and appropriate symmetrization is to be understood, i. e. $(\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{L})(\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{L})$ means (1/2) $\{(\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{L})(\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{L}) + (\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{L})(\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{L})\}$.

we obtain

$$V_S^{(1\pi)}(x, \boldsymbol{q}) = \mu \frac{f^2}{4\pi} (\boldsymbol{\tau}^{(1)} \cdot \boldsymbol{\tau}^{(2)}) \Big\{ V_\sigma(x) (\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)}) + V_T(x) S_{12}$$
$$+ V_q(x) \frac{(\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{q}) (\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{q})}{M^2} + V_{\sigma q}(x) (\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)}) \frac{\boldsymbol{q}^2}{M^2} + V_{Tq}(x) S_{12} \frac{\boldsymbol{q}^2}{M^2}$$
$$+ V_{LL}(x) \Big[(\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)}) \widetilde{\boldsymbol{L}}^2 - (\boldsymbol{\sigma}^{(1)} \cdot \widetilde{\boldsymbol{L}}) (\boldsymbol{\sigma}^{(2)} \cdot \widetilde{\boldsymbol{L}}) \Big] \Big\}, \tag{5·5}$$

where

$$S_{12} = 3 \frac{(\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{r}) (\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{r})}{r^2} - (\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)}),$$

and

$$V_\sigma(x) = \frac{1}{3} \frac{e^{-x}}{x}, \tag{5·6a}$$

$$V_T(x) = \frac{1}{3} \left(1 + \frac{3}{x} + \frac{3}{x^2}\right) \frac{e^{-x}}{x}, \tag{5·6b}$$

$$V_q(x) = \mp \frac{1}{2} \left(1 + \frac{1}{x} + \frac{1}{x^2}\right) \frac{e^{-x}}{x}, \tag{5·6c}$$

$$V_{\sigma q}(x) = V_{Tq}(x) = \mp \frac{1}{6} \left(1 + \frac{3}{x} + \frac{3}{x^2}\right) \frac{e^{-x}}{x}, \tag{5·6d}$$

$$V_{LL}(x) = \pm \frac{1}{2} \left(\frac{\mu}{M}\right)^2 \left(1 + \frac{3}{x} + \frac{3}{x^2}\right) \frac{e^{-x}}{x^3}, \tag{5·6e}$$

the upper (lower) sign corresponds to the case of the $ps(pv)$-coupling. Okubo and Marshak[6] previously calculated the nonstatic corrections up to the order of $g^2 M^{-2}$ by means of M^{-1}-expansion and obtained the corresponding expressions as above. δ-function terms are not written explicitly in Eqs. (5·6a)~(5·6e). In calculating the matrix elements of the potential in the given angular momentum state, we obtain the same results as calculated directly in the momentum representation, if we take into account the contributions from these δ-function terms.

§ 6. Comparison with experiment

To compare the nonstatic effects with experiments, we need some knowledge on TPEP, especially because the differences in OPEP due to the ps- and pv-couplings disappear in the Born approximation (although we may not rely upon the quantitative detail of the TPEP, we may use the knowledge on TPEP qualitatively).

We have given in this paper matrix elements of the OPEP both on and off the energy shell according to the prescription described in § 3, and will give the TPEP in the paper II[22] using the same prescription, which may be thought to

give the qualitative features of the correct TPEP in Region II, although the contributions from the resonances in the pion-nucleon and the pion-pion scatterings are to be added. Therefore, it is very interesting to *compare the differences in the OPEP due to the ps- and pv-couplings with experiments, under some reasonable restrictions on the potential in Region II.* Otsuki, Tamagaki and Watari have investigated this problem. Their results, which will be given in detail in the paper III,[7a] seem to show that the *pv*-coupling is preferable and that it is difficult to interpret the experimental data by the *ps*-coupling only.

Although one might think at first sight the differences appeared in Eqs. (5·6a) ~ (5·6e) due to the *ps*- and *pv*-couplings are too small to be detected by experimental data available at present, this is not necessarily the case. For example, Hamada[8] has recently analyzed the *p-p* scattering data semi-phenomenologically including the quadratic spin-orbit potential.* Although we must take into account the contributions from $\boldsymbol{q}^2$- and $(\boldsymbol{\sigma}^{1)}\cdot\boldsymbol{q})(\boldsymbol{\sigma}^{2)}\cdot\boldsymbol{q})$-terms to discuss the differences due to the *ps*- and *pv*-couplings, it would be also interesting to see the effects of the $\boldsymbol{L}$-dependent potentials in the OPEP.

First, Hamada[5] adjusted the central potential so as to fit the zero energy scattering parameters and the experimental phase shifts for the $^1S_0$ state at 310 Mev. The central potential thus obtained, however, gives $\delta(^1D_2)\approx 0.264$ at 310 Mev, which is too large compared with No. 1 Solution of MacGregor et al.,[31] $\delta(^1D_2)\approx 0.207$, and does not agree with the experimental differential cross section. To remedy this deficiency, one will have to introduce terms like quadratic spin-orbit potential, since the tensor and the ordinary spin-orbit potentials vanish in the ^1E-state. In fact, Hamada has found that the quadratic spin-orbit potential is manifestly required in the ^1E-state ($^1D_2$ and $^1G_4$), the precise form of which was chosen to be

$$V_{LL}^{H}(a)=-0.0054\mu\frac{e^{-x}}{x^3}(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{L})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{L}) \tag{6·1a}$$

or

$$V_{LL}^{H}(b)=-0.00155\mu\frac{e^{-x}}{x}\left(1+8\frac{e^{-x}}{x}+6\frac{e^{-2x}}{x^2}\right)(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{L})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{L}). \tag{6·1b}$$

Using the formula

$$(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{L})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{L})=(\boldsymbol{L}\cdot\boldsymbol{S})^2-\delta(J,L)L(L+1), \tag{6·2}$$

we have

$$V_{LL}^{H}(a)=0.0054\mu L(L+1)\frac{e^{-x}}{x^3} \tag{6·3a}$$

and

$$V_{LL}^{H}(b)=0.00155\mu L(L+1)\frac{e^{-x}}{x}\left(1+8\frac{e^{-x}}{x}+6\frac{e^{-2x}}{x^2}\right) \tag{6·3b}$$

* The authors are indebted to Drs. Otsuki, Tamagaki and Watari for calling their attention to Hamada's paper and helpful discussions.

respectively for the ^1E-state.

These expressions should be compared with the following term, which is included in the nonstatic OPEP, Eq. (5·5),

$$\begin{aligned}\Delta V &= \mu \frac{f^2}{4\pi} (\boldsymbol{\tau}^{(1)} \cdot \boldsymbol{\tau}^{(2)}) V_{LL}(x) \cdot [(\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)}) \boldsymbol{L}^2 - (\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{L})(\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{L})] \\ &= \pm \mu \frac{f^2}{4\pi} (\boldsymbol{\tau}^{(1)} \cdot \boldsymbol{\tau}^{(2)}) \frac{1}{2} \left(\frac{\mu}{M}\right)^2 \frac{e^{-x}}{x^3} \left(1 + \frac{3}{x} + \frac{3}{x^2}\right) \\ &\times [(\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)}) \boldsymbol{L}^2 - (\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{L})(\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{L})], \qquad (6\cdot4)\end{aligned}$$

where the upper (lower) sign corresponds to the $ps(pv)$-coupling. For the ^1E-state, Eq. (6·4) turns into

$$\begin{aligned}\Delta V &= \mp \mu \frac{f^2}{4\pi} \left(\frac{\mu}{M}\right)^2 L(L+1) \left(1 + \frac{3}{x} + \frac{3}{x^2}\right) \frac{e^{-x}}{x^3} \\ &= \mp 0.0018 \mu L(L+1) \left(1 + \frac{3}{x} + \frac{3}{x^2}\right) \cdot \frac{e^{-x}}{x^3}. \qquad (6\cdot5)\end{aligned}$$

The form of the right-hand side of this equation is very similar to that of Eq. (6·3a) and the magnitude is also nearly equal at $x \cong 1$ (the impact parameter of D-wave at 310 Mev is about $1\mu^{-1}$).

Apparently, the pv-coupling is the one that gives the same sign as Hamada's and the ps-coupling gives the different one. If one takes into account ΔV, Eq. (6·5), for the ps-coupling, the angular distribution of the high energy p-p scattering would definitely disagree with experiment.

Hamada[8] has also examined the experimental phase shifts in the ^3O-state and found that the fit to the experimental differential cross section at 310 Mev is noticeably improved by including a quadratic spin-orbit potential. We see that in this case too the pv-coupling gives the same sign and order of magnitude as Hamada's potential while the ps-coupling gives a different sign.

It will be in order here to note that the quadratic spin-orbit potentials due to two-poin-exchange processes[22] are small compared with Eq. (6·4) at $x \approx 1$, and have the same sign as those in OPEP for the ps- and pv-couplings respectively.

§ 7. Conclusions and discussions

i) We have first described the outline of our method and the general properties of the potential in the momentum space.

ii) Then, following the FST method, we have obtained the nonstatic OPEP with full recoil in the momentum space and showed that it can be transformed into a local potential in x-space in a good approximation for two-nucleon problems at energies below nearly 300 Mev.

iii) For momentum transfer small compared with 3μ, or for distances large

compared with $(3\mu)^{-1}$, the nonstatic OPEP is determined unambiguously (although some physical assumptions are used for the case of the *pv*-coupling), and is given by the result of the second order perturbation calculation with the observed coupling constant.

iv) Nonstatic parts of the OPEP, Eqs. (5·6c)$\sim$(5·6e), have different signs for different types of the coupling between the pion and the nucleon. This is the first definite difference obtained from the theory unambiguously for the observable results derived from the *ps*- and *pv*-couplings.

v) Otsuki, Tamagaki and Watari[7a] have compared the nonstatic OPEP with experimental data, assuming the appropriate potentials, which seems to be reasonable from the standpoint of the pion theory, in Region II. Their results seem to show that the case of the *pv*-coupling is preferable, although more elaborate investigations would be necessary to establish the above statement conclusively, which might give a link between the theory of the strong interactions and the theory of the structure of elementary particles.

vi) The differences between the two potentials derived from the *ps*- and *pv*-couplings, respectively, have arisen because of the nonconservation of energy for the matrix elements of the potential operator. They cannot be obtained by purely dispersion theoretic approaches, since they will be included in the subtraction terms.

The authors would like to express their gratitude to Professor M. Taketani for his stimulating discussions and important suggestions throughout this work. They also thank Dr. S. Otsuki, Dr. R. Tamagaki and Dr. W. Watari for their helpful discussions on many aspects of the two-nucleon problems. One of the authors (N. H.) is indebted to the Yomiuri Yukawa Fellowship for the financial aid. Part of this work has been supported by the Yukawa Yomiuri Fund.

Appendix

Transformation of a potential from p-space to x-space

In § 5 we have used several formulas in the course of the transformation of the nonstatic OPEP from p-space to x-space.

i) First we will show that the operator $\boldsymbol{q}$ is $(-i)$ times the gradient on $\psi(\boldsymbol{r})$, the two-nucleon wave function.

Let us start from the Schrödinger equation in the x-space with the potential which is in general nonlocal,

$$\left[E+\frac{1}{M}\nabla^2\right]\psi(\boldsymbol{r})=\int<\boldsymbol{r}|V|\boldsymbol{r}'>d\boldsymbol{r}'\psi(\boldsymbol{r}'), \qquad (A\cdot1)$$

where $\boldsymbol{r}$ and $\boldsymbol{r}'$ are the relative coordinates after and before the potential operates respectively, and the center-of-mass coordinate has already been separated. As will be discussed in detail in reference 7), $<\boldsymbol{r}|V|\boldsymbol{r}'>$ may be written in the following way:

$$<\boldsymbol{r}|V|\boldsymbol{r}'>=\int\frac{d\boldsymbol{q}d\boldsymbol{k}}{(2\pi)^6}e^{i\boldsymbol{q}(\boldsymbol{r}-\boldsymbol{r}')}e^{i\boldsymbol{k}(\boldsymbol{r}+\boldsymbol{r}')/2}V(\boldsymbol{\sigma}^{(1)},\boldsymbol{\sigma}^{(2)},\boldsymbol{k},\boldsymbol{q}). \tag{A·2}$$

Writing Eq. (A·2) in the following form,

$$\begin{aligned}<\boldsymbol{r}|V|\boldsymbol{r}'>&=\int\frac{d\boldsymbol{q}d\boldsymbol{k}}{(2\pi)^6}e^{i\boldsymbol{k}(\boldsymbol{r}+\boldsymbol{r}')/2}V\left(\boldsymbol{\sigma}^{(1)},\boldsymbol{\sigma}^{(2)},\boldsymbol{k},-i\frac{\partial}{\partial(\boldsymbol{r}-\boldsymbol{r}')}\right)\cdot e^{i\boldsymbol{q}(\boldsymbol{r}-\boldsymbol{r}')}\\&=\int\frac{d\boldsymbol{k}}{(2\pi)^3}e^{i\boldsymbol{k}(\boldsymbol{r}+\boldsymbol{r}')/2}V\left(\boldsymbol{\sigma}^{(1)},\boldsymbol{\sigma}^{(2)},\boldsymbol{k},-i\frac{\partial}{\partial(\boldsymbol{r}-\boldsymbol{r}')}\right)\delta(\boldsymbol{r}-\boldsymbol{r}'),\end{aligned} \tag{A·3}$$

and taking the matrix element, we obtain

$$\begin{aligned}&\iint\psi^*(\boldsymbol{r})\,d\boldsymbol{r}<\boldsymbol{r}|V|\boldsymbol{r}'>d\boldsymbol{r}'\psi(\boldsymbol{r}')\\&=\iint\psi^*(\boldsymbol{r})\,d\boldsymbol{r}\left[V\left(\boldsymbol{\sigma}^{(1)},\boldsymbol{\sigma}^{(2)},\frac{\boldsymbol{r}+\boldsymbol{r}'}{2},-i\frac{\partial}{\partial(\boldsymbol{r}-\boldsymbol{r}')}\right)\delta(\boldsymbol{r}-\boldsymbol{r}')\right]d\boldsymbol{r}'\psi(\boldsymbol{r}')\\&=\int\psi^*(\boldsymbol{r})V\left(\boldsymbol{\sigma}^{(1)},\boldsymbol{\sigma}^{(2)},\boldsymbol{r},\frac{-i}{2}(\vec{\boldsymbol{\nabla}}+\overleftarrow{\boldsymbol{\nabla}})\right)\psi(\boldsymbol{r})\,d\boldsymbol{r},\end{aligned} \tag{A·4}$$

where use has been made of

$$\begin{aligned}&\iint\psi^*(\boldsymbol{r})\,d\boldsymbol{r}\left[-i\frac{\partial}{\partial(\boldsymbol{r}-\boldsymbol{r}')}\delta(\boldsymbol{r}-\boldsymbol{r}')\right]d\boldsymbol{r}'\psi(\boldsymbol{r})\\&=\iint\left[-i\frac{\partial}{\partial\boldsymbol{s}}\delta(\boldsymbol{s})\right]d\boldsymbol{s}\,d\boldsymbol{R}\,\psi^*\left(\boldsymbol{R}+\frac{\boldsymbol{s}}{2}\right)\psi\left(\boldsymbol{R}-\frac{\boldsymbol{s}}{2}\right)\\&=\iint\delta(\boldsymbol{s})\,d\boldsymbol{s}\,d\boldsymbol{R}\,i\frac{\partial}{\partial\boldsymbol{s}}(\psi^*\psi)\\&=\iint\delta(\boldsymbol{r}-\boldsymbol{r}')\,d\boldsymbol{r}\,d\boldsymbol{r}'\frac{i}{2}\left[\left(\frac{\partial}{\partial\boldsymbol{r}}\psi^*(\boldsymbol{r})\right)\psi(\boldsymbol{r}')-\psi^*(\boldsymbol{r})\frac{\partial}{\partial\boldsymbol{r}'}\psi(\boldsymbol{r}')\right]\\&=\int\psi^*(\boldsymbol{r})\frac{-i}{2}(\vec{\boldsymbol{\nabla}}+\overleftarrow{\boldsymbol{\nabla}})\psi(\boldsymbol{r})\,d\boldsymbol{r}.\end{aligned} \tag{A·5}$$

Comparison of Eq. (A·2) with Eq. (A·4) shows that one may transform $V(\boldsymbol{\sigma}^{(1)},\boldsymbol{\sigma}^{(2)},\boldsymbol{k},\boldsymbol{q})$ by the formula Eq. (5·3),

$$V(\boldsymbol{x},\boldsymbol{q})=\int\frac{d\boldsymbol{k}}{(2\pi)^3}e^{i\boldsymbol{k}\boldsymbol{r}}V(\boldsymbol{\sigma}^{(1)},\boldsymbol{\sigma}^{(2)},\boldsymbol{k},\boldsymbol{q}), \tag{A·6}$$

where $\boldsymbol{q}$ in $V(\boldsymbol{x},\boldsymbol{q})$ is to be understood as $(-i)$ times the gradient operator on the two-nucleon wave function

$$\psi^*(\boldsymbol{r})\boldsymbol{q}\psi(\boldsymbol{r})=\frac{(-i)}{2}\psi^*(\boldsymbol{r})(\vec{\boldsymbol{\nabla}}+\overleftarrow{\boldsymbol{\nabla}})\psi(\boldsymbol{r}). \tag{A·7}$$

ii) Derivation of Eq. (5·4b)

$$B \equiv \int \frac{d\boldsymbol{k}}{(2\pi)^3} e^{i\boldsymbol{k}\boldsymbol{r}} (\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{k} \times \boldsymbol{q}) (\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{k} \times \boldsymbol{q}) V_4(k)$$
$$= -(\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\nabla} \times \boldsymbol{q}) (\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{\nabla} \times \boldsymbol{q}) V_4(r),$$

where $\boldsymbol{q}$ is a differential operator on the wave function. Making use of Eq. (5·4a) we rewrite the above equation as

$$B = -\left[\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^2 V_4(r)\right](\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{r} \times \boldsymbol{q}) (\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{r} \times \boldsymbol{q})$$
$$+\left[\frac{1}{r}\frac{\partial}{\partial r} V_4(r) - \frac{1}{3}\delta(\boldsymbol{r})\right]\{(\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{q}) (\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{q}) - (\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)}) \boldsymbol{q}^2\}. \quad \text{(A·8)}$$

Transforming Eq. (A·8) we obtain

$$B = -\left[\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^2 V_4(r)\right]\frac{1}{2}\left\{(\boldsymbol{\sigma}^{(1)} \cdot \widetilde{\boldsymbol{L}}) (\boldsymbol{\sigma}^{(2)} \cdot \widetilde{\boldsymbol{L}}) + (\boldsymbol{\sigma}^{(2)} \cdot \widetilde{\boldsymbol{L}}) (\boldsymbol{\sigma}^{(1)} \cdot \widetilde{\boldsymbol{L}})\right\}$$
$$+\left[\frac{1}{r}\frac{\partial}{\partial r} V_4(r) - \frac{1}{3}\delta(\boldsymbol{r})\right]\{(\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{q}) (\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{q}) - (\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)}) \boldsymbol{q}^2\}, \quad \text{(A·9)}$$

where

$$(\boldsymbol{\sigma}^{(1)} \cdot \widetilde{\boldsymbol{L}}) (\boldsymbol{\sigma}^{(2)} \cdot \widetilde{\boldsymbol{L}}) = (\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{L}) (\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{L}) - \frac{1}{4}(\boldsymbol{x} \cdot \vec{\boldsymbol{\nabla}} - \overleftarrow{\boldsymbol{\nabla}} \cdot \boldsymbol{x})$$
$$+\frac{1}{4}\{(\boldsymbol{\sigma}^{(1)} \cdot \vec{\boldsymbol{\nabla}}) (\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{x}) - (\boldsymbol{\sigma}^{(1)} \cdot \overleftarrow{\boldsymbol{\nabla}}) (\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{x})\}, \quad \text{(A·10)}$$

and $\boldsymbol{L}$ satisfies usual commutation relations

$$[L_i, L_j] = iL_k \quad (i, j, k \text{ is a cyclic permutation of } x, y, z)$$

for angular momentum operators.

iii) Derivation of Eq. (5·4c)

$$\int \frac{d\boldsymbol{k}}{(2\pi)^3} e^{i\boldsymbol{k}\boldsymbol{r}} (\boldsymbol{k} \times \boldsymbol{q})^2 V_4(k) = -(\boldsymbol{\nabla} \times \boldsymbol{q})^2 V_4(r) = -\sum_{(i,j,k)}{}' (\nabla_j q_k - \nabla_k q_j)^2 V_4(r)$$
$$= -\left[\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^2 V_4(r)\right]\sum_{(i,j,k)} (x_j q_k - x_k q_j)^2 - \left[\frac{1}{r}\frac{\partial}{\partial r} V_4(r) - \frac{1}{3}\delta(\boldsymbol{r})\right]\sum_{(i,j,k)} (q_j^2 + q_k^2)$$
$$= -\left[\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^2 V_4(r)\right]\widetilde{\boldsymbol{L}}^2 - 2\left[\frac{1}{r}\frac{\partial}{\partial r} V_4(r) - \frac{1}{3}\delta(\boldsymbol{r})\right]\boldsymbol{q}^2, \quad \text{(A·11)}$$

where

$$\widetilde{\boldsymbol{L}}^2 = \boldsymbol{L}^2 - \frac{1}{2}(\boldsymbol{x} \cdot \vec{\boldsymbol{\nabla}} - \overleftarrow{\boldsymbol{\nabla}} \cdot \boldsymbol{x}).$$

References

1) M. Taketani and S. Machida, Prog. Theor. Phys. **24** (1960), 1317, the preceding paper.
2) A review may be found in S. Machida and T. Toyoda, Suppl. Prog. Theor. Phys. No. 3 (1956), Part III.
3) M. Sugawara and S. Okubo, Phys. Rev. **117** (1960), 605, 611.
4) S. Okubo and S. Sato, Prog. Theor. Phys. **21** (1959), 383.
5) N. Tzoar, R. Raphael and A. Klein, Phys. Rev. Letters **2** (1959), 433; **3** (1959), 145.
6) S. Okubo and R. E. Marshak, Ann. Phys. **4** (1958), 166.
7) J. Goto and S. Machida, Prog. Theor. Phys., to be published.
7a) S. Otsuki, R. Tamagaki and W. Watari, Prog. Theor. Phys., to be published.
8) T. Hamada, Prog. Theor. Phys. **24** (1960), 220; **24** (1960), 1033.
9) S. Tani, Prog. Theor. Phys. **6** (1951), 267.
L. L. Foldy and S. A. Wouthuysen, Phys. Rev. **78** (1950), 29.
10) M. L. Goldberger, Phys. Rev. **84** (1951), 929.
11) See, e. g. H. P. Stapp, T. J. Ypsilantis and N. Metropolis, Phys. Rev. **105** (1957), 302.
L. Puzikov, R. Ryndin and J. Smorodinsky, Nuclear Phys. **3** (1957), 436.
12) N. Fukuda, K. Sawada and M. Taketani, Prog. Theor. Phys. **12** (1954), 156.
See also reference 2).
12a) Y. Nambu, Prog. Theor. Phys. **3** (1948), 444.
L. Van Hove, Phys. Rev. **75** (1949), 1519.
H. Enatsu, Prog. Theor. Phys. **5** (1950), 102.
13) N. Hoshizaki, I. Lin and S. Machida, to be published.
14) F. J. Dyson, Phys. Rev. **73** (1948), 929.
K. M. Case, Phys. Rev. **76** (1949), 1.
J. M. Berger, L. L. Foldy and R. K. Osborn, Phys. Rev. **87** (1952), 1065.
15) S. N. Gupta, Phys. Rev. **117** (1960), 1146.
16) K. Hiida, J. Iwadare and S. Machida, Prog. Theor. Phys. **15** (1956), 189.
17) H. Lehmann, Nuovo Cimento **11** (1954), 342.
18) S. Sakata, Prog. Theor. Phys. **16** (1956), 686.
19) S. Machida, Prog. Theor. Phys. **18** (1957), 467.
20) P. Signell, Prog. Theor. Phys. **22** (1959), 492.
21) P. Cziffra, M. H. MacGregor, M. J. Moravisik and H. P. Stapp, Phys. Rev. **114** (1959), 880.
M. H. MacGregor, M. J. Moravcsik and H. P. Stapp, Phys. Rev. **116** (1959), 1284.
22) N. Hoshizaki and S. Machida, to be published.

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Time Dependent Problems of the Localized Lattice Vibration

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The time dependent problems of the vibrational motion are investigated for the cases of infinitely extended one-dimensional lattice which contains one or two impurity atoms (isotopes). Starting from the equations of motion of these systems, we derive the integral equations which show various time dependent properties of the lattice vibration of these perturbed one-dimensional lattices. Namely, the asymptotic solutions of these integral equations represent the localized vibration which is preserved by the impurity atom when its mass is smaller than that of base atoms. The integral equations are actually solved by means of the perturbation calculation and also by the use of Laplace transforms, and the behaviors of the lattice vibration especially the capture of the vibrational energy by the impurity atoms, are examined.

§ 1. Introduction

Recently considerable progress has been made in the theory of lattice vibrations of crystalline solids containing defects such as the impurities and holes. The detailed mathematical investigations of monatomic lattices have been made by Montroll and Potts,[1] and by Lifshitz,[2] using the method of Green's function. The corresponding investigation of the electronic levels in solids has been made by Koster and Slater[3] by the use of the Wannier function. Later, Montroll and his collaborators extended their method to study the vibration of diatomic lattices.[4] Hori and Asahi[5] have presented a method of solving the one-dimensional problem which is an application of the method of transfer matrices originally utilized by Kerner[6] for the problem of the electronic energy band.

Results obtained by various authors show that a local defect causes the appearance of the vibrational modes whose frequencies are discrete and isolated from the main continuum of the modes of unperturbed lattice as well as the shifts of in-band frequencies by the amount of order $O(1/N)$, N being the total number of lattice points. The vibrations of these isolated modes are localized around the lattice defects, which are analogous to the bound states of electrons around the impurity centers.

So far only the stationary properties of the imperfect lattice vibration have been considered, and various methods of approach developed are devoted to solve

the time independent equations of motion. It seems therefore to be interesting to study the problem from a slightly different point of view by explicitly solving the time dependent equations of motion with a suitable initial condition. The present paper represents an attempt to study this time dependent problem with particular attention to the states corresponding to the localized modes.

We deal with such a simple model that a one-dimensional monatomic lattice with nearest neighbor interaction contains one or two isotopic impurity atoms. Our basic procedure is to excite the vibrational motion by making the displacement of an atom at an initial time $t=0$ while all other atoms remain at their equilibrium positions, and then to study the behavior of the motion of each atom as a function of time. The corresponding investigation of the propagation of electronic excitation in pure insulators was made by Merrifield.[7] A general description of the temporal evolution of such a time dependent problem in quantum mechanical motion in the random lattice was made by Anderson[8] in order to give a proper basis for a quantum mechanical theory of transport problem. In the case of lattice vibration, the vibrational energy is transferred throughout the crystal, and for a regular lattice the equipartition of the vibrational energy over all lattice sites is achieved in the limit of $t\to\infty$ (it is infinitesimal for a lattice of infinite dimension). For the imperfect lattice, however, the free propagation wave is scattered by the lattice defects and under certain circumstances it is expected that even for $t\to\infty$ the amplitudes of the vibration around the impurities can remain finite. The solution representing these states seems to elucidate the aspect of the localized vibrational modes from a different point of view.

In § 2 we present preliminary accounts of the equations of motion for the unperturbed lattice by introducing the propagator. It will be also shown that the trace of the propagator is equivalent to the Fourier transform of the frequency spectrum. In § 3 the time dependent equations of motion for imperfect lattices are transformed to the integral equations, which provide us with the general description of the non-stationary properties of the lattice vibration. In the following two sections the approximate procedures of solving the integral equations are given. In § 4 the asymptotic solutions surviving for $t\to\infty$ are obtained, which are actually connected with the localized vibrational modes. In § 5 the first order perturbation calculations are carried out and the solutions explicitly yield the time evolution of the vibrational motion propagating throughout the host crystal and being disturbed by the impurity atoms. § 6 is devoted to the discussion of the Laplace transforms of the integral equations, which are adequate for getting formally exact solutions of our integral equations. Lastly in § 7 we give a brief discussion about a pair of isotopic impurity atoms.

§ 2. Unperturbed lattice

In this section we shall study the system of one-dimensional harmonically

coupled oscillators which consists of N identical atoms of mass m. The equations of motion can be written in the form

$$\frac{d^2u_n}{dt^2}=\mathfrak{D}(n)\cdot u_n, \tag{1}$$

where u_n is the deviation of the coordinate of nth particle from its equilibrium position and $\mathfrak{D}(n)$ denotes the difference operator defined by

$$\begin{aligned}\mathfrak{D}(n)\cdot u_n&=-\frac{K}{m}(u_n-u_{n+1})-\frac{K}{m}(u_n-u_{n-1})\\&=\alpha^2(u_{n+1}+u_{n-1}-2u_n).\end{aligned} \tag{2}$$

The parameter α is $\sqrt{K/m}$, where K denotes the force constant of the interaction acting between the neighbouring particles. The solutions $u_n(t)$ of Eqs. (1), which satisfy the initial conditions that at $t=0$ the displacements of the atoms from their equilibrium positions have given values $\{u_n(0)\}$ and their initial velocities are all zero, can be expressed in the form

$$u_n(t)=\sum_m \rho(n, m\,;t)u_m(0), \tag{3}$$

where the propagator matrix $\rho(n, m\,;t)$ satisfies the differential equations

$$\frac{d^2}{dt^2}\rho(n, m\,;t)=\mathfrak{D}(n)\cdot\rho(n, m\,;t) \tag{4}$$

and also the initial conditions

$$\begin{aligned}&\rho(n, m\,;\ 0)=\delta_{n,m},\\&\left[\frac{d}{dt}\rho(n, m\,;\ t)\right]_{t=0}=0.\end{aligned} \tag{5}$$

As is well known, the solutions of Eqs. (4) for infinitely extended lattice are given by the Bessel functions.[9] This can be easily seen by introducing the generating function defined by

$$K(z, t)=\sum_{n=-\infty}^{\infty}z^{n-m}\rho(n, m\,;\ t) \tag{6}$$

which obviously satisfies the initial conditions

$$\begin{aligned}&K(z,\ 0)=1,\\&[dK(z,\ t)/dt]_{t=0}=0.\end{aligned} \tag{7}$$

By multiplying the both sides of Eq. (4) by z^{n-m} and adding over all n, we have the differential equation

$$\frac{d^2}{dt^2}K(z,\ t)=\{\alpha(\sqrt{z}-\frac{1}{\sqrt{z}})\}^2\ K(z, t). \tag{8}$$

The solution of Eq. (8) is given by

$$K(z,\ t)=A\ \exp\left\{\alpha\left(\sqrt{z}-\frac{1}{\sqrt{z}}\right)t\right\}+B\ \exp\left\{-\alpha\left(\sqrt{z}-\frac{1}{\sqrt{z}}\right)t\right\} \tag{9}$$

where constants A and B are determined by the initial conditions (7) and we find that $A=B=1/2$, hence we have

$$K(z,\ t)=\cosh\left\{\alpha\left(\sqrt{z}-\frac{1}{\sqrt{z}}\right)t\right\}. \tag{10}$$

Now comparing Eq. (10) with the generating function of Bessel function

$$\exp\left\{\frac{1}{2}x\left(\xi-\frac{1}{\xi}\right)\right\}=\sum_{n=-\infty}^{\infty}\xi^n\ J_n(x),$$

we obtain the coefficient of z^{n-m} in Eq. (10) as

$$\rho(n,\ m\ ;\ t)=J_{2(n-m)}(2\alpha t). \tag{11}$$

Here the propagator $\rho(n,\ m\ ;t)$ depends on the positions n and m only through the distance $|n-m|$; because of the relation $J_{-k}(x)=(-1)^kJ_k(x)$.

It obviously follows also that the solution which satisfies the cyclic boundary condition, that is, $\rho(n+N,\ m\ ;\ t)=\rho(n,\ m\ ;\ t)$, can be given by

$$\sum_{k=-\infty}^{\infty}J_{2(n-m+kN)}(2\alpha t). \tag{12}$$

Lastly we shall refer to the relation which connect the function $\rho(n,\ m\ ;t)$ with the time independent problems of the lattice vibration. Let us consider the eigenvalue problem

$$\mathfrak{D}(n)\ v_n=-\lambda v_n, \tag{13}$$

with the cyclic boundary condition. We denote the eigenvalues and the eigenvectors of Eqs. (13) by λ_ν and $v_n^{(\nu)}$ respectively. Clearly (13) is the equations which determine the normal modes of the lattice vibration and λ_ν is related to the angular frequency by $\lambda_\nu=\omega_\nu^2$. Using these eigenvectors, the propagator can be expressed as

$$\rho(n,\ m\ ;\ t)=\sum_\nu A_m^{(\nu)}(t)\,v_n^{(\nu)}, \tag{14}$$

where the functions $A_m^{(\nu)}(t)$ satisfy the differential equations

$$\frac{d^2}{dt^2}A_m^{(\nu)}(t)=-\lambda_\nu\ A_m^{(\nu)}(t)=-\omega_\nu^2\ A_m^{(\nu)}(t). \tag{15}$$

The solutions of these equations are

$$A_m^{(\nu)}(t)=a_m^{(\nu)}\ e^{i\omega_\nu t}+b_m^{(\nu)}\ e^{-i\omega_\nu t}$$

and the constants $a_m^{(\nu)}$ and $b_m^{(\nu)}$ are determined by the initial conditions (5). Then it follows that $a_m^{(\nu)}=b_m^{(\nu)}=v_m^{(\nu)*}/2$ and we have

$$\rho(n, m; t) = \sum_{\nu} v_m^{(\nu)*} \cos(\omega_\nu t) v_n^{(\nu)}. \tag{16}$$

This means that the propagator can be also directly calculated by using the solutions of eigenvalue problem (13). Actually the solutions of Eq. (13) are

$$\begin{aligned} v_n^{(\nu)} &= N^{-1/2} \exp(i2\pi\nu n/N) \\ \omega_\nu^2 &= 4\alpha^2 \sin^2(\nu\pi/N). \end{aligned} \qquad (\nu = 1, 2, \cdots, N). \tag{17}$$

From Eqs. (16) and (17), we get

$$\begin{aligned} \rho(n, m; t) &= \sum_{\nu=1}^{N} N^{-1} e^{i2\pi\nu(n-m)/N} \cos\{2\alpha t \sin(\nu\pi/N)\} \\ &= \frac{1}{\pi} \int_0^{\pi} e^{i2\theta(n-m)} \cos\{2\alpha t \sin\theta\} \, d\theta \\ &= \frac{1}{\pi} \int_0^{\pi} \cos\{2(n-m)\theta\} \cos\{2\alpha t \sin\theta\} \, d\theta, \end{aligned} \tag{18}$$

where the sum has been replaced by an integral, assuming N to be sufficiently large. The last integral is just Jacobi's integral representation of the Bessel function $J_{2(n-m)}(2\alpha t)$.

Further, if we take the trace of Eq. (16), we obtain

$$\begin{aligned} \sum_n \rho(n, n; t) &= \sum_n \sum_\nu v_n^{(\nu)*} \cos(\omega_\nu t) v_n^{(\nu)} \\ &= \sum_\nu \cos(\omega_\nu t) = \int_0^{\infty} \cos(\omega t) f(\omega) \, d\omega, \end{aligned} \tag{19}$$

where again the sum has been replaced by an integral by taking a sufficiently large value of N and introducing the frequency distribution function $f(\omega)$. Thus we know that the trace of propagator $\rho(n, m; t)$ is the Fourier cosine transform of the frequency distribution function $f(\omega)$. When $N \gg 1$, $\rho(n, m; t)$ can be approximated by the central term ($k=0$) in Eq. (12) only, so that from Eqs. (11) and (19) we have

$$N J_0(2\alpha t) = \int_0^{\infty} \cos(\omega t) f(\omega) \, d\omega. \tag{20}$$

The inverse transformation of Eq. (20) gives the frequency spectrum[10]

$$\begin{aligned} f(\omega) &= \frac{2N}{\pi} \int_0^{\infty} J_0(2\alpha t) \cos(\omega t) \, dt \\ &= \begin{cases} \dfrac{2N}{\pi} (4\alpha^2 - \omega^2)^{-1/2}, & 0 < \omega < 2\alpha \\ 0 \quad ; & \omega > 2\alpha. \end{cases} \end{aligned} \tag{21}$$

§ 3. Integral equation of the time dependent problem

Here we shall consider the one-dimensional lattice in which several atoms are replaced by the isotopic impurity atoms of mass M, then the equations of motion can be written in the form

$$\frac{d^2u_n}{dt^2}=\mathfrak{D}(n)\cdot u_n+\overset{*}{\mathcal{V}}(n)\cdot u_n, \tag{22}$$

where $\overset{*}{\mathcal{V}}(n)$ represents the difference operator

$$\begin{aligned}\overset{*}{\mathcal{V}}(n)\cdot u_n&=\varepsilon\alpha^2\sum_l\delta_{n,l}(u_{n+1}+u_{n-1}-2u_n)\\&=\varepsilon\sum_l\delta_{n,l}\,\mathfrak{D}(l)\cdot u_l\end{aligned} \tag{23}$$

in which l's denote the sites of impurity atoms and $\varepsilon=(m/M)-1$. For the propagator we have also the differential equations

$$\frac{d^2}{dt^2}\rho(n,m\,;\,t)=\mathfrak{D}(n)\cdot\rho(n,m\,;\,t)+\overset{*}{\mathcal{V}}(n)\cdot\rho(n,m\,;\,t) \tag{24}$$

and the initial conditions are again given by Eqs. (5). Now we assume the solution of Eqs. (24) to be of the form

$$\rho(n,m\,;\,t)=\rho_0(n,m\,;\,t)+F(n,m\,;\,t), \tag{25}$$

where $\rho_0(n,m:t)$ is the solution for unperturbed lattice, that is,

$$\frac{d^2}{dt^2}\rho_0(n,m\,;\,t)=\mathfrak{D}(n)\cdot\rho_0(n,m\,;\,t). \tag{26}$$

Then it follows that the function $F(n,m\,;t)$ should satisfy the equations

$$\frac{d^2}{dt^2}F(n,m\,;\,t)-\mathfrak{D}(n)\cdot F(n,m:\,t)=\overset{*}{\mathcal{V}}(n)\cdot\rho(n,m\,;\,t), \tag{27}$$

with the initial conditions

$$\begin{aligned}&F(n,m\,;\,0)=0,\\&\left[\frac{d}{dt}F(n,m\,;\,t)\right]_{t=0}=0.\end{aligned} \tag{28}$$

Using the eigenvectors of Eq. (13), if we put

$$F(n,m\,;\,t)=\sum_\nu a_m^{(\nu)}(t)\,e^{i\omega_\nu t}v_n^{(\nu)}, \tag{29}$$

we have

$$\frac{d^2}{dt^2}a_m^{(\nu)}(t)+2i\omega_\nu\frac{d}{dt}a_m^{(\nu)}(t)=e^{-i\omega_\nu t}\sum_k v_k^{(\nu)*}\overset{*}{\mathcal{V}}(k)\cdot\rho(k,m\,;\,t). \tag{30}$$

Further we write

$$\frac{d}{dt}a_m^{(\nu)}(t)=b_m^{(\nu)}(t)\,e^{-2i\omega_\nu t}, \tag{31}$$

then Eqs. (30) are reduced to

$$\frac{d}{dt}b_m^{(\nu)}(t)=e^{i\omega_\nu t}\sum_k v_k^{(\nu)*}\,{}^{t}\vartheta\,(k)\cdot\rho(k,\,m\,;\,t),$$

so we obtain

$$b_m^{(\nu)}(t)=\sum_k\int_0^t e^{i\omega_\nu t'}\,v_k^{(\nu)*}\,{}^{t}\vartheta\,(k)\rho(k,\,m\,;\,t')\,dt', \tag{32}$$

where the additive constant of integration is zero because of the initial condition (28). From Eqs. (31) and (32), we have

$$a_m^{(\nu)}(t)=\sum_k\int_0^t e^{-2i\omega_\nu t'}\,dt'\int_0^{t'}e^{i\omega_\nu t''}\,v_k^{(\nu)*}\,{}^{t}\vartheta\,(k)\cdot\rho(k,\,m\,;t'')\,dt''. \tag{33}$$

By means of a partial integration, Eq. (33) is seen to be equal to

$$a_m^{(\nu)}(t)=-\frac{1}{2i\omega_\nu}\Big\{e^{-2i\omega_\nu t}\int_0^t e^{i\omega_\nu t'}\,v_k^{(\nu)*}\,{}^{t}\vartheta\,(k)\cdot\rho(k,\,m\,;\,t')\,dt'$$

$$-\int_0^t e^{-i\omega_\nu t'}\,v_k^{(\nu)*}\,{}^{t}\vartheta\,(k)\cdot\rho(k,\,m\,;\,t')\,dt'\Big\}. \tag{34}$$

Therefore we obtain

$$F(n,\,m\,;\,t)=\sum_\kappa\int_0^t\sum_\nu v_k^{(\nu)*}\{(e^{i\omega_\nu(t-t')}-e^{-i\omega_\nu(t-t')})/2i\omega_\nu\}\,v_n^{(\nu)}$$

$$\times\,{}^{t}\vartheta\,(k)\cdot\rho(k,\,m\,;t')\,dt'$$

$$=\sum_k\int_0^t\sum_\nu v_k^{(\nu)*}\frac{\sin\{\omega_\nu(t-t')\}}{\omega_\nu}v_n^{(\nu)}\,{}^{t}\vartheta\,(k)\cdot\rho(k,\,m\,;t')\,dt', \tag{35}$$

and according to Eq. (16)

$$F(n,\,m\,;\,t)=\sum_k\int_0^t dt'\int_0^{t-t'}d\tau\rho_0(n,\,k\,;\,\tau)\,{}^{t}\vartheta\,(k)\cdot\rho(k,\,m\,;\,t') \tag{36}$$

Thus we finally obtain the integral equations

$$\rho(n,\,m\,;\,t)=\rho_0(n,\,m\,;\,t)+\sum_k\int_0^t dt'\int_0^{t-t'}d\tau\rho_0(n,\,k\,;\,\tau)\,{}^{t}\vartheta\,(k)\cdot\rho(k,\,m\,;\,t'). \tag{37}$$

§ 4. Asymptotic solution of the integral equation

In the perfect lattice the vibrational motion which is initially excited at some

point in the lattice spreads over all lattice points and especially in the case of infinitely extended perfect linear lattice the vibrational amplitude of every atom diminishes to zero as $t\to\infty$. In fact, the propagator for the infinitely extended unperturbed lattice has a property that as $t\to\infty$

$$\rho_0(n, m; t) = J_{2(n-m)}(2\alpha t)\to 0 \tag{38}$$

for all n. However, when the lattice contains some impurity atoms, we encounter different situations. Namely in some cases the impurity atom can capture the vibrational energy and maintain a non-vanishing stationary localized vibration.

In order to study this localized stationary vibration, we shall treat the infinitely extended linear lattice which contains one isotopic impurity atom located at the lattice point l and we shall be concerned only with the asymptotic solutions of our integral equations. It is readily seen that the integral equations (37) can be rewritten in the form

$$\begin{aligned}\rho(n, m; t) &= \rho_0(n, m; t) + \sum_k \int_0^t dt' \int_0^{t'} d\tau \rho_0(n, k; t-t')\, \mathcal{V}(k) \cdot \rho(k, m; \tau) \\ &= \rho_0(n, m; t) + \varepsilon\alpha^2 \int_0^t dt' \rho_0(n, l; t-t') \int_0^{t'} d\tau \{\rho(l+1, m; \tau) \\ &\quad + \rho(l-1, m; \tau) - 2\rho(l, m; \tau)\}. \end{aligned} \tag{39}$$

Here we assume that in the limit of $t\to\infty$ there exists a stationary vibration which is asymptotically attained irrespective of the initial state. We denote it by

$$\rho(n, m; t) = a_n \exp(i\omega t). \tag{40}$$

Inserting (40) into (39) and using the fact expressed by Eq. (38), we have

$$\begin{aligned} a_n e^{i\omega t} &= \varepsilon\alpha^2 \int_0^t dt'\, J_{2(n-l)}(t-t') \int_0^{t'} d\tau (a_{l+1} + a_{l-1} - 2a_l) e^{i\omega\tau} \\ &= \frac{\varepsilon\alpha^2}{i\omega}(a_{l+1} + a_{l-1} - 2a_l) \int_0^t dt' J_{2(n-l)}(2\alpha(t-t'))\,(e^{i\omega t'} - 1). \end{aligned} \tag{41}$$

The result of integration, for sufficiently large value of t, is given by the equations[10]

$$\begin{aligned} \int_0^t J_{2(n-l)}(2\alpha(t-t'))\, e^{i\omega t'}\, dt' &= e^{i\omega t} \int_0^t J_{2(n-l)}(2\alpha\tau)\, e^{-i\omega\tau}\, d\tau \\ &\to e^{i\omega t} \int_0^\infty J_{2(n-l)}(2\alpha\tau)\, e^{-i\omega\tau}\, d\tau \\ &= -i(-1)^{n-l}(2\alpha)^{2|n-l|} \{\sqrt{\omega^2 - 4\alpha^2}(\omega + \sqrt{\omega^2 - 4\alpha^2})^{2|n-l|}\}^{-1} e^{i\omega t} \end{aligned} \tag{42}$$

and

$$\int_0 J_{2(n-l)}(2\alpha(t-t'))\,dt'=\frac{1}{\alpha}\sum_{m=0}^{\infty}J_{2(n-l+m)+1}(2\alpha t)\to 0. \tag{43}$$

In Eq. (42) we assumed that $\omega>2\alpha$. Actually it has been made clear by many authors that the characteristic frequency of the localized vibration always appears above the top of the frequency continuum of unperturbed lattice. Therefore from Eqs. (41), (42) and (43), we obtain the equations

$$a_n=\varepsilon(-1)^{n-l+1}\frac{\alpha^2}{\omega\sqrt{\omega^2-4\alpha^2}}\left(\frac{2\alpha}{\omega+\sqrt{\omega^2-4\alpha^2}}\right)^{2|n-l|}(a_{l+1}+a_{l-1}-2a_l) \tag{44}$$

which determine the characteristic frequency ω and the amplitudes $\{a_n\}$ of the localized vibration. It is obvious that the solution of Eq. (44) has such a symmetry that $a_{n-l}=a_{-(n-l)}$. Consequently, from Eq. (44) for $n=l$ and $l+1$,

$$\begin{aligned}a_l&=-2\varepsilon\frac{\alpha^2}{\omega\sqrt{\omega^2-4\alpha^2}}(a_{l+1}-a_l),\\ a_{l+1}&=2\varepsilon\frac{\alpha^2}{\omega\sqrt{\omega^2-4\alpha^2}}\left(\frac{2\alpha}{\omega+\sqrt{\omega^2-4\alpha^2}}\right)^2(a_{l+1}-a_l).\end{aligned} \tag{45}$$

Thus we can have non-vanishing amplitudes, provided that the condition

$$2\varepsilon\frac{\alpha^2}{\omega\sqrt{\omega^2-4\alpha^2}}\left\{\left(\frac{2\alpha}{\omega+\sqrt{\omega^2-4\alpha^2}}\right)^2+1\right\}=1 \tag{46}$$

is satisfied. This can be simplified and we get

$$\omega-\sqrt{\omega^2-4\alpha^2}=\sqrt{\omega^2-4\alpha^2}/\varepsilon. \tag{47}$$

The main fact revealed by Eq. (47) is that the real value of ω can be obtained only when $\varepsilon>0$ or $M<m$ In this case we find

$$\omega=\frac{1+\varepsilon}{\sqrt{1+2\varepsilon}}2\alpha. \tag{48}$$

As for the amplitudes, from Eqs. (44) and (45)

$$a_n=(-1)^{n-l}\left(\frac{2\alpha}{\omega+\sqrt{\omega^2-4\alpha^2}}\right)^{2|n-l|}a_l \tag{49}$$

and using Eq. (48), we have

$$a_n=(-1)^{n-l}(1+2\varepsilon)^{-|n-l|}a_l \tag{50}$$

which shows the localization of the amplitudes which has a peak at the lighter impurity atom.

We have seen that there exists a stationary localized vibration when the mass of isotopic impurity atom is smaller than that of basic atoms. Therefore the vibrational energy which is introduced by exciting some part of the linear lattice, can be

captured at the impurity atom in a form of the localized vibration. This time dependent behavior will be actually demonstrated by the calculations given in the next section.

§ 5. Perturbation calculation

In order to see the time dependent behavior of the propagation of the lattice vibration, we shall evaluate the solutions of the integral equations (39) by a perturbation procedure. Namely, assuming that the parameter ε is sufficiently small, we shall be concerned only with the first order quantities in ε.

In the first stage of successive approximation of Eqs. (39), we have

$$\rho(n, m; t) = \rho_0(n, m; t) + \varepsilon \int_0^t dt' \int_0^{t'} d\tau \rho_0(n, l; t-t') \mathfrak{D}(l) \cdot \rho_0(l, m; \tau). \tag{51}$$

Using Eqs. (4) and (5), this can be rewritten in the form

$$\begin{aligned}\rho(n, m; t) &= \rho_0(n, m; t) + \varepsilon \int_0^t dt' \int_0^{t'} d\tau \rho_0(n, l; t-t') \frac{d^2}{d\tau^2} \rho_0(l, m; \tau) \\ &= \rho_0(n, m; t) + \varepsilon \int_0^t dt' \rho_0(n, l; t-t') \frac{d}{dt'} \rho_0(l, m; t'). \end{aligned} \tag{52}$$

Now we can use the recurrence formula of the Bessel function

$$\frac{d}{dt'} \rho_0(l, m; t') = \frac{d}{dt'} J_{2(l-m)}(2\alpha t') = \alpha \{ J_{2(l-m)-1}(2\alpha t') - J_{2(l-m)+1}(2\alpha t') \}.$$

Thus the second term of Eqs. (52) reduces to

$$\begin{aligned}&\varepsilon \int_0^t dt' \rho_0(n, l; t-t') \frac{d}{dt'} \rho_0(l, m; t') \\ &= \varepsilon\alpha \int_0^t dt' J_{2(n-l)}(2\alpha(t-t')) \{ J_{2(l-m)-1}(2\alpha t') - J_{2(l-m)+1}(2\alpha t') \}. \end{aligned} \tag{53}$$

In order to calculate these integrals, we shall take the origin of lattice coordinates so as to be $m=0$ and assume that $l>0$. Then using the relations[11]

$$J_{-\mu}(x) = (-1)^{\mu} J_{\mu}(x), \tag{54}$$

$$\int_0^x J_{\mu}(y) J_{\nu}(x-y) dy = 2 \sum_{k=0}^{\infty} (-1)^k J_{\mu+\nu+2k+1}(x), \quad (\mu, \nu > -1) \tag{55}$$

and

$$\sum_{i=0}^{\infty} (-1)^i J_{2i}(x) = (\cos x + J_0(x))/2, \tag{56}$$

we have

$$\int_0^t J_{2(n-l)}(2\alpha(t-t'))\,J_{2l-1}(2\alpha t')\,dt'$$

$$=\frac{1}{\alpha}\sum_{k=0}^{\infty}(-1)^k\,J_{2(|n-l|+l)+2k}(2\alpha t)$$

$$=\frac{1}{\alpha}(-1)^{|n-l|+l}\left\{\sum_{i=0}^{\infty}(-1)^i J_{2i}(2\alpha t)-\sum_{i=0}^{|n-l|+l-1}(-1)^i J_{2i}(2\alpha t)\right\}$$

$$=\frac{1}{\alpha}(-1)^{|n-l|+l}\left\{\frac{\cos(2\alpha t)+J_0(2\alpha t)}{2}-\sum_{i=0}^{|n-l|+l-1}(-1)^i J_{2i}(2\alpha t)\right\}, \quad (57)$$

and similarly we have

$$\int_0^t J_{2(n-l)}(2\alpha(t-t'))\,J_{2l+1}(2\alpha t')\,dt'$$

$$=\frac{1}{\alpha}(-1)^{|n-l|+l+1}\left\{\frac{\cos(2\alpha t)+J_0(2\alpha t)}{2}-\sum_{i=0}^{|n-l|+l}(-1)^i J_{2i}(2\alpha t)\right\}. \quad (58)$$

Thus from Eqs. (53), (57) and (58), we obtain

$$\varepsilon\int_0^t dt'\rho_0(n,l;t-t')\frac{d}{dt'}\rho_0(l,0;t')$$

$$=\varepsilon(-1)^{|n-l|+l}\{\cos(2\alpha t)+J_0(2\alpha t)+(-1)^{|n-l|+l}J_{2(|n-l|+l)}(2\alpha t)-2\sum_{i=0}^{|n-l|+l}(-1)^i J_{2i}(2\alpha t)\}. \quad (59)$$

Especially when $l=0$, the calculation of integral (53) becomes simpler because of the relation (54), that is,

$$\varepsilon\int_0^t dt'\rho_0(n,0;t-t')\frac{d}{dt'}\rho_0(0,0;t')$$

$$=\varepsilon\alpha\int_0^t dt'\,J_{2|n|}(2\alpha(t-t'))\{J_{-1}(2\alpha t')-J_1(2\alpha t')\}$$

$$=-2\varepsilon\alpha\int_0^t dt'\,J_{2|n|}(2\alpha(t-t'))\,J_1(2\alpha t')$$

$$=-2\varepsilon\sum_{k=0}^{\infty}(-1)^k J_{2(|n|+k+1)}(2\alpha t)$$

$$=\varepsilon(-1)^{|n|}\{\cos(2\alpha t)+J_0(2\alpha t)-2\sum_{i=0}^{|n|}(-1)^i J_{2i}(2\alpha t)\}. \quad (60)$$

Thus we have obtained an approximate solution of our integral equations in the forms, for $l>0$,

$$\rho(n,\,0\,;\,t)=J_{2n}(2\alpha t)+\varepsilon(-1)^{|n-l|+l}\{\cos(2\alpha t)+J_0(2\alpha t)$$
$$+(-1)^{|n-l|+l}J_{2(|n-l|+l)}(2\alpha t)-2\sum_{i=0}^{|n-l|+l}(-1)^i J_{2i}(2\alpha t)\}\,, \tag{61}$$

and for $l=0$,

$$\rho(n,\,0\,;\,t)=J_{2n}(2\alpha t)+\varepsilon(-1)^{|n|}\{\cos(2\alpha t)+J_0(2\alpha t)$$
$$-2\sum_{i=0}^{|n|}(-1)^i J_{2i}(2\alpha t)\}\,, \tag{62}$$

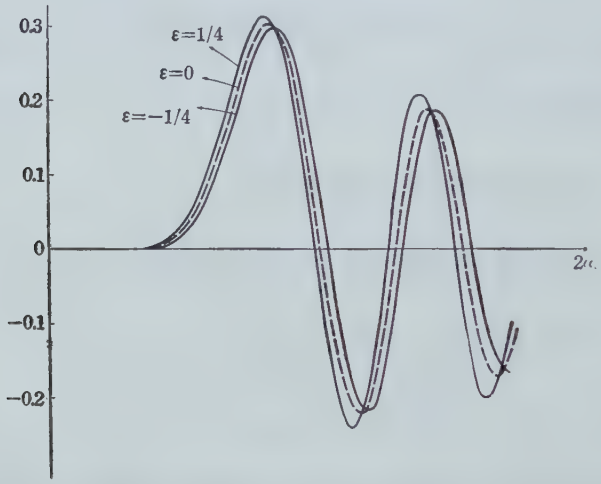

Fig. 1. The motion of the impurity atom $\rho(l,\,0\,;\,t)$ with $l=5$ and $\varepsilon=0,\ 1/4,\ -1/4$.

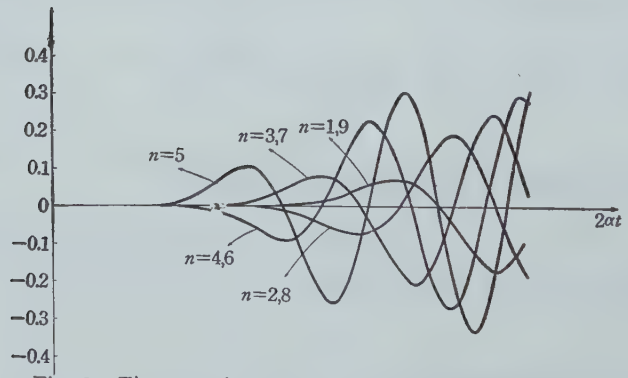

Fig. 2. The perturbation vibration Eq. (59) with $l=5$.

where $l=0$ means that the system starts from such an initial state that the impurity atom is displaced by unit distance from its equilibrium position and all other atoms sit on their equilibrium positions. In these expressions the cosine term implies the existence of the stationary vibration, but we cannot say much about

the stationary localized vibration using these solutions which were obtained by the perturbation calculation, because our approximation goes wrong for a large value of time t. On this point, however, we shall give further discussions in the next section.

Eq. (61) with $n=l$, that is, the actual motion of the impurity atom is graphically shown in Fig. 1. Fig. 2 shows the perturbation term (59).

§ 6. Method of the Laplace transform

In this section we shall try to solve the integral equations (37), for the lattice with a single isotopic impurity atom at the site $l(\geq 0)$, by means of the Laplace transformation. At first let us denote the Laplace transforms of the propagators by

$$g_{|n-m|}(s)=\int_0^\infty e^{-st}\rho_0(n, m; t)\,dt=\int_0^\infty e^{-st}J_{2(n-m)}(2\alpha t)\,dt$$

$$=\frac{1}{\sqrt{4\alpha^2+s^2}}\left(\frac{2\alpha}{\sqrt{4\alpha^2+s^2}+s}\right)^{|n-m|} \tag{63}$$

and

$$G(n, m; s)=\int_0^\infty e^{-st}\rho(n, m; t)\,dt. \tag{64}$$

If we calculate the Laplace transforms of both sides of Eq. (37) with $m=0$, we have

$$G(n, 0; s)=g_{|n|}(s)+\varepsilon\alpha^2 H(n, l; s)\{G(l+1, 0; s)$$
$$+G(l-1, 0; s)-2G(l, 0; s)\}, \tag{65}$$

where

$$H(n, l; s)=\int_0^\infty e^{-st}\left\{\int_0^t \rho_0(n, l; \tau)\,d\tau\right\}dt$$

$$=\int_0^\infty d\tau\rho_0(n, l; \tau)\int_\tau^\infty dt\, e^{-st}$$

$$=\frac{1}{s}g_{|n-l|}(s). \tag{66}$$

Consequently, Eq. (65) can be rewritten in the form

$$G(n, 0; s)=g_{|n|}(s)+\frac{\varepsilon\alpha^2}{s}g_{|n-l|}(s)\{G(l+1, 0; s)$$
$$+G(l-1, 0; s)-2G(l, 0; s)\}. \tag{67}$$

Next, introducing the functions

$$X_n(s)=G(n,\,0\,;\,s)-g_{|n|}(s), \tag{68}$$

we get

$$\begin{aligned}X_n(s)&=\frac{\varepsilon\alpha^2}{s}g_{|n-l|}(s)\,\{X_{l+1}(s)+X_{l-1}(s)-2X_l(s)\\&\qquad+g_{|l+1|}(s)+g_{|l-1|}(s)-2g_{-l|}(s)\}\\&=\varepsilon g_{|n-l|}(s)\,\{sg_{|l|}(s)-\delta_{l,0}\}\\&\qquad+\frac{\varepsilon\alpha^2}{s}g_{|n-l|}(s)\,\{X_{l+1}(s)+X_{l-1}(s)-2X_l(s)\},\end{aligned} \tag{69}$$

where we have used the relation

$$g_{|l+1|}(s)+g_{|l-1|}(s)-2g_{|l|}(s)=\int_0^\infty e^{-st}\{\rho_0(l+1,\,0\,;\,t)+\rho_0(l-1,\,0\,;\,t)-2\rho_0(l\;0\,;\,t)\}\,dt$$

$$=\frac{1}{\alpha^2}\int_0^\infty e^{-st}\frac{d^2}{dt^2}\rho_0(l,\,0\,;\,t)\,dt=\frac{s}{\alpha^2}[e^{-st}\rho_0(l,\,0\,;\,t)]_0^\infty+\frac{s^2}{\alpha^2}\int_0^\infty e^{-st}\rho_0(l,\,0\,;\,t)\,dt$$

$$=\frac{s^2}{\alpha^2}g_{|l|}(s)-\frac{s}{\alpha^2}\delta_{l,0}.$$

Now it is seen that the solutions of Eq. (69) have such a symmetry that $X_{l+k}(s)=X_{l-k}(s)$, because $g_{|n-l|}(s)$ depends on the distance $|n-l|$ only, so that we get

$$X_n(s)=\varepsilon g_{|n-l|}(s)\,\{sg_{|l|}(s)-\delta_{l,0}\}+2\frac{\varepsilon\alpha^2}{s}g_{|n-l|}(s)\,\{X_{l+l}(s)-X_l(s)\}. \tag{70}$$

If we put $n=l$ and $l+1$ in Eq. (70), we obtain two equations which can be used to evaluate the solutions $X_l(s)$ and $X_{l+1}(s)$ as functions of s. Actually we obtain

$$X_n(s)=\frac{\varepsilon g_{|n-l|}(s)\,\{sg_{|l|}(s)-\delta_{l,0}\}}{1-2(\varepsilon\alpha^2/s)\,\{g_1(s)-g_0(s)\}}. \tag{71}$$

Substituting (63) into (71), we finally obtain

$$\begin{aligned}X_n(s)=\varepsilon\left(\frac{2\alpha}{\sqrt{4\alpha^2+s^2}+s}\right)^{2|n-l|}&\left\{\frac{s}{\sqrt{4\alpha^2+s^2}}\left(\frac{2\alpha}{\sqrt{4\alpha^2+s^2}+s}\right)^{2l}-\delta_{l,0}\right\}\\&\times\frac{1}{(1+\varepsilon)\sqrt{4\alpha^2+s^2}-\varepsilon s}.\end{aligned} \tag{72}$$

Thus if we calculate the inverse Laplace transform of Eq. (72), we can obtain the exact solution of our problem, but the calculation is too complicated to have a simple analytical expression. However, it is interesting to see the relations

which can be derived from Eq. (72). Namely from Eq. (72) we have

$$X_n(s)=\left(\frac{2\alpha}{\sqrt{4\alpha^2+s^2}+s}\right)^{2|n-l|}X_l(s). \tag{73}$$

If we use here Eqs. (63), (64), (68) and the relation

$$\left\{\frac{2\alpha}{\sqrt{4\alpha^2+s^2}+s}\right\}^{2|n|}=2|n|\int_0^\infty e^{-st}\frac{J_{2n}(2\alpha t)}{t}dt, \tag{74}$$

we obtain the inverse Laplace transform of Eq. (73) in the form

$$\rho(n,0\,;t)=J_{2n}(2\alpha t)-J_{2(|n-l|+l)}(2\alpha t)+2|n-l|\int_0^t\rho(l,0\,;t-t')\frac{J_{2(n-l)}(2\alpha t')}{t'}dt'. \tag{75}$$

We shall consider from now on a special case in which the impurity atom sits on the site $l=0$, in other words, the case in which the impurity atom is initially displaced by a unit distance while all other atoms remain at their equilibrium positions. In this case from Eq. (72) we have

$$\begin{aligned}X_0^{(0)}(s)&=\varepsilon\frac{s-\sqrt{4\alpha^2+s^2}}{\sqrt{4\alpha^2+s^2}}\,\frac{1}{(1+\varepsilon)\sqrt{4\alpha^2+s^2}-\varepsilon s}\\&=\frac{1}{(1+\varepsilon)\sqrt{4\alpha^2+s^2}-\varepsilon s}-\frac{1}{\sqrt{4\alpha^2+s^2}},\end{aligned} \tag{76}$$

or

$$G^{(0)}(0,0\,;\,s)=1/\{(1+\varepsilon)\sqrt{4\alpha^2+s^2}-\varepsilon s\}, \tag{77}$$

where the superscript (0) indicates the position of the impurity atom $l=0$. Now from Eq. (72) we have for $l>0$

$$X_l(s)=\left(\frac{2\alpha}{\sqrt{4\alpha^2+s^2}+s}\right)^{2l}\{X_0^{(0)}(s)+\varepsilon G^{(0)}(0,0\,;s)\};$$

or using (63) and (68),

$$G(l,0\,;s)=(1+\varepsilon)\left(\frac{2\alpha}{\sqrt{4\alpha^2+s^2}+s}\right)^{2l}G^{(0)}(0,0\,;s). \tag{78}$$

The inverse transformation of Eq. (78) gives the relation

$$\rho(l,0\,;t)=2l(1+\varepsilon)\int_0^t\rho^{(0)}(0,0\,;t-t')\frac{J_{2l}(2\alpha t')}{t'}dt'. \tag{79}$$

Thus it has been shown that if we know a propagator $\rho^{(0)}(0,0\,;t)$ we can derive all solutions using the relations (75) and (79).

We can easily see that when $\varepsilon>0$ the denominator of Eq. (77) has a pure

imaginary root $s=i\omega$ and $\omega>2\alpha$. This again implies the fact that the lighter isotopic impurity atom can give rise to the stationary localized vibration. In fact, Eq. (77) can be rewritten in the form

$$G^{(0)}(0,0;s)=\frac{(1+\varepsilon)\sqrt{4\alpha^2+s^2}+\varepsilon s}{(1+\varepsilon)^2(4\alpha^2+s^2)-\varepsilon^2 s^2}$$

$$=\frac{(1+\varepsilon)\sqrt{4\alpha^2+s^2}}{4(1+\varepsilon)^2\alpha^2+(1+2\varepsilon)s^2}+\frac{\varepsilon}{1+2\varepsilon}\frac{s}{\frac{(1+\varepsilon)^2}{1+2\varepsilon}4\alpha^2+s^2}. \tag{80}$$

Here we can see that the inverse transform of the second term of Eq. (80) gives

$$\frac{\varepsilon}{1+2\varepsilon}\cos\left\{\frac{1+\varepsilon}{\sqrt{1+2\varepsilon}}2\alpha t\right\}. \tag{81}$$

This corresponds to the stationary localized vibration the frequency of which was already obtained in Eq. (48). Moreover if we expand Eq. (80) in power series of ε, the cosine term of $\rho^{(0)}(0,0;t)$ given by Eq. (62) is obtained as the first order term of (81) and all other terms of Eq. (62) result from the first term of Eq. (80).

§ 7. A pair of the isotopic impurity atoms

Lastly we shall consider the case in which the linear lattice contains two isotopic impurity atoms of equal mass M at the positions l and $-l$. In order to obtain the characteristic frequencies of this binary localized vibration, we shall study the asymptotic solutions of our integral equations just in the same way as that we used in § 4. In this case our integral equations with $m=0$ become

$$\rho(n,0;t)=\rho_0(n,0;t)+\varepsilon\int_0^t dt'\rho_0(n,l;t-t')\int_0^{t'}d\tau\mathfrak{D}(l)\cdot\rho(l,0;\tau)$$

$$+\varepsilon\int_0^t dt'\rho_0(n,-l;t-t')\int_0^{t'}d\tau\mathfrak{D}(-l)\cdot\rho(-l,0;\tau). \tag{82}$$

Again assuming the asymptotic solution of Eq. (82) in the form (40), we have for $t\to\infty$

$$a_n e^{i\omega t}=\frac{\varepsilon\alpha^2}{i\omega}(a_{l+1}+a_{l-1}-2a_l)\int_0^t dt' J_{2(n-l)}(2\alpha(t-t'))\,(e^{i\omega t'}-1)$$

$$+\frac{\varepsilon\alpha^2}{i\omega}(a_{-l+1}+a_{-l-1}-2a_{-l})\int_0^t dt' J_{2(n+l)}(2\alpha(t-t'))\,(e^{i\omega t'}-1), \tag{83}$$

and making use of Eqs. (42) and (43), we get

$$a_n=\varepsilon(-1)^{n-l+1}\frac{\alpha^2}{\omega\sqrt{\omega^2-4\alpha^2}}\left(\frac{2\alpha}{\omega+\sqrt{\omega^2-4\alpha^2}}\right)^{2|n-l|}(a_{l+1}+a_{l-1}-2a_l)$$
$$+\varepsilon(-1)^{n+l+1}\frac{\alpha^2}{\omega\sqrt{\omega^2-4\alpha^2}}\left(\frac{2\alpha}{\omega+\sqrt{\omega^2-4\alpha^2}}\right)^{2|n+l|}(a_{-l+1}+a_{-l-1}-2a_{-l}) \tag{84}$$

provided that $\omega>2\alpha$. Now, corresponding to the symmetric ($a_{-n}=a_n$) and antisymmetric ($a_{-n}=-a_n$) vibrational modes, we can rewrite Eq. (84) as

$$a_n=\varepsilon(-1)^{n+l+1}\frac{\alpha^2}{\omega\sqrt{\omega^2-4\alpha^2}}\left\{\left(\frac{2\alpha}{\omega+\sqrt{\omega^2-4\alpha^2}}\right)^{2|n-l|}\pm\left(\frac{2\alpha}{\omega+\sqrt{\omega^2-4\alpha^2}}\right)^{2|n+l|}\right\}$$
$$\times(a_{l+1}+a_{l-1}-2a_l), \tag{85}$$

where the plus and the minus sign correspond to the symmetric and the antisymmetric case respectively. The solutions of Eqs. (85) can be obtained by studying a set of three equations for which $n=l$, $l+1$ and $l-1$. Thus it follows that ω should satisfy the equations

$$\varepsilon\frac{\alpha^2}{\omega\sqrt{\omega^2-4\alpha^2}}\left[2\left\{\left(\frac{2\alpha}{\omega+\sqrt{\omega^2-4\alpha^2}}\right)^2+1\right\}\pm\frac{\omega^2}{\alpha^2}\left(\frac{2\alpha}{\omega+\sqrt{\omega^2-4\alpha^2}}\right)^{4l}\right]=1. \tag{86}$$

To write this equation in a simpler form, we introduce the variable $z=\omega/2\alpha(>1)$. Thus it turns out to be

$$\left(1+\frac{1}{\varepsilon}\right)\sqrt{z^2-1}-z=\pm z(z+\sqrt{z^2-1})^{-4l}. \tag{87}$$

For sufficiently large value of l the right-hand side of Eq. (87) becomes negligibly small and the equation is reduced to Eq. (47) which gives the frequency of the localized mode of a single isotopic impurity atom.

In order to have an approximate solution of Eq. (87) for large l, we shall write

$$z=z_0+\delta \tag{88}$$

where z_0 is the solution of Eq. (47), that is,

$$z_0=(1+\varepsilon)/\sqrt{1+2\varepsilon} \tag{89}$$

and δ is a small correction. If we evaluate only the first order correction, we obtain

$$\delta=\pm\frac{\varepsilon^2(1+\varepsilon)}{(1+2\varepsilon)^{3/2}}\left(\frac{\varepsilon}{\sqrt{1+2\varepsilon}}\right)^{4l} \tag{90}$$

or

$$\omega=\frac{(1+\varepsilon)}{\sqrt{1+2\varepsilon}}\left\{1\pm\frac{\varepsilon^2}{1+2\varepsilon}\left(\frac{\varepsilon}{\sqrt{1+2\varepsilon}}\right)^{4l}\right\}2\alpha, \tag{91}$$

where the plus and the minus sign correspond to the symmetric and antisymmetric localized vibrational mode respectively.

Finally we shall study a special case in which the pair of impurity atoms has the closest distance. If they are situated at sites 0 and 1, it is easily seen that we have, instead of Eq. (84), the equation

$$a_n=\varepsilon(-1)^{n+1}\frac{\alpha^2}{\omega\sqrt{\omega^2-4\alpha^2}}\left(\frac{2\alpha}{\omega+\sqrt{\omega^2-4\alpha^2}}\right)^{2|n|}(a_1+a_{-1}-2a_0)$$

$$+\varepsilon(-1)^n\frac{\alpha^2}{\omega\sqrt{\omega^2-4\alpha^2}}\left(\frac{2\alpha}{\omega+\sqrt{\omega^2-4\alpha^2}}\right)^{2|n-1|}(a_0+a_2-2a_1). \tag{92}$$

Now it is again easily verified that there exist two types of solutions, for which

i) Symmetric case $a_{-n}=a_{n+1}$

$$a_n=\varepsilon(-1)^n\frac{1}{4z\sqrt{z^2-1}}\left\{\left(\frac{1}{z+\sqrt{z^2-1}}\right)^{2|n-1|}-\left(\frac{1}{z+\sqrt{z^2-1}}\right)^{2|n|}\right\}(a_2-a_1) \tag{93}$$

ii) Antisymmetric case $a_{-n}=-a_{n+1}$

$$a_n=\varepsilon(-1)^n\frac{1}{4z\sqrt{z^2-1}}\left\{\left(\frac{1}{z+\sqrt{z^2-1}}\right)^{2|n-1|}+\left(\frac{1}{z+\sqrt{z^2-1}}\right)^{2|n|}\right\}(a_2-3a_1). \tag{94}$$

By letting n be 1 and 2 in each case, we can derive the secular equations in the form

$$\frac{z}{\sqrt{z^2-1}}\{1\mp(z-\sqrt{z^2-1})^2\}=1+\frac{1}{\varepsilon} \tag{95}$$

where the upper and the lower sign refer to case i) and ii) respectively. The solutions of Eqs. (95) are given by

case i) $$z=(1+\varepsilon)/2\sqrt{\varepsilon}, \qquad \varepsilon>1 \tag{96}$$

case ii) $$z=\frac{\sqrt{(1+\varepsilon)(1+9\varepsilon)}-1-\varepsilon}{2\sqrt{2\varepsilon}}\left\{\sqrt{(1+\varepsilon)(1+9\varepsilon)}-(1+3\varepsilon)\right\}^{-1/2}. \tag{97}$$

The solution of the case i) gives a lower level and that of case ii) an upper level, each of which is splitted from the degenerate level when the two impurity atoms are infinitely separated. Thus, in this case, the frequency of the symmetric localized mode is lower than that of the antisymmetric localized mode. However, it can be shown that the solution of lower level does exist only when $\varepsilon>1$ or $M<m/2$. Therefore it may be concluded that the stationary localized vibration corresponding to the lower impurity level does not exist when $m/2<M<m$ (Fig. 3). In these cases we must reformulate our theory of the stationary state in such a way that it can explicitly take account of the scattering processes of the free incoming wave, the frequency of which lies around the maximum frequency of the unperturbed lattice.

Further, it is easily shown that the secular equation (95) can be generalized

to the case in which the relative distance between two impurity atoms is $p(=1, 2, \cdots)$, that is,

$$\frac{z}{\sqrt{z^2-1}}\{1\pm(-1)^p(z-\sqrt{z^2-1})^{2p}\}=1+\frac{1}{\varepsilon}, \tag{98}$$

where the upper and the lower sign correspond to the symmetric and the antisymmetric vibrational mode respectively. By analyzing this equation (98), we can prove the fact that when p is even the antisymmetric mode gives the lower frequency and if p is odd the symmetric mode becomes the lower level, and these localized vibrational modes corresponding to the lower levels can exist only when

$$\varepsilon>\frac{1}{2p-1}. \tag{99}$$

Therefore if

$$\frac{(2p-1)}{2p}m<M<m, \tag{100}$$

we have only one localized mode with frequency higher than that of a single impurity atom, otherwise we have two impurity levels, corresponding to the symmetric and antisymmetric binary localized vibrational modes.

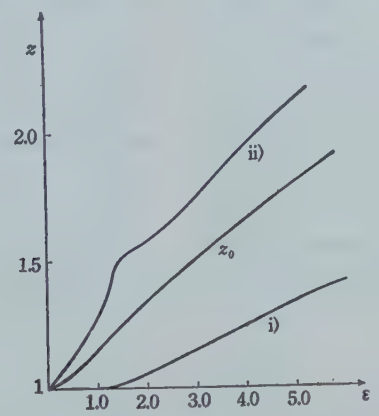

Fig. 3. The characteristic frequencies of the localized vibration associated with the closest pair of impurity atoms.

References

1) E. W. Montroll and R. B. Potts, Phys. Rev. **100** (1955), 525; **102** (1956), 72.
2) I. M. Lifshitz, Nuovo Cimento, Supple. AI **3** (1956), 716.
3) G. H. Koster and J. C. Slater, Phys. Rev. **95** (1954), 1167; **96** (1954), 1206.
4) P. Mazur, E. W. Montroll and R. B. Potts, J. Wash. Acad. Sci. **46** (1956), 2.
A. A. Maradudin, P. Mazur, E. W. Montroll and G. H. Weiss, Rev. Mod. Phys. **30** (1958), 175.
5) J. Hori and T. Asahi, Prog. Theor. Phys. **17** (1957), 523.
6) E. H. Kerner, Phys. Rev. **95** (1954), 687; Proc. Phys. Soc. London **69A** (1956), 234.
7) R. E. Merrifield, J. Chem. Phys. **28** (1958), 647.
8) P. W. Anderson, Phys. Rev. **109** (1958), 1492.
9) E. Schrödinger, Ann. der Phys. **44** (1914), 916.
G. Klein and I. Prigogine, Physica **19** (1953), 1053.
10) Bateman, *Tables of Integral Transforms* (McGraw-Hill, 1954), Vol. 1.
11) Bateman, *Higher Transcendental Functions* (McGraw-Hill, 1954), Vol. 2.

Letters to the Editor

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On the Scattering of K^+ by Nucleon at Low Energies

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According to the older data,[1] the elastic K^+-nucleon ordinary scattering cross sections are almost constant up to about 2 Bev, while the charge exchange one rises remarkably even at low energies. Many authors[5)~10)] have tried to explain the two features in marked contrast, assuming the K-baryon interactions of Yukawa type with or without $KK\pi$ or $KK\pi\pi$ interaction, but no sufficient success was obtained.

Now the recent data[2)~4)] show that the charge exchange scattering cross section is rather energy-insensitive. Thus the S-wave interaction predominates in every K^+-nucleon scattering, in contrast to the case of π-nucleon one. In this situation, it is important to reflect whether or not the elementary K-baryon interaction with *ps-ps* coupling can explain the low energy scattering data. The point is, no doubt, how to treat the S-wave interaction or so called core term. In the case of π-nucleon scattering, as was shown by Chew et al.,[12] the Born term in the dispersion relation is largely reduced by the P-wave contribution in the (3, 3) state. In our case there is no such large P-wave contribution. Thus it will be of some interest to see to what extent the simple second-order perturbation ap-

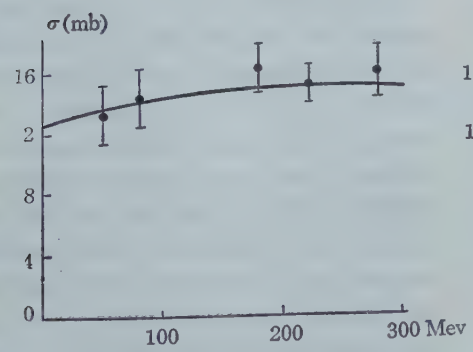

Fig. 1. $\sigma(K^+ + p \to K^+ + p)$.

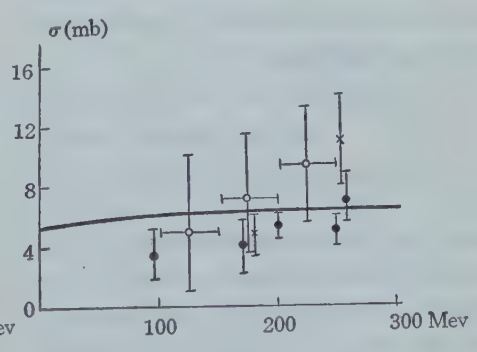

Fig. 2. $\sigma(K^+ + n \to K^0 + P)$.

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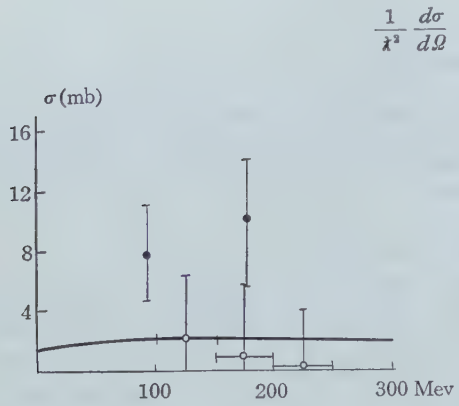

Fig. 3. $\sigma(K^+ + n \to K^+ + n)$.

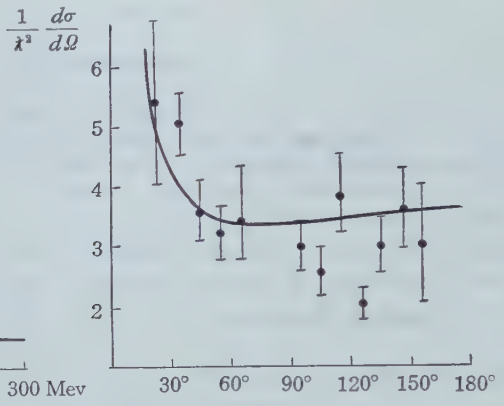

Fig. 4. $d\sigma/d\Omega(K^+ + p \to K^+ + p)$ at 225 Mev

● : ref. 2) ⊕ : ref. 3) ✕ : ref. 4)

proximation can reproduce the experiment. It has been pointed out by some authors[6),9)] that the perturbation calculation would fit the experimental data near the zero energy. Especially, D. Amati and B. Vitale[10)] carried out the second-order perturbation calculation in practice. However, they neglected the recoil effect, and the cross sections thus obtained decrease smoothly with energies in contradiction to the experimental ones. We carried out the calculation taking this recoil effect into account. It will be shown that the results are much improved. The calculation is simple; the scattering amplitudes are

$$M \propto \begin{cases} g_A^2 + g_\Sigma^2 & \text{for } K^+ + p \to K^+ + p \\ 2g_\Sigma^2 & \text{for } K^+ + n \to K^+ + n \\ g_A^2 - g_\Sigma^2 & \text{for } K^+ + n \to K^0 + p \end{cases} \quad (1)$$

except for the common function. (The mass difference between Λ and Σ is neglected, the resulting error being a few percent.) The results are given in Figs. 1, 2 and 3, where K^+-nucleon elastic total cross sections *vs* energy are plotted in comparison with the experimental data.[2)~4)] Fig. 4 shows the differential cross section for K^+-p scattering, including Coulomb part. The coupling constants chosen are

$$g_A^2 + g_\Sigma^2 = 4.4, \quad g_A^2/g_\Sigma^2 = 4.5 \quad (2)$$

which agree with those obtained by Sugano and Komatsuzawa from dispersion relation.[8)]

We see that the fit is rather quite good considering the roughness of our approximation, and any other coupling such as $KK\pi$, $KK\pi\pi$ or $(K^*\tau_\alpha\partial_\mu K - \partial_\mu K^*\tau_\alpha K)(\pi \times \partial_\mu \pi)_\alpha$ interaction[11)] seems to be dispensable. The reason for the reasonable fit is not clear. This fit may be only apparent or be due to the smallness of higher order corrections[9)] and the possible enlarged range of S-wave interaction. In this connection, it is remarked that Kawarabayashi and Miyazawa[13)] explained the smallness of π-nucleon S-wave scattering, assuming the core term of long range and taking into account the cancellation due to π-π interaction. In the case of

K^+-nucleon interaction, there may be no interaction such as π-π in their paper, and the S-wave occurring from the direct K^+-nucleon interaction will then be fully effective. If its range is really considerably large, our Born approximation will turn out to be quite valid.

As for K^--nucleon scattering, there is large absorption even at low energies, in contrast to the K^+-nucleon case. We may not be able to treat this process so easily as K^+-nucleon scattering.

The authors should like to express their thanks to Prof. Nakabayashi, Prof. Sato, Dr. Akiba and Dr. Takahashi for their helpful discussions.

1) C. Ceolin, N. Dallaporta, L. Guerriero, I. Laboragine, G. A. Salandin and L. Taffara, Nuovo Cimento **13** (1959), 818. This contains further references.
2) Proceedings of the Ninth Conference on High Energy Nuclear Physics, Kiev, 1959.
3) D. F. Davis, N. Kwak and M. F. Kaplon, Phys. Rev. **117** (1960), 846.
4) L. S. Rodberg and R. M. Thaler, Phys. Rev. Letters **4** (1960), 372.
5) H. P. Stapp, Phys. Rev. **106** (1957), 134.
6) G. Bialkowski and A. Jurewicz, Phys. Rev. **116** (1959), 1269.
7) K. Igi, Prog. Theor. Phys. **19** (1958), 238; **20** (1958), 403.
8) R. Sugano and A. Komatsuzawa, Prog. Theor. Phys. **23** (1960), 287.
9) R. Sugano, Prog. Theor. Phys. **22** (1959), 381. There are some errors in Eq. (6.6); the correct ones are
$V_0(m_K)=0.30(g_\Lambda^2-3g_\Sigma^2)+0.03(g_\Lambda^2+g_\Sigma^2)$
$V_1(m_K)=-0.30(g_\Lambda^2+g_\Sigma^2)+0.01(g_\Lambda^2+5g_\Sigma^2)$
(Private communication from Dr. Sugano).
10) D. Amati and B. Vitale, Nuovo Cimento **5** (1957), 435.
11) K. Igi, taking into account the $(K^*\tau_\alpha\partial_\mu K-\partial_\mu K^*\tau_\alpha K)(\Pi\times\partial_\mu\Pi)_\alpha$, fitted the experiments (preprint).
12) G. F. Chew, M. L. Goldberger, F. E. Low and Y. Nambu, Phys. Rev. **106** (1957), 1337.
13) K. Kawarabayashi and H. Miyazawa, Prog. Theor. Phys. **23** (1960), 490.

Electromagnetic Vertex for $\Sigma^0\to\Lambda^0+\gamma$

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August 20, 1960

The vertex part $\Gamma_\mu(\Sigma^0\Lambda^0)$ which describes the decay of Σ^0 into a Λ^0 and a photon is generally written as the sum of three terms which are proportional to γ_μ, k_μ and $\sigma_{\mu\nu}$, where k_μ is the momentum transfer of the photon.[1] It has been shown that the first two terms do not contribute to the real photon emission and that the Pauli-moment-like term $\sigma_{\mu\nu}$ gives the major contribution even to the process including virtual photons.[1),2)] Also the possibility has been pointed out that the observed mass of Σ^0, which is slightly lower than the arithmetic mean of the masses of Σ^+ and Σ^-, may be explained by taking into account the electromagnetic self-energy due to the vertex $\Gamma_\mu(\Sigma^0\Lambda^0)$, especially the part $\sigma_{\mu\nu}$.[3)] In the present report several relations of the vertex $\Gamma_\mu(\Sigma^0\Lambda^0)$ will be examined, which are rigorously true to all orders in the strong interactions.

The charge independent strong interactions of baryons with π and K mesons, H_π and H_K, are assumed to be represented by the doublet form of Gell-Mann[4)] and Pais.[5)] The minimal electromagnetic interactions of baryons and mesons are written in three parts,[6)] $H_{em}=H_a+H_b+H_c$, where

$$
\left.\begin{aligned}
H_a &= \frac{ie}{2}(\bar{N}_1\gamma_\mu N_1 - \bar{N}_4\gamma_\mu N_4)A_\mu \\
&\quad + \frac{ie}{2}(\bar{K}\partial_\mu K - \partial_\mu \bar{K}K)A_\mu, \\
H_b &= \frac{ie}{2}(\bar{N}_2\gamma_\mu N_2 - \bar{N}_3\gamma_\mu N_3)A_\mu \\
&\quad + \frac{ie}{2}(\bar{K}\tau_3\partial_\mu K - \partial_\mu \bar{K}\tau_3 K)A_\mu, \\
H_c &= \frac{ie}{2}\sum_{i=1}^{4}\bar{N}_i\gamma_\mu\tau_3 N_i A_\mu \\
&\quad + ie\bar{\pi}T_3\partial_\mu\pi A_\mu,
\end{aligned}\right\} \quad (1)
$$

$N_1=\begin{pmatrix} p \\ n \end{pmatrix}$, $N_2=\begin{pmatrix} \Sigma^+ \\ (\Lambda^0-\Sigma^0)/\sqrt{2} \end{pmatrix}$, $N_3=\begin{pmatrix} (\Lambda^0+\Sigma^0)/\sqrt{2} \\ \Sigma^- \end{pmatrix}$ and $N_4=\begin{pmatrix} \Xi^0 \\ \Xi^- \end{pmatrix}$. We shall consider, following Tanaka's argument,[6] the transformation of the conventional isospin rotation,

$$
\begin{aligned}
\text{(I)}: \quad & N_1\to e^{i(\pi/2)T_2}N_1, \quad N_2\to -e^{i(\pi/2)T_2}N_3, \\
& N_3\to e^{i(\pi/2)T_2}N_2, \quad N_4\to e^{i(\pi/2)T_2}N_4, \\
& \pi\to e^{i\pi T_2}\pi, \quad K\to e^{i(\pi/2)T_2}K,
\end{aligned}
$$

under which H_π and H_K are invariant, and H_{em} is also invariant if $A_\mu\to A_\mu$ in H_a and $A_\mu\to -A_\mu$ in H_b and H_c, and next consider the second transformation,

$$
\begin{aligned}
\text{(II)}: \quad & N_1\to e^{i(\pi/2)T_2}N_1, \quad N_2\to e^{i(\pi/2)T_2}N_2, \\
& N_3\to e^{i(\pi/2)T_2}N_3, \quad N_4\to e^{i(\pi/2)T_2}N_4, \\
& \pi\to e^{i\pi T_2}\pi, \quad K\to K,
\end{aligned}
$$

under which H_π and H_K are invariant and H_{em} is also invariant if $A_\mu\to A_\mu$ in H_a and H_b, and $A_\mu\to -A_\mu$ in H_c. The electromagnetic vertex defined as the expectation value of the time-ordering operator in the physical vacuum may be written as the sum of the contributions from H_a, H_b and H_c, i. e. $\Gamma_\mu(\cdots)=\langle\cdots\rangle_a+\langle\cdots\rangle_b+\langle\cdots\rangle_c$. Combining the results to which the transformations (I) and (II) lead, we obtain the relations,

$$
\left.\begin{aligned}
&\langle\Sigma^+\rangle_a=\langle\Sigma^-\rangle_a=\langle\Sigma^0\rangle_a=\langle\Lambda^0\rangle_a \\
&\quad \langle\bar{\Sigma}^0\Lambda^0\rangle_a=\langle\bar{\Lambda}^0\Sigma^0\rangle_a=0, \\
&\langle\Sigma^+\rangle_b=-\langle\Sigma^-\rangle_b=-\langle\bar{\Sigma}^0\Lambda^0\rangle_b \\
&\quad =-\langle\bar{\Lambda}^0\Sigma^0\rangle_b; \quad \langle\Sigma^0\rangle_b=\langle\Lambda^0\rangle_b=0, \\
&\langle\Sigma^+\rangle_c=-\langle\Sigma^-\rangle_c=\langle\bar{\Sigma}^0\Lambda^0\rangle_c=\langle\bar{\Lambda}^0\Sigma^0\rangle_c; \\
&\quad \langle\Sigma^0\rangle_c=\langle\Lambda^0\rangle_c=0,
\end{aligned}\right\} \quad (2)
$$

where $\langle\Sigma^+\rangle_a=\langle T(\bar{\Sigma}^+\Sigma^+A_\mu)\rangle_0^a$, $\langle\bar{\Sigma}^0\Lambda^0\rangle_a=\langle T(\bar{\Sigma}^0\Lambda^0A_\mu)\rangle_0^a$, etc. Thus we have

$$
\left.\begin{aligned}
&\Gamma_\mu(\Sigma^+)=\langle\Sigma^0\rangle_a-\langle\bar{\Sigma}^0\Lambda^0\rangle_b+\langle\bar{\Sigma}^0\Lambda^0\rangle_c, \\
&\Gamma_\mu(\Sigma^-)=\langle\Sigma^0\rangle_a+\langle\bar{\Sigma}^0\Lambda^0\rangle_b-\langle\bar{\Sigma}^0\Lambda^0\rangle_c, \\
&\Gamma_\mu(\Sigma^0)=\Gamma_\mu(\Lambda^0)=\langle\Sigma^0\rangle_a=\langle\Lambda^0\rangle_a^{6)}, \\
&\Gamma_\mu(\bar{\Sigma}^0\Lambda^0)=\Gamma_\mu(\bar{\Lambda}^0\Sigma^0)=\langle\bar{\Sigma}^0\Lambda^0\rangle_b \\
&\quad +\langle\bar{\Sigma}^0\Lambda^0\rangle_c.
\end{aligned}\right\} \quad (3)
$$

In the above derivation, we have neglected the mass difference between Λ and Σ, and have assumed the even relative parity between them. If we assume, in addition to the above symmetry, the symmetry between nucleon and Ξ, we get $\langle\Sigma^0\rangle_a=\langle\Lambda^0\rangle_a=0$,[7] but $\langle\bar{\Sigma}^0\Lambda^0\rangle_b\neq 0$ and $\langle\bar{\Sigma}^0\Lambda^0\rangle_c\neq 0$. Even if the nucleon and Ξ symmetry does not strictly hold, the magnetic moments μ_{Σ^0} and μ_{Λ^0} are thus expected to be small enough, while the magnetic-moment-like coefficient $\mu_{\Sigma^0\Lambda^0}$ may be appreciably large. Furthermore if the pair effects are neglected as in the Chew model, $\langle\cdots\rangle_c$ is the only contribution including

Table 1

Coupling of $(\Sigma\Lambda\pi)$	$B_1^{(\pi)}(\Sigma\Sigma\Lambda)$	$B_2^{(\pi)}(\Sigma\Sigma\Lambda)$
1	0.392	−0.572
$i\gamma_5$	−0.237	0.183

Table II

Coupling of (ΣNK) or $(\Sigma\Xi K)$	Coupling of (ΛNK) or $(\Lambda\Xi K)$	$B_1^{(K)}(\Sigma N\Lambda)$	$B_2^{(K)}(\Sigma N\Lambda)$	$B_1^{(K)}(\Sigma\Xi\Lambda)$	$B_2^{(K)}(\Sigma\Xi\Lambda)$
1	1	0.660	−0.679	0.341	−0.348
1	$i\gamma_5$	0.469	−0.413	0.266	−0.233
$i\gamma_5$	1	0.445	−0.377	0.255	−0.222
$i\gamma_5$	$i\gamma_5$	−0.255	0.111	−0.184	0.107

the direct π meson current.[7] Therefore if the pair effects are not predominant and if $g_\pi^2 \gg g_K^2$, it follows from (3) that

$$\begin{aligned} &\mu_{\Sigma^0\Lambda^0} \simeq \mu_{\Sigma^+} \simeq -\mu_{\Sigma^-}\,; \\ &|\mu_{\Sigma^0\Lambda^0}| \gg |\mu_{\Sigma^0}| \simeq |\mu_{\Lambda^0}|\,. \end{aligned} \tag{4}$$

Finally, the lowest-order perturbation result of $\mu_{\Sigma^0\Lambda^0}$ is listed in Tables I and II, for the purpose of reference, where B_1 and B_2 denote the contributions from the baryon and meson currents respectively, though B_1 calculated by the perturbation theory might be overestimated as in the case of nucleon. $\mu_{\Sigma^0\Lambda^0}$ in hyperon magnetons is given by

$$\begin{aligned} \mu_{\Sigma^0\Lambda^0} = &-\frac{g_{\Sigma\Sigma\pi}}{\sqrt{4\pi}}\cdot\frac{g_{\Sigma\Lambda\pi}}{\sqrt{4\pi}}\cdot\frac{2}{\pi} \\ &\times[B_1^{(\pi)}(\Sigma\Sigma\Lambda) - B_2^{(\pi)}(\Sigma\Sigma\Lambda)] \\ &+\frac{g_{\Sigma NK}}{\sqrt{4\pi}}\cdot\frac{g_{\Lambda NK}}{\sqrt{4\pi}}\cdot\frac{1}{\pi} \\ &\times[B_1^{(K)}(\Sigma N\Lambda) - B_2^{(K)}(\Sigma N\Lambda)] \\ &+\frac{g_{\Sigma\Xi K}}{\sqrt{4\pi}}\cdot\frac{g_{\Lambda\Xi K}}{\sqrt{4\pi}}\cdot\frac{1}{\pi} \\ &\times[B_1^{(K)}(\Sigma\Xi\Lambda) - B_2^{(K)}(\Sigma\Xi\Lambda)]. \end{aligned} \tag{5}$$

Relative signs of the coupling constants and the coupling types (1 or $i\gamma_5$) are both indeterminate at present. The observed mean life of Σ^0($\lesssim 0.1\times10^{-10}$ sec[8]) provides a too lower limit, $|\mu_{\Sigma^0\Lambda^0}| \gtrsim 10^{-4}$ in hyperon magnetons. The above study, however, seems to suggest that the reasonable magnitude of $|\mu_{\Sigma^0\Lambda^0}|$ is a few hyperon magnetons as well as $|\mu_{\Sigma^\pm}|$, and such conjecture does not contradict the standard deviation of the recent measurement of the mass of Σ^0($\sim \pm 1.0\, m_e$[8]).

1) G. Feldman and T. Fulton, Nuclear Phys. **8** (1958), 106.
2) G. Feinberg, Phys. Rev. **109** (1958), 1019.
3) H. Katsumori, Prog. Theor. Phys. **24** (1960), 35.
4) M. Gell-Mann, Phys. Rev. **106** (1957), 1296.
5) A. Pais, Phys. Rev. **112** (1958), 624.
6) K. Tanaka, to be published. We are grateful to Dr. K. Tanaka for sending us a preprint of his paper.
7) G. Feinberg and R. E. Behrends, Phys. Rev. **115** (1959), 745.
8) L. W. Alvarez, Proc. 9th Conference on High Energy Nuclear Physics (Kiev, 1959), Part III, 1.

The Analyticity Properties of the Bethe-Salpeter Amplitude

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As is well known, the Bethe-Salpeter equation has abnormal solutions which

are never met in the case of non-relativistic Schrödinger equation. They correspond to the degree of freedom as to the relative time. Ohnuki, Takao and Umezawa,[1] and Ohnuki and Watanabe[2] investigated this situation, and showed that, in contrast to the normal solutions, the abnormal ones do not belong to the eigen state of the Hamiltonian of the system in question. With the anticipation that the normal and abnormal solutions should have some differences also in their analyticity properties, we have examined them. In the case that the mass of intermediary boson is zero, we have found no difference among them.

We consider the bound states of two bosons, whose masses are assumed to be equal, interacting through boson field with zero mass. The Bethe-Salpeter equation in the ladder approximation is as follows:

$$[(p+\eta)^2+1]\cdot[(p-\eta)^2+1]\chi(p) = \frac{\lambda}{\pi}\cdot\int\frac{d^4k\chi(k)}{(p-k)^2-i\epsilon}, \tag{1}$$

where $\eta=(0, E/2)$, E being the total energy of the system. We have assumed that the bound state is at rest. According to Wick and Cutkosky[3] the eigenvalue problem Eq. (3) can be reduced to a Sturm Liouville type:

$$\begin{aligned} &g_n''(z)+2(n-1)z(1-z^2)^{-1}g_n'(z)\\ &\quad -n(n-1)(1-z^2)^{-1}g_n(z)\\ &\quad +\lambda(1-z^2)^{-1}(1-\eta^2+\eta^2z^2)^{-1}\\ &\quad \times g_n(z)=0, \end{aligned} \tag{2}$$

where $n(>0)$ stands for a principal quantum number of the bound state. The boundary condition for $g_n(z)$ is

$$g_n(\pm 1)=0. \tag{2a}$$

From Eq. (2) we see

$$g_n(z)=P\begin{Bmatrix} \pm 1, & \pm\sqrt{1-\eta^2}/\eta, & \infty \\ 0, & 0, & -n, \quad z \\ n, & 1, & -n+1 \end{Bmatrix}. \tag{3}$$

We may remark here that the singularities at $z=\pm 1$ are common to the normal and the abnormal solutions. With this $g_n(z)$, we can represent the partial wave amplitude $\chi_l^m(p)$ in the following form[3]: For simplicity we here restrict ourselves to the case $n=l+1$,

$$\chi_l^m(p) = \int_{-1}^{1}\frac{g_m(z)\,dz}{[p^2-p_0Ez+1-(E^2/4)-i\epsilon]^{n+2}} \times \mathcal{Y}_l^m(\boldsymbol{p}). \tag{4}$$

According to Eqs. (2a) and (3), the power series expansion of $g_n(z)$ around $z=\pm 1$ is $g_n(z)=(z\pm 1)^n+\cdots$. As is well known, this series does not contain the logarithmic term, which can be ascertained by the direct computation. Therefore,

$$\frac{d^{n-1}g_n(z)}{dz^{n-1}}\Big|_{z=\pm 1}=0, \quad \frac{d^ng_n(z)}{dz^n}\Big|_{z=\pm 1}=n!. \tag{5}$$

Thus, we can perform partial integration of (4), obtaining the following formula:

$$\chi_l^m(p)=\frac{(-)^n}{n+1}\cdot\left(\frac{1}{p_0E}\right)^{n+1} \times\left[\frac{1}{p^2+1-p_0E-(E^2/4)-i\epsilon} - \frac{1}{p^2+1+p_0E-(E^2/4)-i\epsilon}\right.$$

$$-\frac{1}{n!}\int_{-1}^{1}\frac{1}{p^2+1-p_0Ez-(E^2/4)-i\epsilon}\times\frac{d^{n+1}g_n}{dz^{n+1}}dz\Big]\mathcal{Y}_l^m(\boldsymbol{p}) \qquad (6a)$$

if $p_0=0$,

$$\chi(\boldsymbol{p})=\frac{1}{(\boldsymbol{p}^2+1-E^2)^{n+2}}\cdot\int_{-1}^{1}g_n(z)\,dz\,. \qquad (6b)$$

Eq. (8) tells us about the analyticity property of $\chi_l^m(p)$ in the p_0-plane as follows: the poles at $p_0=-(E/2)\pm\sqrt{\boldsymbol{p}^2+1}$, $p_0=(E/2)\pm\sqrt{\boldsymbol{p}^2+1}$, the cuts extending from $-(E/2)+\sqrt{\boldsymbol{p}^2+1}$ to $(E/2)+\sqrt{\boldsymbol{p}^2+1}$ and from $-(E/2)-\sqrt{\boldsymbol{p}^2+1}$ to $+(E/2)-\sqrt{\boldsymbol{p}^2+1}$. This is shown in Fig. 1.

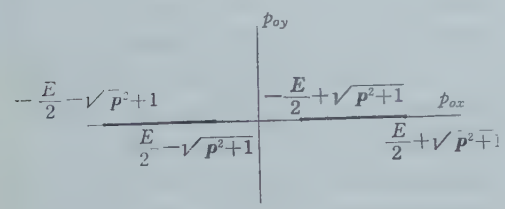

Fig. 1. p_0-plane.

When E becomes zero, the four poles degenerate into two and branch points vanish. This property can also be ascertained by solving Eq. (3) directly. The solution is $\chi(p)=(1+p^2)^{-n-2}\cdot F(-n,-n+2;2;-p^2)$, where F is the hypergeometric series with finite terms and the singularities are involved only in the first factor.

Remembering the remark under Eq. (3), we can thus conclude that the normal and abnormal solutions are not different in analyticity. In this note we have assumed that the masses of the bosons bound together are equal; this assumption is, however, immaterial, because the integral representation like Eq. (6) can also be obtained in the case of unequal masses.

Now, the Bethe-Salpeter amplitude is originally defined as

$$\chi(x)=e^{-iP\cdot X}\langle 0|T(\phi(x_1)\phi(x_2))|B\rangle, \qquad (7)$$

where P denotes the total momentum and $X=1/2\cdot(x_1+x_2)$, $x=x_1-x_2$. When we break up the product in Eq. (7) by the insertion of the intermediate state assumed to have positive energy, we obtain

$$\chi(p)=\frac{1}{2}\cdot\Big\{\frac{\phi(\boldsymbol{p})}{\sqrt{\boldsymbol{p}^2+1}-p_0-(E/2)-i\epsilon}+\frac{\phi(-\boldsymbol{p})}{\sqrt{\boldsymbol{p}^2+1}+p_0-(E/2)-i\epsilon}\Big\}+\int_{\omega_0}^{\infty}\Big\{\frac{\sigma(\boldsymbol{p},\omega)}{\omega-p_0-(E/2)-i\epsilon}+\frac{\sigma(-\boldsymbol{p},\omega)}{\omega+p_0-(E/2)-i\epsilon}\Big\}d\omega, \qquad (8)$$

where ϕ, and σ are certain functions, E is the total energy of the bound system, and $\omega_0=\sqrt{\boldsymbol{p}^2+1}$. Thus, the analyticity property is as follows: poles at $p_0=(E/2)-\sqrt{\boldsymbol{p}^2+1}$, $-(E/2)+\sqrt{\boldsymbol{p}^2+1}$, the cuts extending from $\omega_0-E/2$ to ∞, and $-\omega_0+E/2$ to $-\infty$.

When we compare this result with that obtained by the ladder approximation, we see that the latter has the extra-poles at $p_0=(E/2)+\sqrt{\boldsymbol{p}^2+1}$, and $p_0=-(E/2)-\sqrt{\boldsymbol{p}^2+1}$. A remark should be made here that such extra-poles are excluded in Eq. (8) because the intermediate states are limited to the positive energy ones.

The author is indebted to Dr. Y. Ohnuki for the most helpful discussions.

1) Y. Ohnuki, Y. Takao, and H. Umezawa, Prog. Theor. Phys. **23** (1960), 273.
2) Y. Ohnuki and K. Watanabe, to be published.
3) G. C. Wick, Phys. Rev. **96** (1954), 1124.
R. E. Cutkosky, Phys. Rev. **96** (1954), 1135.

Note on the $D(\gamma, p)n$ Reaction at the γ-ray Energy of 164 Mev

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September 14, 1960

The differential cross section and the proton polarization have been computed for the γ-ray energy of 164 Mev. For the deuteron wave function and the interaction operator of $E1$, $E2$ and spin flip $M1$ we have used the same one as that in our previous work[1] but a different energy.

Two types of the interaction operators for each transition are used in order to determine retardation effect. The radial dependences of $E1$, $E2$ and spin flip $M1$ transition operator including higher power terms or the retardation effect are written in the second column of the Table I referred to as (A). First terms of the expansion of (A) in terms of κx are the conventional transition operators and are written in the third column referred to as (B).

The final state wave functions are approximated by the asymptotic form

Table I. Radial dependence of the interaction operator The retardation effect is involved in (A). (B) is the conventional transition operator.

Interaction	(A)	(B)
$E1$-transition	$(3/\kappa)\{2j_1(\kappa x/2) - (\kappa x/2)j_2(\kappa x/2)\}$	x
$E2$-transition	$(20/\kappa^2)\{3j_2(\kappa x/2) - (\kappa x/2)j_3(\kappa x/2)\}$	x^2
spin-flip $M1$-transition	$j_0(\kappa x/2)$	1

$$v_{lJ}(kx) = kx\, j_l(kx)\cos\delta_{lJ} - kx\, n_l(kx)\sin\delta_{lJ},$$

except for the region $x \lesssim 1.5$. A set of phase shifts used is given by the analysis of Gammel and Thaler at 310 Mev.[2),3)] The full account of the tensor coupling in the final states for $E1$ and $E2$ transition is involved. The obtained results are shown in Table II and Fig. 1.

Comparing the results due to the interactions (A) and (B) in Table II, the retardation effect gives the total cross section larger than the one of the conventional interaction. The retardation effect really gives the larger cross section $\sigma_T(E1)$ for the $E1$ transition, but we get larger cross sections due to the conventional interaction for $E2$ and spin flip $M1$ transition. This is an unexpected fact. Both angular distributions (Fig. 1) are similar but (B) shows a little strong forward asymmetry depending on the larger $E2$ transition amplitudes.

Total cross sections and angular distributions do not agree with experimental data, and have rather small value.[4)] Lately T. Akiba[5)] has estimated

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Table II. Transition amplitudes and cross sections in each transition. Calculations due to the interaction (A) (second column) and the interaction (B) (third column) are compared.

	Interaction	(A)	(B)
(E1)	deuteron $\to {}^3P_0=I_{10}$	0.0723	0.0462
	deuteron $\to {}^3P_1=I_{11}$	0.406	0.375
	deuteron $\to {}^3P_2=I_{12}$	0.113	0.0739
	deuteron $\to {}^3F_2=I_{32}$	0.211	0.232
	$\sigma_T(E1)$	35.2 μb	32.3 μb
	$d\sigma(E1-E1)/d\Omega$	$2.255+0.814\sin^2\theta$	$2.414+0.230\sin^2\theta$
(E2)	deuteron $\to {}^3S_1=I_{01}$	-0.243	-0.297
	deuteron $\to {}^3D_1=I_{21}$	0.387	0.449
	deuteron $\to {}^3D_2=I_{22}$	0.276	0.267
	deuteron $\to {}^3D_3=I_{23}$	0.0914	0.214
	deuteron $\to {}^3G_3=I_{43}$	0.552	0.628
	$\sigma_T(E2)$	1.19 μb	1.87 μb
	$d\sigma(E2-E2)/d\Omega$	$0.0996+0.134\sin^2\theta-0.178\sin^4\theta$	$0.0751+0.587\sin^2\theta-0.596\sin^4\theta$
	$d\sigma(E1-E2)/d\Omega$	$0.168\cos\theta+2.16\sin^2\theta\cos\theta$	$0.182\cos\theta+2.62\sin^2\theta\cos\theta$
(M1)	deuteron $\to {}^1S_0=I_s$	0.0307	0.0323
	deuteron $\to {}^1S_2=I_d$	0.108	0.115
	$\sigma_T(M1)$	2.47 μb	2.77 μb
	$d\sigma(M1-M1)/d\Omega$	$0.0392+0.236\sin^2\theta$	$0.0443+0.264\sin^2\theta$
	σ_T	38.9 μb	36.9 μb

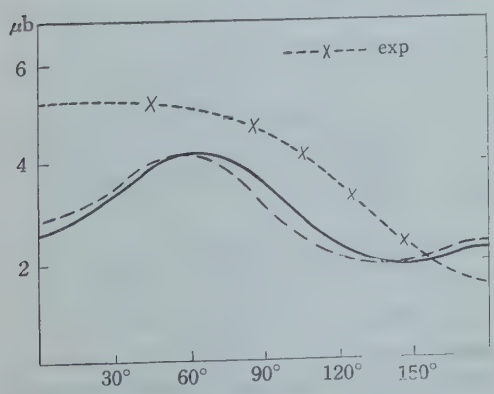

Fig. 1. Angular distribution at $\hbar\nu=164$ Mev. The soild line denotes the result due to (A), the dotted line denotes the result due to (B).

the virtual meson effect at $\hbar\nu=160$ Mev without the final state interaction using the Hulthèn function for the deuteron state and he has found that the cross section due only to the virtual meson effect for $E1$ transition is about $8\sim13\mu$b at the energy.[6] If we add our total cross section and the increment due to the virtual meson effect, the total cross section may approach the scope of the experimental value.

It was thought that the wave function inside the nuclear force range had little contribution for $E1$ transition. However, the shorter wavelength of the final state at the proton energy 310 Mev makes a first maximum of the waves around $x\sim1$. Therefore the contributions of the inner region of the

nuclear force to the overlap integrals are very important and the approximation of the final wave function in this region may have the major role of the $E1$ transition. However, we do not know the reliable features of waves except the asymptotic form. Then we must content ourselves with the approximation which modifies the asymptotic form by analogy of the previous work.[1] The approximation $v_{lJ}=kx \times j_l(kx)\cos\delta_{lJ}(x\leqq 1)$ gives rise to the values of overlap integrals about 10 to 15% smaller than the present approximation.

For the variety of sets of phase shifts for p-p scattering, the set of Gammel and Thaler[2] or of Stapp's solution No. 1 gives the largest cross section. This is the same conclusion as that of de Swart and Marshak.[7] We have also calculated the proton polarization but it may be of no use except for the comparison of (A) and (B) because the differential cross section is out of fit with experiment.

The author would like to express his gratitude to Professor M. Kobayasi for his encouragement.

1) J. Iwadare and M. Matsumoto, Prog. Theor Phys. **24** (1960), 797.
2) J. L. Gammel and R. M. Thaler, Phys. Rev **107** (1957), 291, 1337.
3) We use the same notation as that in reference 1).
4) W. Zernik, M. L. Rustgi and G. Breit, Phys. Rev. **114** (1959), 1358.
They obtained $\sigma_T=48\mu$b without the retardation.
5) T. Akiba, Prog. Theor. Phys. **24** (1960), 370.
6) T. Akiba, private communication.
7) J. J. de Swart and R. E. Marshak, Physica **25** (1959), 1001.

Collision Broadening of the Landau Levels

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September 14, 1960

A tentative method is proposed in order to avoid the divergence of the scattering probability of a carrier by the static impurity potential in a magnetic field at the bottom of each Landau subband. The conventional assumptions such as "uncertainty cutoff"[1] and "collision cutoff"[2] are not used. The unit volume of the crystal is taken, a spherical mass is assumed, and all the spin-effects are ignored.

Applying the multiple scattering method for the amplitude correlation function[3] to the case in the presence of a magnetic field, we have for the density of states

$$\rho(E)=\frac{i}{2\pi}\sum_{\alpha}\psi_\alpha^*(\boldsymbol{r})\psi_\alpha(\boldsymbol{r}) \times\{1/[b_\alpha(E+i\eta)-\langle O_\alpha(E+i\eta)\rangle] -1/[b_\alpha(E-i\eta)-\langle O_\alpha(E-i\eta)\rangle]\}, \tag{1}$$

$$\alpha\equiv(n,l,k),\quad b_\alpha(E)=E-\bar{E}_\alpha,$$

$$\langle\alpha|O|\alpha'\rangle=\delta_{\alpha\alpha'}\langle\alpha|\sum_{i=1}^{N}t_i|\alpha\rangle,$$

where $\psi_\alpha(\boldsymbol{r})$ is a function of the carrier coordinate $\boldsymbol{r}$ and denotes a Landau state with a magnetic quantum number n, an azimuthal quantum number l, and a wave number component along the magnetic field k, t_i is the scattering

matrix due to i-th impurity, N is the number of the impurities, and $\langle\ \rangle$ means the averaging over all the impurity positions. In deriving Eq. (1) an approximation $\langle F/(b-O)\rangle \approx \langle F\rangle/(b-\langle O\rangle)$ was assumed and the fourth and the higher orders of the scattering matrix was ignored. $F-1$ is the incoherent part of the Möller wave matrix.[3]

Separating the real and the imaginary parts of the diagonal operator O_α as

$$\langle O_\alpha(E+i\eta)\rangle = \Delta_\alpha(E,\eta) \mp i\hbar\Gamma'_\alpha(E,\eta)/2,$$

we obtain

$$\rho(E) = (1/2\pi\lambda)^2 \left(\frac{1}{\pi}\right) \lim_{\eta\to+0} \sum_{n,k} (\hbar\Gamma_{nk}/2) /[(E-E_{nk}-\Delta_{nk})^2 + (\hbar\Gamma_{nk}/2)^2], \quad \lambda^2 = \hbar c/eH. \tag{2}$$

Using the perturbation method and the screened Coulomb impurity potential, the scattering probability in the quantum limit is given by

$$\Gamma'_{ok} = (2\pi/\hbar)\,(4\pi e^2/\kappa)^2\,(\lambda/2)^2\,(N/2\pi) \times \rho_0'(\epsilon_z)\,(1/\gamma)[\{1+\gamma\exp(\gamma)\,\mathrm{Ei}(-\gamma)\} + (\gamma/\gamma_0)\,\{1+\gamma_0\exp(\gamma_0)\,\mathrm{Ei}(-\gamma_0)\}], \quad \gamma = (4\epsilon_z+\epsilon_s)/\hbar\omega,\ \gamma_0 = \epsilon_s/\hbar\omega, \quad \epsilon_s = \hbar^2/2m^* r_s^2,\ \epsilon_z = \hbar^2 k^2/2m^*, \tag{3}$$

where r_s is the Debye length, ω is the cyclotron frequency, κ is the dielectric constant, and $\rho_0'(\epsilon_z) = (1/2\pi\hbar)(m^*/2\epsilon_z)^{1/2}$ is the usual density of states. Putting Eq. (3) into Eq. (2) in place of Γ_{nk}, we obtain the density of states at (o, l, k), in which the broadening effect is adequately taken into account. Next we substitute the obtained density of states except the factor $1/(2\pi\lambda)^2$, which comes from the degeneracy of the center of cyclotron orbit, into Eq. (3) in place of $\rho_0'(\epsilon_z)$ and obtain

$$\Gamma_{ok} = (1/2\pi\hbar)\,\mathrm{J}(\epsilon_z)\,\epsilon_s^2(\hbar\omega+\epsilon_s) \times \{1+(1+4\epsilon_z/\epsilon_s)\,(1+4\epsilon_z/(\epsilon_s+\hbar\omega))\} /(4\epsilon_z+\epsilon_s)\,(4\epsilon_z+\epsilon_s+\hbar\omega),$$

$$\mathrm{J}(\epsilon_z) = \int_0^\infty du f(u)\,\{1+f(u)\}/[\{1+f(u)\}^2 + a^5 b^2 K^2 f^2(u)\,u\,(u-\epsilon_z/a)^2], \quad a=\epsilon_s/4,\ \ b=(\epsilon_s+\hbar\omega)/4 \tag{4}$$

$$K = 16\kappa^2(2m^*)^{1/2}/\pi\hbar^4 N\hbar\omega, \quad f(u) = (1+u)\,(1+au/b),$$

where the approximation $1+\gamma\exp(\gamma) \times \mathrm{E}_1(-\gamma) \approx 1/(1+\gamma)$ is used.

From Eq. (4) we can readily obtain the half-width of the broadened level of the bottom ($\epsilon_z=0$) of the sub-band and compare the result with the experimental values in the following: "Dingle's temperature" found experimentally ($H=2\times10^4$ Oe) for InAs is 14.7°K,[4] while the calculated value is 13.5°K assuming $H=5\times10^4$ Oe, $N=4\times10^{17}$/cc. and $r_s=5\times10^{-7}$ cm. The half-width of the I.M.O.-absorption in Ge inferred from the observation is 1×10^{-3} eV[5] ($H=4.66\times10^4$ Oe) and the width due to the phonon scattering has been calculated to be 3.5×10^{-4} eV[6] ($H=5\times10^4$ Oe), while the calculated values due to the impurity scattering are 1.3×10^{-4} eV for c-band and 2.6×10^{-4} eV for v-band assuming $H=5\times10^4$ Oe, $N=2\times10^{15}$/cc, and $r_s=1\times10^{-6}$ cm. The total value 7.4×10^{-4} eV may account for the order of the experimental value.

Very recently, Kubo et al.[7] have

calculated the broadening with a method of approximating the integral equation by an algebraic cubic equation assuming the Gaussian impurity potential. Our method, which may be called the "iteration method" for solving the same integral equation, will be discussed in detail in a forthcoming paper, in comparison with the work of Kubo et al.

1) B. Davydov and I. Pomeranchuk, J. of Phys U.S.S.R. **11** (1940), 147.
2) E. N. Adams and T. D. Holstein, J. Phys. Chem. Solids **10** (1959), 254.
3) P. R. Weiss and E. Abrahams, Phys. Rev. **111** (1958), 722.
4) R. J. Sladek, Phys. Rev. **110** (1958), 817.
5) E. Burstein, G. S. Picus, R. F. Wallis and F. Blatt, Phys. Rev. **113** (1959), 15.
6) T. Ohta and T. Miyakawa, Prog. Theor. Phys. **22** (1959), 893.
T. Ohta, M. Nagae and T. Miyakawa, Prog. Theor. Phys. **23** (1960), 229.
7) R. Kubo, N. Hashitsume and S. Miyake, presented at the 15th Annual Meeting of the Physical Society of Japan, April, 1960.

Application of ρ-Meson Theory to 5 Bev π^--p Data

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The angular distribution of protons of π^--p inelastic collision at 5 Bev suggests a sharp tendency to forward scattering and this fact is favorable for the existence of some isobaric states such as Fig. 1 and Fig. 2. Therefore, using Takeda's ρ-meson[1] or the author's (1, 1) nucleoneum,[2] we have calculated Feynman diagram (Fig. 1) and applied it to the 5 Bev data.[3]

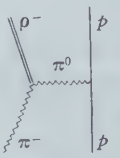

Fig. 1

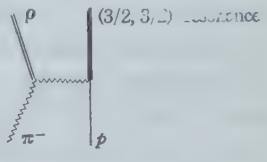

Fig. 2

Following Takeda's theory, if we put $m_\rho = 4\mu$ (m_ρ is ρ-meson mass.), the momentum p^* of the recoiled proton is estimated as

$$1.3 \leq p^* \leq 1.4 \text{ Bev/c. (c.m.s.)}$$

The remarkable difference between inelastic 2-prong and 4-prong events is the existence of the anomalous group of the recoiled protons in the region (A)

$$1.2 \leq p^* \leq 1.4 \text{ Bev/c}, \qquad (1)$$

$$-1.0 \leq \cos\theta^* \leq -0.9, \qquad (2)$$

where θ^* is the scattering angle at barycentric system. Since the Feynman diagram (Fig. 1) contributes as 2-prong inelastic scattering, we may consider that this anomalous group results from this diagram.

It is easily shown that the main angular dependent part of the differential cross section for this diagram is given by

$$|F_\rho\left(\left(\frac{k-h}{2}\right)^2\right) F_N(h^2)|^2, \quad h = k - k',$$

where k and k' represent the four vector of π and ρ, and F_ρ and F_N represent the form factor of the vertex π-π-ρ and π-N-N, respectively. Therefore,

we must consider that $F_N(-x^2)$ (or $F_\mu(-x^2)$) may damp in high energy region $x \geq 4.4\mu$ (or $x \geq 3.6\mu$) in order to get the desired angular distribution (2).

From this form factor, the total cross section becomes

$$1.7 \sim 1.9 \text{ mb.}$$

This gives rather good agreement with the experimental cross section in the region (A)

$$1.3 \text{ mb.} \quad (8 \text{ events})$$

Thus, ρ-meson theory gives a explanation to group (A) of the data, though we determined the form factors from the experiments.

Concerning to p^*, the best fit is given if we assume

$$m_\rho = 6\mu.$$

In conclusion we remark that the events of group (A) must be of the two pion production, if our description is true.

1) G. Takeda et al., Prog. Theor. Phys. **24** (1960), 529.
2) H. Nakamura, Prog. Theor. Phys. **24** (1960), 271.
3) W. B. Fowler et al., Phys. Rev. **108** (1957), 850.

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